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Maximum Betweenness Centrality: Approximability and Tractable Cases

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Universität Würzburg

A Centrality Problem

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- computer network
- transportation network

This network can be modeled by a graph.

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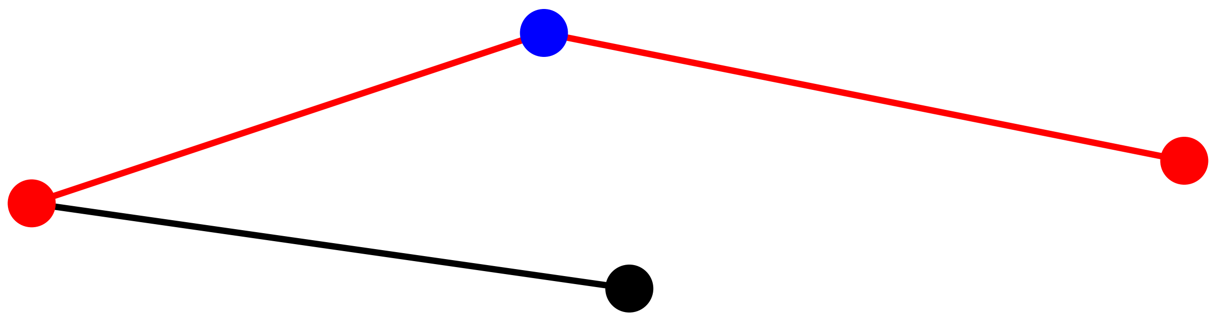
This network can be modeled by a graph.

- Occupy some of the nodes.
- As much communication as possible should be detected.

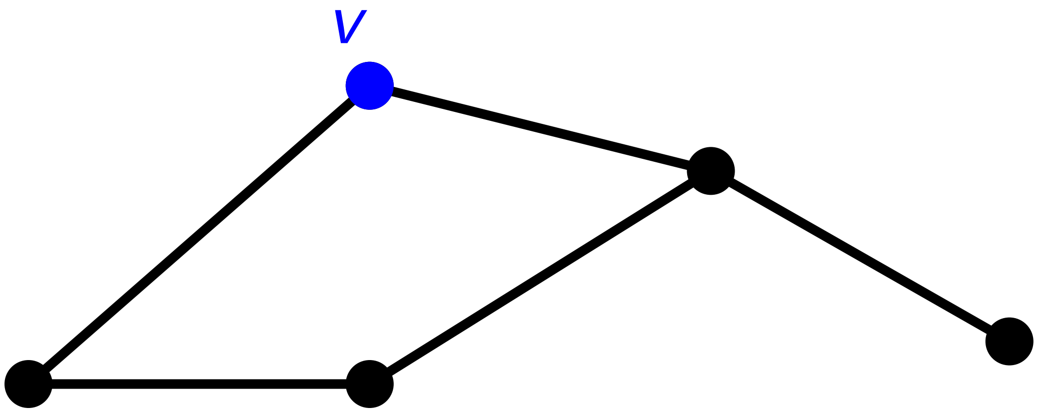


Overview

- Maximum Betweenness Centrality
- Approximating MBC
- APX-Completeness
- MBC on Trees
- Conclusion

Betweenness Centrality

Given a graph $G = (V, E)$ and a node $v \in V$



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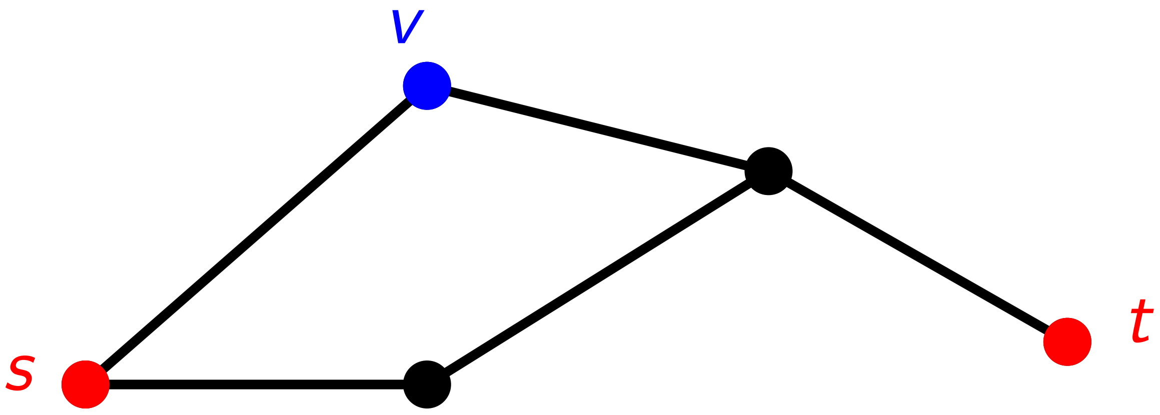

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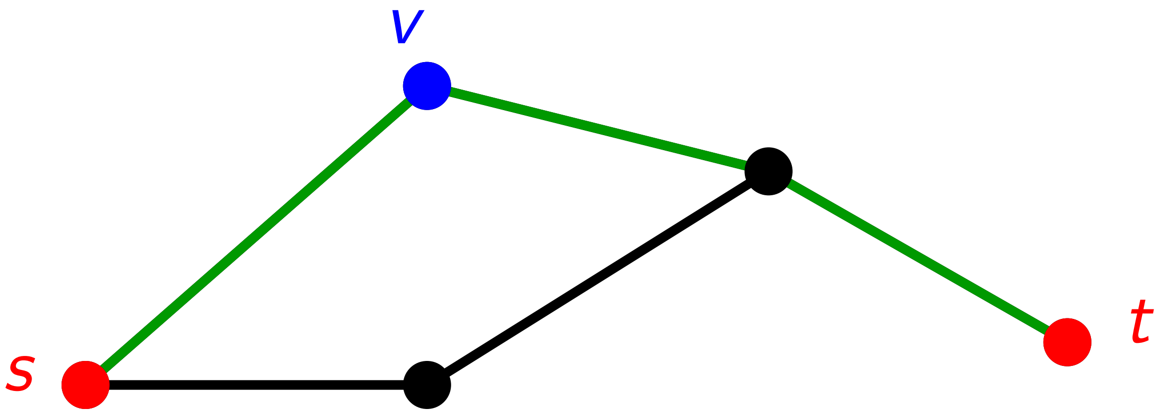

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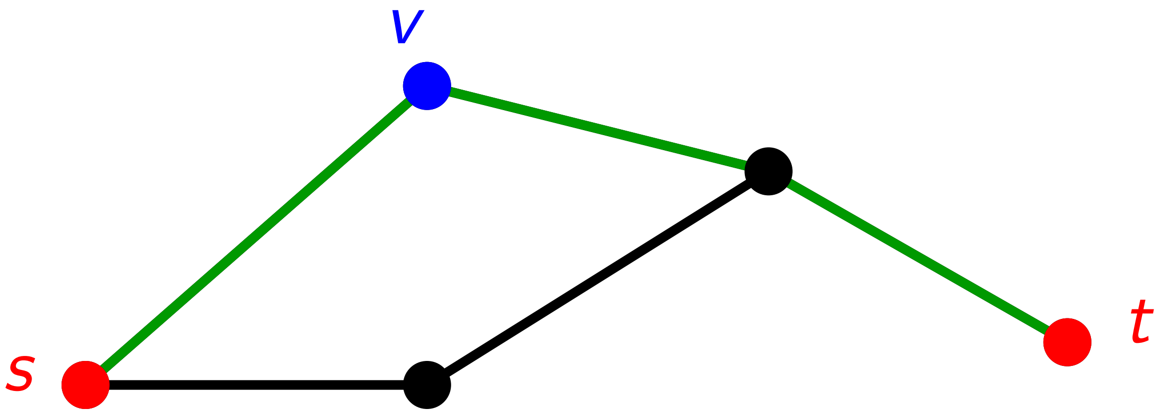

