Fine Hierarchy of Regular Aperiodic ω-Languages

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Abstract

We develop a theory of regular aperiodic ω-languages in parallel with the theory around the Wagner hierarchy. In particular, we characterize the Wadge degrees of regular aperiodic ω-languages, find an effective version of the Wadge reducibility adequate for this class of languages and prove "aperodic analogs" of the Büchi-Landweber determinacy theorem and of Landweber's description of regular open and regular G_δ sets.

Keywords: Automaton, acceptor, transducer, regular aperiodic ω-language, Wagner hierarchy, reducibility.

1 Introduction

This paper is devoted to the theory of infinite behavior of computing devices that is of primary importance for theoretical and practical computer science. More exactly, we consider topological aspects of this theory in the simplest case of finite automata.

A series of papers in this direction culminated with the paper [Wag79] giving in a sense the finest possible topological classification of regular ω-languages (i.e., of subsets of X^ω for a finite alphabet X recognized by finite automata) known as the Wagner hierachy. In particular, K. Wagner completely described the (quotient structure of the) preorder $(\mathcal{R}; \leq_{CA})$ formed by the class $\mathcal{R}$ of regular subsets of X^ω and the reducibility by functions continuous in the Cantor topology on X^ω (note that in descriptive set theory CA-reducibility is widely known as the Wadge reducibility denoted by $\leq_W$ but we follow the notation $\leq_{CA}$ from [Wag79] for the sake of uniformity, because we will use also several versions of this notation).

In [Se94a, Se95, Se98] the Wagner hierarchy of regular ω-languages was related to the Wadge hierarchy and to the author's fine hierarchy [Se95a, Se96]. This provided new proofs of results in [Wag79] and yielded some new results on the Wagner hierarchy. See also an alternative approach [CP97, CP99, DR0?].

Later some results from [Wag79, Se98] were extended to more complicated computing devices. In particular, the Wadge degrees of deterministic context-free ω-languages, of ω-languages recognized by deterministic Turing machines, and of infinite tree lanquages recognized by deterministic tree automata were determined respectively in [D03, Se03, M06]. Note that in all these three cases some important properties of the Wagner hierarchy are either false or still open.

In this paper, we develop a complete analog of the theory from [Wag79, Se98] for the class $\mathcal{A}$ of regular aperodic ω-languages. The class $\mathcal{A}$ is certainly the most important subclass of $\mathcal{R}$ which has several

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remarkable characterizations and plays a noticible role in the field of specification and verification of finite-state systems. To explain our results, let us recall some results on the Wagner hierarchy in more details. In [Wag79] the following results (among others) were established:

1) The structure $(\mathcal{R}; \leq_{CA})$ is almost well-ordered with order type ω^ω, i.e. there are $A_\alpha \in \mathcal{R}$, $\alpha < \omega^\omega$, such that $A_\alpha <_{CA} A_\alpha \oplus \overline{A}_\alpha <_{CA} A_\beta$ for $\alpha < \beta < \omega^\omega$ and any regular set is CA-equivalent to one of sets $A_\alpha, \bar{A}_\alpha, A_\alpha \oplus \overline{A}_\alpha (\alpha < \omega^\omega)$.

2) CA-reducibility coincides on $\mathcal{R}$ with DA-reducibility, i.e. the reducibility by functions computed by deterministic asyncronous finite transducers, and $\mathcal{R}$ is closed under DA-reducibility.

3) Any level $\mathcal{R}_\alpha = \{C : C \leq_{DA} A_\alpha\}$ of the Wagner hierarchy is decidable (even in polynomial time, as observed in [KPB95, WY95]).

In [Se98] the following additional facts (among others) about the Wagner hierarchy were established:

4) Any class $\mathcal{R}_\alpha$ has a natural set-theoretic description in terms of classes $\mathcal{L}_0$ of regular open sets and $\mathcal{L}_1$ of regula F_σ sets. In particular, there is a Boolean term $t(x_1, \ldots, x_n, y_1, \ldots, y_n)$ with $\mathcal{R}_\alpha = t(\mathcal{L}_0, \mathcal{L}_1)$, where $t(\mathcal{L}_0, \mathcal{L}_1)$ is the set of values of t when $x_1, \ldots, x_n$ range over $\mathcal{L}_0$ and $y_1, \ldots, y_n$ range over $\mathcal{L}_1$.

5) For every term t as above, the set $t(\mathcal{L}_0, \mathcal{L}_1)$ coincides with one of classes $\mathcal{R}_\alpha$ or their duals.

6) If a regular set R is represented as $R = t(B_1, \ldots, B_n, C_1, \ldots, C_n)$ for some term t as above, open sets $B_1, \ldots, B_n$ and F_σ-sets $C_1, \ldots, C_n$, then $R = t(B'_1, \ldots, B'_n, C'_1, \ldots, C'_n)$ for some $B'_1, \ldots, B'_n \in \mathcal{L}_0$ and $C'_1, \ldots, C'_n \in \mathcal{L}_1$.

In this paper we will show that the sets A_α from 1) may be choosen from $\mathcal{A}$, and thus the Wadge degrees (as well as DA-degrees) of sets in $\mathcal{A}$ are the same as those of sets in $\mathcal{R}$. We will find a reducibility $\leq_{AA}$ related to the class $\mathcal{A}$ in exactly the same way as DA-reducibility is related to $\mathcal{R}$. Thus, we obtain analogs of 1)—3). We also show that classes $\mathcal{A}_\alpha = \{C : C \leq_{AA} A_\alpha\}$ of the fine hierarchy of regular aperiodic languages have properties 4)—6), if we take the classes $\mathcal{K}_0$ of regular aperiodic open and $\mathcal{K}_1$ of regular aperiodic F_σ-sets in place of the classes $\mathcal{L}_0$ and $\mathcal{L}_1$, respectively. We obtain also some facts of independent interest, e.g. "aperodic analogs" of the Büchi-Landweber determinacy theorem and of Landweber's description of regular open and regular G_δ sets. Note that these results are actually strengthenings of corresponding results about regular sets, hence this paper formally subsumes many results from [Wag79, Se98].

The rest of the paper is organized as follows. In Section 2 we collect notation and known facts we will rely upon. In Sections 3 and 4 we prove some necessary facts about aperiodic automata (known also as counter-free automata). In Section 5 we prove the "aperiodic analog" of the Büchi-Landweber determinacy theorem. In Section 6 we establish some facts about regular aperiodic sets in the Borel hierarchy. Section 7 deals with the fine hierarchy of regular aperiodic sets, while Section 8 — with reducibilities on such sets. We conclude in Section 9 with mentioning a couple of open questions.

2 Notation and Reminder

We use standard set-theoretic notation. For a set S, $P(S)$ is the class of subsets of S. For a class $\mathcal{C} \subseteq P(S)$, $\check{\mathcal{C}}$ is the dual class $\{\overline{C} : C \in \mathcal{C}\}$ and $B(\mathcal{C})$ is the Boolean closure of $\mathcal{C}$.

We assume the reader to be acquainted with the notion of ordinal and with arithmetic operations on ordinals (see e.g. [KM67]). Actually, we consider only ordinals less than ω^ω that is the supremum of ordinals ω^m, $m < \omega$, where $\omega = \{0, 1, \ldots\}$ is the first infinite ordinal. Recall that ω^ω is the order type of finite sequnces $(k_0, \ldots, k_n)$ of natural numbers $k_0 \geq \cdots \geq k_n$, ordered lexicografically. Every non-zero ordinal $\alpha < \omega^\omega$ is uniquely representable in the form $\alpha = \omega^{k_0} + \cdots + \omega^{k_n}$ with $\omega > k_0 \geq \ldots \geq k_n$.

Fix a finite alphabet X containing more than one symbol (for simplicity we may assume that $X = \{x | x < k\}$ for a natural number $k > 1$, so $0, 1 \in X$). Let X^* and X^ω denote respectively the sets of all words and of all ω-words (i.e. sequences $\alpha : \omega \to X$) over X. The empty word is denoted by ε. Let $X^+ = X^* \setminus \{\varepsilon\}$ and $X^{\leq\omega} = X^* \cup X^\omega$. For $n < \omega$, let X^n be the set of words of length n. Sets of words $X^{\leq n}$ and $X^{>n}$ are defined in the same way. For $X = \{0, 1\}$ we write 2^* in place of X^*, 2^ω in place of X^ω and so on.

We use some almost standard notation concerning words and ω-words, so we are not too casual in reminding it here. For $w \in X^*$ and $\xi \in X^{\leq\omega}$, $w \sqsubseteq \xi$ means that w is a substring of ξ, $w \cdot \xi = w\xi$ denote the concatenation, $l = |w|$ is the length of $w = w(0) \cdots w(l-1)$. For $w \in X^*, W \subseteq X^*$ and $A \subseteq X^{\leq\omega}$, let $w \cdot A = \{w\xi : \xi \in A\}$ and $W \cdot A = \{w\xi : w \in W, \xi \in A\}$. For $k, l < \omega$ and $\xi \in X^{\leq\omega}$, let $\xi[k, l) = \xi(k) \cdots \xi(l-1)$ and $\xi[k] = \xi[0, k)$. For $u \in X^*$ and $n < \omega$, u^n denote the concatenation of n copies of the word u. Our notation does not distinguish a word of length 1 and the corresponding letter.

Note that usually we work with the fixed alphabet X but sometimes we are forced to consider several alphabets simultaniously; in this case we denote the alphabets by Y, Z, possibly with indices, and include the alphabets in the corresponding notation. The "fixed-alphabet mode" is the default one.

The set X^ω carries the Cantor topology with the open sets $W \cdot X^\omega$, where $W \subseteq X^*$. Continuous functions in this topology are called also *CA-functions*. A *CS-function* is a function $f : X^\omega \to Y^\omega$ satisfying $f(\xi)(n) = \phi(\xi[n+1])$ for some $\phi : X^\omega \to X$. A *delayed CS-function* is a function $f : X^\omega \to Y^\omega$ satisfying $f(\xi)(n) = \phi(\xi[n])$ for some $\phi : X^\omega \to X$. Every delayed CS-function is a CS-function, and every CS-function is a CA-function. In descriptive set theory CS-functions are known as Lipschitz functions. All three classes of functions are closed under composition.

For arbitrary alphabets $X_0, \ldots, X_{n-1}$, $n > 0$, there is a natural homeomorphism between the spaces $X_0^\omega \times \cdots \times X_{n-1}^\omega$ and $(X_0 \times \cdots \times X_{n-1})^\omega$ sending $(\xi_0, \ldots, \xi_{n-1})$ to the ω-word $\xi_0 \times \cdots \times \xi_{n-1}$ defined by $(\xi_0 \times \cdots \times \xi_{n-1})(i) = (\xi_0(i), \ldots, \xi_{n-1}(i))$, $i < \omega$. This reduces consideration of continuous functions with many variables to consideration of those with one variable. There is also a natural continuous fuction $f : X_0^\omega \times \cdots \times X_{n-1}^\omega \to (X_0 \cup \cdots \cup X_{n-1})^\omega$ defined by $f(\xi_0, \ldots, \xi_{n-1})(i) = \xi_0(i) \cdots \xi_{n-1}(i)$.

In the particular case when $X_0, \ldots, X_{n-1}$ are one and the same alphabet X the function f is a homeomorphism between n-th cartesian power of X^ω and X^ω. This gives rise to a bijective coding of n-tuples $(\xi_0, \ldots, \xi_{n-1})$ of elements of X^ω by elements $\langle \xi_0, \ldots, \xi_{n-1} \rangle = f(\xi_0, \ldots, \xi_{n-1})$ of X^ω defined by

$$\langle \xi_0, \ldots, \xi_{n-1} \rangle (n \cdot i + r) = \xi_r(i), \quad r < n.$$

The corresponding "projections" $p_r(\langle \xi_0, \ldots, \xi_{n-1} \rangle) = \xi_r$, $r < n$, are CA-functions on X^ω.

Let $\mathcal{B}$ denote the class of Borel subsets of X^ω, i.e. the least class containing the open sets and closed under complementation and countable union. Borel sets are organized in a hierarchy the lowest levels of which are as follows: G and F are the classes of open and closed sets, respectively; G_δ (F_σ) is the class of countable intersections (unions) of open (resp. closed) sets; $G_{\delta\sigma}$ ($F_{\sigma\delta}$) is the class of countable unions (intersections) of G_δ- (resp. of F_σ-) sets, and so on. In the modern notation of hierarchy theory, $\mathbf{\Sigma}^0_1 = G$, $\mathbf{\Sigma}^0_2 = F_\sigma$, $\mathbf{\Sigma}^0_3 = G_{\delta\sigma}$, $\mathbf{\Sigma}^0_4 = F_{\sigma\delta\sigma}$ and so on, $\mathbf{\Pi}^0_n$ is the dual class for $\mathbf{\Sigma}^0_n$, and $\mathbf{\Delta}^0_n = \mathbf{\Sigma}^0_n \cap \mathbf{\Pi}^0_n$. The sequence $\{\mathbf{\Sigma}^0_{n+1}\}_{n<\omega}$ is known as *the finite Borel hierarchy*. It may be in a natural way extended on all countable ordinals. The resulting sequence called *the Borel hierarchy* exhausts the class $\mathcal{B}$. For any $n > 0$, the class $\mathbf{\Sigma}^0_n$ contains $\emptyset, X^\omega$ and is closed under countable unions and finite intersections, while the class $\mathbf{\Delta}^0_n$ is a Boolean algebra. For any $n > 0$, $\mathbf{\Sigma}^0_n \cup \mathbf{\Pi}^0_n \subseteq \mathbf{\Delta}^0_{n+1}$, and $\mathbf{\Sigma}^0_n \not\subseteq \mathbf{\Pi}^0_n$.

