

Outside-Obstacle Representations with All Vertices on the Outer Face

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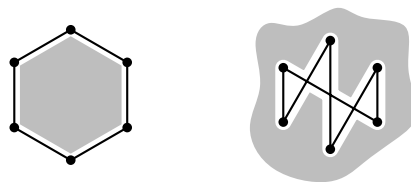
Abstract

An *obstacle representation* of a graph G consists of a set of polygonal obstacles and a drawing of G as a *visibility graph* with respect to the obstacles: vertices are mapped to points and edges to straight-line segments such that each edge avoids all obstacles whereas each non-edge intersects at least one obstacle. Obstacle representations have been investigated quite intensely over the last few years. Here we focus on *outside-obstacle representations* that use only one obstacle in the outer face of the drawing. It is known that every outerplanar graph admits such a representation [Alpert, Koch, Laison; DCG 2010]. We strengthen this result by showing that every partial 2-tree has an outside-obstacle representation. We also consider a restricted version of outside-obstacle representations where the vertices lie on a regular polygon. We construct such regular representations for partial outerpaths, partial cactus graphs, and partial grids.

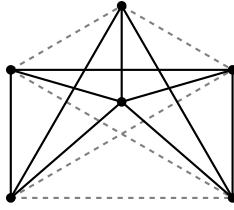
1 Introduction

Recognizing graphs that have a certain type of geometric representation is a well-established field of research dealing with, for example, interval graphs, unit disk graphs, coin graphs, and visibility graphs. Given a set \mathcal{C} of *obstacles* (in our case: polygons) and a set P of points in the plane, the *visibility graph* $G_{\mathcal{C}}(P)$ has a vertex for each point in P and an edge pq for any two points p and q in P that can *see* each other, that is, the line segment \overline{pq} that connects p and q does not intersect any obstacle in \mathcal{C} . An *obstacle representation* of a graph G consists of a set \mathcal{C} of obstacles in the plane and a mapping of the vertices of G to a set P of points such that $G = G_{\mathcal{C}}(P)$. Drawing the edges of the visibility graph as straight-line segments allows us to differentiate between two types of obstacles: *outside* obstacles lie in the outer face of the drawing, and *inside* obstacles lie in the complement of the outer face; see Fig. 1.

Every graph trivially admits an obstacle representation: take an arbitrary drawing and “fill” each face with an obstacle. However, this can lead to a large number of obstacles. Hence, it makes sense to consider the optimization problem of finding an obstacle representation



■ **Figure 1** Two representations of C_6 : with an inside obstacle (left) and an outside obstacle (right).



■ **Figure 2** The wheel graph W_6 admits an outside-obstacle representation – but not a convex one (see full version [6]). Non-edges are dashed.

with the minimum number of obstacles. For a graph G , the *obstacle number* $\text{obs}(G)$ is the smallest number of obstacles that suffice to represent G as a visibility graph.

In this paper, we focus on *outside-obstacle representations*, that is, obstacle representations with a single outside obstacle and without any inside obstacles. For such a representation, it suffices to specify the position of the vertices. The outside obstacle is simply the whole outer face of the straight-line drawing of the graph. We also consider three special types: In a *convex outside-obstacle representation*, the vertices must be in convex position; in a *circular outside-obstacle representation*, the vertices must lie on a circle; and in a *regular outside-obstacle representation*, the vertices must form a regular n -gon.

In general, the class of graphs representable by outside obstacles is not closed under taking subgraphs, but the situation is different for graphs admitting an outside-obstacle representation that is *reducible*, meaning that all of its edges are incident to the outer face:

► **Observation 1.** *If a graph G admits a reducible outside-obstacle representation, then every subgraph of G also admits such a representation.*

Previous Work. The notion of the obstacle number of a graph has been introduced by Alpert et al. [1]. They also introduced inside-obstacle representations, i.e., representations without an outside-obstacle. They characterized the class of graphs that have an inside-obstacle representation with a single convex obstacle and showed that every outerplanar graph has an outside-obstacle representation. They showed, for any $m \leq n$, that $\text{obs}(K_{m,n}^*) \leq 2$, where $K_{m,n}^*$ with $m \leq n$ is the complete bipartite graph minus a matching of size m . They also proved that $\text{obs}(K_{5,7}^*) = 2$. Pach and Sariöz [10] showed that $\text{obs}(K_{5,5}^*) = 2$. More recently, Berman et al. [3] suggested some necessary conditions for a graph to have obstacle number 1, which they used to find a *planar* 10-vertex graph that has no 1-obstacle representation.