Group Betweenness Centrality

Given a graph $G = (V, E)$ and a ~~node $v \in V$~~ set $C \subseteq V$

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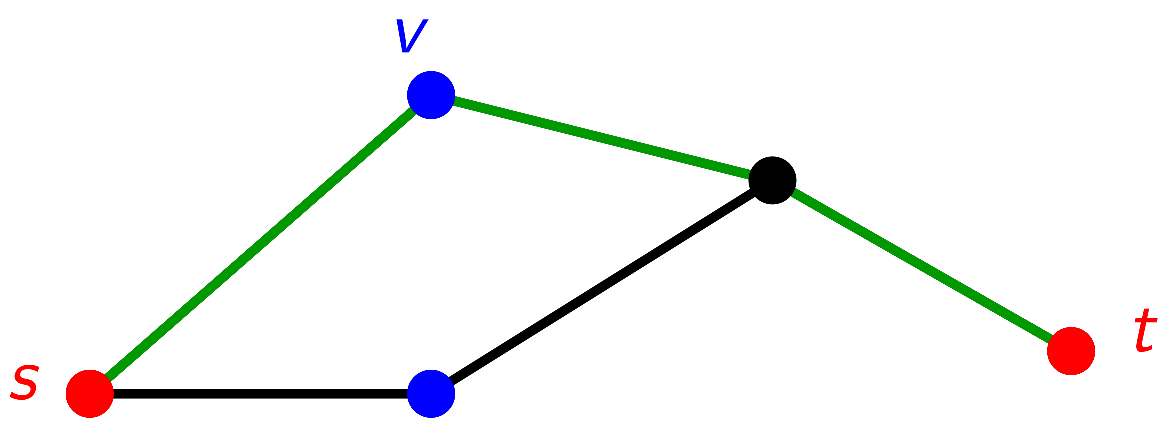

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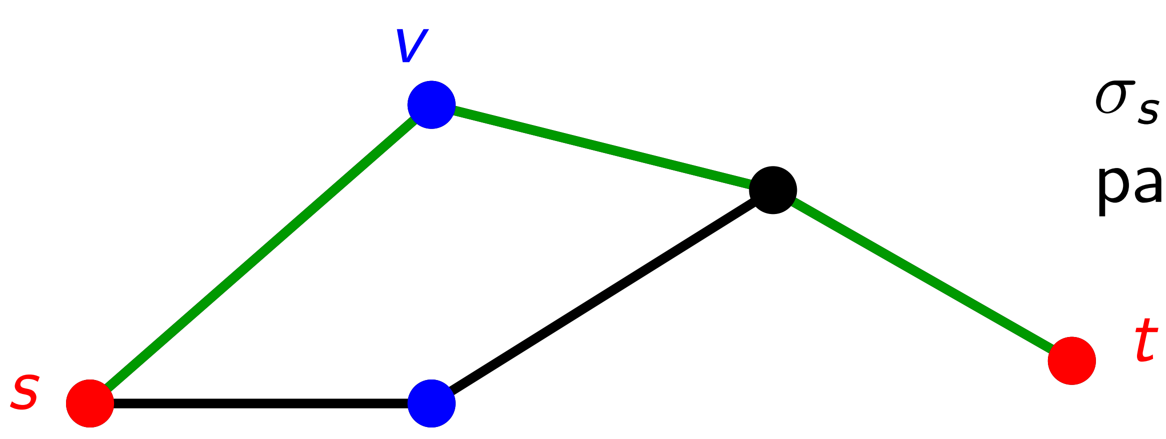
$$\frac{\sigma_{s,t}(C)}{\sigma_{s,t}}$$



$\sigma_{s,t}$, $\sigma_{s,t}(C)$: #shortest s–t paths (using a node of C)

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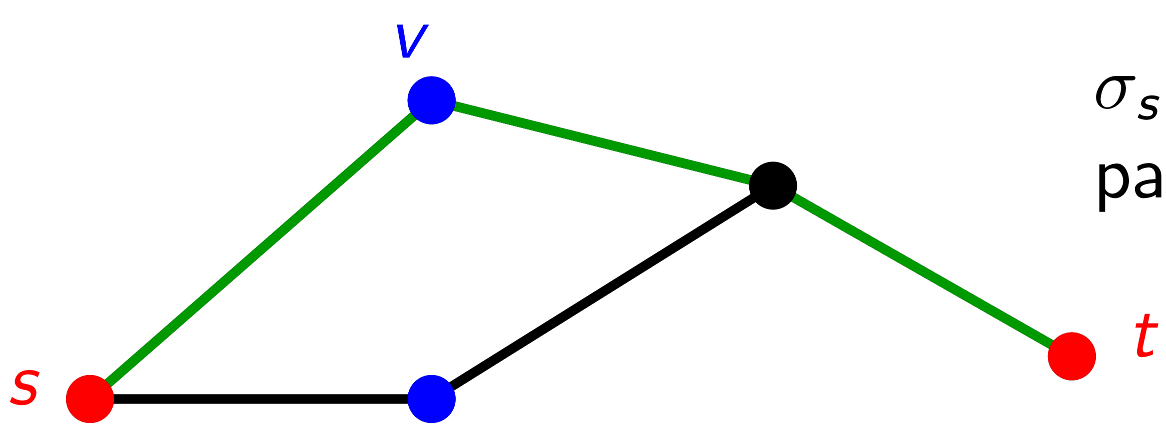
$$\mathrm{GBC}(C) := \sum_{s,t \in V \mid s \neq t} \frac{\sigma_{s,t}(C)}{\sigma_{s,t}}$$



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Previous Results

Theorem. [Brandes, 2001]
The Shortest Path Betweenness Centrality of all nodes can be computed in $O(nm)$ time.

Theorem. [Puzis et. al., 2007]
The Group Betweenness Centrality of one set $C \subseteq V$ can be computed in $O(n^3)$ time.

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Method: iteratively add nodes, $O(n^2)$ update time for each step

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Theorem. [Puzis et al., 2007]
(unit-cost) MBC is NP-hard.

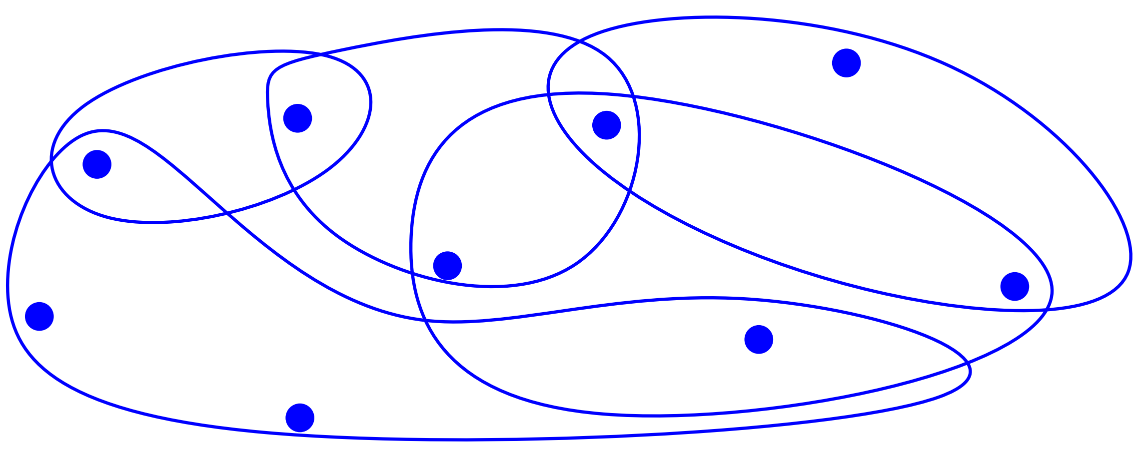
Theorem. [Dolev et al., 2009]
A simple greedy-algorithm computes a $(1 - 1/e)$-approximation for *unit-cost* MBC in $O(n^3)$ time.

