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Master thesis

Optimal Straight-Line Drawings of Non-Planar Graphs

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Contents

Introduction

Graph Drawing is an interesting subfield of Graph Theory. It is related to different aspects of science such as social network analysis, cartography, linguistics, and bioinformatics.

Why it is so useful to research this area of mathematics and computer science? Graphs are used for solving different problems. Computational problems in industry are such. For example: traffic organization, social relations, artificial intelligence and so on. One real-world problem solved by graph theory is finding the shortest path between two given nodes. Shortest path algorithms are widely used due to their generality. We can find an optimal solution for given problem using graphs.

One of the questions in this field is: which drawing of graph is the best for a given application. An *(undirected) graph* is a pair $G = (V, E)$ where V is a set of vertices and $E \subseteq \{\{u, v\} \subseteq V \mid u \neq v\}$ is a set of edges.

The drawing of a graph is a pictorial representation of the vertices and edges of a graph.

An abstract graph can have many different concrete drawings. For instance, the complete graph K_4 with four vertices can be drawn in many ways, see Fig. 1. But, for the sake of convenience sometimes we shall identify a graph with its fixed drawing.

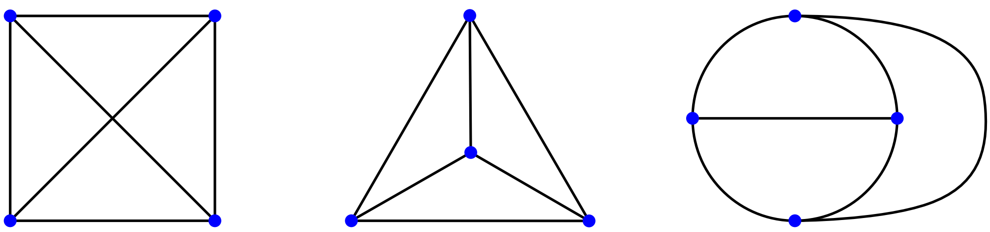


Figure 1: Different drawings of K_4.

A drawing of a given graph can be evaluated by many different quality measures depending on concrete purpose of the visual representation and different layouts optimizing values of these measures are devised. For example, Hoffmann et al. [1] compared different styles of graph drawing and Marc van Kreveld [2] studied right-angled crossing (RAC) drawings of planar graphs.

In this paper we shall consider only finite simple graphs, that is the graphs with finitely many vertices and edges and without loops and multiple edges. And all graph drawings are crossing-free, that is drawings without intersection of its edges (excluding the only case of an intersection of endpoints of two edges incident to the same vertex, at that vertex).

Basic Information about Planar Graphs. *Planar graph* is a graph that can be drawn on a plane without intersection of edges. Many people investigated it. There was a question which graphs are planar. In 1930 Kazimierz Kuratowski proved that a graph is planar iff neither K_5 nor $K_{3,3}$ is its minor, see Fig. 2 (A graph H is called a *minor* of the graph G if H can be obtained from G by deleting edges and vertices and by contracting edges).

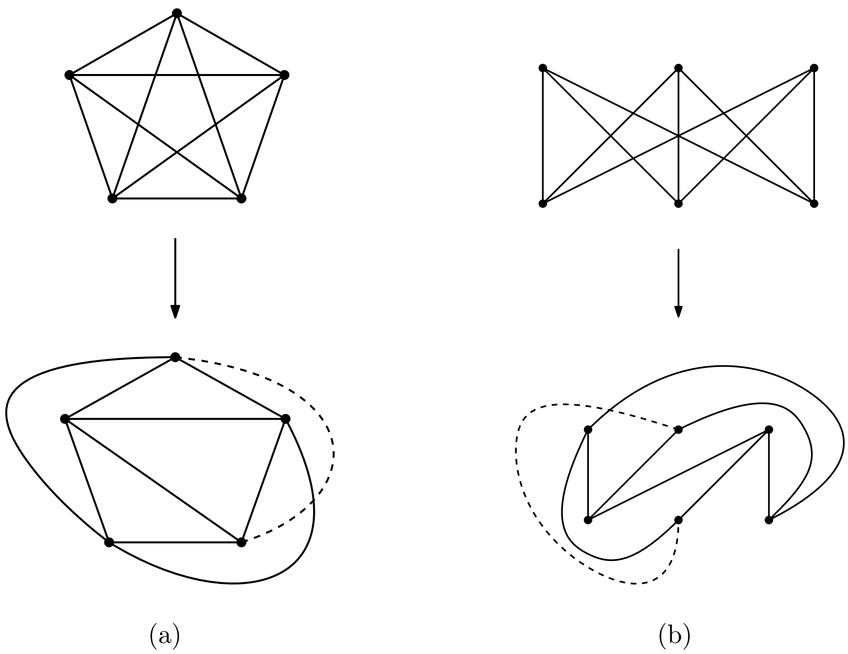


Figure 2: Non-planar graphs: (a) K_5 and (b) $K_{3,3}$.

Also we can quickly test whether a given graph G is planar or not. Hopcroft and Tarjan (1974) showed that it takes $O(n)$ time to test whether G is planar, where n is the number of nodes.

The following theorem is fundamental for graph drawing. Klaus Wagner (1936), Fáry (1948), and Stein (1951) proved independently that every planar graph can be drawn with straight-line edges. But in the same year 1936, Koebe showed that every planar graph can be represented as the contact graph of disks (coin graph). Koebe's circle packing theorem implies Wagner's. Indeed, if we connect the centers of disks in a coin graph G by straight-line segments, we obtain a straight-line drawing of G, see Fig. 3.

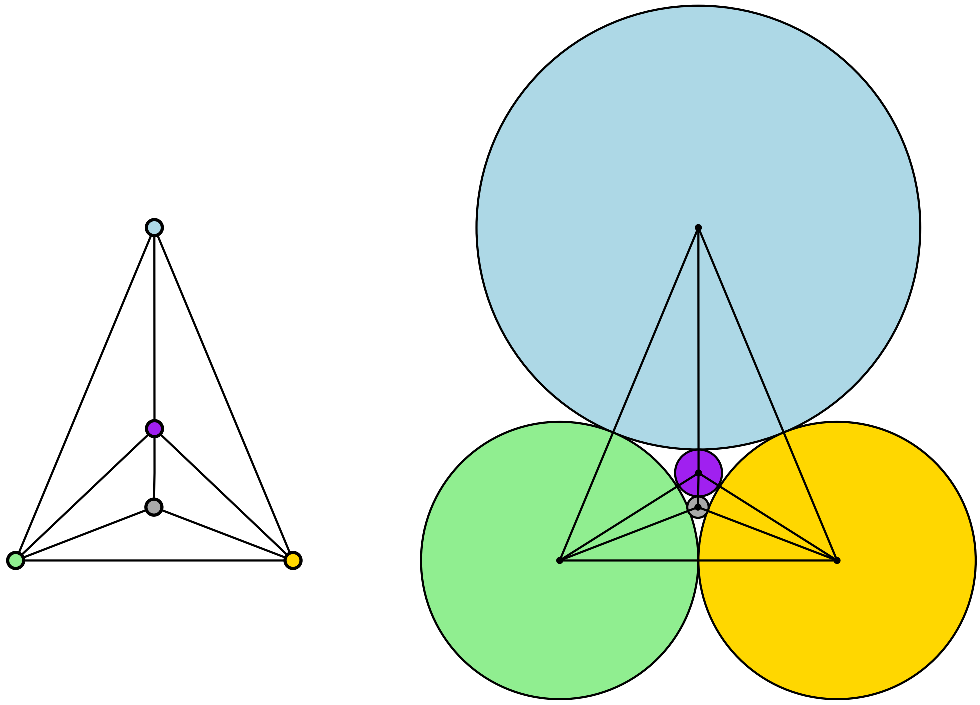


Figure 3: A circle packing for a five-vertex planar graph.