Obviously, any n -vertex graph has obstacle number $O(n^2)$. Balko et al. [2] improved this to $O(n \log n)$. For the lower bound, Dujmović and Morin [5] showed that there are n -vertex graphs whose obstacle number is $\Omega(n/(\log \log n)^2)$, improving on previous results [1, 8, 9].

Chaplick et al. [4] proved that the class of graphs with an inside-obstacle representation is incomparable with the class of graphs with an outside-obstacle representation. They showed that any graph with at most seven vertices has an outside-obstacle representation, which does not hold for a specific graph with eight vertices.

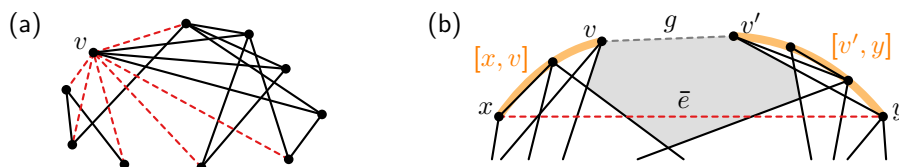
Our Contribution. We first establish two combinatorial conditions for convex outside-obstacle representations (see Section 2) that we later use to establish our main results. In particular, we introduce a necessary condition that can be used to show that a given graph does *not* admit a convex representation as, e.g., the graph in Fig. 2. We construct *regular* reducible outside-obstacle representations for outerpaths, grids, and cacti; see Section 3. Finally, we strengthen the result of Alpert et al. [1] about outside-obstacle representations of

outerplanar graphs by showing that every (partial) 2-tree admits a reducible outside-obstacle representation with all vertices on the outer face; see Section 4. We remark that outerplanar graphs and series-parallel graphs are partial 2-trees.

2 Conditions for Convex Outside-Obstacle Representations

We start with a sufficient condition. Suppose that we have a convex outside-obstacle representation of a graph G . Let σ be the clockwise circular order of the vertices of G along the convex hull. If all neighbors of a vertex v of G are consecutive in σ , we say that v has the *consecutive-neighbors property*, which implies that all non-edges incident to v trivially intersect the outer face in the immediate vicinity of v ; see Fig. 3a.

► **Lemma 2** (Consecutive-neighbors property). *For a graph G , a circular vertex order σ admits a convex outside-obstacle representation if a subset of $V(G)$ that covers all non-edges has the consecutive-neighbors property.*



■ **Figure 3** (a) Vertex v has the consecutive-neighbors property; (b) gap g is a candidate gap for \bar{e} .