Approximating MBC

- Reduce MBC to (budgeted) Maximum Coverage.
- Use existing results for Maximum Coverage.
- implicit reduction

(budgeted) Maximum Coverage

Input: set S, weight function $w\colon S \to \mathbb{R}_0^+$
family $\mathcal{F}$ of subsets of S;
costs $c'\colon \mathcal{F} \to \mathbb{R}_0^+$ and a budget $b \geq 0$



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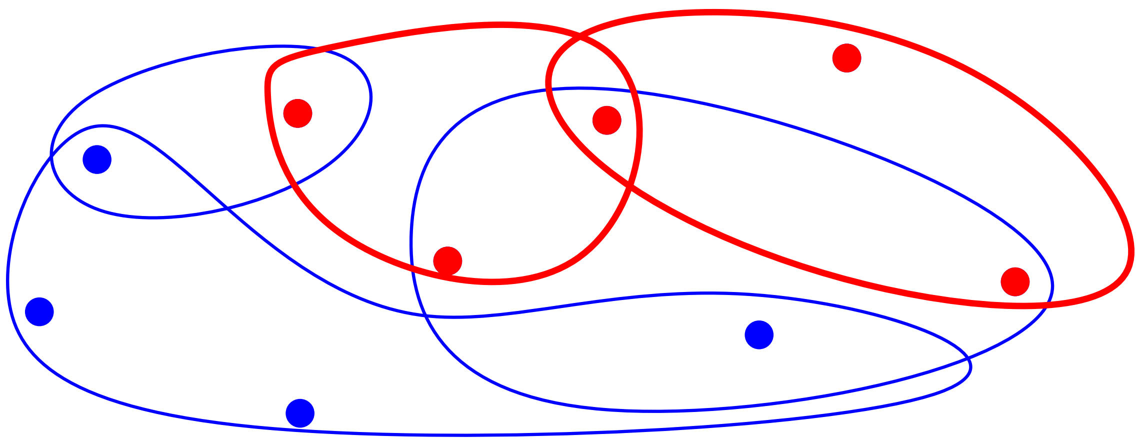

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Approximation Algorithms for MBC

$U := V$

while $U \neq \emptyset$ **do**

$\quad u =$ node with maximal $\frac{\text{GBC}(C+u)-\text{GBC}(C)}{c(u)}$

$\quad$ **if** $c(C+u) \leq b$ **then**

$\quad\quad C := C + u$

$\quad U := U - u$

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Theorem. [Dolev et al., 2009]
$(1-1/e)$-approximation for *unit-cost* MBC in $O(n^3)$ time.

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better approximation for arbitrary costs?

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Approximation Algorithms for MBC

Extended greedy approach

```
H := ∅
foreach C ⊆ V with |C| ≤ 3 and c(C) ≤ b do
    U := V \ C
    while U ≠ ∅ do
        u = node with maximal (GBC(C+u) − GBC(C)) / c(u)
        if c(C + u) ≤ b then
            C := C + u
        U := U − u
    if GBC(C) > GBC(H) then H := C
return H
```

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Input: Graph $G = (V, E)$, number $k \leq n = |V|$

Task: find a set $C \subseteq V$ with $|C| = k$ maximizing the number of covered edges

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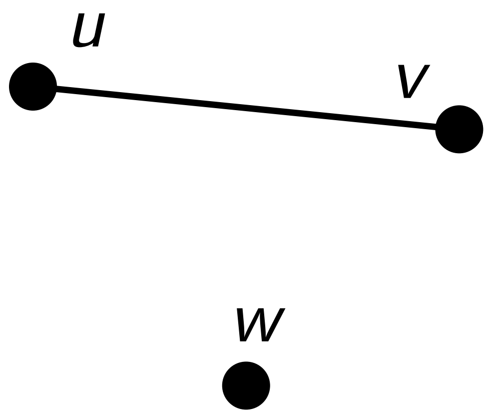

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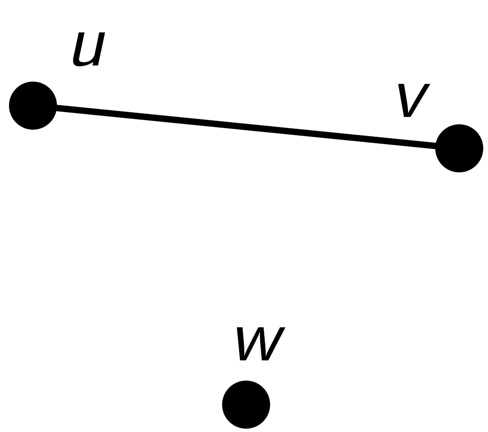

MBC

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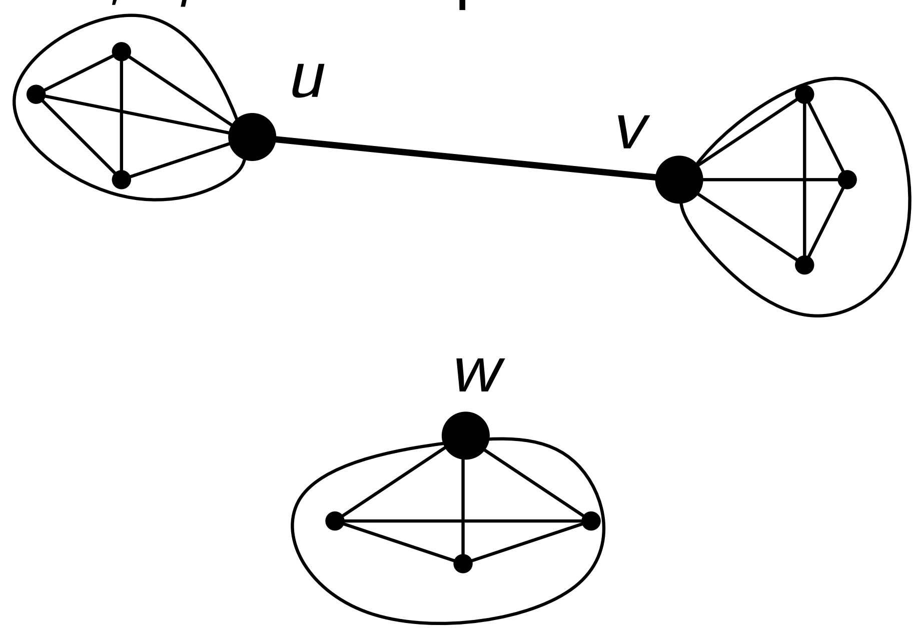

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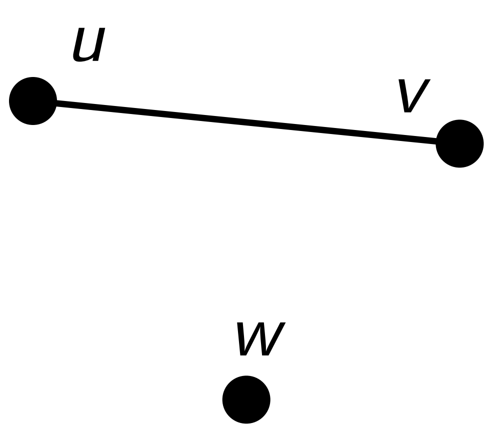

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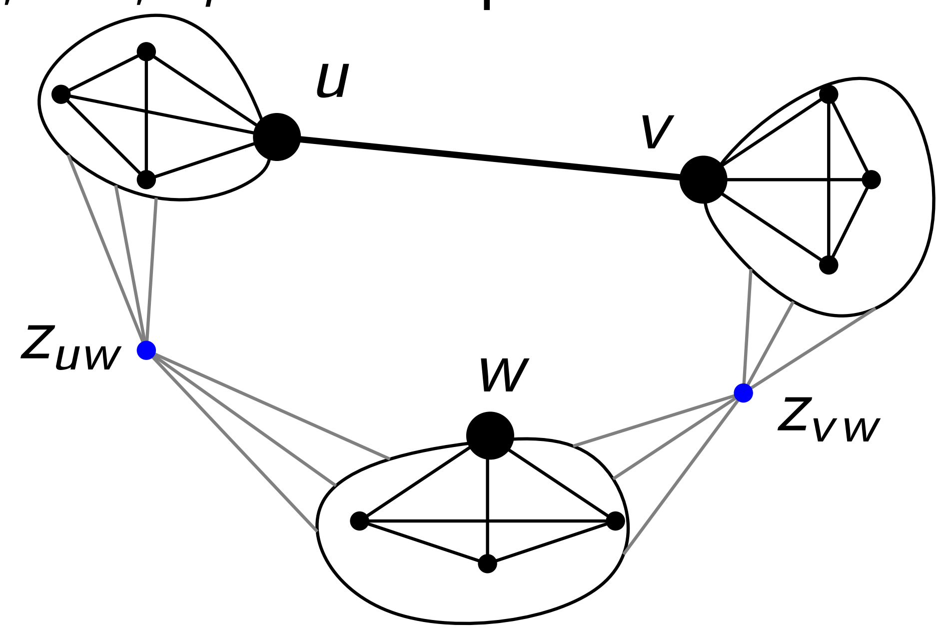