Unfortunately, coin graphs are not always applicable for real visual representations, because they may contain very small circles, which are hard to see.

From the other hand, in 1990 Schnyder showed that any planar graph with $n \geq 3$ vertices has a straight-line drawing with vertices placed at the nodes of a grid of size $(n-2) \times (n-2)$.

So, straight-line drawings of planar graphs are rather investigated. Dealing with non-planar graphs, first of all we remark that each graph can be drawn straight-line in space ($\mathbb{R}^3$) without crossings. In fact, any placement of vertices with no four at one plane generates such a drawing. Such generic drawings do not reveal any structural information about the graph.

My master thesis consists of an introduction, three chapters and conclusions.

The first chapter describes some information about drawing graphs on few objects, including straight-line drawings on few lines or few planes. There are shown some examples of optimal drawings of certain graphs. There is shown the connection of this paper with the work of various scientists.

The second chapter contains information about platonic graphs. There are shown the minimum numbers of straight lines which cover all vertices of platonic graphs (π_2^1 and π_3^1). It is proved that these values are optimal.

The third chapter contains two examples of graphs with unbounded π_2^1. The first example Ex. 3.1 show that the value of $\pi_2^1(G)$ is unbounded on a class of graphs of treewidth 3, and the second one Ex. 3.2 show that the value of $\pi_2^1(G)$ is unbounded on a class of graphs of treewidth 4 and maximal vertex degree 9.

The purpose of this project is to investigate graphs with unbounded π_2^1. This topic has many open problems and there are shown example with bounded maximum vertex degree. Also there is investigated placing vertices on few straight lines in two- and three-dimensional spaces for Platonic graphs, which are 1-skeletons of Platonic solids. It interested me because it is connected to the new Project Sigma, which was created by Lviv and German mathematicians. It includes Covering of Platonic Graphs with Few Circles.

Chapter 1

Drawing Graphs on Few Objects

Three-dimensional graph drawing on the grid has been surveyed by Wood [3] and by Dujmović and Whitesides [4]. For example, Dujmović [5], improving on a result of Di Battista et al. [6], showed that any planar graph can be drawn into a three-dimensional grid of volume $O(n \log n)$.

One of the problems is to study drawings whose edge sets are represented (or covered) by as few objects as possible. The type of objects that have been used are straight-line segments [7, 8] and circular arcs [9]. The idea behind this objective is to keep the visual complexity of a drawing low for the observer. For example, consider dodecahedron, see Fig. 1.1. It has 20 vertices and 30 edges.

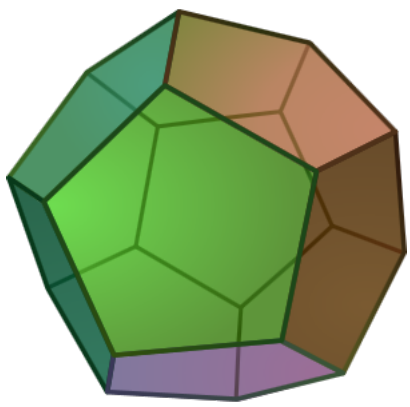


Figure 1.1: [10] A dodecahedron in $\mathbb{R}^3$.

Schulz [9] showed how to draw the dodecahedron using just 10 circular arcs, which is optimal, see Fig. 1.2(a). Scherm in her bachelor thesis [11] showed how to draw the dodecahedron using 10 straight lines, see Fig. 1.2(b). Can we reduce this number using a few types of objects? For example, Scherm showed a drawing of the dodecahedron using just 8 objects: 3 circles and 5 straight lines, see Fig. 1.2(c). Combining several types of objects, we can reduce the number of these objects on which we draw a graph. It is a new area for research: drawing graphs on different objects.

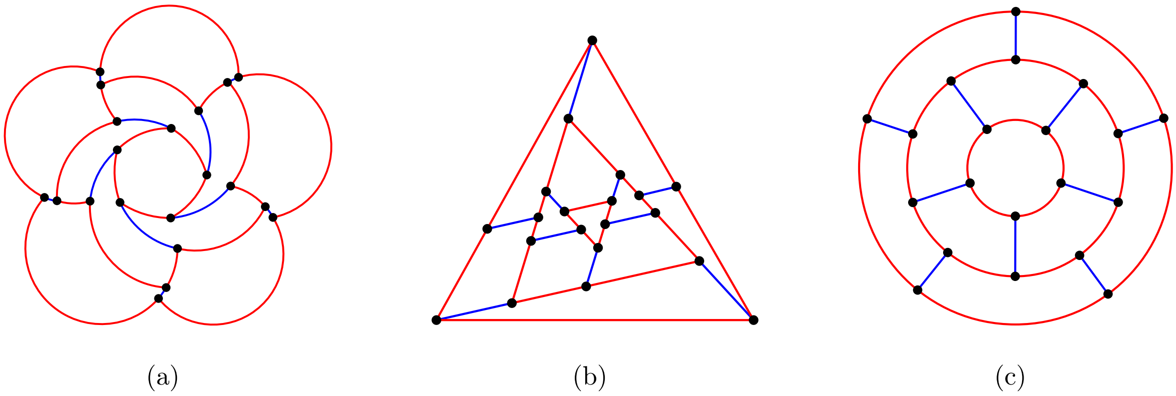


Figure 1.2: Planar drawings of the dodecahedron: (a) on 10 circular arcs; (b) on 10 straight lines; (c) on 3 circles and 5 straight lines.

Drawing Graphs on Few Lines or Few Planes. From now all considered graph drawings are straight-line.

Another problem is to study graph drawings whose edges or only vertices are covered by as few straight lines or planes as possible. Chaplick, Fleszar, Lipp, Ravsky, Verbitsky, and Wolff [12] have introduced this problem, which is defined as follows.