By *automaton* (over X) we mean a triple $\mathcal{M} = (Q, X, f)$ consisting of a finite non-empty set Q of states, an input alphabet X and a transition function $f : Q \times X \to Q$. The transition function is naturally extended to the function $f : Q \times X^* \to Q$ defined by induction $f(q, \varepsilon) = q$ and $f(q, u \cdot x) = f(f(q, u), x)$, where $u \in X^*$ and $x \in X$. Similarly, we may define the function $f : Q \times X^\omega \to Q^\omega$ by $f(q, \xi)(n) = f(q, \xi[n])$. Note that in this paper we consider only deterministic finite automata. A *partial automaton* is defined in the same way, only now $f : Q \times X \rightharpoonup Q$ is a partial function.

Automata equiped with appropriate additional structures are used as acceptors (devises accepting words or ω-words) and transducers (devices computing functions on words or ω-words). A *word acceptor* is a triple $(\mathcal{M}, i, F)$ consisting of an automaton $\mathcal{M}$, an initial state i of $\mathcal{M}$ and a set of final states $F \subseteq Q$. Such an acceptor recognizes the language

$$L(\mathcal{M}, i, F) = \{u \in X^* : f(i, u) \in F\}.$$

Languages recognized by such acceptors are called *regular*. The languages recognized by partial automata are defined in the same way and coincide with the regular languages.

For the case of ω-words, there are several notions of acceptors of which we will use only three. A *Büchi acceptor* has the form $(\mathcal{M}, i, F)$ as above and recognizes the set

$$L_\omega(\mathcal{M}, i, F) = \{\xi \in X^* : In(f(i,\xi)) \cap F \neq \emptyset\},$$

where $In(f(i,\xi))$ is the set of states which occur infinitely often in the sequence $f(i,\xi) \in Q^\omega$. A *Muller acceptor* has the form $(\mathcal{M}, i, \mathcal{F})$, where $\mathcal{M}, i$ are as above and $\mathcal{F} \subseteq P(Q)$; it recognizes the set

$$L_\omega(\mathcal{M}, i, \mathcal{F}) = \{\xi \in X^\omega : In(f(i,\xi)) \in \mathcal{F}\}.$$

A *Mostowski acceptor* (known also as Rabin chain acceptor or pairity acceptor) has the form $(\mathcal{M}, i, \Omega)$, where $\mathcal{M}, i$ are as above and $\Omega = (E_1, F_1, \ldots, E_n, F_n)$ for some $E_1 \subseteq F_1 \subseteq \cdots \subseteq E_n \subseteq F_n \subseteq Q$; it recognizes the set

$$L_\omega(\mathcal{M}, i, \Omega) = \{\xi \in X^\omega : \exists k(In(f(i,\xi)) \cap E_k = \emptyset \wedge In(f(i,\xi)) \cap F_k \neq \emptyset)\}.$$

It is well known that Muller and Mostowski acceptors recognize the same ω-languages; these are called *regular ω-languages* or just regular sets. The class $\mathcal{R}$ of all regular ω-languages is a proper subclass of $B(\mathbf{\Sigma}^0_2)$ that in turn is a proper subclass of $\mathbf{\Delta}^0_3$. Büchi acceptors recognize a smaller class of sets, namely exactly the regular $\mathbf{\Pi}^0_2$-sets.

A *syncronous transducer* (over alphabets X, Y) is a tuple $\mathcal{T} = (Q, X, Y, f, g, i)$, also written as $\mathcal{T} = (\mathcal{M}, Y, g, i)$, consisting of an automaton $\mathcal{M}$ as above, an initial state i and an output function $g : Q \times X \to Y$. The output function is extended to the function $g : Q \times X^* \to Y^*$ defined by induction

$$g(q, \varepsilon) = \varepsilon, \quad g(q, u \cdot x) = g(q, u) \cdot g(f(q, u), x),$$

and to the function $g : Q \times X^\omega \to Y^\omega$ defined by

$$g(q, \xi) = g(q, \xi(0)) \cdot g(f(q, \xi(0)), \xi(1)) \cdot g(f(q, \xi[0,2)), \xi(2)) \cdots . \tag{1}$$

In other notation, $g(q,\xi) = lim_n g(q, \xi[n])$. The transducer $\mathcal{T}$ *computes* the function $g_{\mathcal{T}} : X^\omega \to Y^\omega$ defined by $g_{\mathcal{T}}(\xi) = g(i, \xi)$. If the output function is of the form $g : Q \to Y$ (i.e., does not really depend on the second argument), then $\mathcal{T}$ is called a *delayed syncronous transduser*.

An *asyncronous transducer* (over alphabets X, Y) is defined as a syncronous transducer with only one exeption: this time the output function g maps $Q \times X$ into Y^*. As a result, the value $g(q, \xi)$ defined as in (1) is in $Y^{\leq\omega}$, and the function $g_{\mathcal{T}}$ maps X^ω into $Y^{\leq\omega}$. Nevertheless, we usually consider the case when $g_{\mathcal{T}}$ maps X^ω into Y^ω; this condition is easily described in terms of $\mathcal{T}$.

Functions computed by syncronous (delayed syncronous, asyncronous) transducers are called DS-functions (respectively delayed DS-functions and DA-functions). As is well known, all tree classes of functions are closed under composition, and every DS-function (delayed DS-function, DA-function) is a CS-function (delayed CS-function, CA-function)

We will study several reducibilities on subsets of X^ω. For $A, B \subseteq X^\omega$, A is said to be *CA-reducible* to B (in symbols $A \leq_{CA} B$), if $A = g^{-1}(B)$ for some CA-function $g : X^\omega \to X^\omega$. The relations $\leq_{DA}$, $\leq_{CS}$ and $\leq_{DS}$ on $P(X^\omega)$ are defined in the same way but using the other three classes of functions. The introduced relations on $P(X^\omega)$ are preorderings. The CA-reducibility is widely known as Wadge reducibility, and CS-reducibility as Lipschitz reducibility. The other two reducibilities are effective automatic versions of these. By $\equiv_{CA}$ we denote the induced equivalence relation which gives rise to the corresponding quotient partial ordering. Following a well established jargon, we call this ordering the strudure of CA-degrees. The same applies to the other reducibilities (and to reducibilities to be introduced later). In the "alphabet-dependent mode", we say that $A \subseteq X^\omega$ is CA-reducible to $B \subseteq Y^\omega$, if $A = g^{-1}(B)$ for some CA-function $g : X^\omega \to Y^\omega$. Sometimes such variations are also of use.

The operation $A \oplus B = \{0 \cdot \alpha, i \cdot \beta | 0 < i < k, \alpha \in A, \beta \in B\}$ on subsets of X^ω, $X = \{0, \ldots, k-1\}$, induces the operation of least upper bound in the structures of degrees under all four reducibilities introduced above. Any level of the Borel hierarchy is closed under CA-reducibility (and thus under all four reducibilities) in the sense that every set reducible to a set in the level is itself in that level. The class $\mathcal{R}$ is closed under DA- and DS-reducibilities but is not closed under CA- and CS-reducibilities. Every $\mathbf{\Sigma}$-level $\mathcal{C}$ (and also every $\mathbf{\Pi}$-level) of the Borel hierarchy has a CA-complete set C which means that $\mathcal{C} = \{A : A \leq_{CA} C\}$.

More detailed information related to the introduced notions may be found in many sources including [Sta97, Th90, Th96, TB70, PP04].

3 Aperiodic Automata and Acceptors

Here we collect some facts on the regular aperiodic sets, the main object of this paper, and on the closely related aperiodic automata (known also as counter-free automata).

Aperiodic languages were characterized in several ways, in particular as: languages defined by extended regular star-free expressions; languages of words described by first-order sentences of a natural signature; languages of words satisfying a formula of linear time temporal logic; languages recognized by aperiodic acceptors. Similar characterizations exist also for regular aperiodic ω-languages (see e.g. [Th79, Th90, Th96] and references therein). It is well-known (and follows from the mentioned characterizations) that classes of regular aperiodic languages and ω-languages are closed under Boolean operations.

Let us illustrate the logical characterization by proving a fact which will be used later. First recall some relevant terminology. Relate to any alphabet X the signature $\sigma_X = \{<, Q_x : x \in X\}$ where $<$ is a binary predicate symbol and Q_x are unary predicate symbols. Relate to any $\xi \in X^\omega$ the structure $\tilde{\xi} = (\omega; \sigma_X)$ of signature σ_X where $<$ is the usual order on ω and $Q_x(i) \leftrightarrow \xi(i) = x$ for all $i < \omega$ and $x \in X$. The logical desription (essntially due to McNaughton and Papert) states that $A \subseteq X^\omega$ is regular aperiodic iff $A = L_\phi$ for some sentence ϕ of σ_X, where $L_\phi = \{\xi \in X^\omega : \tilde{\xi} \models \phi\}$.

Proposition 3.1 *Let X, Y be disjoint alphabets and $A \subseteq (X \times Y)^\omega$ be regular aperiodic. Then the language $A' = \{\langle \xi, \eta \rangle : \xi \in X^\omega, \eta \in Y^\omega, \xi \times \eta \in A\}$ is a regular aperiodic subset of $(X \cup Y)^\omega$. The same applies to any finite sequence of pairwise disjoint alphabets.*

Proof. Let θ be a sentence of $\sigma_{X \times Y}$ satisfying $A = L_\theta$. It suffices to find a sentence $\tilde{\theta}$ of $\sigma_{X \cup Y}$ satisfying $A' = L_{\tilde{\theta}}$. Since we use an induction on formulas, we are forced to employ also formulas with free variables, i.e. we want to relate to any formula $\phi(v_1, \ldots v_n)$ of $\sigma_{X \times Y}$ with free variables among $v_1, \ldots v_n$ a formula $\tilde{\phi}(v_1, \ldots v_n)$ of $\sigma_{X \cup Y}$ such that

$$\xi \times \eta \models \phi(i_1, \ldots i_n) \text{ iff } \widetilde{\langle \xi, \eta \rangle} \models \tilde{\phi}(2i_1, \ldots 2i_n) \tag{2}$$

for all $i_1, \ldots i_n < \omega$.

Note that in defining the formula $\tilde{\phi}$ below we use not only symbols from $\sigma_{X \cup Y}$ but also symbols 0 and +1 with the usual interpretation; this is possible since these new symbols are easily definable in terms of symbols in $\sigma_{X \cup Y}$. Relate first to any formula $\phi(v_1, \ldots v_n)$ of $\sigma_{X \times Y}$ a formula $\phi'(v_1, \ldots v_n)$ of $\sigma_{X \cup Y}$ by induction as follows:

if ϕ is $v_l < v_k$ or $v_l = v_k$ then ϕ' is ϕ;

if ϕ is $Q_{(x,y)}(v_l)$ then ϕ' is $Q_x(v_l) \wedge Q_y(v_l + 1)$;

if ϕ is $\phi_1 \wedge \phi_2$ then ϕ' is $\phi_1' \wedge \phi_2'$ and similarly for the other Boolean connectives;

if ϕ is $\exists v \phi_1$ then ϕ' is $\exists v(Q_X(v) \wedge \tilde{\phi}_1)$, where $Q_X(v)$ is $\bigvee_{x \in X} Q_x(v)$.

We did not consider $\forall$ since it may be expressed through $\exists$ and $\neg$. Finally, let

$$\tilde{\phi} = Q_X(0) \wedge \forall v(Q_X(v) \leftrightarrow \neg Q_X(v+1)) \wedge \phi'.$$

By induction on formulas it is easy to check the property (2). $\square$

For our paper, the characterization in terms of aperiodic acceptors is the most relevant. Let us recall the corresponding definition from [MP71].

Definition 3.2 *An automaton $\mathcal{M} = (Q, X, f)$ is aperiodic if for all $q \in Q$, $u \in X^+$ and $n > 0$ the equality $f(q, u^n) = q$ implies $f(q, u) = q$. This is clearly equivalent to say that for all $q \in Q$ and $u \in X^+$ there is $m < \omega$ with $f(q, u^{m+1}) = f(q, u^m)$. An acceptor (or a transducer) is aperiodic if so is the corresponding automaton.*

The next basic fact is essentially proved in [MP71]. The reader not familiar with aperiodic languages could take this fact as their definition.

