Some Hierarchies and Reducibilities on Regular Languages

Victor L. Selivanov*
A.P. Ershov Institute of Informatics Systems
Siberian Division of the Russian Academy of Sciences
vseliv@nspu.ru

Abstract

We discuss some known and introduce some new hierarchies and reducibilities on regular sets. We establish some facts on the corresponding degree structures and relate the reducibilities to hierarchies. As an application, we characterize regular languages whose leaf-language classes (in the balanced model) are contained in the polynomial hierarchy. For any reducibility we try to give some motivation and interesting open questions, in a hope to convince the reader that study of these reducibilities is important for automata theory and computational complexity.

1 Introduction

The notion of hierarchy appeared in descriptive set theory as a classification tool for characterizing complexity of sets studied in analysis. The notion of reducibility appeared in computability theory and plays a central role in classification of undecidable problems.

Afterwards, different notions of hierarchies and reducibilities were employed in different branches of computation theory and of definability theory (examples in descriptive set theory are Borel hierarchy and Wadge reducibility, in complexity theory — polynomial-time hierarchy and polynomial-time m-reducibility, in finite model theory — logical hierarchies and reducibilities and so on). Some of these hierarchies and reducibilities turned out to be also quite important for the corresponding fields.

Hierarchies in automata theory (e.g. dot-depth hierarchy) were introduced long ago. More recently, people began to consider reducibilities inducing nontrivial degree structures on regular sets (i.e., on languages recognized by finite automata) [BCS92, Ve93, SW04, Gl05].

In this paper, we continue to discuss some known and introduce some new hierarchies and reducibilities on regular sets. We establish some facts on the corresponding degree structures and relate the reducibilities to hierarchies. As an application, we characterize regular languages whose leaf-language classes (in the balanced model of leaf language definability) are contained (uniformly on oracles) in the polynomial hierarchy. For any reducibility we try to give some motivation and interesting open questions, in a hope to convince the reader that study of these reducibilities is important for automata theory and complexity theory.

*Supported by the Alexander von Humboldt Foundation.

We use (mostly without definitions here) some standard terminology and notation from computability theory, automata theory and complexity theory, say terminology on reducibilities and degrees, notation of languages by regular expressions or the concept of a polynomial-time nondeterministic Turing machine. Letters A, B will denote alphabets which are always assumed to contain at least two symbols. By A^+ we denote the set of all non-empty words over A, and by A^* the set of all words (including the empty word ε). For any k, let A^k denote the set of words over A of length k; notations $A^{\leq k}$ and $A^{>k}$ are defined in the same manner. Since usually we work with a fixed alphabet A, we normally do not mention the alphabet explicitely. The length of a word w is denoted $|w|$. For every $i < n$, $w(i)$ denotes the i-th letter of $|w|$ (thus, we start the numbering of letters in w with 0). By $\#_a(w)$ we denote the number of occurencies of the letter a in the word w.

Since we use the logical approach to regular languages [B60, MP71, Th82], we work mostly with languages of non-empty words $L \subseteq A^+$. Correspondingly, the complement $\overline{L}$ of such a language L is defined by $\overline{L} = A^+ \setminus L$. As usual, $P(A^+)$ denotes the power set of A^+. By $P'(A^+)$ we denote the class of all non-trivial (i.e. distinct from $\emptyset$ and A^+) languages. By $\mathcal{R}$ ($\mathcal{R}'$) we denote the class of all regular (resp., regular non-trivial) languages over A. For a class $\mathcal{K}$ of languages, let $\mathrm{BC}(\mathcal{K})$ be the boolean closure of $\mathcal{K}$, i.e., the closure of $\mathcal{K}$ under union and complement. By $co(\mathcal{K})$ we denote the class of complements of languages in $\mathcal{K}$.

This paper is closely related to [SW04, Gl05, Se01, GS01, St94], and we often refer to results and proofs there. Reading of our paper would become much easier with these sources at hand.

In Section 2 we remind some known and state some new facts related to the logical approach to automata theory. In Section 3 we consider hierarchies of regular languages induced by the quantifier-alternation hierarchies of first order formulas. Sections 4—6 are devoted to difference hierarchies over levels of the quantifier-altenation hierarchies. In Section 7 we discuss two important reducibilities closely related to so called leaf language approach to complexity classes which is described in detail in a recent survey [Wa04]. In Section 8 we consider some versions of the quantifier-free reducibility [SW04] and show their close relationship to the introduced hierarchies. In Section 9 we present some results on first-order reducibilities. We conclude in Section 10 with mentioning some other reducibilities and open questions.