Definition 1.1. Let $1 \le l < d$, and let G be a graph. The *l-dimensional affine cover* number of G in $\mathbb{R}^d$, denoted by $\rho^l_d(G)$, is defined as the minimum number of l-dimensional planes in $\mathbb{R}^d$ such that G has a drawing that is contained in the union of these planes. $\pi^l_d(G)$, the *weak* l-dimensional affine cover number of G in $\mathbb{R}^d$, is defined similarly to $\rho^l_d(G)$, but under the weaker restriction that the vertices (and not necessarily the edges) of G are contained in the union of the planes. Finally, the *parallel* affine cover number, $\bar{\pi}^l_d(G)$, is a restricted version of $\pi^l_d(G)$, in which the planes are parallel. They consider only straight-line and crossing-free drawings.

As an example, Fig. 1.3 shows how to draw K_6 in $\mathbb{R}^3$ such that all edges lie only on four planes. This is optimal, that is, $\rho^2_3(K_6) = 4$.

Chaplick et al. remarked that for any suitable combination of l and d, it holds that $\pi^l_d(G) \le \bar{\pi}^l_d(G)$ and $\pi^l_d(G) \le \rho^l_d(G)$. Also for any graph G, if $l' \le l$ and $d' \le d$ then $\pi^l_d(G) \le \pi^{l'}_{d'}(G)$, $\rho^l_d(G) \le \rho^{l'}_{d'}(G)$, and $\bar{\pi}^l_d(G) \le \bar{\pi}^{l'}_{d'}(G)$.

They also obtained the following collapsing theorem.

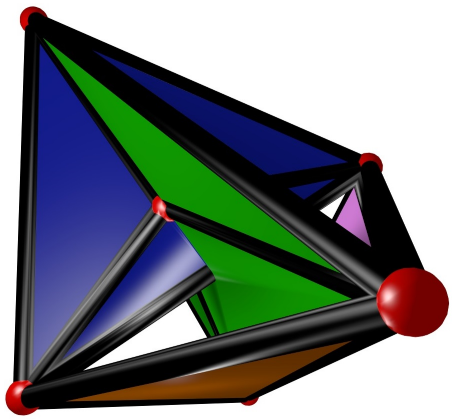


Figure 1.3: K_6 can be drawn straight-line and crossing-free on four planes [12].

Theorem 1.1. *For any integers $1 \le l < 3 \le d$ and for any graph G, it holds that $\pi_d^l(G) = \pi_3^l(G)$, $\bar{\pi}_d^l(G) = \bar{\pi}_3^l(G)$, and $\rho_d^l(G) = \rho_3^l(G)$.*

It means that Project Rho is about the following nine characteristics: π_2^1, π_3^1, π_3^2, $\bar{\pi}_2^1$, $\bar{\pi}_3^1$, $\bar{\pi}_3^2$, ρ_2^1, ρ_3^1, ρ_3^2.

In my master thesis I focus on weak affine cover numbers of some classes of planar graphs. So I should mention some facts which were proved before.

Chaplick et al. related the affine cover numbers to standard combinatorial characteristics of graphs and to parameters that have been studied earlier in graph drawing.

A *linear forest* is a forest whose connected components are paths. The *linear vertex arboricity* $\operatorname{lva} G$ of a graph G equals the smallest size r of a partition $V(G) = V_1 \cup \cdots \cup V_r$ such that every V_i induces a linear forest.

They proved such theorem:

Theorem 1.2. *For any planar graph G, it holds that $\pi_3^1(G) = \operatorname{lva}(G)$.*

This characterization implies that $\pi_3^1(G)$ is linearly related to the chromatic number of the graph G.

Chapter 2

Drawing Platonic Graphs

In three-dimensional space, a *Platonic solid* is a regular, convex polyhedron. It is constructed by congruent regular polygonal faces with the same number of faces meeting at each vertex. Five solids meet those criteria, see Fig. 2.1: *Tetrahedron* (a), *Hexahedron* (or *Cube*) (b), *Octahedron* (c), *Dodecahedron* (d) and *Icosahedron* (e).

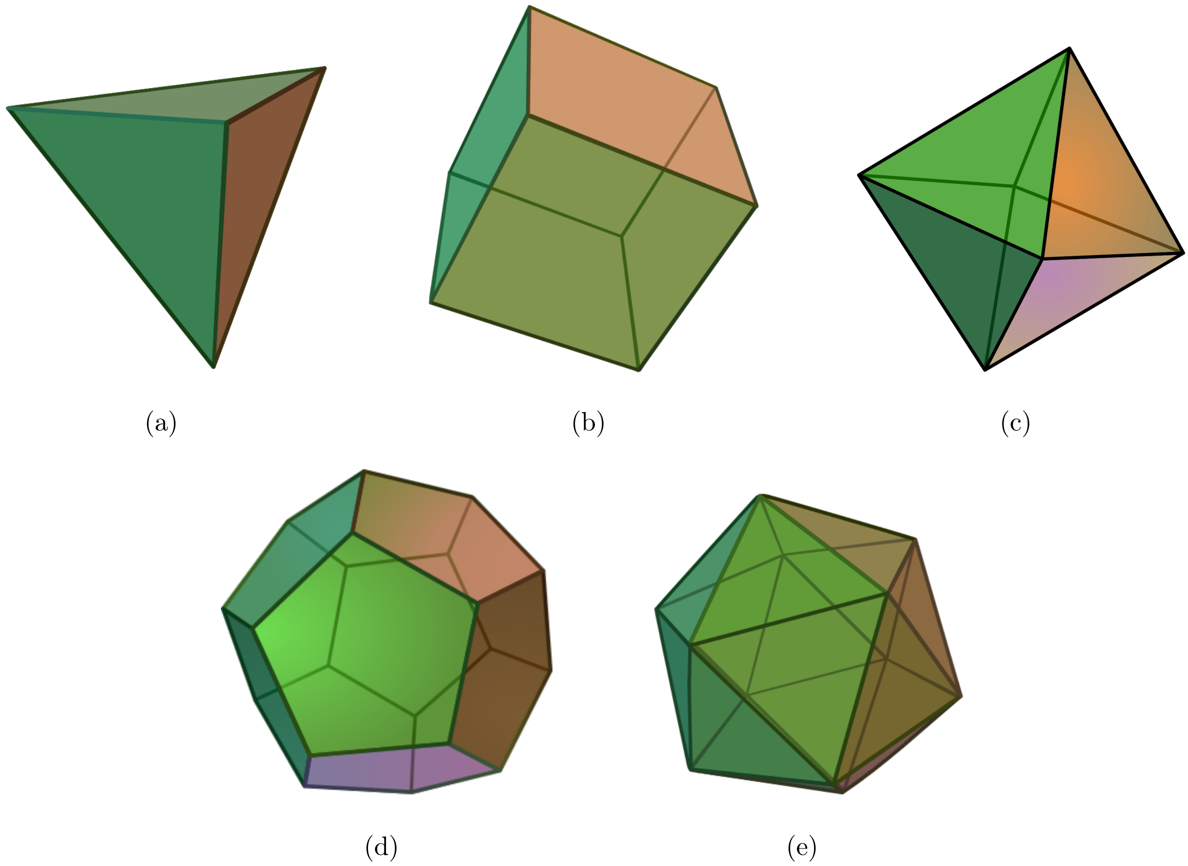


Figure 2.1: Platonic solids in $\mathbb{R}^3$ [10]: (a) Tetrahedron; (b) Cube; (c) Octahedron; (d) Dodecahedron; (e) Icosahedron.