Proposition 3.3 *(i) A regular language $A \subseteq X^*$ is aperiodic iff it is recognized by an aperiodic acceptor.*

(ii) A regular ω-language $A \subseteq X^\omega$ is aperiodic iff it is recognized by an aperiodic Muller acceptor.

We call a partial automaton $\mathcal{M} = (Q, X, f)$ *aperiodic* if for all $q \in Q$ and $u \in X^+$ there is $m < \omega$ such that either $f(q, u^{m+1})$ is undefined or $f(q, u^{m+1}) = f(q, u^m)$.

Proposition 3.4 *A regular language is aperiodic iff it is recognized by a partial aperiodic acceptor.*

Proof. One direction is trivial. For the other direction, let $\mathcal{M} = (Q, X, f)$ be a partial aperiodic acceptor; it suffices to find an aperiodic acceptor $\mathcal{M}'$ recognizing the same language. This is obtained by taking a new sink state s and by extending f to the total function f' satisfying $f'(q, x) = s$ whenever $f(q, x)$ was udefined. $\square$

Next we show that the class of aperiodic automata is closed under some operations. First consider the well known product operation. Relate to any automata $\mathcal{M}_k = (Q_k, X_k, f_k)$, $k < 2$, the automaton $\mathcal{M} = (Q, X, f)$ called the *product* of $\mathcal{M}_0$ and $\mathcal{M}_1$ and defined by

$$Q = Q_0 \times Q_1,\ X = X_0 \times X_1,\ f((q_0, q_1), (x_0, x_1)) = (f_0(q_0, x_0), f_1(q_1, x_1)).$$

For the particular case $X_0 = X_1 = X$ we can define a version of the product automaton over the alphabet X by $f((q_0, q_1), x)) = (f_0(q_0, x), f_1(q_1, x))$. Of course, both versions of product may be defined in the same way for any finite number k of automata, $2 < k < \omega$.

Proposition 3.5 *The class of aperiodic automata is closed under both kinds of product.*

Proof. Both assertions are proved similarly, so we consider only the second kind of product. For simplicity of notations consider only the case $k = 2$. Let $q = (q_0, q_1) \in Q$ and $u \in X^+$. Since $\mathcal{M}_k$ is aperiodic, there are numbers m_k, $k < 2$, with $f_k(q_k, u^{m_k+1}) = f_k(q_k, u^{m_k})$. For the number $m = max(m_0, m_1)$ we then have $f_k(q_k, u^{m+1}) = f_k(q_k, u^{m_k}) = f_k(q_k, u^m)$, $k < 2$, hence

$$f(q, u^{m+1}) = (f_0(q_0, u^{m+1}), f_1(q_1, u^{m+1}) = (f_0(q_0, u^m), f_1(q_1, u^m) = f(q, u^m).$$

$\square$

Next we consider an exotic operation which will be used only in Section 6. Relate to any automaton $\mathcal{M} = (Q, X, f)$ and sets $T_0, T_1 \subseteq Q$ the automaton $\mathcal{M}' = (Q', X, f')$ defined by

$$Q' = (\{0, 1\} \times (T_0 \cup T_1)) \cup (\{2\} \times \overline{(T_0 \cup T_1)})$$

and

$$f'((k, q), x) = \begin{cases} (2, f(q, x)) & \text{if } f(q, x) \notin T_0 \cup T_1 \\ (0, f(q, x)) & \text{if } f(q, x) \in T_0 \setminus T_1 \\ (1, f(q, x)) & \text{if } f(q, x) \in T_1 \setminus T_0 \\ (k, f(q, x)) & \text{if } f(q, x) \in T_0 \cap T_1 \wedge\ k < 2 \\ (0, f(q, x)) & \text{if } f(q, x) \in T_0 \cap T_1 \wedge\ k = 2. \end{cases}$$

Proposition 3.6 *The class of aperiodic automata is closed under the operation from the preceiding paragraph.*

Proof. Let $(k,q) \in Q'$ and $u \in X^+$. Since $\mathcal{M}$ is aperiodic, $f(q,u^{m+1}) = f(q,u^m)$ for some m. W.l.o.g. we may assume that $m = 0$, i.e. $f(q,u) = q$ (otherwise, take $f(q,u^m)$ in place of q). Now consider the following four cases.

Case 1. $q \notin T_0 \cup T_1$.

By definition of f', $f'((j,q),u) = (2,q)$ for every $j < 3$. Therefore, $f'((k,q),u^2) = (2,q) = f'((k,q),u)$, as desired.

Case 2. $q \in T_0 \setminus T_1$.

By definition of f', $f'((j,q),u) = (0,q)$ for every $j < 3$. Therefore, $f'((k,q),u^2) = (0,q) = f'((k,q),u)$, as desired.

Case 3. $q \in T_1 \setminus T_0$.

This case is similar to the previous one.

Case 4. $q \in T_0 \cap T_1$.

Let $n < 3$ be the unique number satisfying $f'((k,q),u) = (n,q)$. Since $q \in T_0 \cap T_1$, by definition of f' we actually have $n < 2$. It suffices to show that $f'((n,q),u) = (n,q)$. Let $u = x_0 \cdots x_l$, where $x_0,\ldots,x_l \in X$. For any $i \le l$, let $q_i = f(q, x_0 \cdots x_{i-1})$ and $k_i, n_i < 3$ be the unique numbers satisfying

$$(k_i,q_i) = f'(((k,q),x_0\cdots x_{i-1}) \text{ and } (n_i,q_i) = f'(((n,q),x_0\cdots x_{i-1}).$$

Then

$$f'((n,q),u) = f'((n_l,q_l),x_l) \text{ and } f'((k,q),u) = f'((k_l,q_l),x_l) = (n,q) = (n_0,q_0),$$

hence it suffices to show that $f'((n_l,q_l),x_l) = (n_0,q_0)$. If $q_1,\ldots,q_l \in T_0 \cap T_1$ then, by definition of f', $n_0 = \cdots = n_l < 2$, therefore $f'((n_l,q_l),x_l) = (n_0,q_0)$.

Now assume that $q_i \notin T_0 \cap T_1$ for some $i \in \{1,\ldots,l\}$. By definition of f', k_i and n_i depend only on q_i (and does not depend on k_{i-1} or n_{i-1}), hence $k_i = n_i$. Therefore,

$$(k_j,q_j) = f'((k_i,q_i),x_i\cdots x_{j-1}) = f'((n_i,q_i),x_i\cdots x_{j-1}) = (n_j,q_j)$$

for all $j \in \{i,\ldots,l\}$. Thus, $f'((n_l,q_l),x_l) = f'((k_l,q_l),x_l) = (n_0,q_0)$. □

Next consider an operation used earlier by several authors (see e.g. remarks in [Th96], Theorem 6.5). Relate to any automaton $\mathcal{M} = (Q,X,f)$ the automaton $\mathcal{M}' = (Q',X,f')$ defined as follows. W.l.o.g. we may assume that $Q = \{1,\ldots,n\}$. Let $S(Q)$ be be the set of permutations of Q; the permutations will be written as "vectors" $v = (v(1),\ldots,v(n))$. Let $Q' = Q \times S(Q)$ and for $(h,v) \in Q'$ and $x \in X$ let $f'((h,v),x) = (k,w)$ be the unique element of Q' satisfying

$$f(v(n),x) = v(k) \text{ and } w = (v(1),\ldots,v(k-1),v(k+1),\ldots,v(n),v(k)).$$

Proposition 3.7 *The class of aperiodic automata is closed under the operation from the preceiding paragraph.*

Proof. Let $(h,v) \in Q'$ and $u \in X^+$. Since $\mathcal{M}$ is aperiodic, $f(v(n),u^{m+1}) = f(v(n),u^m)$ for some m. As in the previous proof, w.l.o.g. we may assume that $m = 0$, i.e. $f(v(n),u) = v(n)$. It suffices to show that $f'((h,v),u^2) = f'((h,v),u)$. Let $u = x_0\cdots x_l$, where $x_0,\ldots,x_l \in X$. For any $i \le l+1$, let (h_i,v_i) and (k_i,w_i) be the unique elements of Q' satisfying

$$(h_i,v_i) = f'(((h,v),x_0\cdots x_{i-1}) \text{ and } (k_i,w_i) = f'(((h_{l+1},v_{l+1}),x_0\cdots x_{i-1}).$$

Since $(k_0, w_0) = f'(((h, v), u)$ and $(k_{l+1}, w_{l+1}) = f'((k_0, w_0), u)$, it remains to show that $(k_{l+1}, w_{l+1}) = (k_0, w_0)$.

By definition of f',

$$w_0(n) = f(v_0(n), u) = f(v(n), u) = v(n),\ w_{l+1}(n) = f(w_0(n), u) = f(v(n), u) = v(n)$$

and

$$f(v_i(n), x_i) = v_{i+1}(n),\ f(w_i(n), x_i) = w_{i+1}(n)$$

for all $i \leq l$. Thus, $w_{l+1}(n) = w_0(n)$ and $(v_0(n), \ldots, v_l(n)) = (w_0(n), \ldots, w_l(n))$. Omitting repetitions in the last sequence we obtain a sequence $(s_0, \ldots, s_{m-1})$, where m is the number of elements in $\{v_0(n), \ldots, v_l(n)\} = \{w_0(n), \ldots, w_l(n)\}$, $0 < m \leq n$.

By definition of f', the last m elements of vectors v_l and w_l are exactly $s_0, \ldots, s_{m-1}$, in this order. Since the first $n - m$ elements of vectors v_l and w_l are not influenced by the function f' (when computing $w_0, \ldots, w_l$) at all, we actually have $v_l = w_l$. Again by definition of f', k_0 and k_{l+1} are positions of the same element $w_0(n) = w_{l+1}(n)$ in the same vector $v_l = w_l$, thus

$$(k_0, w_0) = f'(((h_l, v_l), x_l) = f'(((k_l, w_l), x_l) = (k_{l+1}, w_{l+1}).$$

□

We conclude this section with "aperiodic analog" of a well known fact of automata theory.

Proposition 3.8 *A regular omega-language is aperiodic iff it is recognized by an aperiodic Mostowski acceptor.*

Proof. In the easy direction, we have to relate to any Mostowski aperiodic acceptor $(\mathcal{M}, i, \Omega)$, $\Omega = (E_1, F_1, \ldots, E_n, F_n)$, an aperiodic Muller acceptor recognizing the same language. It is obvious that the acceptor $(\mathcal{M}, i, \mathcal{F})$, where

$$\mathcal{F} = \{S \subseteq Q : \exists k \leq n(S \cap E_k = \emptyset \wedge S \cap F_k \neq \emptyset)\},$$

has the desired properties.

In the hard direction, we have to relate to any Muller aperiodic acceptor $(\mathcal{M}, i, \mathcal{F})$ an aperiodic Mostowski acceptor recognizing the same language A. Assuming w.l.o.g. that $Q = \{1, \ldots, n\}$ and $i = 1$, let $\mathcal{M}'$ be the automaton from the previous proposition, $i' = (1, (2, \ldots, n, 1))$ and $\Omega = (E_1, F_1, \ldots, E_n, F_n)$, where

$$E_1 = \{(n, v) : (\{v(n)\} \notin \mathcal{F})\},\ F_1 = E_1 \cup \{(n, v) : (\{v(n)\} \in \mathcal{F})\}$$

and, for every $i \in \{1, \ldots, n-1\}$,

$$E_{i+1} = F_i \cup \{(n - i + 1, v) : (\{v(i+1), \ldots, v(n)\} \notin \mathcal{F})\}$$

and

$$F_{i+1} = E_{i+1} \cup \{(n - i + 1, v) : (\{v(i+1), \ldots, v(n)\} \in \mathcal{F})\}.$$

By Proposition 3.6, $\mathcal{M}'$ is aperiodic. By the argument in the proof of Theorem 6.5 in [Th96] (that theorem is about tree automata but the argument works also for the word automata), $(\mathcal{M}', i', \Omega)$ recognizes the language A. □

4 Aperiodic Transducers

Here we establish some results on functions $g_{\mathcal{T}}$ computed by aperiodic transducers $\mathcal{T}$ (see Section 2). Though we are mostly interested in functions from X^ω to Y^ω, sometimes we are forced to consider a more general case of functions from X^ω to $Y^{\leq\omega}$.

Definition 4.1 *A function $h : X^\omega \to Y^{\leq\omega}$ is called an AA-function (an AS-function) if it is computed by an aperiodic asyncronous (respectively, aperiodic syncronous) transducer $\mathcal{T}$ over X, Y, i.e. $h = g_{\mathcal{T}}$.*

We start with a couple of examples of AA-functions.

Proposition 4.2 *Let $Y_0, \ldots, Y_n$ be pairwise disjoint alphabets, $Y = Y_0 \cup \cdots \cup Y_n$ and for every $k \leq n$ let $p_k : Y^\omega \to Y_k^{\leq\omega}$ be the function deleting all letters not in Y_k.*

(i) Any of $p_0, \ldots, p_n$ is an AA-function.