2 Regular Languages and Logic

Relate to any alphabet $A = \{a, \ldots\}$ the signature $\sigma = \{\leq, Q_a, \ldots, \perp, \top, p, s\}$, where $\leq$ is a binary relation symbol, Q_a (for any $a \in A$) is a unary relation symbol, $\perp$ and $\top$ are constant symbols, and p, s are unary function symbols. A word $u = u_0 \ldots u_n \in A^+$ may be considered as a structure $\mathbf{u} = (\{0, \ldots, n\}; <, Q_a, \ldots)$ of signature σ, where $<$ has its usual meaning, $Q_a (a \in A)$ are unary predicates on $\{0, \ldots, n\}$ defined by $Q_a(i) \leftrightarrow u_i = a$, the symbols $\perp$ and $\top$ denote the least and the greatest elements, while p and s are respectively the predecessor and successor functions on $\{0, \ldots, n\}$ (with $p(0) = 0$ and $s(n) = n$). For a sentence ϕ of σ, let $L_\phi = \{u \in A^+ | \mathbf{u} \models \phi\}$. Sentences ϕ, ψ are treated as equivalent when $L_\phi = L_\psi$. In [MP71] it was shown that the class of $fo\sigma$-definable languages (i.e., languages of the form L_ϕ, where ϕ ranges through first-order sentences of σ), coincides with the class of regular aperiodic languages (known also as star-free languages).

We will consider also some enrichments of the signature σ. Namely, for any positive integer d let τ_d be the signature $\sigma \cup \{P_d^0, \ldots, P_d^{d-1}\}$, where P_d^r is the unary predicate true on the positions of a word which are equivalent to r modulo d. By an $fo\tau_d$-definable language we mean any language of the form L_ϕ, where ϕ is a first-order sentence of signature τ_d. Note that signature τ_1 is essentially the same as σ because P_1^0 is the valid predicate. In contrast, for $d > 1$ $fo\tau_d$-definable languages need not be aperiodic. E.g., the sentence $P_2^1(\top)$ defines the language L consisting of all words of even length which is known to be non-aperiodic. It is easy to see that predicates $P_d^1, \ldots, P_d^{d-1}$ maybe eliminated from formulas of signature τ_d (e.g. $P_d^1(x)$ is equivalent to $P_d^0(s(x))$). Nevertheless, for technical reasons we will not remove them from τ_d. We are also interested in signature $\tau = \cup_d \tau_d$.

Let us formulate a precise characterization of the introduced classes of languages in terms of syntactic monoids and homomorphisms (see [St94]). For a language L, let $M(L)$ be its syntactic monoid and $\eta_L : A^* \to M(L)$ the canonical syntactic homomorhism. We denote the semigroup operation on $M(L)$ by $\cdot$. As is well-known, L is regular iff $M(L)$ is finite. A language L is called *aperiodic* if there is no non-trivial group $(G;\cdot) \subseteq (M(L);\cdot)$. A language L is called *quasiaperiodic* [St94] if there is no non-trivial group $(G;\cdot) \subseteq (\eta_L(A^d);\cdot)$, for every $d > 0$. We call a language L *d-quasiaperiodic* (for any fixed $d > 0$), if there is no non-trivial group $(G;\cdot) \subseteq (\eta_L((A^d)^+);\cdot)$. Note that L is aperiodic iff L is 1-quasiaperiodic.

Theorem 2.1 *1. A regular language L is qausiaperiodic iff it is $fo\tau$-definable.*

2. A regular language L is d-quasiaperiodic iff it is $fo\tau_d$-definable.

3. A regular language L is aperiodic iff it is $fo\sigma$-definable.

For the proof of 1 see Theorem VI.4.1 in [St94]. The assertion 2 seems formally new though its proof is implicitely contained in the proof of that Theorem VI.4.1. 3 is a particular case of 2 and is a classical result of automata theory.