A *Platonic graph* is a graph that has one of the Platonic solids as its skeleton. All of them are regular, polyhedral, and also Hamiltonian graphs.

A *regular graph* is a graph where every vertex has the same degree i.e. the same number of neighbors. Obviously, Tetrahedron, Cube and Dodecahedron are 3-regular (see Fig. 2.2 (a), (c), (e) accordingly), Octahedron is 4-regular and Icosahedron is 5-regular (see Fig. 2.3 (a), (c) accordingly).

A *polyhedral graph* is the undirected graph formed from the vertices and edges of a convex polyhedron. The polyhedral graphs are the 3-vertex-connected planar graphs.

A connected graph G is said to be *k-vertex-connected* (or *k-connected*) if it has more than k vertices and remains connected whenever fewer than k vertices are removed.

A *Hamiltonian cycle* is a graph cycle (i.e., closed loop) through a graph that visits each vertex exactly once. A graph possessing a Hamiltonian cycle is said to be a *Hamiltonian graph*.

Because these graphs are planar we can find values π_2^1 and π_3^1 for them.

2.1 Placing Vertices of Platonic Graphs on Few Lines in $\mathbb{R}^2$ (π_2^1)

Theorem 2.1. *(a)* $\pi_2^1(G) = 2$*, if* G *is Tetrahedron, Cube or Dodecahedron;*

(b) $\pi_2^1(G) = 3$*, if* G *is Octahedron or Icosahedron.*

Proof. (a) See Fig. 2.2

(b) Let G be Octahedron or Icosahedron. Then $\pi_2^1(G) \leq 3$, see Fig. 2.3.

On the other hand $\pi_2^1(G) \geq 3$. Indeed, assume that there exists a plane drawing of G such that all vertices of G are covered by two straight lines l_1 and l_2. Since the outer face of G is a triangle, one of these straight lines, say l_1 contains two vertices of the outer face. Thus l_1 contains no other vertices of G, and all of them are placed on l_2. But this is impossible since the subgraph, induced on these vertices is not a linear forest (in fact, it even contains a triangle), a contradiction.

Hence, $\pi_2^1(G) = 3$.

□

Corollary 2.1. $\pi_3^1(G) = 2$*, if* G *is Tetrahedron, Cube or Dodecahedron.*

Proof. This follows from the fact that $\pi_3^1(G) \leq \pi_2^1(G)$. □

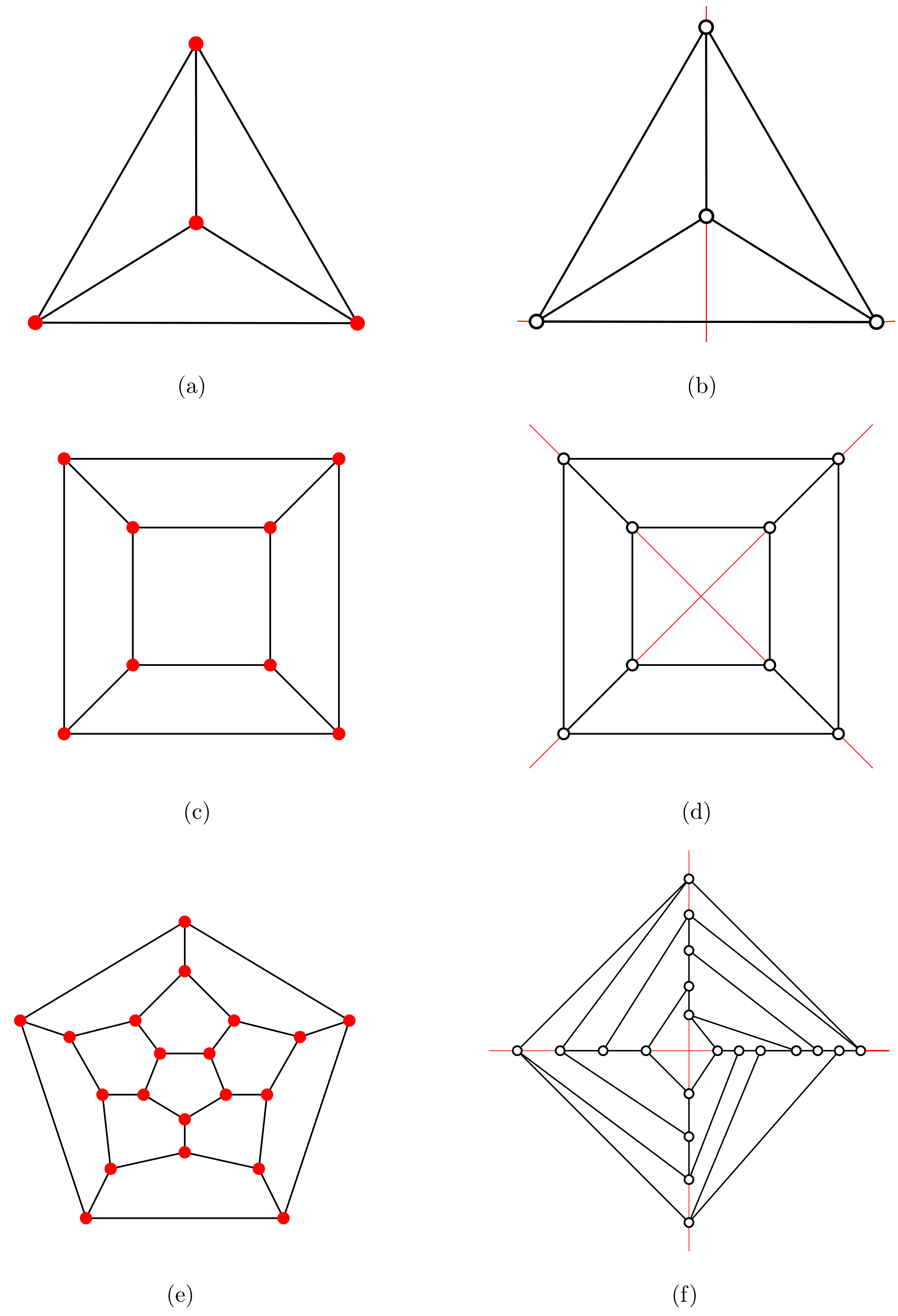


Figure 2.2: Tetrahedron graph (a) and placing its vertices on two straight lines (b);
Cube graph (c) and placing its vertices on two straight lines (d);
Dodecahedron graph (e) and placing its vertices on two straight lines (f).