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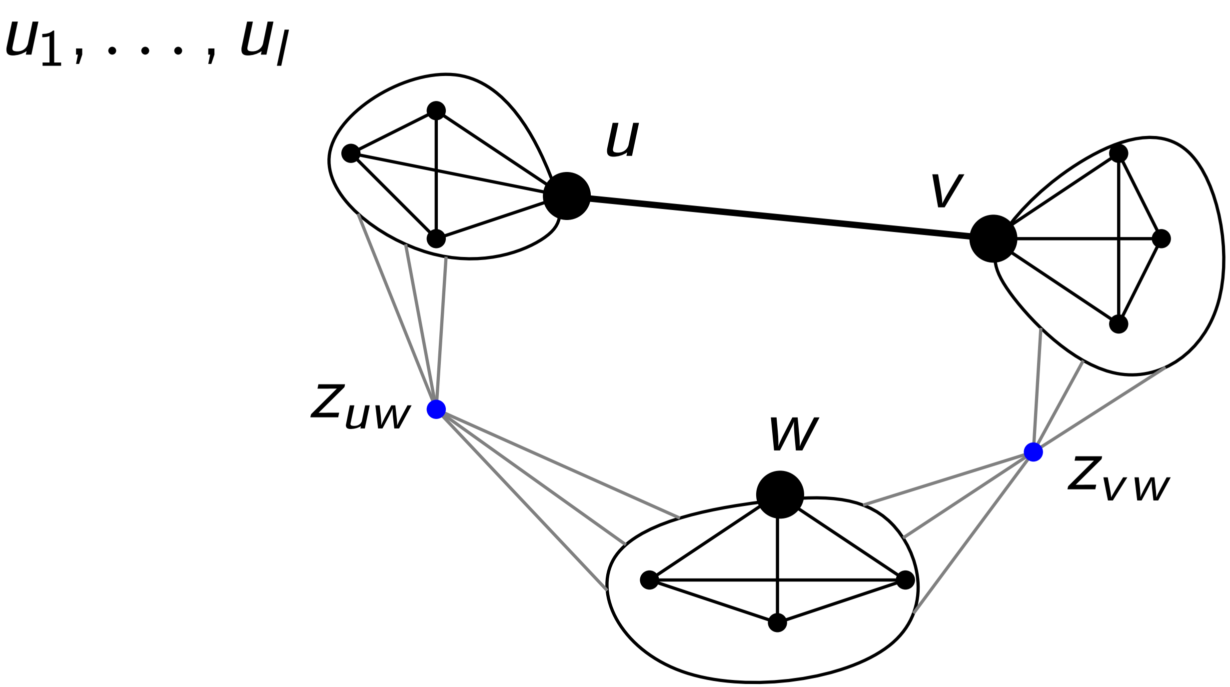
- Only paths between copies of distinct nodes are essential (for large l):
  - u covers shortest path for all l^2 pairs (u_i, v_j)
  - number of other pairs only linear in l



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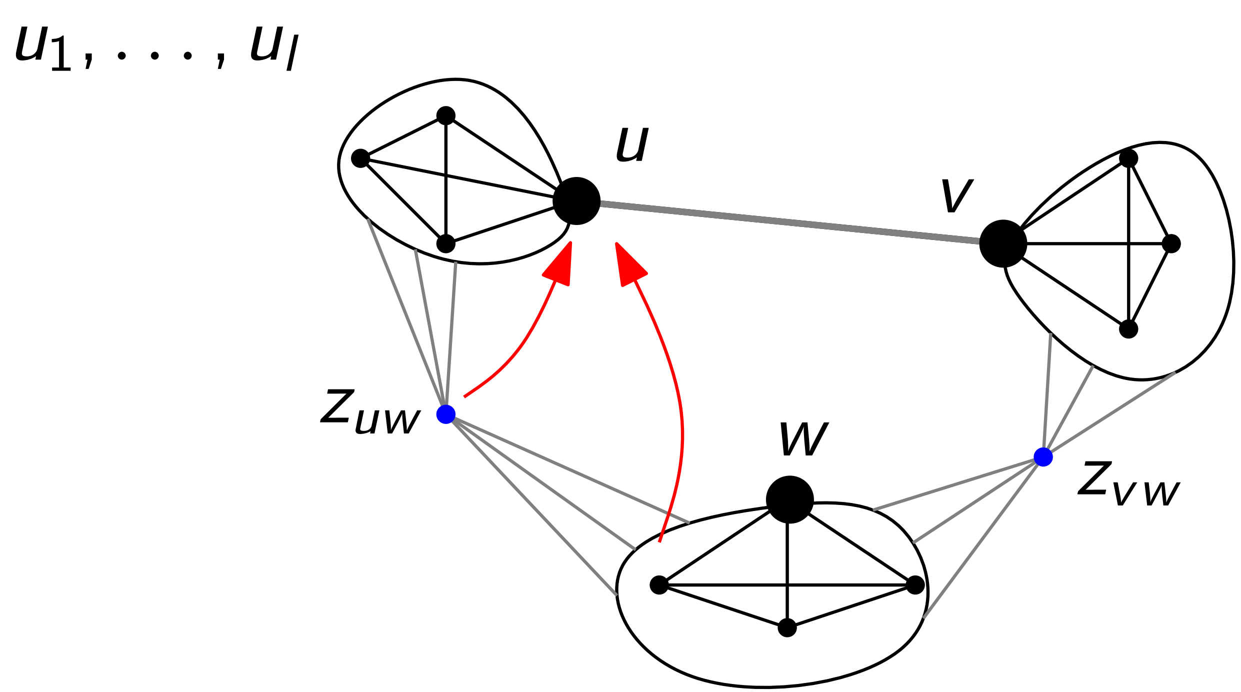

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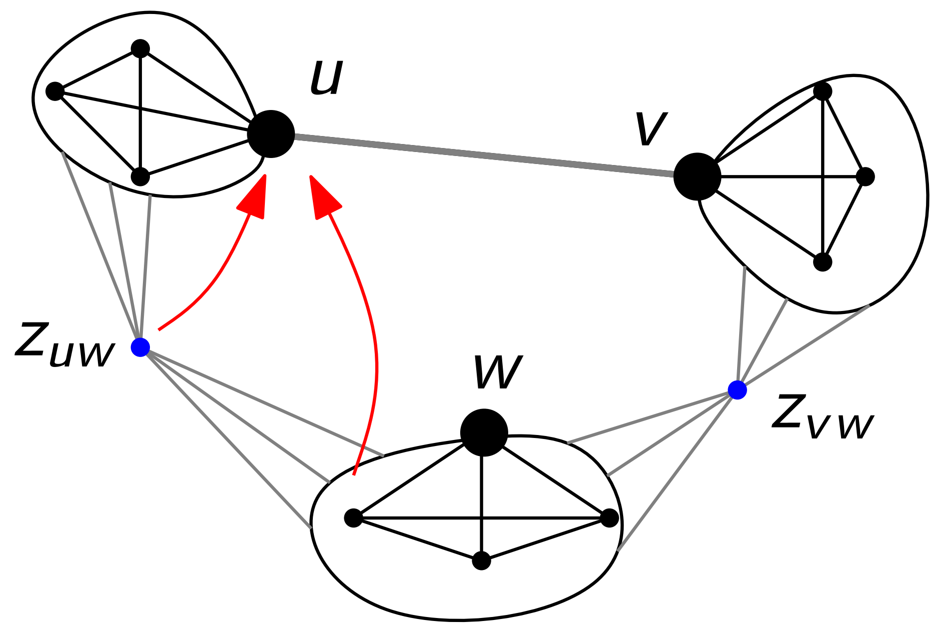
- $\mathrm{GBC}(C) \approx l^2 \times$ #covered edges in G