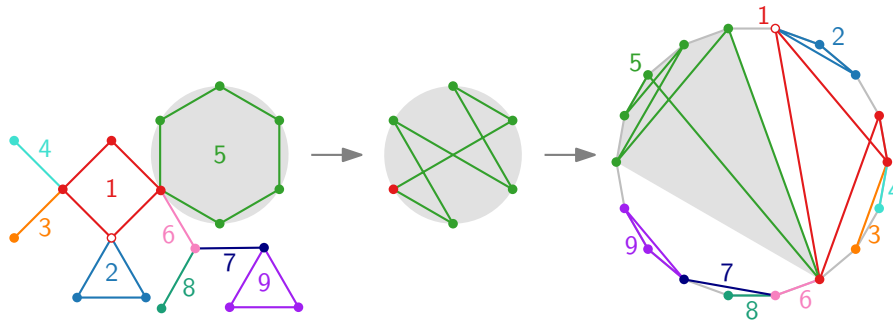
Next, we derive a necessary condition. For any two consecutive vertices v and v' on the convex hull that are not adjacent in G , we say that the line segment $g = \overline{vv'}$ is a *gap*. Then the *gap region* of g is the inner face of $G + \overline{vv'}$ incident to g ; see the gray region in Fig. 3b. (We consider the gap region open, but add to it the relative interior of the line segment $\overline{vv'}$, so that the non-edge vv' actually intersects its own gap region.) Observe that each non-edge $\bar{e} = xy$ that intersects the outer face has to intersect some gap region in an outside-obstacle representation. For vertices a and b , the set $[a, b] \subseteq V(G)$ consists of a and b and all vertices that succeed a and precede b in σ . Suppose that g lies between x and y with respect to σ , that is, $[v, v'] \subseteq [x, y]$. We say that g is a *candidate gap* for \bar{e} if there is no edge that connects a vertex in $[x, v]$ and a vertex in $[v', y]$. (Otherwise \bar{e} cannot intersect the gap region of g .)

► **Lemma 3** (Gap condition). *For a graph G , a circular vertex order σ admits a convex outside-obstacle representation only if there exists a candidate gap for each non-edge of G .*

It remains an open problem whether the gap condition is also sufficient. We can use the gap condition for no-certificates. To this end, we derived a SAT formula from the following expression, which checks the gap condition for every non-edge of a graph G :

$$\bigwedge_{xy \notin E(G)} \left[\bigvee_{v \in [x, y]} \left(\bigwedge_{u \in [x, v], w \in (v, y]} uw \notin E(G) \right) \vee \bigvee_{v \in [y, x]} \left(\bigwedge_{u \in [y, v], w \in (v, x]} uw \notin E(G) \right) \right]$$

We have used this formula to test whether all small cubic graphs (with up to 16 vertices) admit convex outside-obstacle representations. The only counterexample we found was the Petersen graph. The so-called Blanusa snarks, the Pappus graph, the dodecahedron, and the generalized Petersen graph $G(11, 2)$ satisfy the gap condition. The latter three graphs do admit convex outside-obstacle representations [7]. This motivates the following conjecture.



■ **Figure 4** Constructing a reducible regular outside-obstacle representation of a cactus.

► **Conjecture 4.** *Every connected cubic graph except the Peterson graph admits a convex outside-obstacle representation.*

The smallest graph (and only graph with six vertices) that does not satisfy the gap condition is the wheel graph W_6 with six vertices (see the full version [6]). Obviously, W_6 does not admit a *convex* outside-obstacle representation, but it does admit a (non-convex) outside-obstacle representation; see Fig. 2.

3 Regular Outside-Obstacle Representations

In this section, we show that some graph classes admit regular outside-obstacle representations. A *cactus* is a connected graph where every edge is contained in at most *one* simple cycle. An *outerpath* is an outerplanar graph that admits a drawing whose weak dual is a path. The constructions for the following result are rather simple; see Figs. 4–6.

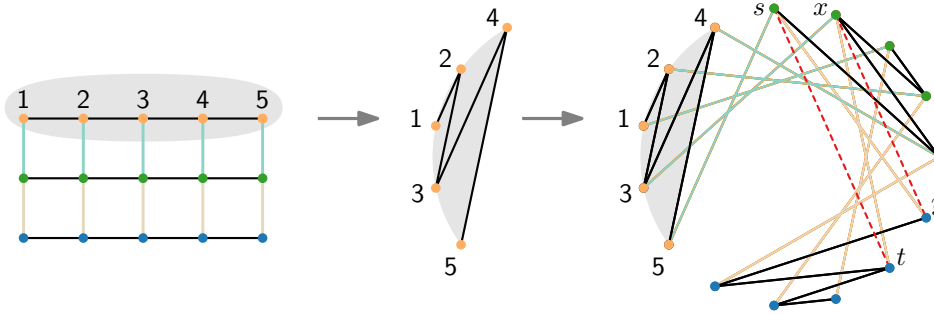
► **Theorem 5.** *The following graphs have reducible regular outside-obstacle representations:*
1. every cactus; **2.** every grid; **3.** every outerpath.