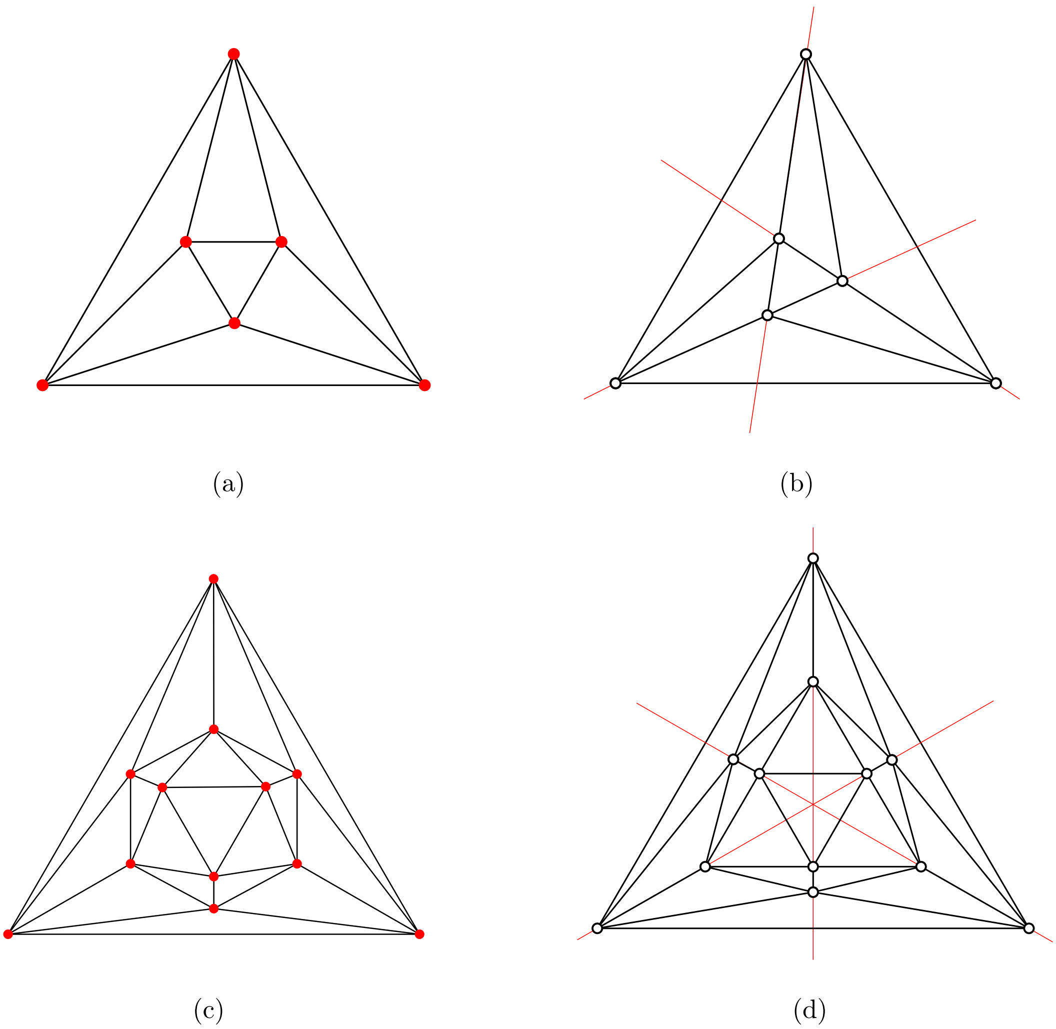


Figure 2.3: Octahedron graph (a) and placing its vertices on three straight lines (b); Icosahedron graph (c) and placing its vertices on three straight lines (d).

2.2 Placing Vertices of Platonic Graphs on Few Lines in $\mathbb{R}^3$ (π_3^1)

In the previous section we showed that $\pi_2^1(G)$ for platonic graphs equals two or three. But $\pi_3^1(G)$ for any platonic graph equals two. Corollary 2.1 shows it for Tetrahedron, Cube and Dodecahedron. So the task is to prove that for Octahedron and Icosahedron.

Theorem 2.2. $\pi_3^1(G) = 2$, *if* G *is Octahedron or Icosahedron.*

Proof. By theorem 1.2 $\pi_3^1(G) = \text{lva}(G)$. Obviously that $\text{lva}(G) = 2$ for Octahedron and Icosahedron, see Fig. 2.4. Each blue and yellow set of vertices induces a linear forest.

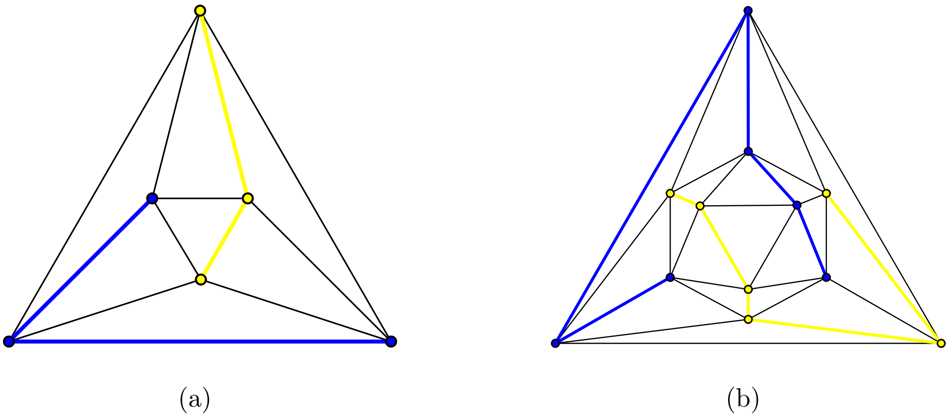


Figure 2.4: Sets of vertices which induce linear forests: (a) of Octahedron; (b) of Icosahedron.

Fig. 2.5 show drawings of Octahedron and Icosahedron on two skew lines in three-dimensional space.

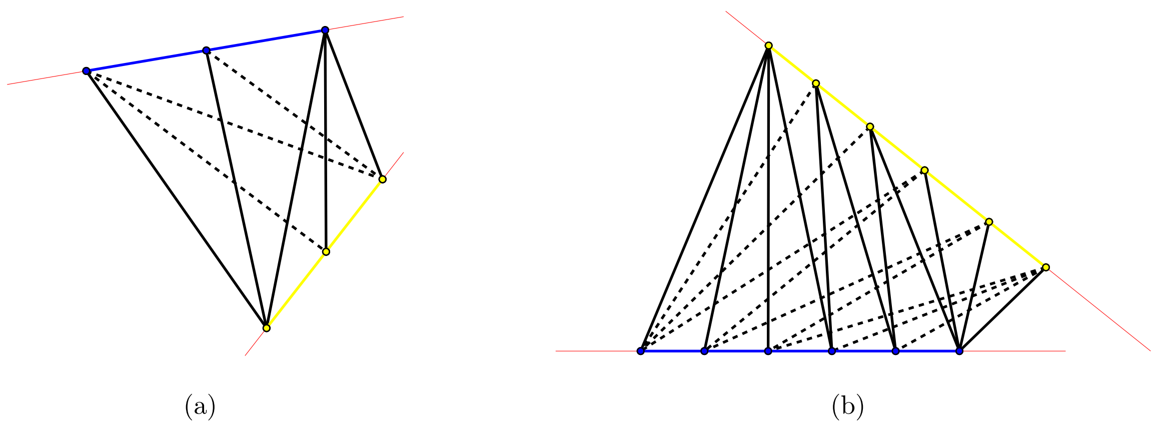


Figure 2.5: Placing vertices of Octahedron (a) and Icosahedron (b) on two skew lines in $\mathbb{R}^3$.