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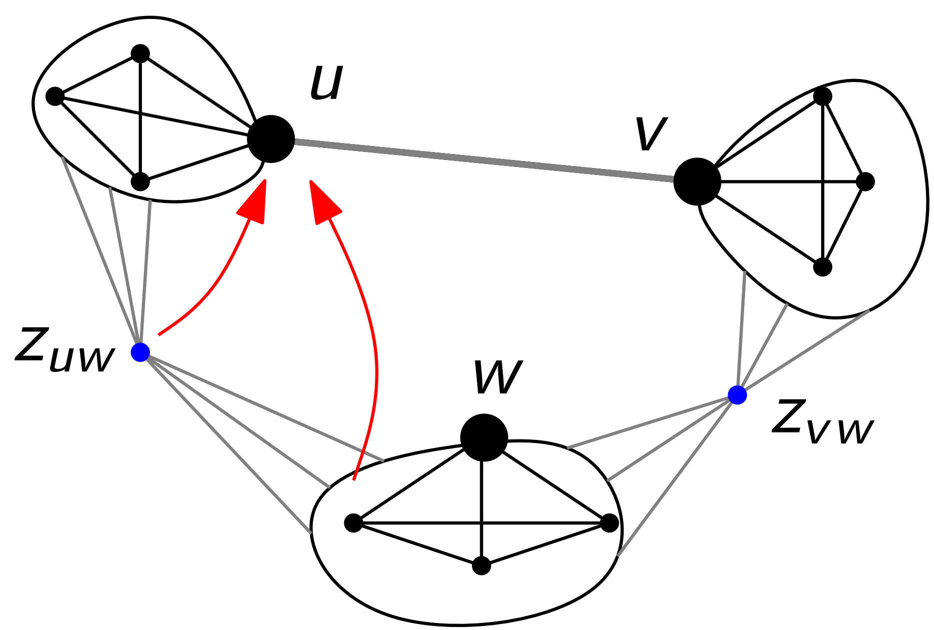
- C approximative solution for MBC $\Rightarrow$ C approximative solution for Maximum Vertex Cover



MBC is APX-complete

Theorem. [Petrank, 1994]
Maximum Vertex Cover is APX-complete.

Theorem. (Unit-cost) Maximum Betweenness Centrality is APX-complete.

Not much hope for a PTAS

MBC on Trees

- For tree $T = (V, E)$: Exactly one (shortest) path between each pair of nodes.

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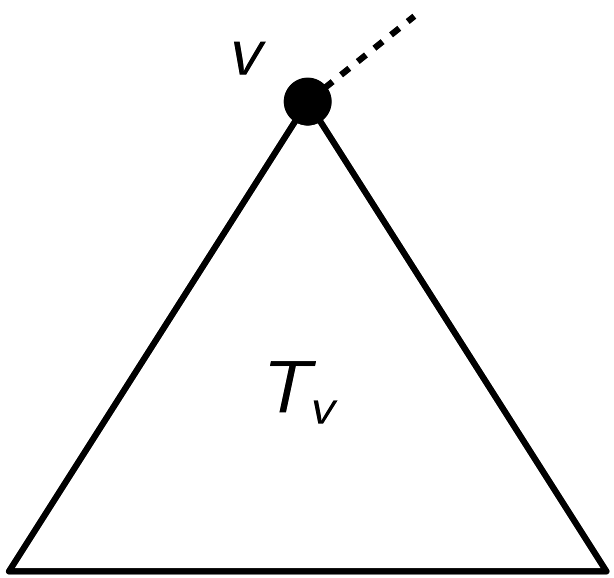

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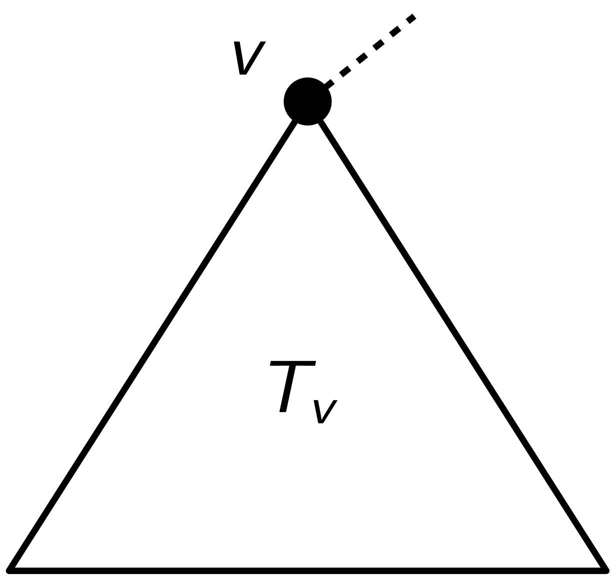
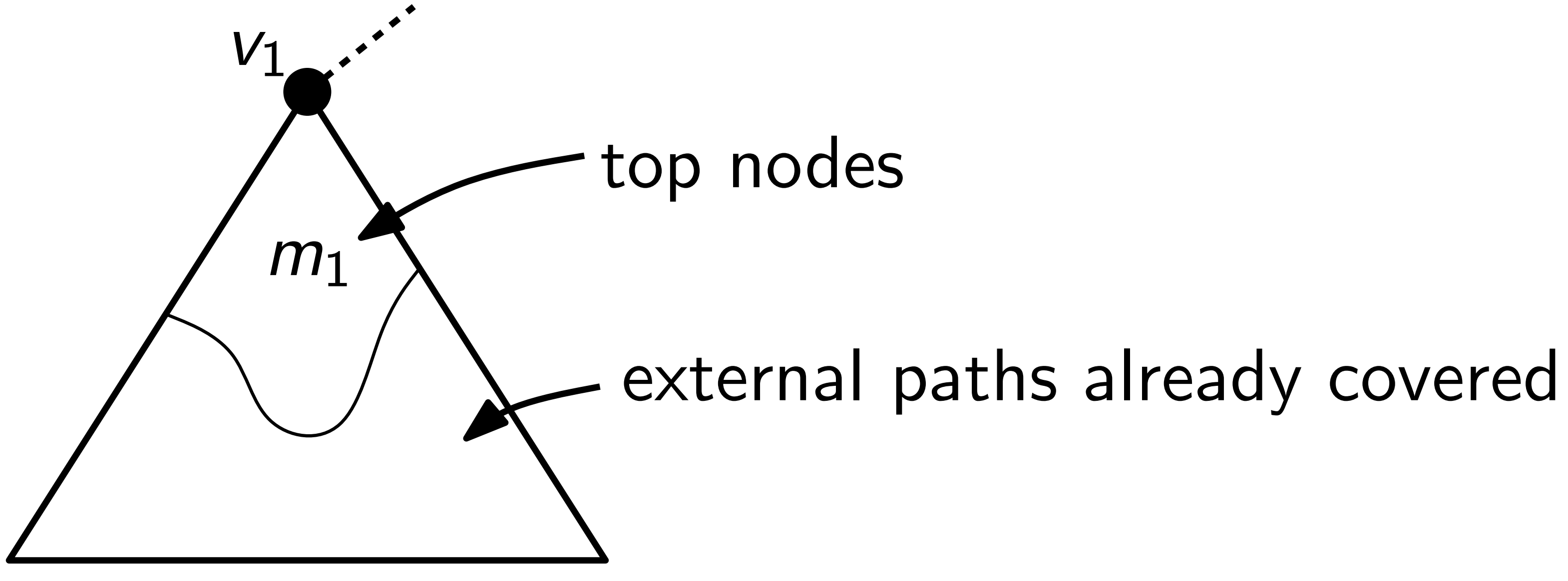


- $\mathrm{GBC}_v(C) = $ #internal paths in T_v covered by C
- Some paths from T_v to nodes outside might already be covered.