Corollary 2.2 *The classes of languages from the last theorem are decidable.*

Words correspond bijectively to (isomorphism types of) finite models of the theory CLO^{τ_d} of signature τ_d with the following axioms:

- $<$ is a linear ordering,
- any element satisfies exactly one of the predicates $Q_a (a \in A)$,
- $\forall x(\bot \leq x \leq \top)$,
- $\forall x(p(x) \leq x \wedge \neg\exists y(p(x) < y < x))$,
- $\forall x(x \leq s(x) \wedge \neg\exists y(x < y < s(x)))$,
- $\forall x(x > \bot \to p(x) < x)$,
- $\forall x(x < \top \to x < s(x))$,
- $P_d^0(\bot)$,
- any element satisfies exactly one of the predicates $P_d^0, \ldots, P_d^{d-1}$,
- $\forall x < \top(P_d^r(x) \to P_d^{r+1}(s(x)))$ for $0 \leq r < d-1$,
- $\forall x < \top(P_d^{r-1}(x) \to P_d^0(s(x)))$.

Sometimes it is technically more convenient to consider 'relational' versions of the signature τ_d (and of the theory CLO^{τ_d}). E.g., one could take sigature $\tau'_d = \{\leq, Q_a, \ldots, P^r_d, \bot, \top, S\}$, where $S(x,y)$ is a binary relation symbol interpreted as 'x is an immediate predecessor of y'. The both languages are clearly equivalent (i.e., they do not change the classes $\Sigma^{\tau_d}_n$), and we will use any of them when appropriate.

For any set $\mathcal{P}$ of positive integers, let $FO + MOD(\mathcal{P})$ denote the class of languages defined by σ-sentences using modulo counting quantifiers $\exists^{(q,r)}$ with moduli q in $\mathcal{P}$, along with the usual first order quantifiers. It is known (see e.g. [St94]) that the class $FO + MOD = FO + MOD(\{1, 2, \ldots\})$ consists exactly of languages with solvable syntactic monoid. More information on the logical approach may be found in [Th82, St94, Se01].

3 Quantifier Alternation Hierarchies

In this section we consider hierarchies of regular languages induced by quantifier-alternation hierarchy of first order formulas in prenex normal form. For any signature θ we get the corresponding hierarchy called θ-hierarchy.

For $n > 0$, let Σ^σ_n be the class of all languages L_ϕ, where ϕ ranges through the Σ_n-sentences of σ. Let $\Pi^\sigma_n = \mathrm{co}(\Sigma^\sigma_n)$ and $\Delta^\sigma_n = \Sigma^\sigma_n \cap \Pi^\sigma_n$. In [Th82] it was shown that σ-hierarchy essentially coincides with the dot-depth hierarchy which is a popular object of automata theory.

If we take the smaller signature $\rho = \{<, Q_a, \ldots\}$, we get the ρ-hierarchy $\{\Sigma^\rho_n\}$ known as the Straubing-Therien hierarchy.

Levels of the τ_d-hierarchy are denoted $\Sigma^{\tau_d}_n, \Pi^{\tau_d}_n, \Delta^{\tau_d}_n$. Of course, τ_d-hierarchy exhausts the class of τ_d-definable languages, and τ_1-hierarchy coincides with the σ-hierarchy. To our knowledge, τ_d-hierarchy for $d > 1$ was not so far considered in the literature.

One could also consider similar hierarchies of languages for signatures like $\tau_{d_1} \cup \cdots \cup \tau_{d_k}$, but actually, as the following proposition shows, we do not get new hierarchies in this way.

Proposition 3.1 *For all $n, c, e > 0$ there holds $\Sigma^{\tau_c \cup \tau_e}_n = \Sigma^{\tau_d}_n$, where d is the least common multiple of c, e.*

Proof. For the inclusion from left to right, we have to show that predicates P^0_c and P^0_e are easily (say, by quantifier-free formulas) expressible through the predicates P^r_d. Indeed, let $d = ck$. Then $P^0_c(x)$ is equivalent to $P^0_d(x) \vee \cdots \vee P^{k-1}_d(x)$, and similarly for e.

For the converse inclusion, let a be greatest common divisor of c, e. Then $d = c'e$, where $c' = c/a$. Since, by the previous paragraph, $P^0_{c'}$ is easily expressible through P^0_c, if suffices to express P^0_d through $P^0_{c'}$ and P^0_e. Since c' and e have no common divisors, it holds $P^0_d(x) \leftrightarrow P^0_{c'}(x) \wedge P^0_e(x)$. $\square$

By $\Sigma^\tau_n, \Pi^\tau_n, \Delta^\tau_n$ we denote levels of the τ-hierarchy. From the last proposition we immediately get the following relation between the introduced hierarchies.