□

Chapter 3

Graphs with Unbounded π_2^1

Related work. All graphs considered in this section are planar, unless the otherwise is stated. For graph G, $\pi_2^1(G)$ is at most as large as $\rho_2^1(G)$ but it can be much smaller. For instance, for the nested-triangles graph $T_k = C_3 \times P_k$ shown in Fig. 3.1, $\rho_2^1(T_k) \geq n/2$, see [12, Theorem 3.10], whereas $\pi_2^1(T_k) \leq 3$. Usually it is hard to find non-trivial lower bounds for $\pi_2^1(G)$ and one of main open problems in this topic is whether $\pi_2^1(G) = o(n)$ for all graphs G, see [12, Problem 3.4]. Even with some restrictions imposed on G the problem remains open and there are only two known examples of graph families with unbounded π_2^1.

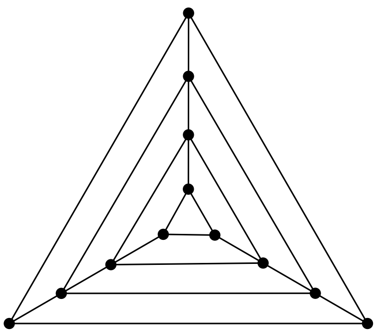


Figure 3.1: The nested-triangles graph T_k.

The first example was constructed in Section 3 of the paper "On collinear sets in straight-line drawings" [13] using iterative face triangulations, see Fig. 3.2. To describe it we need the following definitions.

A graph (non necessarily planar) is *k-connected* if it has more than k vertices and stays connected after removal of any k vertices. 3-connected planar graphs are called *polyhedral*. A planar graph G is *maximal* if adding an edge between any two non-adjacent vertices of G violates planarity. Maximal planar graphs on more than 3 vertices are 3-connected. Clearly, all facial cycles

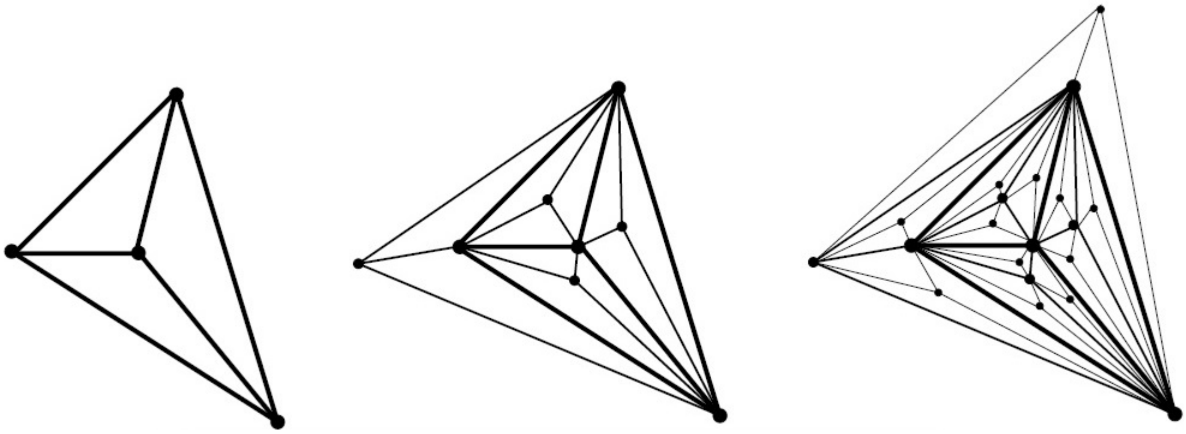


Figure 3.2: The beginning of the sequence $G_1 = K_4, G_2, G_3$, [13, Fig.1].

in such graphs have length 3. By this reason maximal planar graphs are also called *triangulations*. *Cubic* graph is a graph in which all vertices have degree three. The *circumference* of a graph G, denoted by $c(G)$, is the length of a longest cycle in G. The *shortness exponent* of a class of graphs $\mathcal{G}$ is the limit inferior of quotients $\log c(G)/\log v(G)$ over all $G \in \mathcal{G}$. Let σ denotes the shortness exponent of the class of cubic polyhedral graphs. It is known that $0.753 < \sigma \le \frac{\log 22}{\log 23} = = 0.985\ldots$. [13, Theorem 3.3] implies that for each $\varepsilon > 0$ there is a sequence of triangulations G with $\pi_2^1(G) = \Omega(n^{1-\sigma-\varepsilon})$.

The second example is used in [12, Theorem 3.3], stating that there are infinitely many triangulations G with $\Delta(G) \le 12$ and $\pi_2^1(G) \ge n^{0.01}$. I guess that 0.01 here can be replaced by $1-\sigma_{12}-\varepsilon$ for each $\varepsilon > 0$, where σ_{12} denotes the shortness exponent for the class of cubic 3-connected graphs with each face incident to at most 12 edges (this parameter is well defined by the Whitney theorem). It is known [14] that $\sigma_{12} \le \frac{\log 26}{\log 27} = 0.988\ldots$.

We are interested in examples with bounded maximum vertex degree $\Delta(G)$ because an other basic open problem in this topic, whether $\pi_2^1(G) = o(n)$ for a graph G with $\Delta(G) = O(1)$. It is even not known whether $\pi_2^1(G) = O(1)$ provided $\Delta(G) = 3$. Note that the proof of Theorem 3.3 from [12] cannot be extended to graphs of maximum vertex degree 6 because the shortness exponent of the cubic 3-connected graphs with each faces incident to at most 6 edges is known to be equal to 1 [15].

Another thread concerns boundedness of π_2^1 is graphs of bounded treewidth. Recall that a graph has threewidth at most k provided it is a subgraph of a k-tree. The class of *k-trees* (consisting of not necessarily planar graphs) is defined recursively as follows. The complete graph K_k is a k-tree; if G is a k-tree and H is obtained from G by adding a new vertex and connecting it to a k-clique of G then H is a k-tree. Observe that 1-trees are exactly trees.