MBC on Trees



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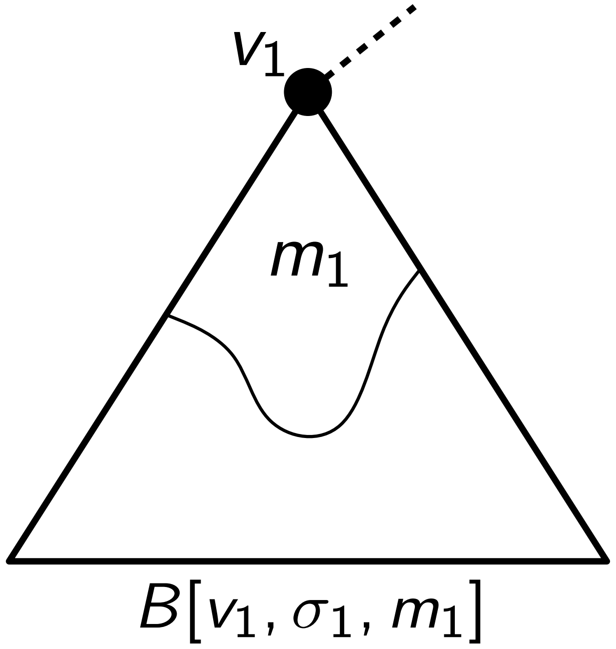

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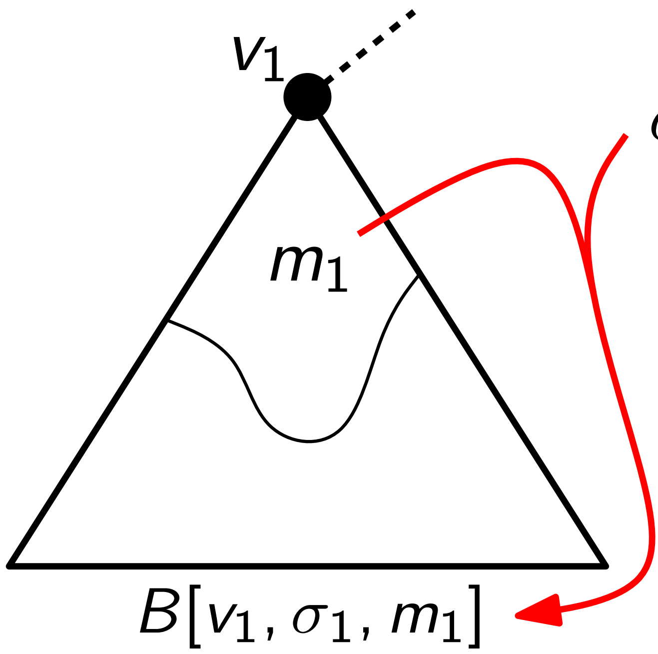

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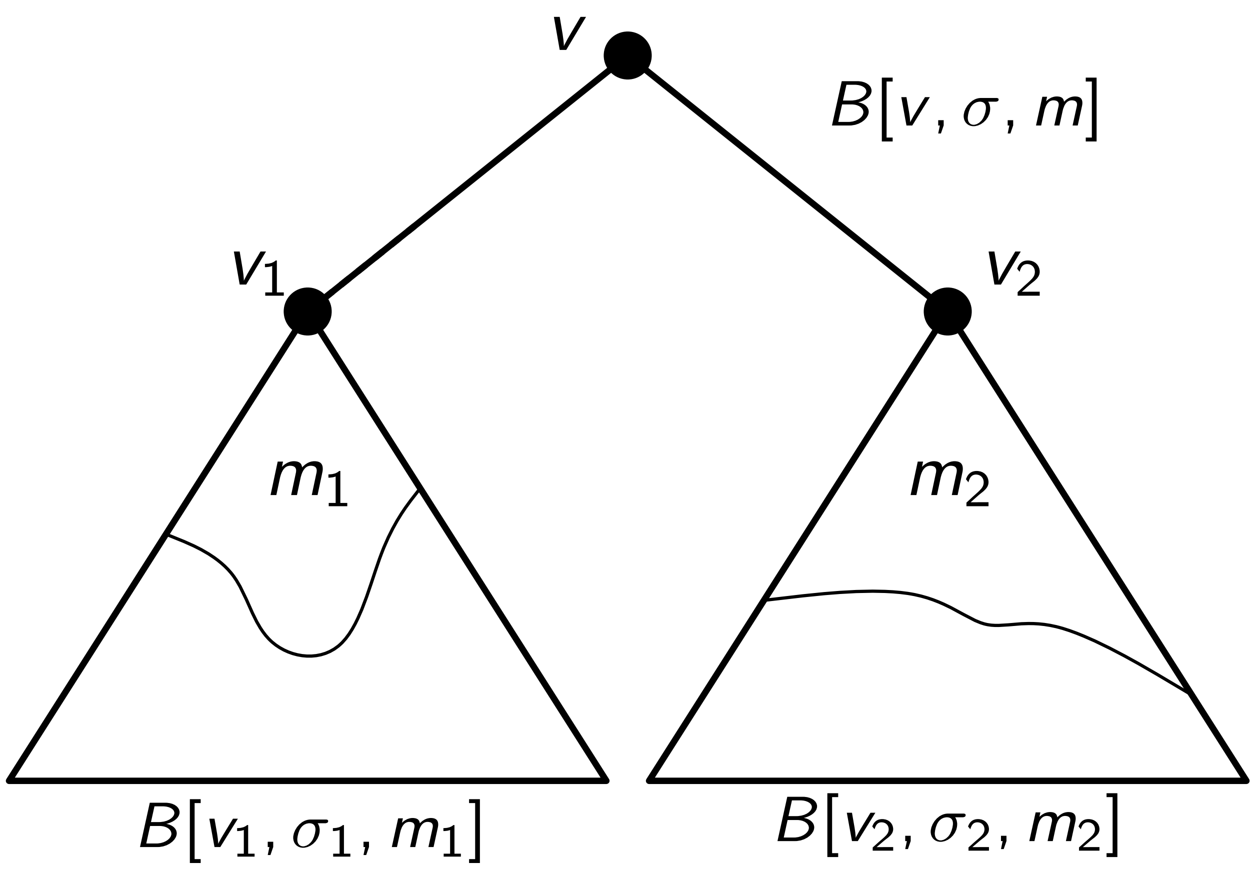
cost of cheapest set $C \subseteq V$ providing these values σ_1, m_1

MBC on Trees



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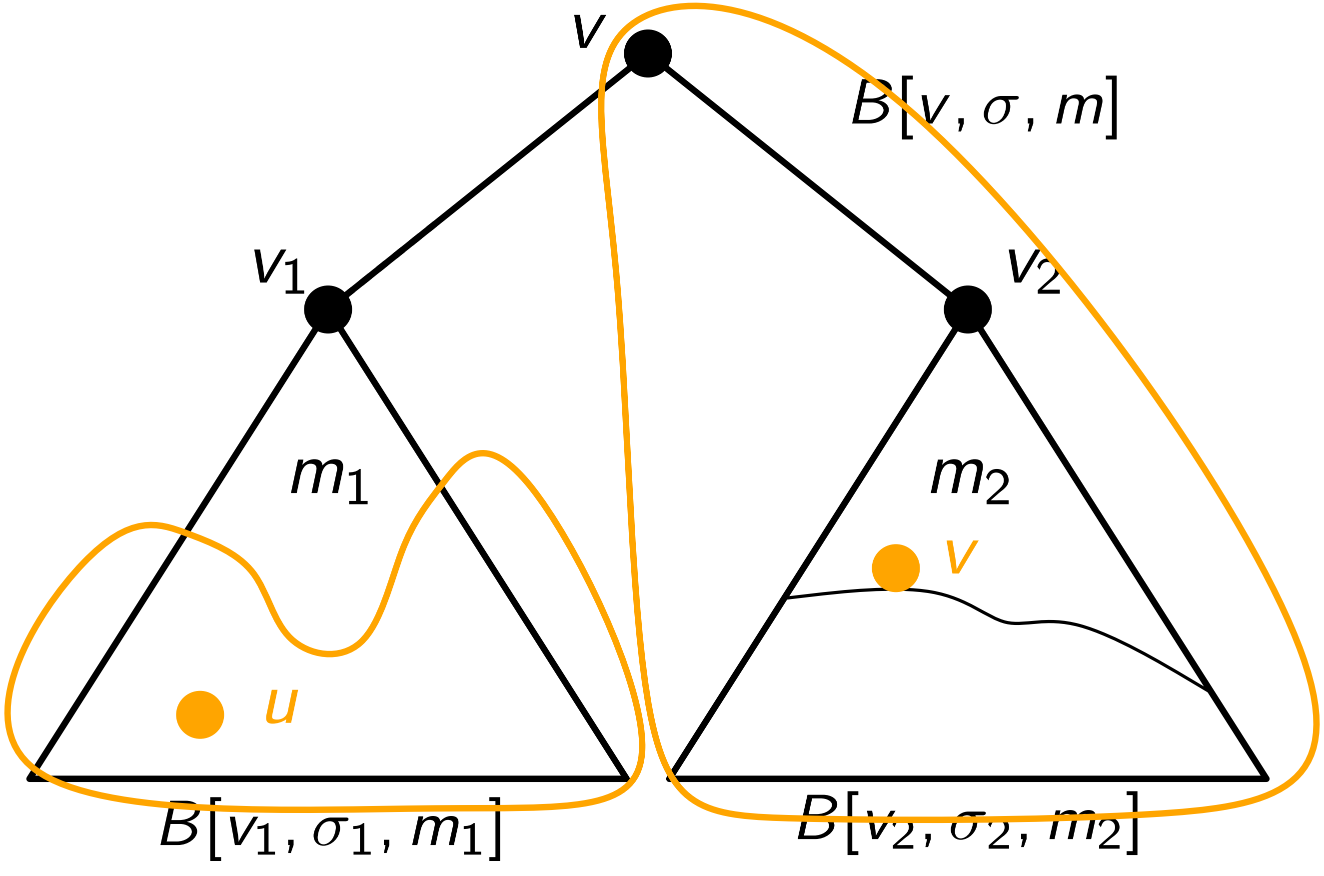