Corollary 3.2 *For any $n > 0$ there holds $\Sigma^\tau_n = \cup_d \Sigma^{\tau_d}_n$.*

It turns out that methods developed in [Th82, Shu98, SS00] for the dot-depth and Straubing-Therien hierarchies (based on the Ehrenfeucht-Fraisse games) generalize in a straightforward way to the introduced hierarchies. E.g., any of these hierarchies does not collapse:

Theorem 3.3 *For any $n > 0$ there holds $\Sigma_n^\rho \not\subseteq \Pi_n^\tau$. In particular, $\Sigma_n^\tau \not\subseteq \Pi_n^\tau$ and $\Sigma_n^{\tau_{\rm d}} \not\subseteq \Pi_n^{\tau_{\rm d}}$ for all $n, d > 0$.*

We conclude this section with a characterization of the class $\Sigma_1^{\tau_{\rm d}}$ in terms of regular expressions.

Theorem 3.4 *1. $\Sigma_1^{\tau_{\rm d}}$ coincides with the class of languages L_ϕ, where ϕ is a finite disjucntion of atomic sentences and sentences of the form $\exists x_1 < \cdots < x_n\psi$, where ψ is a conjunction of atomic formulas of signature τ_d.*

2. $\Sigma_1^{\tau_{\rm d}}$ coincides with the class of finite unions of languages of the form $w_0(A^d)^*w_1\cdots(A^d)^*w_n$, where $n \geq 0$ and $w_i \in A^+$.

Proofhint. (1) One direction is obvious, the other one is proved by an elimination procedure similar to that in the proof of Lemma 2.5 in [Se01].

(2) One direction is easy (one can write down in a straightforward way a Σ_1-sentence ϕ of signature τ_d such that $w_0(A^d)^*w_1\cdots(A^d)^*w_n = L_\phi$). The other direction uses the representation of (1) and a development of the mentioned elimination procedure. For an alternative proof (without logical circles!) see Corollary 8.6 below. $\square$

4 Abstract Difference Hierarchy

In this section we make a couple of general remarks about the difference hierarchy (DH).

Let S be any set. By a *base in S* we mean any class $\mathcal{L}$ of subsets of S which is closed under union and intersection and contains $\emptyset$ and S as elements. For any $k < \omega$, let $\mathcal{L}(k)$ be the class of all sets of the form $\bigcup_i(L_{2i} \setminus L_{2i+1})$, where $L_0 \supseteq L_1 \supseteq \cdots$ is a descending sequence of sets from $\mathcal{L}$ and $L_i = \emptyset$ for $i \geq k$. The sequence $\{\mathcal{L}(k)\}_{k<\omega}$ is known as the *difference hierarchy over $\mathcal{L}$*. As is well-known, $\mathcal{L}(k) \cup \mathrm{co}(\mathcal{L}(k)) \subseteq \mathcal{L}(k+1)$ for every k, and the class $\bigcup_k \mathcal{L}(k)$ coincides with the boolean closure $BC(\mathcal{L})$ of $\mathcal{L}$.

We will need the following relation between DH's.

Proposition 4.1 *Let $f : T \to S$ be a surjection, $\mathcal{L}$ a base in S, $\mathcal{M}$ a base in T, and $f(M) \in \mathcal{L}$ for all $M \in \mathcal{M}$. Then $f^{-1}(L) \in \mathcal{M}(k)$ implies $L \in \mathcal{L}(k)$ for all $L \subseteq S$ and $k < \omega$.*

Proof. Let $f^{-1}(L) \in \mathcal{M}$, then $f^{-1}(L) = \bigcup_i(M_{2i} \setminus M_{2i+1})$ for some $M_0 \supseteq M_1 \supseteq \cdots$, $M_k = \emptyset$, $M_i \in \mathcal{M}$. Then $f(M_0) \supseteq f(M_1) \supseteq \cdots$, $f(M_k) = \emptyset$, and $f(M_i) \in \mathcal{L}$ for all i. One easily checks that $L = \bigcup_i(f(M_{2i}) \setminus f(M_{2i+1}))$. Therefore, $L \in \mathcal{L}(k)$. $\square$

The next notion is well known from descriptive set theory.

Definition 4.2 *A base $\mathcal{L}$ is said to have the* separation property*, if for every two disjoint sets $L, K \in \mathcal{L}$ there is a set $M \in \mathcal{L} \cap co(\mathcal{L})$ with $L \subseteq M \subseteq \overline{K}$. We say that M* separates *L from K (note that it is equivalent to say that $\overline{M}$ separates K from L).*

The following easy fact is often of use.