(ii) For any AA-functions $h_k : X^\omega \to Y_k^{\leq\omega}$, $k \leq n$, there is an AA-function $h : X^\omega \to Y^{\leq\omega}$ such that $h_k = p_k \circ h$ for all $k \leq n$.

Proof. (i) is easy, since any p_k is computed by a trivial transducer with one state.

(ii) For every $k \leq n$ let $(\mathcal{M}_k, Y_k, g_k, i_k)$ be an aperiodic asyncronous transduser that computes h_k. Consider the transducer $\mathcal{T} = (\mathcal{M}, Y, g, i)$, where $\mathcal{M}$ is the product of automata $\mathcal{M}_0, \ldots, \mathcal{M}_n$ with the common alphabet X, $i = (i_0, \ldots, i_n)$ and $g((q_0, \ldots, q_n), x) = g_0(q_0, x) \cdots g_n(q_n, x)$. By Proposition 3.5, $\mathcal{M}$ is aperiodic, hence $h = g_{\mathcal{T}}$ is an AA-function. Clearly, $h_k = p_k \circ h$ for all $k \leq n$. $\square$

Next consider a simple coding $e : Y^\omega \to X^\omega$ of ω-words in arbitrary alphabet $Y = \{y_1, \ldots, y_n\}$ by ω-words in alphabet X with $0, 1 \in X$ defined as follows: for every $k \in \{1, \ldots, n\}$ let $\tilde{y}_k = 01^k0$, and for every $\eta \in Y^\omega$ let $e(\eta) = \widetilde{\eta(0)}\widetilde{\eta(1)} \cdots$.

Proposition 4.3 *In notation of the previous paragraph, e is an AA-function and there is an AA-function $r : X^\omega \to Y^{\leq\omega}$ such that $r \circ e$ is the identity function on Y^ω.*

Proof. The function e is computed by a trivial transducer with only one state, hence it is an AA-function. Consider the transducer $\mathcal{T} = (Q, X, Y, f, g, i)$ where $Q = \{0, 1, \ldots, n+2\}$, $i = 0$ and f, g are defined as follows:

$f(0, 0) = 1$ and $g(0, 0) = \varepsilon$;

$f(1, 0) = n + 2$ and $g(1, 0) = \varepsilon$;

$f(k, x) = n + 2$ and $g(k, x) = \varepsilon$ for all $k < n + 2$ and $x \in X \setminus \{0, 1\}$;

$f(n + 2, x) = n + 2$ and $g(n + 2, x) = \varepsilon$ for all $x \in X$;

$f(k, 1) = k + 1$ and $g(k, 0) = \varepsilon$ for all $k \in \{1, \cdots, n\}$;

$f(k, 0) = 0$ and $g(k, 0) = y_{k-1}$ for all $k \in \{2, \cdots, n + 1\}$.

It is easy to check that automaton (Q, X, f) is aperiodic and the AA-function $r = g_{\mathcal{T}}$ has the desired property. $\square$

Next we state an important closure property of the introduced classes of functions.

Proposition 4.4 *The classes of AA-functions and of AS-functions are closed under composition.*

Proof. We consider only the asyncronous case but the proof applies also to the syncronous case. Let $h_k : X_k^\omega \to X_{k+1}^{\leq\omega}$, $k < 2$, be AA-functions computed by aperiodic transducers $\mathcal{T}_k = (Q_k, X_k, X_{k+1}, f_k, g_k, i_k)$. We have to show that composition $h_2 = h_1 \circ h_0 : X_0^\omega \to X_2^{\leq\omega}$ is an AA-function. Define a transducer $\mathcal{T} = (Q, X_0, X_2, f, g, i)$ as follows: $Q = Q_0 \times Q_1$, $i = (i_0, i_1)$ and

$$f((q_0, q_1), x) = (f_0(q_0, x), f_1(q_1, g_0(q_0, x)), \; g((q_0, q_1), x) = g_1(q_1, g_0(q_0, x)).$$

As is well-known and easy to see, the transducer $\mathcal{T}$ computes h_2, hence it remains to show that automaton $\mathcal{M} = (Q, X_0, f)$ is aperiodic.

Let $q = (q_0, q_1)$ and $u \in X_0^+$. Since (Q_0, X_0, f_0) is aperiodic, $f_0(q_0, u^{m+1}) = f_0(q_0, u^m)$ for some m. As above, w.l.o.g. we may assume that $m = 0$, i.e. $f_0(q_0, u) = q_0$. For any $n < \omega$ we have $f(q, u^n) = (q_0, f_1(q_1, g_0(q_0, u^n)))$. Set $v = g_0(q_0, u) \in X_1^*$ and consider two cases. If $v = \varepsilon$ then $f(q, u) = (q_0, f_1(q_1, v)) = q$, hence $\mathcal{M}$ is aperiodic. Now let $v \neq \varepsilon$. Since (Q_1, X_1, f_1) is aperiodic, $f_1(q_1, v^{m+1}) = f_1(q_1, v^m)$ for some m, and again we may assume that $m = 0$, i.e. $f_1(q_1, v) = q_1$. We then have $f(q, u) = (q_0, f_1(q_1, v)) = q$, hence $\mathcal{M}$ is aperiodic. $\square$

The last proposition implies some natural properties of the following reducibilities.

Definition 4.5 *A set $A \subseteq X^\omega$ is AA-reducible (AS-reducible) to a set $B \subseteq Y^\omega$, in symbols $A \leq_{AA} B$ ($A \leq_{AS} B$) if $A = g^{-1}(B)$ for some AA-function (respectively, AS-function) $g : X^\omega \to Y^\omega$. For $X = Y$ we obtain relations $\leq_{AA}$ and $\leq_{AS}$ on $P(X^\omega)$ called AA- and AS-reducibilities.*

Corollary 4.6 *The relations $\leq_{AA}$ and $\leq_{AS}$ on $P(X^\omega)$ are preorders. The corresponding quotient partial orderings of AA- and AS-degrees are upper semilattices under operation induced by the operations $\oplus$ from Section 2.*

Now we relate AA-transducers to regular aperiodic sets. We say that a set is *Büchi aperiodic* if it is recognized by an aperiodic Büchi acceptor.

Proposition 4.7 *Let $h : X^\omega \to Y^{\leq\omega}$ be an AA-function.*

(i) For every $F \subseteq Y$, the set $A_F = \{\xi \in X^\omega : In(h(\xi)) \cap F \neq \emptyset\}$ is Büchi aperiodic.

(ii) For every $\mathcal{F} \subseteq P(Y)$, the set $A_{\mathcal{F}} = \{\xi \in X^\omega : In(h(\xi)) \in \mathcal{F}\}$ is regular aperiodic.

Proof. (i) Let (Q, X, Y, f, g, i) be an AA-transducer that computes h. Let $V = g(Q, X) \subseteq Y^*$ and v_0 be a fixed word from V. Define a function $p : (Q \times V) \times X \to Q \times V$ by $p((q, v), x) = (f(q, x), g(q, x))$. Then the automaton $\mathcal{M} = (Q \times V, X, p)$ is aperiodic. Indeed, let $(q, v) \in Q \times V$ and $u = x_0 \cdots x_l \in X^+$ where $x_0, \ldots, x_l \in X$. Since (Q, X, f) is aperiodic, $f(q, u^{m+1}) = f(q, u^m)$ for some m. As above, we may assume that $m = 0$, i.e. $f(q, u) = q$. From definition of p it follows that $p((q, v), u) = (q, g(f(q, x_0 \cdots x_{l-1}), x_l))$, hence $\mathcal{M}$ is aperiodic. Let $G = \{(q, v) \in Q \times V : \exists k(v(k) \in F)\}$. Then the aperiodic Büchi acceptor $(\mathcal{M}, (i, v_0), G)$ recognizes A_F.

(ii) The set $A_{\mathcal{F}}$ is a Boolean combination of regular aperiodic sets A_F, $F \subseteq Y$, hence it is regular aperiodic. $\square$

Proposition 4.8 *The classes of Büchi aperiodic and of regular aperiodic ω-languages are closed under preimages of AA-functions.*

Proof. Let $h : X^\omega \to Y^{\leq\omega}$ be an AA-function computed by an AA-transducer $\mathcal{T}_0 = (Q_0, X, Y, f_0, g_0, i_0)$ and let $(\mathcal{M}_1, i_1, F)$ ($(\mathcal{M}_1, i_1, \mathcal{F})$) be an aperiodic Büchi (respectively, Muller) acceptor recognizing a set $A \subseteq Y^\omega$. We have to show that $h^{-1}(A)$ is Büchi (respectively, regular) aperiodic.

Let $p : Y^\omega \to Q_1^\omega$ be the AS-function computed by transducer $\mathcal{T}_1 = (Q_1, X, Q_1, f_1, f_1, i_1)$. Let $\mathcal{T}$ be the AA-transduser constructed from $\mathcal{T}_0$ and $\mathcal{T}_1$ as in the proof of Proposition 4.4 (for $X_0 = X$, $X_1 = Y$, $X_2 = Q_1$ and $g_1 = f_1$). Then $g_{\mathcal{T}} = p \circ h$, hence

$$h^{-1}(A) = \{\xi \in X^\omega : In(g_{\mathcal{T}}(\xi)) \cap F \neq \emptyset\} \text{ (resp., } h^{-1}(A) = \{\xi \in X^\omega : In(g_{\mathcal{T}}(\xi)) \in \mathcal{F}\}.$$

Therefore, $h^{-1}(A)$ is Büchi (respectively, regular) aperiodic. $\square$

Corollary 4.9 *The classes of Büchi aperiodic and of regular aperiodic sets in X^ω are closed under AA- and AS-reducibilities.*

For a function $h : X^\omega \to Y^\omega$, let $G_h = \{\xi \times h(\xi) : \xi \in X^\omega\}$ be the "graph" of h, $G_h \subseteq (X \times Y)^\omega$.

Proposition 4.10 *The grapf of an AS-function is regular aperiodic.*

Proof. Let (Q, X, Y, f, g, i) be an AS-transducer that computes a function $h : X^\omega \to Y^\omega$. Define the automaton $\mathcal{M} = (Q', X \times Y, f'$ by $Q' = Q \cup \{s\}$, where $s \notin Q$, and

$$f'(r, (x, y)) = \begin{cases} f(r, x) & \text{if} \quad r \in Q \wedge g(r, x) = y \\ s & \text{otherwise.} \end{cases}$$

It is easy to see that $\mathcal{M}$ is aperiodic and the Muller acceptor $(\mathcal{M}, i, P(Q))$ recognizes the graph G_h. $\square$

5 Aperiodic Determinacy

Here we prove an "aperiodic version" of the Büchi-Landweber regular determinacy theorem. This may be of independent interest and is also an important technical tool to prove some results below.

We start with recalling some relevant information on Gale-Stewart games. Relate to any set $A \subseteq (X \times Y)^\omega$ the Gale-Steweart game $G(A)$ played by two opponents 0 and 1 as follows. Player 0 chooses a letter $x_0 \in X$, then player 1 chooses a letter $y_0 \in Y$, then 0 chooses $x_1 \in X$, then 1 chooses $y_1 \in Y$ and so on. Each player knows all the previous moves. After ω moves, 0 has constructed a word $\xi = x_0 x_1 \cdots \in X^\omega$ while 1 has constructed a word $\eta = y_0 y_1 \cdots \in Y^\omega$. Player 1 wins this particular play if $\xi \times \eta \in A$, otherwise 0 wins.

A *strategy for player* 1 (0) in the game $G(A)$ is a function $h : X^+ \to Y$ (respectively, $h : Y^* \to X$) that prompts the 1's move (respectively, the 0's move) for any finite string of the opponent's previous moves. It is clear that strategies for 1 (for 0) are in a bijective correspondence with CS-functions $h : X^\omega \to Y^\omega$ (respectively, with delayed CS-functions $h : Y^\omega \to X^\omega$); we identify strategies with the corresponding CS-functions.

A strategy h for player 1 (0) in the game $G(A)$ is *winning* if the player wins each play when following the strategy, i.e. if $\xi \times h(\xi) \in A$ for all $\xi \in X^\omega$ (respectively, $h(\eta) \times \eta \in \overline{A}$ for all $\eta \in Y^\omega$). A set $A \subseteq (X \times Y)^\omega$ is *determined* if one of the players has a winning strategy in $G(A)$. It is interesting and useful to know which sets are determined and, in case of determinacy, how complicated it is to find the winner and how complicated is his winning strategy.

One of the best results of descriptive set theory is Martin determinacy theorem (see e.g. [Ke94]) stating that any Borel set is determined. Note that, since any regular set is Borel, this implies determinacy of regular sets. One of the best results of automata theory is Büchi-Landweber regular determinacy theorem stating that for any regular set A the winner in $G(A)$ may be computed effectively, he has a winning strategy which is a DS-function, and the strategy is also computed effectively.