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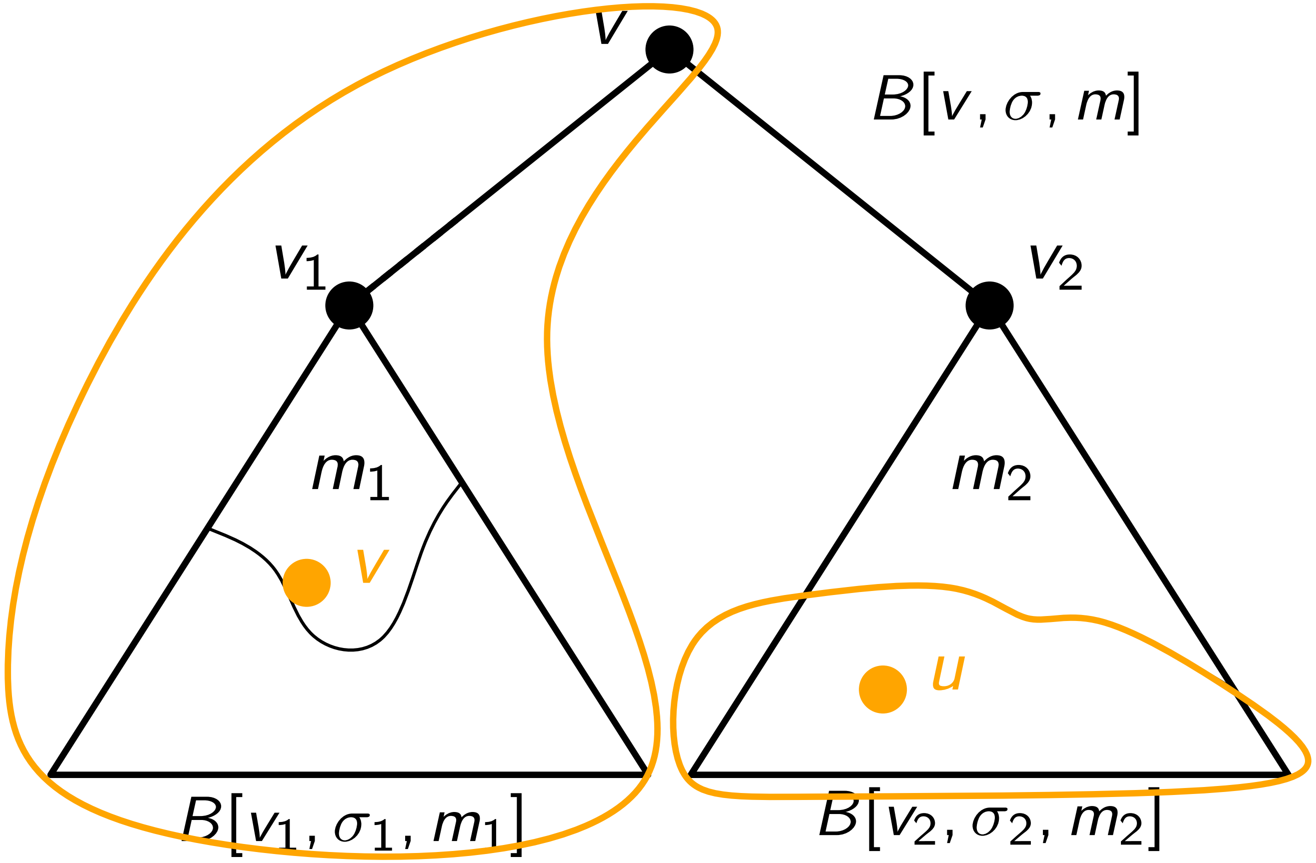

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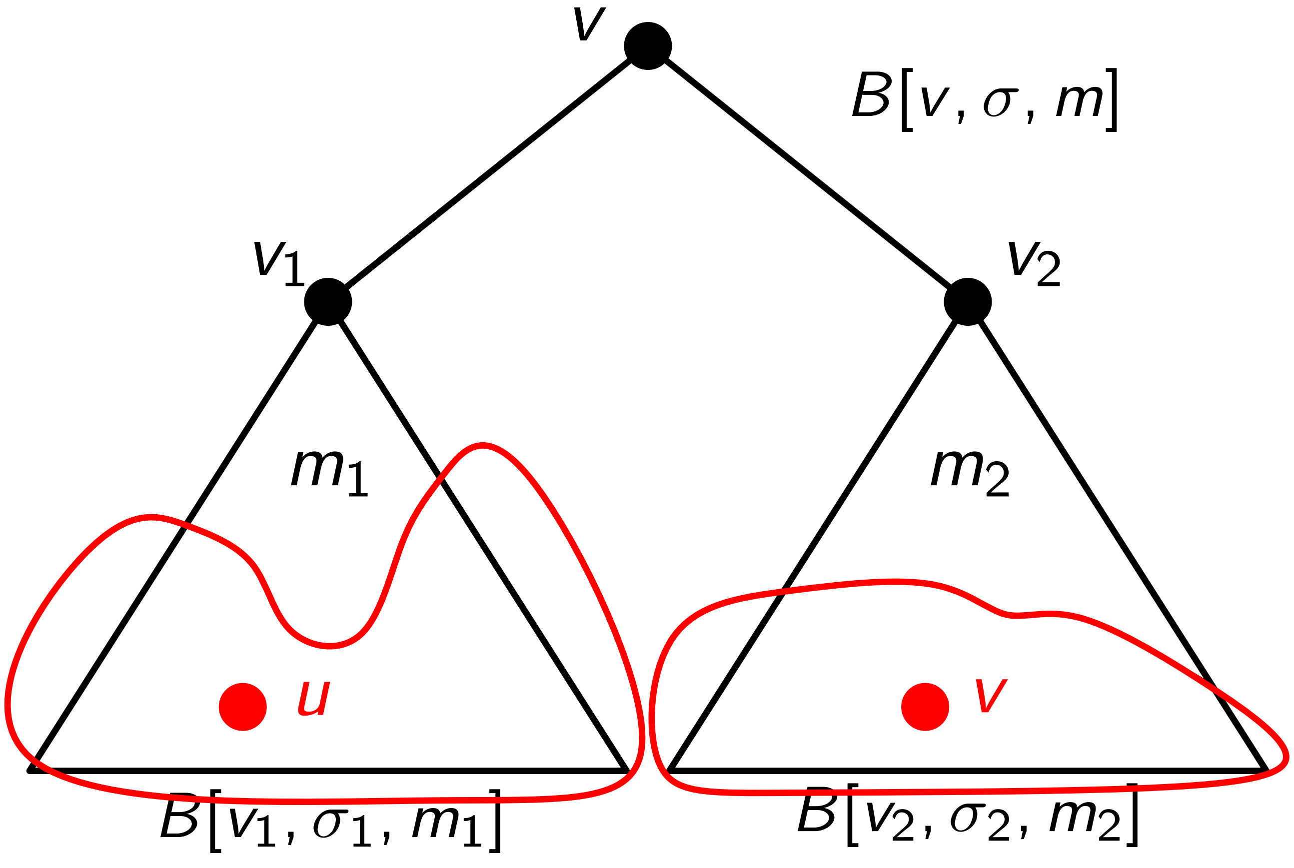