Proposition 4.3 *Let L_i $(i \in I)$ and K_j $(j \in J)$ be finite families of subsets of S and let $M_{i,j}$ separates L_i from K_j for all $i \in I, j \in J$. Then the set $\bigcup_i \bigcap_j M_{i,j}$ separates $L = \cup_i L_i$ from $K = \cup_j K_j$.*

Proof. We have $L_i \subseteq M_{i,j} \subseteq \overline{K}_j$ for all $i \in I, j \in J$. Then $L_i \subseteq \bigcap_j M_{i,j} \subseteq \bigcap_j \overline{K}_j$ for all $i \in I$ Therefore, $L = \bigcup_i L_i \subseteq \bigcup_i \bigcap_j M_{i,j} \subseteq \bigcap_j \overline{K}_j = \overline{K}$. $\square$

We will need the following fact about DH over bases with the separation property.

Proposition 4.4 *Let $\mathcal{L}$ be a base with the separation property. Then for any $k < \omega$ class $\mathcal{L}(k+1) \cap co(\mathcal{L}(k+1))$ coincides with the class of sets of the form $(U \cap L) \cup (\overline{U} \cap K)$, where $U \in \mathcal{L} \cap co(\mathcal{L})$, $L \in \mathcal{L}(k)$ and $K \in co(\mathcal{L}(k))$.*

Proof. Let $M = (U \cap L) \cup (\overline{U} \cap K)$, where $U \in \mathcal{L} \cap co(\mathcal{L})$, $L \in \mathcal{L}(k)$ and $K \in \mathrm{co}(\mathcal{L}(k)))$. One easily checks (even without using the separation property) that $M \in \mathcal{L}(k+1) \cap co(\mathcal{L}(k+1))$.

Simplifying notation, we prove the opposite inclusion only for the typical particular case $k = 2$. Let $M \in \mathcal{L}(k+1) \cap co(\mathcal{L}(k+1))$. Then

$$M = \overline{L}_0 \cup (L_1 \setminus L_2) \text{ and } \overline{M} = \overline{K}_0 \cup (K_1 \setminus K_2), \tag{1}$$

for some sets $L_0 \supseteq L_1 \supseteq L_2$ and $K_0 \supseteq K_1 \supseteq K_2$ from $\mathcal{L}$. Taking complements from (1), we get

$$\overline{M} = (L_0 \setminus L_1) \cup L_2 \text{ and } M = (K_0 \setminus K_1) \cup K_2, \tag{2}$$

hence L_2 and K_2 are disjoint. By the separation property, $L_2 \subseteq U \subseteq \overline{K}_2$ for some $U \in \mathcal{L} \cap co(\mathcal{L})$. Intersecting the first equality in (1) with $\overline{U}$ we get

$$M \cap \overline{U} = (\overline{L}_0 \cap \overline{U}) \cup (L_1 \cap \overline{U}) = (\overline{L}_0 \cup L_1) \cap \overline{U}.$$

Intersecting the second equality in (2) with U we get $M \cap U = (K_0 \setminus K_1) \cap U$. Therefore, $M = (U \cap L) \cup (\overline{U} \cap K)$, where $L = (K_0 \setminus K_1) \in \mathcal{L}(2)$ and $K = (\overline{L}_0 \cup L_1) \in \mathrm{co}(\mathcal{L}(2))$. $\square$

Next we formulate an obvious fact showing that the separation property survives under some operation on bases.

Proposition 4.5 *Let $\{\mathcal{L}_i\}_{i \in I}$ be a family of bases in S such that every $\mathcal{L}_i$ has the separation property and for all $i, j \in I$ there exists a $k \in I$ with $\mathcal{L}_i \cup \mathcal{L}_j \subseteq \mathcal{L}_k$. Then $\mathcal{L} = \cup_i \mathcal{L}_i$ is a base in S with the separation property.*

Relate to any partial order $P = (P; \leq)$ the base $\mathcal{L}$ in P consisting of all upper sets of P, including the empty set. Recall that a set $L \subseteq P$ is upper if $x \in L$ and $x \leq y$ imply $y \in L$. By an *alternating chain* of length k for a set $K \subseteq P$ we mean a sequence $(x_0, \ldots, x_k)$ of elements of P such that $x_0 \leq \cdots \leq x_k$ and $x_i \in K \leftrightarrow x_{i+1} \notin K$ for every $i < k$. Such a chain is called a 1-alternating chain if $x_0 \in K$, otherwise it is called a 0-alternating chain. Variants of the following fact frequently appear when treating DH's of different kind.