To prove the aperiodic version of the Büchi-Landweber theorem, we need some information about games played on bipartite graph. Such games were considered by several authors (see e.g. a discussion in [Th96]). Recall that a *game graph* is a tuple $G = (V_0, V_1, E, c, C)$ where V_0, V_1 are (at most) countable disjoint non-empty sets of vertices, $E \subseteq (V_0 \times V_1) \cup (V_0 \times V_1)$ is a set of edges satisfying $\forall v \in V_i \exists w \in V_{1-i}((v, w) \in E)$ for $i < 2$, C is a finite set and $c : V \to C$, where $V = V_0 \cup V_1$. Games over G are again played by 0 and 1. The set V_i is intended as the set of game positions where it is the turn of player i to move. Winning conditions for such graph games may be specified in different ways.

E.g., by *Mostowski game* we mean a triple (G, v, Ω), where G is a game graph as above, $v \in V_0$ and $\Omega = (E_1, F_1, \ldots, E_n, F_n)$ is a sequence of subsets of C with $E_1 \subseteq F_1 \subseteq \cdots \subseteq E_n \subseteq F_n$. A *play* in a Mostowski game (G, v, Ω) is a sequence $\gamma \in V^\omega$ such that $\gamma(0) = v$ and $(\gamma(k), \gamma(k+1)) \in E$ for all $k < \omega$. Player 1 wins the play γ iff $\exists k (In(c \circ \gamma) \cap E_k = \emptyset \wedge In(c \circ \gamma) \cap F_k \neq \emptyset)$. A *memoryless strategy* for player i is a function $s : V_i \to V_{1-i}$ such that $(v, s(v)) \in E$ for all $v \in V_i$. A play γ *fits* a strategy s for i if $s(\gamma(2k+i)) = \gamma(2k+i+1)$ for all $k < \omega$. A strategy s for i is *winning* if i wins every play that fits the strategy s. For a discussion and proof of the next important result known as memoryless determinacy theorem see e.g. [Th96].

Proposition 5.1 *Let (G, v, Ω) be a Mostowski game as above. Then one of players has a memoryless winning strategy. If V is finite then the winner and some of his memoryless winning strategies may be found effectively.*

Now we are ready to prove the aperiodic version of the Büchi-Landweber theorem.

Theorem 5.2 *For any regular aperiodic set $A \subseteq (X \times Y)^\omega$ the winner of the game $G(A)$ may be computed effectively, he has a winning strategy which is an AS-function, and the strategy is also computed effectively.*

Proof. W.l.o.g. we may assume X and Y to be disjoint. By Proposition 3.1, $A' = \{\langle \xi, \eta \rangle : \xi, \eta \in A\}$ is a regular aperiodic subset of $(X \cup Y)^\omega$. By Proposition 3.8, A' is recognized by an aperiodic Mostowski acceptor $(\mathcal{M}, i, \Omega)$, where $\mathcal{M} = (Q, X \cup Y, f)$ and $\Omega = (E_1, F_1, \ldots, E_n, F_n)$, $E_1 \subseteq F_1 \subseteq \cdots \subseteq E_n \subseteq F_n \subseteq Q$.

Define a finite game graph $G = (V_0, V_1, E, c, C)$ by

$$V_0 = Q \times \{0\},\ V_1 = Q \times \{1\},\ C = Q,\ c(q,k) = q,$$

and

$$E = \{((q,0),(r,1)) : \exists x \in X(f(q,x) = r)\} \cup \{((q,1),(r,0)) : \exists y \in Y(f(q,y) = r)\}$$

and consider the Mostowski game (G, v, Ω), where $v = (i, 0)$. By Proposition 5.1, some player i has a memoryless winning strategy $s : V_i \to V_{1-i}$, and the winner and the strategy are computed effectively.

Let first $i = 1$. Then $\forall q \in Q \exists y \in Y(s(q,1) = (f(q,y),0))$, so there is a function $s' : Q \to Y$ with $s(q,1) = (f(q,s'(q)),0)$. Define a function $q : Q \times X \to Y$ by $g(q,x) = s'(f(q,x))$. Let $h : X^\omega \to Y^\omega$ be the AS-function computed by the aperiodic transducer (Q, X, Y, f_X, g, i), where f_X is the restriction of f to $Q \times X$. Then h is a winning AS-strategy for 1 in $G(A)$, i.e. $\xi \times h(\xi) \in A$ for all $\xi \in X^\omega$. Indeed, let $f(i, \xi \times h(\xi)) = q_0 q_1 \cdots$. Then the play $(q_0,0),(q_1,1),(q_2,0),(q_3,1),\ldots$ in (G, v, Ω) fits the strategy s, hence 1 wins this play which means $\langle \xi, h(\xi) \rangle \in A'$, i.e. $\xi \times h(\xi) \in A$.

Now let $i = 0$. Then $\forall q \in Q \exists x \in X(s(q,0) = (f(q,x),1))$, so there is a function $g : Q \to X$ with $s(q,0) = (f(q,g(q)),1)$. Let $h : X^\omega \to Y^\omega$ be the delayed AS-function computed by the aperiodic transducer (Q, Y, X, f_Y, g, i), where f_Y is the restriction of f to $Q \times Y$. Then h is a winning AS-strategy for 0 in $G(A)$, i.e. $h(\eta) \times \eta \in A$ for all $\eta \in Y^\omega$. Indeed, let $f(i, h(\eta) \times \eta) = q_0 q_1 \cdots$. Then the play $(q_0,0),(q_1,1),(q_2,0),(q_3,1),\ldots$ in (G, v, Ω) fits the strategy s, hence 0 wins this play which means $\langle h(\eta, \eta \rangle \in A'$, i.e. $\in h(\eta) \times \eta \in A$. □

The last result and proof have several interesting corollaries. E.g. the proof works also for the case when A is regular (but not necessatily aperiodic) and yields the Büchi-Landweber theorem in the source form [BL69]. We conclude this section with a couple of other corollaries relevant to furter results of this paper.

Corollary 5.3 *(i) Let $A \subseteq (X \times Y)^\omega$ be regular aperiodic and player i has a winning CS-strategy in the game $G(A)$. Then i has also a winning AS-strategy.*

(ii) Let $B \subseteq X^\omega$ and $C \subseteq Y^\omega$ be regular aperiodic and let $h : X^\omega \to Y^\omega$ be a CS-function satisfying $B = h^{-1}(C)$. Then $B = g^{-1}(C)$ for some AS-function $g : X^\omega \to Y^\omega$.

(iii) Let $B, C \subseteq X^\omega$ be regular aperiodic. Then $B \leq_{AS} C$ or $\overline{C} \leq_{AS} B$.

Proof. (i) Suppose i has no winning AS-strategy in $G(A)$. By Theorem 5.2, player $1-i$ has a winning AS-strategy, hence also a winning CS-strategy in $G(A)$. Thus, both players have winning strategies in $G(A)$, a contradiction.

(ii) Let $A = \{\xi \times \eta : \xi \in B \leftrightarrow \eta \in C\}$. By Proposition 4.2, A is a regular aperiodic subset of $(X \times Y)^\omega$. Since $B = h^{-1}(C)$, h is a winning CS-strategy for 1 in the game $G(A)$. By (i), 1 has also a winning AS-strategy g in $G(A)$. Therefore, $B = g^{-1}(C)$.

(iii) Let A be as in the proof of (ii). If 1 has a winning strategy in $G(A)$ then, by (ii), $B \leq_{AS} C$. Otherwise, 0 has a winning strategy in $G(A)$, hence, by (ii), $\overline{C} \leq_{AS} B$. □

Define subsets K_0, K_1 of 2^ω which play a noticible role in further considerations by

$$K_0 = \{\xi : \exists k(\xi(k) = 1)\} \text{ and } K_1 = \{\xi : \exists l \forall k > l(\xi(k) = 0)\}.$$

Corollary 5.4 *(i) The sets K_0 and $\overline{K}_1$ are Büchi aperiodic.*

(ii) $K_0 \in \mathbf{\Sigma}^0_1 \setminus \mathbf{\Pi}^0_1$ and $K_1 \in \mathbf{\Sigma}^0_2 \setminus \mathbf{\Pi}^0_2$.

(iii) Any $\mathbf{\Sigma}^0_1$-set ($\mathbf{\Sigma}^0_2$-set) $B \subseteq X^\omega$ is CS-reducible to K_0 (respectively, to K_1).

(iv) Any regular $\mathbf{\Sigma}^0_1$-set (regular $\mathbf{\Sigma}^0_2$-set) $B \subseteq X^\omega$ is DS-reducible to K_0 (respectively, to K_1).

(v) Any regular aperiodic $\mathbf{\Sigma}^0_1$-set (regular aperiodic $\mathbf{\Sigma}^0_2$-set) $B \subseteq X^\omega$ is AS-reducible to K_0 (respectively, to K_1).

Proof. The assertion (i) is obvious while (ii)—(iv) are well known. The assertion (v) follows from (iii) and Corollary 5.3(ii). □

Finally, we prove the following "aperiodic version" of a well known characterization of DS-functions (see e.g. [TB70]).

Corollary 5.5 *A CS-function $h : X^\omega \to Y^\omega$ is an AS-function iff its graph G_h is a regular aperiodic subset of $(X \times Y)^\omega$.*

Proof. From left to right, this is Proposition 4.10. Conversely, let $A = G_h$ be regular aperiodic. Then h is a winning strategy for 1 in the game $G(A)$. By Corollary 5.3(iii), there is a winning strategy g for 1 in $G(A)$ which is an AS-function. Since $G_h = G_g$, we have $h = g$. □

6 Aperiodic $\mathbf{\Sigma}^0_n$-Sets

Here we establish some facts on regular aperiodic sets in the Borel hierarchy.

First we recall some well known characterizations of regular $\mathbf{\Sigma}^0_n$-sets from. For a Muller acceptor $(\mathcal{M}, i, \mathcal{F})$, $\mathcal{M} = (Q, X, f)$, let $\mathcal{E} = \{In(f(i, \xi) : \xi \in X^\omega\}$. Thus, $\mathcal{E}$ is the class of strongly connected sets of vertices of the graph of $\mathcal{M} = (Q, X, f)$ reachable from i. Since $(\mathcal{M}, i, \mathcal{F} \cap \mathcal{E})$ recognizes the same ω-language as $(\mathcal{M}, i, \mathcal{F})$ does, we may w.l.o.g. think that $\mathcal{F} \subseteq \mathcal{E}$. Define a preorder $\leq_0$ and a partial order $\leq_1$ on $\mathcal{E}$ as follows: $U \leq_1 V$, if $U \supseteq V$, and $U \leq_0 V$, if for any $q \in U$ there exists a $w \in X^*$ with $f(q, w) \in V$. The next result is due to L. Landweber and K. Wagner, see e.g. [Wag79, Th90].

Proposition 6.1 *Let $(\mathcal{M}, i, \mathcal{F})$, $\mathcal{F} \subseteq \mathcal{E}$, be a Muller acceptor recognizing a set $A \subseteq X^\omega$. Then $A \in \mathbf{\Sigma}^0_1$ ($A \in \mathbf{\Sigma}^0_2$) iff $\mathcal{F}$ is closed upwards in $(\mathcal{F}; \leq_0)$ (respectively, in $(\mathcal{F}; \leq_1)$).*

Our first result is an "aperiodic version" of slightly different Landweber's characterizations of regular $\mathbf{\Sigma}^0_n$-sets (see e.g. [Th90]).

Theorem 6.2 *(i) A regular aperiodic set $A \subseteq X^\omega$ is $\mathbf{\Sigma}^0_1$ iff $A = W \cdot X^\omega$ for some regular aperiodic set $W \subseteq X^*$.*

(ii) A regular aperiodic set $A \subseteq X^\omega$ is $\mathbf{\Pi}^0_2$ iff A is Büchi aperiodic.

Proof. (i) Let A be $\mathbf{\Sigma}^0_1$ and $(\mathcal{M}, i, \mathcal{F})$ be an aperiodic Muller acceptor recognizing A. Let W be the regular aperiodic set recognized by the acceptor $(\mathcal{M}, i, F)$, where $F = \cup\mathcal{F}$. From Proposition 6.1(i) it easily follows that $A = W \cdot X^\omega$, as desired. The opposite implication is trivial (e.g., it follows immediately from the well known characterization of regular aperiodic ω-languages in terms of regular expressions [Th90]).

(ii) Let $A \subseteq X^\omega$ be regular aperiodic and $\mathbf{\Pi}^0_2$. By Corollary 5.4(v), $\overline{A} = g^{-1}(K_1)$ for some AS-function $h : X^\omega \to 2^\omega$, so $A = g^{-1}(\overline{K_1})$. By Corollary 5.4(i), $\overline{K_1}$ is Büchi aperiodic. By Proposition 4.8, so is also the set A, as desired. The opposite implication is trivial. □

Recall that a class $\mathcal{C}$ has the *reduction property*, if for all $C_0, C_1 \in \mathcal{C}$ there are disjoint $C'_0, C'_1 \in \mathcal{C}$ such that $C'_i \subseteq C_i$ for both $i < 2$ and $C_0 \cup C_1 = C'_0 \cup C'_1$; such a pair (C'_0, C'_1) is called a *reduct* of (C_0, C_1).