A *track drawing* [16] of a graph is a plane drawing for which there are parallel lines, called *tracks*, such that every edge either lies on a track or its endpoints lie on two consecutive tracks. We call a graph *track drawable* if it has a track drawing. Observe that any tree is track drawable: two vertices are aligned on the same track iff they are at the same distance from an arbitrarily assigned root. Moreover, any outerplanar graph is track drawable [16]. In Theorem 3.5 [12] it is proved that $\pi_2^1(G) \le 2$ for each track drawable graph G. On the other hand, as is well known, the treewidth of an outerplanar graph is at most 2, and all graphs of treewidth 2 are planar but not all of them are track drawable (for example, the graph consisting of three triangles that share one edge). This suggested the following (still open) problem from [12], whether $\pi_2^1(G) = O(1)$ for all graphs of treewidth 2. This problem is also motivated by an inequality $\pi_2^1(G) \ge n/\bar{v}(G)$, where $\bar{v}(G)$ denote the maximum k such that G has a straight-line plane drawing with k collinear vertices, as Verbitsky [13, Theorem 4.5] showed that $\bar{v}(G) > n/30$ for all n-vertex graphs of treewidth at most 2. Moreover, the linear lower bound was extended to all graph G of treewidth at most 3 by Da Lozzo et al. [17].

Our contribution. Nevertheless, our Example 3.1 will show that the value of $\pi_2^1(G)$ is unbounded on a class of graphs of treewidth 3, whereas Example 3.2 will show that the value of $\pi_2^1(G)$ is unbounded on a class of graphs of treewidth 4 and maximal vertex degree 9. In both examples we recurrently construct a sequence $\{G_i\}$ of triangulations with unbounded values of $\pi_2^1(G_i)$.

We say that two plane graphs are *strongly equivalent* if they are obtainable from one another by a plane homeomorphism, and are *equivalent* if they are obtainable from one another by a plane homeomorphism, up to the choice of the outer face.

We start from a meeting with our old friend.

Example 3.1. As the base of the induction we start from a triangulation $G_1 = K_4$. Let H_1 be a *plane drawing* of G_1. At the induction step $i \ge 1$ we obtain G_{i+1} from G_i by triangulating each face of G_i by a new vertex, see Fig. 3.3.

Similarly, we obtain H_{i+1} from H_i by triangulating each *inner* face of H_i by a new vertex. The construction of G_i shows that its treewidth is 3. On the other hand, $\operatorname{tw}(G_i) \ge \min\deg G_i = 3$. Since G_i is a triangulation, it is 3-connected, so, by Whitney's theorem, all its plane embeddings are equivalent. But, independently of the choice of outer face for this equivalence, for each plane drawing d of G_i there exists a homeomorphism δ of the plane such that $\delta \circ d(G_i)$ contains H_i as

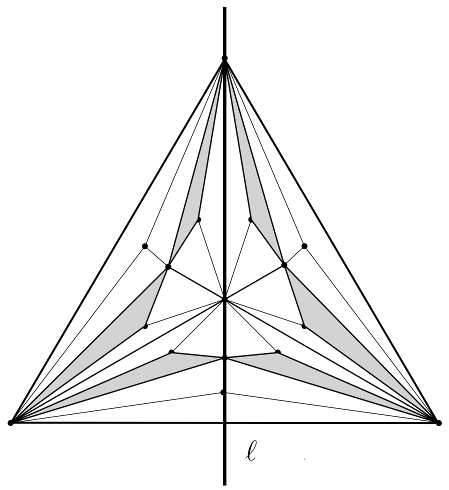


Figure 3.3: Graph G_{i+1} obtained from G_i by triangulating each face of G_i by a new vertex.

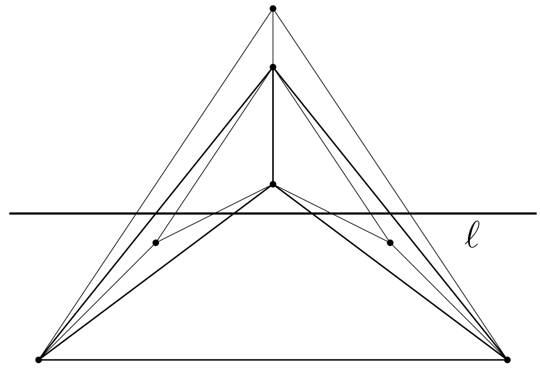


Figure 3.4: Straight line ℓ intersects all faces of the graph.

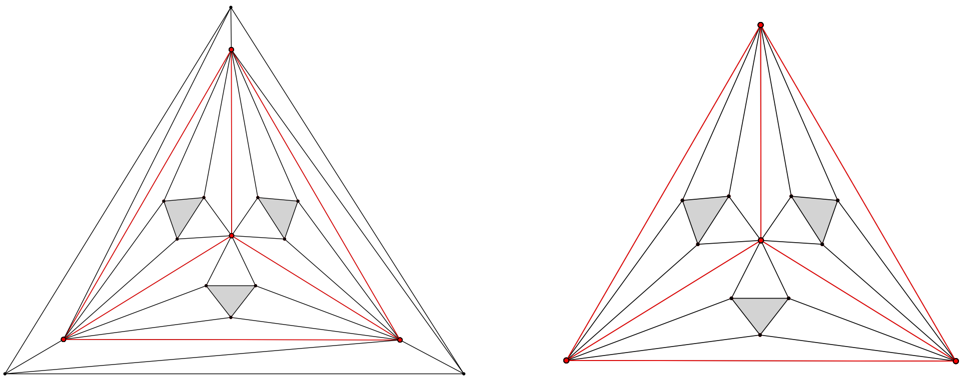
a subgraph. We need at least 2 straight lines to cover all vertices of H_1. If $i \geq 4$ then we can easily check that for each straight line ℓ containing the central vertex of the graph H_i there exists a subgraph strongly equivalent to H_{i-3} (drawn inside of one of the grey triangles at Fig. 3.3) disjoint from ℓ. This observation inductively implies that we need at least $i/3 + 1$ straight lines to cover all vertices of H_i. Thus $\pi_2^1(G_i) \geq i/3 + 1$.

On the other hand, $\bar{v}(G_i) \geq \frac{3}{4}n - 2$ for each $i \geq 2$. Indeed, we can choose a straight-line ℓ and then consecutively draw graphs $G_1, \ldots, G_{i-1}$ in such a manner that ℓ intersects all their faces, see Fig. 3.4. So at the last step we shall be able to place all vertices of G_i on ℓ, which yield the claimed lower bound.

Example 3.2. As the base of the induction we start from a graph G_1, constructed by gluing an octahedron O to each face of a regular tetrahedron T, see Fig. 3.5 (a). It has four special faces, which are disjoint with T (at the drawing they are three grey regular triangles and the outer face). Let H_1 be a *plane* graph, see Fig. 3.5 (b), with three special faces (at the drawing they are three grey regular triangles).

Assume that at the induction step $i \geq 1$ we are given a triangulation G_i and a plane graph H_i with $4 \cdot 3^{i-1}$ and 3^i special faces, respectively. We construct a triangulation G_{i+1} from G_i and a plane graph H_{i+1} from H_i by replacing each of its special faces by a copy of the graph H_1, see Fig. 3.6.