Proof sketch. We sketch our algorithms. For correctness and reducibility, see [6].

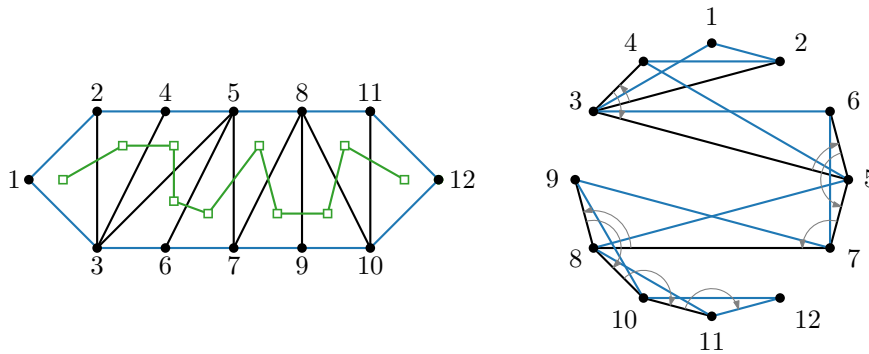
1. Let G be a cactus, and let T be the *block-cut tree* of G , which has a vertex for each *block* (i.e., a biconnected component) and for each cut vertex. There is an edge in the block-cut tree for each pair of a block and a cut vertex that belongs to it. We root T in an arbitrary block vertex. We construct a drawing of G on a circle, starting with the root block and then inserting the other blocks in the order of a BFS traversal of T ; see Fig. 4. We insert the vertices of a block B as an interval between the cut vertex that connects B to its parent in T and its clockwise successor in the circular order. The resulting drawing has the consecutive-neighbors property. Hence, by Lemma 2, it is an outside-obstacle representation.

2. Given the graph $P_k \times P_\ell$ of a square $k \times \ell$ grid, we order the vertices of each copy of the path P_k in a zig-zag manner as shown in Fig. 5. We place the copies one after the other around the circle such that the vertices of each copy form an interval.

3. Let G be an outerpath with n vertices. Since our representation will be reducible, we can assume that G is a *maximal outerpath*, i.e., for any vertex pair $\{u, v\}$, $G + uv$ is not outerplanar. Let $\langle v_1, v_2, \dots, v_n \rangle$ be a *stacking order* of G , that is, for each $i \in \{3, \dots, n\}$, the graph $G_i = G[v_1, v_2, \dots, v_i]$ is a maximal outerpath. Vertex v_j ($3 < j < n$) is incident to an inner edge $v_i v_j$. We place v_j cyclically next to v_i , avoiding the (empty) arc of the circle that corresponds to the previous inner edge; see Fig. 6. ◀



■ **Figure 5** Constructing a reducible regular outside-obstacle representation of the grid $P_5 \square P_3$.



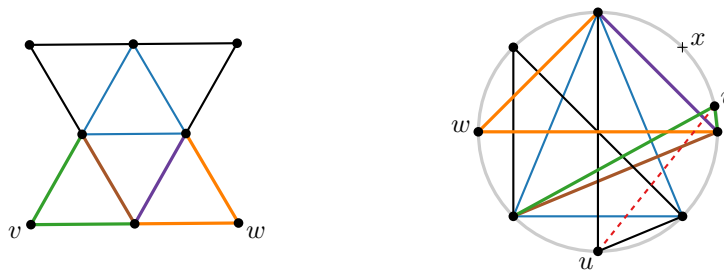
■ **Figure 6** A maximal outerpath and its reducible regular outside-obstacle representation with inner edges (black), outer edges (blue), weak dual (green). Vertices are numbered in stacking order.

Our representations for cacti and outerpaths depend only on the vertex order rather than the exact positions. Hence, for such graphs every cocircular point set is *universal*, i.e., every set of n points on a circle can be used for the vertices of an outside-obstacle representation.