A class $\mathcal{C}$ has the *separation property*, if all disjoint sets $C_0, C_1 \in \mathcal{C}$ are separable by a set $B \in \mathcal{C} \cap \check{\mathcal{C}}$ (i.e. $C_0 \subseteq B \subseteq \overline{C}_1$). It is well known that if a class $\mathcal{C}$ has the reduction property then the dual class $\check{\mathcal{C}}$ has the separation property.

As is well known [Ke94], any level $\mathbf{\Sigma}^0_n$ of the Borel hierarchy has the reduction property. In [Se98] it was shown that classes $\mathcal{L}_n = \mathcal{R} \cap \mathbf{\Sigma}^0_{n+1}$, $n < 2$, have the reduction property (this trivally holds also for $n \geq 2$, because in this case we have $\mathcal{L}_n = \mathcal{R}$ and this class is closed under the Boolean operations). The next result is an "aperiodic version" of the last fact. Let $\mathcal{K}_n = \mathcal{A} \cap \mathbf{\Sigma}^0_{n+1}$, where $\mathcal{A}$ is the class of regular aperiodic ω-languges. Since again $\mathcal{K}_n = \mathcal{A}$ is a Boolean algebra for $n \geq 2$, these classes trivially have the reduction property.

Theorem 6.3 *The classes $\mathcal{K}_0$ and $\mathcal{K}_1$ have the reduction property.*

Proof. Let $C_0, C_1 \in \mathcal{K}_0$. By Theorem 6.2(i), $C_k = W_k \cdot X^\omega$, $k < 2$, for some regular aperiodic sets $W_0, W_1 \subseteq X^*$. Let $C'_k = W'_k \cdot X^\omega$, $k < 2$, where $W'_0, W'_1 \subseteq X^*$ are defined by

$$W'_0 = \{u \in W_0 : \neg \exists v \sqsubset u (v \in W_0 \cup W_1)\}$$

and

$$W'_1 = \{u \in W_1 \setminus W_0 : \neg \exists v \sqsubseteq u (v \in W_0 \cup W_1)\}.$$

It is easy to check that (C'_0, C'_1) is a reduct of (C_0, C_1), hence, by By Theorem 6.2(i), it remains to show that W'_0 and W'_0 are regular aperiodic.

For any $k < 2$, let $(\mathcal{M}_k, i_k, F_k)$, $\mathcal{M}_k = (Q_k, X, f_k)$, be an aperiodic acceptor recognizing W_k. Let $\mathcal{M} = (Q, X, f)$ be the product of automata $\mathcal{M}_0$ and $\mathcal{M}_1$. By Proposition 3.5, the automaton $\mathcal{M}$ is aperiodic. It is clear that W_0 (W_1) is recognized by $(\mathcal{M}, i, F_0 \times Q_1)$ (respectively, by $(\mathcal{M}, i, Q_0 \times F_1)$), where $i = (i_0, i_1)$. Let $\mathcal{M}'$ be the partial automaton obtained from (the graph of) $\mathcal{M}$ by removing all edges with origins (q_0, q_1) such that $q_0 \in F_0$ or $q_1 \in F_1$. It is easy to see that W'_0 (W'_1) is recognized by the partial aperiodic acceptor $(\mathcal{M}', i, F_0 \times Q_1)$ (respectively $(\mathcal{M}', i, (Q_0 \setminus F_0) \times F_1)$. By Proposition 3.4, W'_0 and W'_1 are regular aperiodic.

It remains to prove the reduction property for $\mathcal{K}_1$. Let $C_0, C_1 \in \mathcal{K}_1$. By Theorem 6.2(ii), $\overline{C}_k$, $k < 2$, is recognized by an aperiodic Büchi acceptor $(\mathcal{M}_k, i_k, F_k)$, where $\mathcal{M}_k = (Q_k, X, f_k)$. Let $\mathcal{M} = (Q, X, f)$ be the product of automata $\mathcal{M}_0$ and $\mathcal{M}_1$. By Proposition 3.5, the automaton $\mathcal{M}$ is aperiodic. It is clear that $C_k = \{\xi \in X^\omega : In(f(i, \xi)) \subseteq T'_k\}$ where $k < 2$, $i = (i_0, i_1)$, $T_0 = (Q_0 \setminus F_0) \times Q_1$ and $T_1 = Q_0 \times (Q_1 \setminus F_1)$. Let $\mathcal{M}' = (Q', X, f')$ be the automaton constructed from $\mathcal{M}$ and sets T_0, T_1 as in Proposition 3.6. By that proposition, the automaton $\mathcal{M}'$ is aperiodic. Let $C'_k = \{\xi \in X^\omega : In(f'(i', \xi)) \subseteq T'_k\}$ where $k < 2$, $i' = (i, 0)$ and $T'_k = \{k\} \times T_k$. By Proposition 6.2(ii), $C'_0, C'_1 \in \mathcal{K}_1$, so it remains to check that (C'_0, C'_1) is a reduct of (C_0, C_1).

Since T'_0 and T'_1 are disjoint, so are also C'_0 and C'_1. The inclusion $C'_k \subseteq C_k$, $k < 2$, follows easily from definition of f' (see Proposition 3.6). It remains to show the inclusion $C_0 \cup C_1 \subseteq C'_0 \cup C'_1$. Let $\xi \in C_0 \cup C_1$, then $\xi \in C_0 \setminus C_1$ or $\xi \in C_1 \setminus C_0$ or $\xi \in C_0 \cap C_1$. In the first case we have $In(f(i, \xi)) \subseteq T_0$ and $In(f(i, \xi)) \not\subseteq T_1$. In other words, $r_l \in T_0$ for almost all l and $r_l \in T_0 \setminus T_1$ for infinitely many l, where $r_0 r_1 \cdots = In(f(i, \xi))$. By definition of f', $In(f'(i', \xi)) \subseteq T'_0$, hence $\xi \in C'_0 \subseteq C'_0 \cup C'_1$. The second case is considered similarly. Finally, let $\xi \in C_0 \cap C_1$. Then there is l_0 such that $r_l \in T_0 \cap T_1$ for all $l \geq l_0$. By definition of f', there is $k < 2$ such that $r_l = (k, q_l)$ for all $l \geq l_0$. Therefore, $\xi \in C'_k \subseteq C'_0 \cup C'_1$. $\square$

Corollary 6.4 *The classes $\check{\mathcal{K}}_0$ and $\check{\mathcal{K}}_1$ have the separation property.*

Next we want to establish the "aperiodic version" of a theorem due to L. Staiger and K. Wagner [SW74] but first we recall some facts on the difference hierarchy. Let S be a set and $\mathcal{C} \subseteq P(S)$ a class of sets closed under finite unions and intersections. For any $n < \omega$, let $D_n(\mathcal{C})$ be the n-th level of the difference hierarchy over $\mathcal{C}$ consisting of sets $\cup_i (A_{2i} \setminus A_{2i+1})$, where $A_i \in \mathcal{C}$, $A_0 \supseteq A_1 \supseteq \cdots$ and $A_n = \emptyset$. The next fact is well known.

Proposition 6.5 *In notation of the previous paragraph, $D_n(\mathcal{C}) \cup \check{D}_n(\mathcal{C}) \subseteq D_{n+1}(\mathcal{C})$ for every $n < \omega$, and $\cup_n D_n(\mathcal{C}) = B(\mathcal{C})$.*

For $\xi \in X^\omega$ and $x \in X$, let $\#_x(\xi)$ denote the number of occurences of x in ξ, thus $\#_x(\xi) \leq \omega$. For any $n < \omega$, let

$$B_n = \{\xi \in X^\omega : \#_1(\xi) \geq n\},\ C_{2n} = \bigcup_{i<n}(B_{2i+1} \setminus B_{2i+2}) \text{ and } C_{2n+1} = C_{2n} \cup B_{2n+1}$$

(recall that $0, 1 \in X$).

Proposition 6.6 *For any $n < \omega$, C_n is regular aperiodic and CS-complete (DS-complete, AS-complete) in $D_n(\mathbf{\Sigma}^0_1)$ (respectively in $D_n(\mathcal{L}_0)$, $D_n(\mathcal{K}_0)$).*

Proof. The regular aperiodicity of C_n is obvious. Its CS-completness in $D_n(\mathbf{\Sigma}^0_1)$ is well known and easy to check. Its DS-completness in $D_n(\mathcal{L}_0)$ follows from results in [Wag79]. Its AS-completness in $D_n(\mathcal{K}_0)$ follows from Corollary 5.3(ii). □

Now we are ready to establish the "aperiodic version" of the Staiger-Wagner theorem.

Theorem 6.7 *Every regular aperiodic $\mathbf{\Delta}^0_2$-set is a Boolean combination of open regular aperiodic sets. In symbols, $\mathcal{K}_1 \cap \check{\mathcal{K}}_1 = B(\mathcal{K}_0)$.*

Proof. The inclusion $B(\mathcal{K}_0) \subseteq \mathcal{K}_1 \cap \check{\mathcal{K}}_1$ is obvious. Conversely, let $A \in \mathcal{K}_1 \cap \check{\mathcal{K}}_1$, then $A \in \mathcal{L}_1 \cap \check{\mathcal{L}}_1$. By the Staiger-Wagner theorem, $A \in B(\mathcal{L}_0)$. By Proposition 6.5, $A \in D_n(\mathcal{L}_0)$ for some $n < \omega$. By Proposition 6.6, $A \leq_{DS} C_n$. Since A is regular aperiodic, by Proposition 6.6 we have $A \leq_{DS} C_n$. Therefore, $A \in D_n(\mathcal{K}_0) \subseteq B(\mathcal{K}_0)$. □

We conclude this section with a corollary which is crucial for the subsequent sections. Recall that *a base* (in a set S) is a sequence $\{L_n\}_{n<\omega}$ of sublattices of $(P(S); \cup, \cap, 0, 1)$ satisfying $L_n \cup \check{L}_n \subseteq L_{n+1}$.

Definition 6.8 *(i) A base L is reducible, if every L_n has the reduction property.*

(ii) A base L is interpolable, if for all $n < \omega$ any two disjoint elements $A, B \in \check{L}_{n+1}$ are separable by a Boolean combination of elements of L_n.

In [Se98] we have shown that the base $\{\mathcal{L}_n\}_{n<\omega}$ is reducible and interpolable. From results of this section we obtain

Corollary 6.9 *The base $\{\mathcal{K}_n\}_{n<\omega}$ is reducible and interpolable.*

Proof. The reducibility was established earlier. Since $\mathcal{A} = B(\mathcal{K}_1) = \mathcal{K}_n$ for every $n > 1$, the interpolability propery for levels $\mathcal{K}_n$, $n > 1$, is obvious. It remains to show that any two disjoint sets $A, B \in \check{\mathcal{K}}_1$ are separable by a Boolean combination of elements of $\mathcal{K}_0$. Since, by Corollary 6.4, the class $\check{\mathcal{K}}_1$ has the separation property, A is separated from B by a set in $\mathcal{K}_1 \cap \check{\mathcal{K}}_1$. By Theorem 6.7, A is separated from B by a set in $B(\mathcal{K}_0)$. □

7 Fine Hierarchy

Here we describe some basic properties of the fine hierarcy of regular aperiodic ω-languages which is just the fine hierarchy over the base $\mathcal{K} = \{\mathcal{K}_n\}_{n<\omega}$. For background on the fine hierarchy see e.g. [Se95a, Se96, Se98]. Results of this section are particular cases of the corresponding general facts about the fine hierarchy [Se94, Se95], so the proofs are omitted.

In our definition of the fine hierarchy we use the difference hierarchy from the previuos section and an operation *bisep* on classes of ω-languages defined by

$$bisep(\mathcal{C}, \mathcal{D}_0, \mathcal{D}_1, \mathcal{D}_2) = \{(C_0 \cap D_0) \cup (C_1 \cap D_1) \cup (\overline{C}_0 \cap \overline{C}_1 \cap D_2) : C_i \in \mathcal{C}, D_j \in \mathcal{D}_j, C_0 \cap C_1 = 0\}.$$

Definition 7.1 *The fine hierarchy over $\mathcal{K}$ is the sequence $\{\mathcal{A}_\alpha\}_{\alpha<\omega^\omega}$ defined as follows:*

$\mathcal{A}_n = D_n(\mathcal{K}_0)$ *for* $n < \omega$;

$\mathcal{A}_{\omega^n} = D_n(\mathcal{K}_1)$ *for* $0 < n < \omega$;

$\mathcal{A}_{\beta+\omega^n} = bisep(\mathcal{K}_0, \mathcal{A}_\beta, \check{\mathcal{A}}_\beta, \mathcal{A}_{\omega^n})$ *for* $0 < n < \omega$ *and* β *of the form* $\beta = \omega^n \cdot \beta_1$ *for some* β_1, $0 < \beta_1 < \omega^\omega$;

$\mathcal{A}_{\beta+1} = bisep(\mathcal{K}_0, \mathcal{A}_\beta, \check{\mathcal{A}}_\beta, \mathcal{A}_0)$ *for* $\omega \leq \beta < \omega^\omega$.