Thus the triangulation G_{i+1} and the plane graph H_{i+1} have $4 \cdot 3^i$ and 3^{i+1} special faces, respectively. Using the fact that special faces of G_{i+1} are disjoint with G_i, we can easily show that $\Delta(G_{i+1}) = 9$. Since G_i is a triangulation, it is 3-connected, so, by Whitney's theorem, all its plane



(a) (b)

Figure 3.5: Graphs G_1 (a) and H_1 (b) with special faces (grey regular triangles).

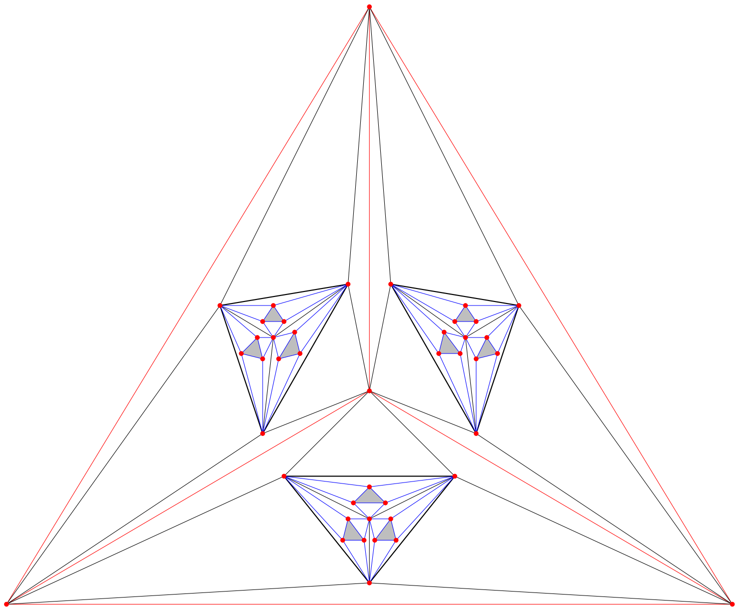


Figure 3.6: The plane graph H_2 from Example 3.2 constructed by replacing each of special faces of graph H_1 by a copy of the graph H_1.

embeddings are equivalent. But, independently of the choice of outer face for this equivalence, for each plane drawing d of G_i there exists a homeomorphism δ of the plane such that $\delta \circ d(G_i)$ contains H_i as a subgraph.

Clearly, $\mathrm{tw}(G_i) \geq \min \deg G_i = 4$. On the other hand, Fig. 3.7 shows a way to construct from K_4 a (not necessarily planar) 4-tree F_1 containing the graph G_1 as a subgraph. Remark that for each grey triangle graph inherited by F_1 from G_1 there exists a vertex adjacent to all vertices of triangle. This allows us to use the same construction for each of grey triangle graph, finally obtaining (not necessarily planar) 4-tree F_2 containing the graph G_2 as a subgraph and so forth. Thus $\mathrm{tw}(G_i) \leq 4$.

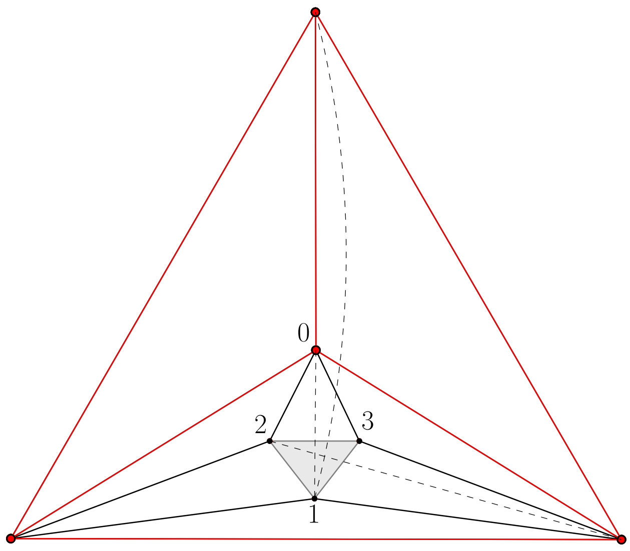


Figure 3.7: A way to construct from K_4 a 4-tree F_1 containing the graph G_1 as a subgraph.

For each i let π_i be a minimum number of straight lines needed to cover all vertices of a plane straight-line graph H_i' strongly equivalent to H_i. As the base of the induction we start from a graph H_1'. Assume that a family $\mathcal{L}$ consisting of π_1 straight lines covers all vertices of H_1'. By a *crossing* we shall call a pair (ℓ, x) where $\ell \in \mathcal{L}$ and x is a vertex of the red K_4 subgraph of H_1' (see Fig. 3.5(b)) covered by ℓ or x is the interior of an inner face of the red K_4 subgraph intersected by ℓ. Let C be the total number of crossings. Each of four vertices of red K_4 subgraph needs at least one crossing, and each of its three inner faces needs at least two crossings, because its interior contains a grey triangle, which needs at least two straight lines to cover all its vertices. Thus $C \geq 4 + 3 \cdot 2 = 10$. On the other hand, it is easy to check that each straight line can participate in at most 3 crossings. Then $3\pi_1 \geq C$. Combining the inequalities, we obtain $\pi_1 \geq 4$.

The induction step is similar. Assume that $i \geq 2$ and a family $\mathcal{L}$ consisting of π_i straight lines covers all vertices of H_i'. By a *crossing* we shall call a pair (ℓ, x) where $\ell \in \mathcal{L}$ and x is a vertex of the red K_4 subgraph of H_i' (see Fig. 3.6) covered by ℓ or x is the interior of an inner face of the red K_4 subgraph intersected by ℓ. Let C be the total number of crossings. Each of four vertices of red K_4 subgraph needs at least one crossing, and each of its three inner faces needs at least π_{i-1} crossings, because its interior contains a subgraph strongly equivalent to H_{i-1} drawn inside a grey triangle. Thus $C \geq 4 + 3\pi_{i-1}$. On the other hand, it is easy to check that each straight line can participate in at most 3 crossings. Then $3\pi_i \geq C$. Combining the inequalities, we obtain $\pi_i \geq \pi_{i-1} + 2$. Induction implies that $\pi_i \geq 2i + 2$, so $\pi_2^1(G_i) \geq 2i + 2$ too.

Conclusion

In this paper I investigated drawing graphs in $\mathbb{R}^2$ and $\mathbb{R}^3$ such that their vertices can be covered by few straight lines (π_2^1 and π_3^1). I found these values for Platonic graphs:

Platonic graph	|V|	|E|	|F|	π_2^1	π_3^1
Tetrahedron	4	6	4	2	2
Cube	8	12	6	2	2
Octahedron	6	12	8	3	2
Dodecahedron	20	30	12	2	2
Icosahedron	12	30	20	3	2

But the main result of this paper is two recurrently constructed triangulations with unbounded values of π_2^1. For the sequence $\{G_i\}$ of triangulations from Example 3.1, which has treewidth 3, it is proved that $\pi_2^1(G_i) \geq i/3 + 1$, and from Example 3.2, which has treewidth 4 and maximal vertex degree 9, it is proved that $\pi_2^1(G_i) \geq 2i + 2$.

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