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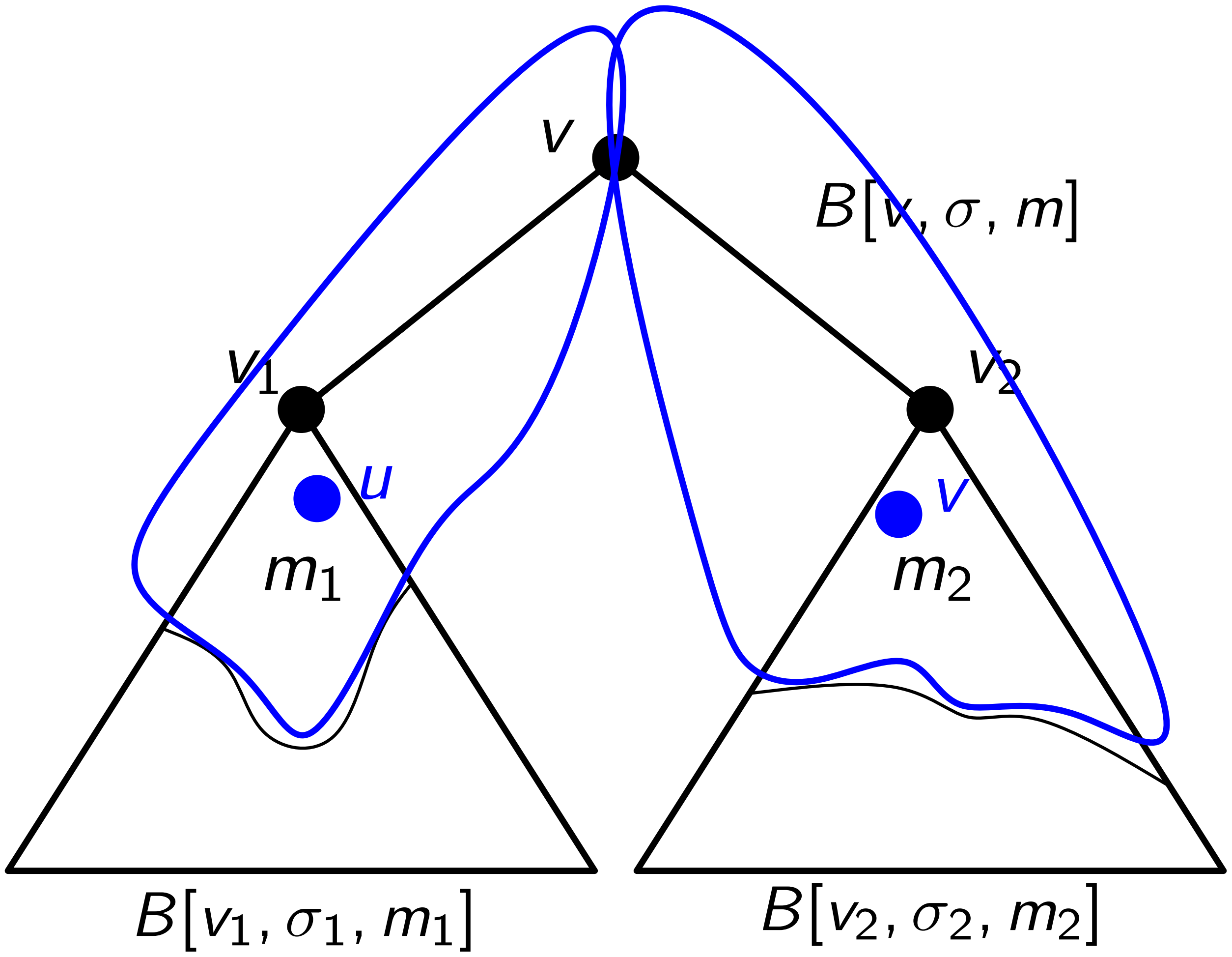
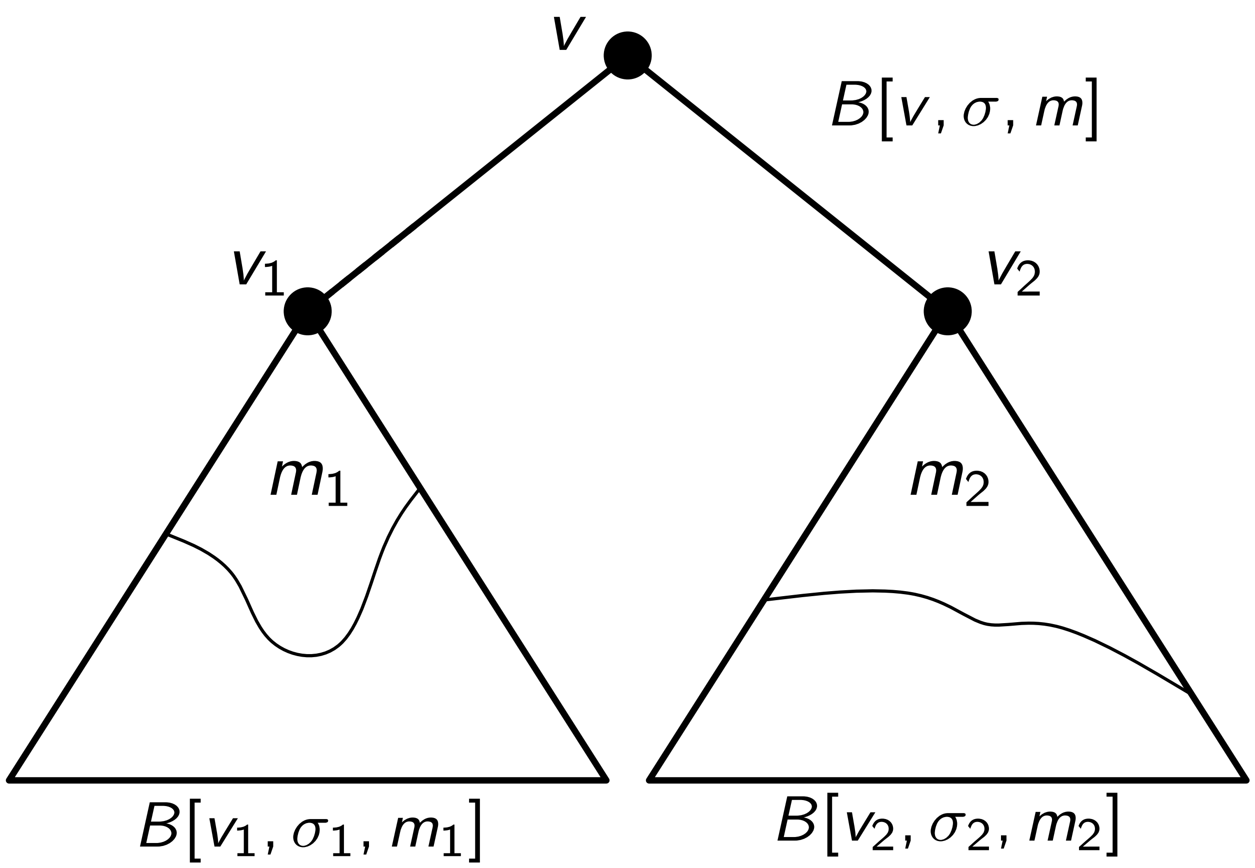


MBC on Trees

Computation of $B[v, \sigma, m]$:

- split m, σ among T_{v_1}, T_{v_2}, v
- $m = 0$ needs special handling

} $O(n^3)$ combinations



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Theorem. MBC can be solved in $O(n^7)$ time on trees.

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- Polynomial-time Algorithm for trees

  Also possible for other classes of graphs?

Conclusion

- Approximation Algorithm for Maximum Betweenness Centrality: tight approximation factor of $1 - 1/e$
- Approximability

$$1 - \frac{1}{e} < \overset{?}{\alpha} < \cancel{1 - \epsilon}$$

not possible for Maximum Coverage

- Polynomial-time Algorithm for trees

  Also possible for other classes of graphs?

Thank you!