The definition uses some ordinal arithmetic as described e.g. in [KM67]. It is correct since every non-zero ordinal $\alpha < \omega^\omega$ is uniquely representable in the form $\alpha = \omega^{n_0} + \cdots + \omega^{n_k}$ for a finite sequence $n_0 \geq \cdots \geq n_k$ of ordinals $< \omega$. Applying the definition we subsequently get $\mathcal{A}_{\omega^{n_0}}, \mathcal{A}_{\omega^{n_0}+\omega^{n_1}}, \ldots, \mathcal{A}_\alpha$. Note that the definition applies to any base in place of the base $\mathcal{K}$ above. For this paper, the fine hierarchy $\{\mathcal{S}_\alpha\}$ over the base $\{\mathbf{\Sigma}^0_{n+1}\}$ and the fine hierarchy $\{\mathcal{R}_\alpha\}$ over the base $\mathcal{L} = \{\mathcal{L}_n\}_{n<\omega}$ are also relevant. The hierarchy $\{\mathcal{R}_\alpha\}$ coincides with the Wagner hierarchy [Se98]. Note that in general levels of the fine hierarchy are numbered by ordinals less than the well known ordinal $\varepsilon_0 = sup\{\omega, \omega^\omega, \omega^{\omega^\omega}, \ldots\}$. Since in our case the base $\mathcal{K}$ has the property $\mathcal{K}_{n+2} = \mathcal{A}$, we have $\mathcal{A}_\alpha = \mathcal{A}$ for ${}_\alpha \geq \omega^\omega$, hence only the levels $< \omega^\omega$ are interesting (and the same applies to the hierarchy $\{\mathcal{R}_\alpha\}$). Note also that our definition here slightly differs from (but is equivalent to) the definition of the fine hierarchy in [Se98].

Proof of the next result uses heavily the fact from the previous section that the base $\mathcal{K}$ is reducible and interpolable.

Theorem 7.2 *(i) For all $\alpha < \beta < \omega^\omega$, $\mathcal{A}_\alpha \cup \check{\mathcal{A}}_\alpha \subseteq \mathcal{A}_\beta$.*

(ii) $\bigcup_{\alpha<\omega^\omega} \mathcal{A}_\alpha = \mathcal{A}$.

(iii) For any limit ordinal $\lambda < \omega^\omega$, $\mathcal{A}_\lambda \cap \check{\mathcal{A}}_\lambda = \bigcup_{\alpha<\lambda} \mathcal{A}_\alpha$.

(iv) For any $\alpha < \omega^\omega$, $\mathcal{A}_{\alpha+1} \cap \check{\mathcal{A}}_{\alpha+1}$ coincides with the class of sets $(A \cap C) \cup (B \cap \overline{C})$ where $A \in \mathcal{A}_\alpha$, $B \in \check{\mathcal{A}}_\alpha$ and $C \in \mathbf{\Delta}^0_1 = \mathcal{K}_0 \cap \check{\mathcal{K}}_0$.

(v) For any $\alpha < \omega^\omega$, $\check{\mathcal{A}}_\alpha$ has the separation property.

Next we give an alternative description of the fine hierarchy $\{\mathcal{A}_\alpha\}$.

Definition 7.3 *(i) A typed boolean term is a term of signature $\{\cup, \cap, ^-, 0, 1\}$ with variables $v^0_n, v^1_n (n < \omega)$. Variables v^0_n are of type 0 while variables v^1_n are of type 1.*

(ii) For a typed boolean term t and a base L, let $t(L_0, L_1)$ be the class of values of t when variables $v^i_n (n < \omega)$ of type i range over L_i, $i < 2$.

The next result shows the close relation of classes $t(\mathcal{K}_0, \mathcal{K}_1)$ to the fine hierarchy $\{\mathcal{A}_\alpha\}$. The result follows from the reducibility of the base $\mathcal{K}$. Similar facts for the bases $\{\mathbf{\Sigma}^0_{n+1}\}$ and $\mathcal{L}$ were established in [Se95a, Se98].

Theorem 7.4 *For every $\alpha < \omega^\omega$ one can effectively find a typed boolean term $t = t_\alpha$ such that $\mathcal{A}_\alpha = t(\mathcal{K}_0, \mathcal{K}_1)$. Conversely, for every typed boolean term t the class $t(\mathcal{K}_0, \mathcal{K}_1)$ coincides with one of classes $\mathcal{A}_\alpha, \check{\mathcal{A}}_\alpha$ for some $\alpha < \omega^\omega$, and the class is effectively computable from t.*

We give a couple of examples. If $t = v^0_0 \cap \overline{v^0_1}$, then $t(\mathcal{K}_0, \mathcal{K}_1) = D_2(\mathcal{K}_0)$. Let $u \triangle v = (u \cap \overline{v}) \cup (v \cap \overline{u})$ be the symmetric difference operartion and let $t = v^0_0 \triangle \cdots \triangle v^0_n$. Then $t(\mathcal{K}_0, \mathcal{K}_1) = D_{n+1}(\mathcal{K}_0)$ for any $n < \omega$ (this follows from a result in [KSW86]).

We conclude this section with a generalization of the last example. This is the "aperiodic version" of Theorem 8.6 in [Se98] that gives a set-theoretic description of a subsequence of $\{\mathcal{R}_\alpha\}_{\alpha<\omega^\omega}$ called in [Wag79] the coarse structure. For classes of ω-languages $\mathcal{C}$ and $\mathcal{D}$, let $\mathcal{C} + \mathcal{D} = \{C \triangle D : C \in \mathcal{C}, D \in \mathcal{D}\}$. The operation $+$ is clearly associative and commutative. For any $m < \omega$, let $m \cdot \mathcal{C} = \mathcal{C} + \cdots + \mathcal{C}$ (m

summands, assuming $0 \cdot \mathcal{C} = \{\emptyset\}$). By the mentioned result in [KSW86], $m \cdot \mathcal{K}_i = D_m(\mathcal{K}_i)$ for all $m < \omega$ and $i < 2$.

Theorem 7.5 *For all $m, n < \omega$, $\mathcal{A}_{\omega^{n+1} \cdot (m+1)} = \mathcal{A}_{\omega^{n+1}} + \mathcal{A}_m = (n+1) \cdot \mathcal{K}_1 + m \cdot \mathcal{K}_0$.*

A principal question about any hierarchy is whether it collapses or not (for the case of the hierarchy $\{\mathcal{A}_\alpha\}$ the non-collapse property means that $\mathcal{A}_\alpha \not\subseteq \check{\mathcal{A}}_\alpha$ for all $\alpha < \omega^\omega$). It is well-known (see e.g. [Wag79, Se98]) that the hierarchies $\{\mathcal{S}_\alpha\}$ and $\{\mathcal{R}_\alpha\}$ do not collapse. The hierarchy $\{\mathcal{A}_\alpha\}$ does not collapse too, as we shall see in the next section.

8 Reducibilities on $\mathcal{A}$

Here we establish the non-collapse property of the fine hierarchy $\{\mathcal{A}_\alpha\}_{\alpha<\omega^\omega}$ from the previous section and describe the structures of AA- and AS-degrees of regular aperiodic sets.

First we will show that for every $\alpha < \omega^\omega$ the class $\mathcal{A}_\alpha$ has an AA-complete set. This result is parallel to the result from [Wag79] that any $\mathcal{R}_\alpha$ has a DA-complete set. Since our proof will be a modification of the corresponding proof of the last fact in [Se98], we start with reproducing a version of the last proof. This version uses an idea from[Se94] and is a bit shorter than the proof in [Se98] since it does not use the induction on α.

By Corollary 6.8 in [Se98], for every $\alpha < \omega^\omega$ there is a typed boolean term $t = t_\alpha$ such that $\mathcal{R}_\alpha = t(\mathcal{L}_0, \mathcal{L}_1)$. Thus, it suffices to construct for any given typed boolean term $t = t(x_1, \ldots, x_n, y_1, \ldots, y_n)$ (where x_i are variables of type 0 and y_i are variables of type 1) a set R_t which is DA-complete in $\mathcal{R}_t = t(\mathcal{L}_0, \mathcal{L}_1)$. Recall that we work with a fixed alphabet X containing 0 and 1.

Let $p_1, \ldots, p_{2n} : X^\omega \to X^\omega$ be the "projections" from Section 2 satisfying $p_i(\langle \xi_1, \ldots, \xi_{2n} \rangle) = \xi_i$. Since p_i are DA-functions, $p_i^{-1}(K_0) \in \mathcal{L}_0$ for $i \in \{1, \ldots, n\}$ and $p_i^{-1}(K_1) \in \mathcal{L}_1$ for $i \in \{n+1, \ldots, 2n\}$ where K_0, K_1 are the sets from Corollary 5.4(iv). Therefore, the set

$$R_t = t(p_1^{-1}(K_0), \ldots, p_n^{-1}(K_0), p_{n+1}^{-1}(K_1), \ldots, p_{2n}^{-1}(K_1))$$

is in $\mathcal{R}_t$. It remains to show that $S \leq_{DA} R_t$ for every $S \in \mathcal{R}_t$. Let $B_1, \ldots, B_n \in \mathcal{L}_0$ and $B_{n+1}, \ldots, B_{2n} \in \mathcal{L}_1$ satisfy $S = t(B_1, \ldots, B_n, B_{n+1}, \ldots, B_{2n})$. Since K_0 and K_1 are DA-complete respectively in $\mathcal{L}_0$ and $\mathcal{L}_1$ by Corollary 5.4(iv), there are DA-functions $f_1, \ldots, f_{2n} : X^\omega \to X^\omega$ such that $B_i = f_i^{-1}(K_0)$ for $i \in \{1, \ldots, n\}$ and $B_i = f_i^{-1}(K_1)$ for $i \in \{n+1, \ldots, 2n\}$. For the DA-function $f(\xi) = \langle f_1(\xi), \ldots, f_{2n}(\xi) \rangle$ we have $p_i \circ f = f_i$ for $i \in \{1, \ldots, 2n\}$ and

$$f^{-1}(R_t) = t(f^{-1}p_1^{-1}(K_0), \ldots, f^{-1}p_{2n}^{-1}(K_1)) = t(B_1, \ldots, B_{2n}) = S,$$

Therefore, $S \leq_{DA} R_t$.

Unfortunately, the proof in the revious paragraph does not work in the aperiodic case because the projections p_i are not AA-functions and so the set R_t need not be aperiodic. In the proof of the next result we use the more subtle coding from Section 4 which requires the work with different alphabets simultaniously, so in the proof of the next result we will use the alphabet-dependent notation.

Theorem 8.1 *For every $\alpha < \omega^\omega$ the class $\mathcal{A}_\alpha$ has an AA-complete set.*

Proof. As above, by Proposition 7.4 it suffices to construct for any given typed boolean term $t = t(x_1, \ldots, x_n, y_1, \ldots, y_n)$ (where x_i are variables of type 0 and y_i are variables of type 1) a set A_t which is AA-complete in $\mathcal{A}_t = t(\mathcal{L}_0, \mathcal{L}_1)$; we have here in mind the fixed alphabet X containing 0 and 1.

For any alphabet Y, let $\mathcal{A}_t(Y)$ be the class of sets $t(B_1, \ldots, B_{2n})$ where $B_1, \ldots, B_n \in \mathcal{K}_0(Y)$ and $B_{n+1}, \ldots, B_{2n} \in \mathcal{K}_1(Y)$. In this proof, by an AA-complete set in $\mathcal{A}_t(Y)$ we mean a set $C \in \mathcal{A}_t(Y)$ such that for any alphabet Z and any $D \in \mathcal{A}_t(Z)$ there is an AA-function $f : Z^\omega \to Y^\omega$ with $D = f^{-1}(C)$.

First we prove the auxiliary assertion that there is an alphabet Y such that $\mathcal{A}_t(Y)$ has an AA-complete set. Let $Y = Y_1 \cup \cdots \cup Y_{2n}$ where $Y_1, \ldots, Y_{2n}$ are some pairwise disjoint binary alphabets, $Y_i = \{y_0^i, y_1^i\}$. Let $p_i : Y^\omega \to Y_i^{\leq\omega}$, $i \in \{1, \ldots, 2n\}$, be the AA-function that erases all letters not in Y_i. Let

$$A_t(Y) = t(p_1^{-1}(K_0^1), \ldots, p_n^{-1}(K_0^n), p_{n+1}^{-1}(K_1^{n+1}), \ldots, p_{2n}^{-1}(K_1^{2n}))$$

where

$$K_0^i = \{\xi \in Y_i^\omega : \exists k(\xi(k) = y_1^i)\} \text{ for } i \in \{1, \ldots, n\}$$

and

$$K_1^i = \{\xi \in Y_i^\omega : \exists k \forall l \geq k(\xi(l) = y_0^i)\} \text{ for } i \in \{n+1, \ldots, 2n\}.$$

By Corollary 5.4(v), the sets K_0 and K_1 are AA-complete respectively in $\mathcal{K}_0(\{0,1\})$ and $\mathcal{K}_1(\{0,1\})$. Thus, $A_t(Y) \in \mathcal{A}_t(Y)$. It remains to AA-reduce any set $S \in \mathcal{A}_t(Z)$ to $A_t(Y)$. Let $B_1, \ldots, B_n \in \mathcal{K}_0(Z)$ and $B_{n+1}, \ldots, B_{2n} \in \mathcal{K}_1(Z)$ satisfy $S = t(B_1, \ldots, B_n, B_{n+1}, \ldots, B_{2n})$. From Corollary 5.4(v) it follows that there are AA-functions $f_i : Z^\omega \to Y_i^\omega$, $i \in \{1, \ldots, 2n\}$ such that $B_i = f_i^{-1}(K_0^i)$ for $i \in \{1, \ldots, n\}$ and $B_i = f_i^{-1}(K_0^i)$ for $i \in \{n+1, \ldots, 2n\}$. For the AA-function $f : Z^\omega \to Y^\omega$ defined by $f(\zeta) = \langle f_1(\zeta), \ldots, f_{2n}(\zeta)\rangle$ we have $p_i \circ f = f_i$ for $i \in \{n+1, \ldots, 2n\}$. Therefore, $f^{-1}(A_t(Y)) = S$, completing the proof of the auxiliary assertion.