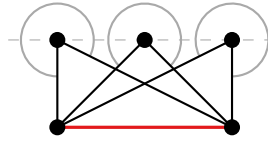
Every graph with up to six vertices – except for the graph in Fig. 2 – admits a regular outside-obstacle representation [6]. The 8-vertex outerplanar graph in Fig. 7, however, does not admit any regular outside-obstacle representation [6].

4 Outside-Obstacle Representations for Partial 2-Trees

The graph class of 2-trees is recursively defined as follows: K_2 is a 2-tree. A graph obtained from a 2-tree G by adding a new vertex x with exactly two neighbors u, v that are adjacent



■ **Figure 7** An outerplanar graph G' and a circular outside-obstacle representation of G' . The dashed red non-edge uv will stop intersecting the outer face of G' if we move v towards the point x .



■ **Figure 8** Step (1) in the proof of Theorem 6.

in G is a 2-tree. We say that x is *stacked* on the edge uv . The edges xu and xv are called the *parent edges* of x . For the full proof of the following theorem, see [6].

► **Theorem 6.** *Every 2-tree admits a reducible outside-obstacle representation with all vertices on the outer face.*

Proof sketch. Every 2-tree T can be constructed through the following iterative procedure: (1) We start with one edge, called the *base* edge and mark its vertices as *inactive*. We stack any number of vertices onto the base edge and mark them as *active*. During the entire procedure, every present vertex is marked either as active or inactive. Moreover, once a vertex is inactive, it remains inactive for the remainder of the construction. (2) We pick one active vertex v and stack any number of vertices onto each of its two parent edges. All the new vertices are marked as active and v is marked as inactive. (3) If there are active vertices remaining, repeat step (2). We construct a drawing of T by geometrically implementing this iterative procedure, so that after every step of the algorithm the present part of the graph is realized as a straight-line drawing satisfying the following invariants:

- (i) Each vertex v not incident to the base edge is associated with an open circular arc C_v that lies completely in the outer face and whose endpoints belong to the two parent edges of v . Moreover, v is located at the center of C_v and the parent edges of v are below v .
- (ii) Each non-edge passes through the circular arc of at least one of its incident vertices.
- (iii) For each active vertex v , the region R_v enclosed by C_v and the two parent edges of v is *empty*, meaning that R_v is not intersected by any edges, vertices, or circular arcs.
- (iv) Every vertex is incident to the outer face.

It is easy to see that once the procedure terminates with a drawing that satisfies invariants (i)–(iv), we have indeed obtained the desired representation (in particular, the combination of invariants (i) and (ii) implies that each non-edge passes through the outer face).

Construction. To carry out step (1), we draw the base edge horizontally and place the stacked vertices on a common horizontal line above the base edge, see Fig. 8. Circular arcs that satisfy the invariants are now easy to define. Suppose we have obtained a drawing Γ of the graph obtained after step (1) and some number of iterations of step (2) such that Γ is equipped with a set of circular arcs satisfying the invariants (i)–(iv). We describe how to carry out another iteration of step (2) while maintaining the invariants. Let v be an active vertex. By invariant (i), both parent edges of v are below v . Let e_ℓ and e_r be the left and right parent edge, respectively. Let $\ell_1, \ell_2, \dots, \ell_i$ and r_1, r_2, \dots, r_j be the vertices stacked onto e_ℓ and e_r , respectively. We refer to $\ell_1, \ell_2, \dots, \ell_i$ and r_1, r_2, \dots, r_j as the *new* vertices; the vertices of Γ are called *old*. We place all the new vertices on a common horizontal line h that intersects R_v above v , see Fig. 9. The vertices $\ell_1, \ell_2, \dots, \ell_i$ are placed inside R_v , to the right of the line $\overline{e_\ell}$ extending e_ℓ . Symmetrically, r_1, r_2, \dots, r_j are placed inside R_v , to the left of the line $\overline{e_r}$ extending e_r . We place $\ell_1, \ell_2, \dots, \ell_i$ close enough to e_ℓ and r_1, r_2, \dots, r_j

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