Now it suffices to show that for any alphabet X with $0, 1 \in X$ there are a set $A_t(X) \in \mathcal{A}_t(X)$ and an AA-function $e : Y^\omega \to X^\omega$ such that $A_t(Y) = e^{-1}(A_t(X))$. Let e, r be functions from Proposition 4.3 and let $A_t(X) = r^{-1}(A_t(Y))$. From Proposition 4.3 it follows that $A_t(X)$ and e have the desired property. $\square$

Let A_α be the AA-complete set in $\mathcal{A}_\alpha$ constructed in the last proof. It has also some additional nice properties related to the hierarchies $\{\mathcal{S}_\alpha\}$ and $\{\mathcal{R}_\alpha\}$ from the previous section.

Corollary 8.2 *Let $\alpha < \omega^\omega$.*

(i) The set A_α is DA-complete in $\mathcal{R}_\alpha$ and CA-complete in $\mathcal{S}_\alpha$.

(ii) The set $A_\alpha \oplus \bar{A}_\alpha$ is AA-complete in $(\mathcal{A}_{\alpha+1} \cap \check{\mathcal{A}}_{\alpha+1}) \setminus (\mathcal{A}_\alpha \cup \check{\mathcal{A}}_\alpha)$ and DA-complete in $(\mathcal{R}_{\alpha+1} \cap \check{\mathcal{R}}_{\alpha+1}) \setminus (\mathcal{R}_\alpha \cup \check{\mathcal{R}}_\alpha)$.

Proof. (i) Since for the hierarchies $\{\mathcal{S}_\alpha\}$ and $\{\mathcal{R}_\alpha\}$ analogs of Theorem 7.4 hold true [Se98], the proof of Theorem 8.1 applies uniformly to the both reducibilities.

(ii) The assertion follows from (i), Theorem 8.1, Theorem 7.1(iv) and the similar fact on the Wagner hierarchy. $\square$

The non-collapse of the fine hierarchy from the previous section is also an easy corollary.

Corollary 8.3 *The hierarchy $\{\mathcal{A}_\alpha\}$ does not collapse, i.e. $\mathcal{A}_\alpha \not\subseteq \check{\mathcal{A}}_\alpha$ for all $\alpha < \omega^\omega$.*

Proof. Suppose the contrary: $\mathcal{A}_\alpha \subseteq \check{\mathcal{A}}_\alpha$ for some $\alpha < \omega^\omega$. Then $A_\alpha \in \check{\mathcal{A}}_\alpha \subseteq \check{\mathcal{S}}_\alpha$. Since $\check{\mathcal{S}}_\alpha$ is closed under CA-reducibility, $\mathcal{S}_\alpha \subseteq \check{\mathcal{S}}_\alpha$. This contradicts to a well-known property of the Wadge hierarchy (see e.g. [Se98]). $\square$

The relation of the hierarchy $\{\mathcal{A}_\alpha\}$ to AA-reducibility is even tighter than Theorem 8.1 and Corollary 8.2 suggest.

Theorem 8.4 *For every $\alpha < \omega^\omega$, $\mathcal{A}_\alpha \setminus \check{\mathcal{A}}_\alpha = \{C : C \equiv_{AA} A_\alpha\}$ and $(\mathcal{R}_{\alpha+1} \cap \check{\mathcal{R}}_{\alpha+1}) \setminus (\mathcal{R}_\alpha \cup \check{\mathcal{R}}_\alpha) = \{C : C \equiv_{AA} A_\alpha \oplus \bar{A}_\alpha\}$.*

Proof. We have seen that $A_\alpha \in \mathcal{A}_\alpha \setminus \check{\mathcal{A}}_\alpha$, so for proving the first equality it remains to show that $C \equiv_{AA} A_\alpha$ for any $C \in \mathcal{A}_\alpha \setminus \check{\mathcal{A}}_\alpha$. The reduction $C \leq_{AA} A_\alpha$ follows from Theorem 8.1. It remains to show that $A_\alpha \leq_{AA} C$. Suppose the contrary, then $A_\alpha \not\leq_{AS} C$. By Corollary 5.3(iii), $C \leq_{AS} \overline{A}_\alpha$, hence $C \leq_{AA} \overline{A}_\alpha$ and $C \in \check{\mathcal{A}}_\alpha$. A contradiction.

The second equality is checked in the same way. $\square$

Let us summarize some facts on AA-degrees of regular aperiodic sets established above. The last assertion (v) in the next corollary refines Theorem 32 in [KPB95].

Corollary 8.5 *(i) For any $\alpha < \omega^\omega$, $A_\alpha \not\leq_{AA} \overline{A}_\alpha$.*

(ii) For all $\alpha < \beta < \omega^\omega$, $A_\alpha \oplus \overline{A}_\alpha <_{AA} A_\beta$.

(iii) Any regular aperiodic ω-language is AA-equivalent to exactly one of the sets A_α, $\overline{A}_\alpha$, $A_\alpha \oplus \overline{A}_\alpha$ $(\alpha < \omega^\omega)$.

(iv) The relations $\leq_{AA}$, $\leq_{DA}$ and $\leq_{CA}$ coincide on $\mathcal{A}$.

(v) For every $R \in \mathcal{R}$ there is $A \in \mathcal{A}$ with $R \equiv_{DA} A$ (and hence $R \equiv_{CA} A$). In particular, the Wadge degrees of regular aperiodic sets coincide with the Wadge degrees of regular sets.

Another interesting corollary states interesting relationships between hierarchies $\{\mathcal{A}_\alpha\}$ and $\{\mathcal{S}_\alpha\}$ parallel to the relationships between hierarchies $\{\mathcal{R}_\alpha\}$ and $\{\mathcal{S}_\alpha\}$ established in [Se98].

Corollary 8.6 *(i) For any $\alpha < \omega^\omega$, $\mathcal{A}_\alpha = \mathcal{A} \cap \mathcal{S}_\alpha = \mathcal{A} \cap \mathcal{R}_\alpha$.*

(ii) Let $t = t(x_1, \ldots, x_n, y_1, \ldots, y_n)$ be a typed boolean term, where x_i are variables of type 0 and y_i are variables of type 1. Let A be a regular aperiodic set such that $A = t(B_1, \ldots, B_n, C_1, \ldots, C_n)$ for some $B_1, \ldots, B_n \in \mathbf{\Sigma}^0_1$ and $C_1, \ldots, C_n \in \mathbf{\Sigma}^0_2$. Then $A = t(B'_1, \ldots, B'_n, C'_1, \ldots, C'_n)$ for some $B'_1, \ldots, B'_n \in \mathcal{K}_0$ and $C'_1, \ldots, C'_n \in \mathcal{K}_1$.

From the last result and the well known facts that classes $\mathcal{A}$ and $\mathcal{R}_\alpha$ (for every $\alpha < \omega^\omega$) are decidable (i.e. from a given Muller automaton one can effectively determine wether the corresponding language belongs to the class) we immediately obtain

Corollary 8.7 *For any $\alpha < \omega^\omega$, the class $\mathcal{A}_\alpha$ is decidable.*

We conclude this section with a characterization of the structure $(\mathcal{A}; \leq_{AS})$ similar to the characterization of the structure $(\mathcal{R}; \leq_{DS})$ in [Wag79, Se98]. To this end, we define for all $\alpha < \omega^\omega$ and $n < \omega$ the regular aperiodic set

$$A^n_\alpha = 0^{n+1} \cdot A_\alpha \cup (\cup\{u \cdot \overline{A}_\alpha : u \in X^{k+1}, u \neq 0^{n+1}\}).$$

Theorem 8.8 *Let $\alpha < \omega^\omega$, $n < \omega$ and $C \in \mathcal{A}$.*

(i) $C \leq_{AA} A_\alpha$ iff $C \leq_{AS} A_\alpha$.

(ii) $\mathcal{A}_\alpha \setminus \check{\mathcal{A}}_\alpha = \{C : C \equiv_{AS} A_\alpha\}$.

(iii) $A^n_\alpha \equiv_{AA} A_\alpha \oplus \bar{A}_\alpha$, $A^n_\alpha <_{AS} A^{n+1}_\alpha$ and $A^n_\alpha \equiv_{AS} \overline{A}^n_\alpha$.

(iv) $C_\alpha \equiv_{AA} A_\alpha \oplus \overline{A}_\alpha$ iff $C_\alpha \equiv_{AS} A^k_\alpha$ for a unique $k < \omega$.

(v) Analogs of (i)—(iv) hold true for $\mathcal{R}$ in place of $\mathcal{A}$ and DS-reducibility in place of AS-reducibility.

Proof. The assertion (i) follows from Corollary 8.5, Corollary 5.3(ii) and the well known property of the Wadge and Lipschitz reducibilities that $C \leq_{CA} A_\alpha$ iff $C \leq_{CS} A_\alpha$. (ii) follows from (i) and Theorem 8.4. (iii) is proved as the corresponding assertion in Theorem 9.1 of [Se98] for DS-reducibility. (iv) follows from (iii). (v) follows from Corollary 8.2(i) and Theorem 9.1 of [Se98]. $\square$

Let us summarize some facts on AS-degrees of regular aperiodic sets established above.

Corollary 8.9 *(i) For all $\alpha < \beta < \omega^\omega$ and $n < \omega$, $A_\alpha, \overline{A}_\alpha <_{AS} A^n_\alpha <_{AS} A^{n+1}_\alpha <_{AS} A_\beta, \overline{A}_\beta$.*

(ii) Any regular aperiodic ω-language is AS-equivalent to exactly one of the sets A_α, $\bar{A}_\alpha$, A^n_α $(\alpha < \omega^\omega, n < \omega)$.

(ii) The relations $\leq_{AS}$, $\leq_{DS}$ and $\leq_{CS}$ coincide on $\mathcal{A}$.

(iv) For every $R \in \mathcal{R}$ there is $A \in \mathcal{A}$ with $R \equiv_{DS} A$ (and hence $R \equiv_{CS} A$). In particular, the Lipschitz degrees of regular aperiodic sets coincide with the Lipschitz degrees of regular sets.

9 Conclusion

In this paper we answered most natural questions on the *topological* fine hierarchy of regular aperiodic ω-languages. Along with the topological classification, there are some alternative classifications of regular aperiodic ω-languages. E.g., one could consider the *logical* fine hierarchy of regular aperiodic ω-languages which is the fine hierarchy over the base $\{L_n\}$ where L_n is the class of regular aperiodic ω-languages defined by the first-order Σ^0_{n+1}-sentences as explained in Section 2. Some related classifications of regular aperiodic languages of finite words were discussed in [Se02]. We plan to consider relations of the topological fine hierarchy of regular aperiodic ω-languages to some alternative classifications in a separate paper.

There are some other natural open questions related to results of this paper. E.g., consider the structures $(B(\mathbf{\Sigma}^0_2); \mathbf{\Sigma}^0_2, \mathbf{\Sigma}^0_1, \cup, \cap, ^-)$, $(\mathcal{R}; \mathcal{L}_1, \mathcal{L}_0, \cup, \cap, ^-)$ and $(\mathcal{A}; \mathcal{K}_1, \mathcal{K}_0, \cup, \cap, ^-)$ where $\mathbf{\Sigma}^0_2, \mathbf{\Sigma}^0_1, \mathcal{L}_1, \ldots$ are treated as unary predicates on the corresponding universes. Results of this paper and of [Se98] (in particular, Corollary 8.6) show some striking similarities between the three structures which suggest that all the structures may turn out to be elemantary equivalent. It may even turn out that any of the two last structures is an elementary substructure of the previous one. At this time the author does not know whether this is really the case.

Another natural open question is to understand the structures $(\mathcal{R}; \leq_{AA})$ and $(\mathcal{R}; \leq_{AS})$. These structures may turn out to be more complicated than the structures discussed in the previous section. In particular, we do not know whether these structures are almost well ordered.

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