Proposition 4.6 *Let $P = (P; \leq)$ be a partial order and $\mathcal{L}$ the base of upper sets in P. For all $K \subseteq P$ and $k < \omega$, $K \in \mathcal{L}(k)$ iff K has no 1-alternating chains of length k.*

Proof. Let first $K \in \mathcal{L}(k)$, then $\bigcup_i (L_{2i} \setminus L_{2i+1})$, where $L_0 \supseteq L_1 \supseteq \cdots$ is a descending sequence of upper sets and $L_k = \emptyset$. Suppose that $(x_0, \ldots, x_k)$ a 1-alternating chain of length k for K. Since $x_0 \in K$, we have $x_0 \in L_0$. Since L_0 is upper and $x_0 \leq x_1$, $x_1 \in L_0$. Since $x_1 \notin K$, $x_1 \in L_1$. Continuing in this manner, we get $x_k \in L_k = \emptyset$, a contradiction.

In the opposite direction, let K has no 1-alternating chain of length k. For any $i < \omega$, let L_i be the set of all $x \in P$ such that there is a 1-alternating chain $(x_0, \ldots, x_i)$ for K with $x_i \leq x$. Then $L_0 \supseteq L_1 \supseteq \cdots$ is a descending sequence of upper sets and $L_k = \emptyset$. It remains to check that

$K = \bigcup_i (L_{2i} \setminus L_{2i+1})$. First we check the inclusion from left to right. Let $x \in K$. Then $x \in L_0$. It suffices to show that $x \in L_{2i+1}$ implies $x \in L_{2i+2}$. Let $x \in L_{2i+1}$, then there is a 1-alternating chain $(x_0, \ldots, x_{2i+1})$ for K with $x_{2i+1} \leq x$. Then $(x_0, \ldots, x_{2i+1}, x)$ is a 1-alternating chain for K, hence $x \in L_{2i+2}$.

It remains to show that if $x \in P \setminus K$ then $x \notin \bigcup_i (L_{2i} \setminus L_{2i+1})$. If $x \notin L_0$, we are done. Now let $x \in L_0$. It suffices to show that $x \in L_{2i}$ implies $x \in L_{2i+1}$, and this is done exactly as in the preceiding paragraph. □

We conclude this section with a result about the DH in well partial orders (wpo). Recall that a partial order $(P; \leq)$ is called a *wpo* if it has neither infinite descending chains nor infinite antichains. By an *ω-alternating chain* for a set $K \subseteq P$ we mean an ω-sequence $(x_0, x_1, \ldots)$ of elements of P such that $x_0 \leq x_1 \leq \cdots$ and $x_i \in K \leftrightarrow x_{i+1} \notin K$ for every $i < \omega$.

Proposition 4.7 *Let $P = (P; \leq)$ be a wpo and $\mathcal{L}$ the base of upper sets in P. For all $K \subseteq P$, $K \in BC(\mathcal{L})$ iff K does not have ω-alternating chains.*

Proof. From left to right, the assertion follows from the last proposition and the equality $BC(\mathcal{L}) = \cup_k \mathcal{L}(k)$. It remains to show that for any $K \in P \setminus BC(\mathcal{L})$ there is an ω-alternating chain. By the last proposition, there are alternating chains for K of arbitrary finite length.

Let ω^* be the set of all finite sequences of natural numbers, including the empty sequence ε. Let $P \cup \{\bot\}$ be the partial order obtained by adjoining a new smallest element $\bot$ to P. We construct a partial function $u : \omega^* \to P \cup \{\bot\}$ as follows. Let $u(\varepsilon) = \bot$ and suppose, by induction on $|\sigma|, \sigma \in \omega^*$, that $u(\sigma)$ is already defined. If $|\sigma|$ is even then find $m \in \omega$ and elements $v_0, \ldots, v_{m-1} \in P$ (if any) which enumerate without repetitions the $\leq$-minimal elements in $X = \{v \in K | u(\sigma) \leq v\}$. Note that, since P is a wpo, the set X is finite.) Then we set $u(\sigma i) = v_i$ for $i < m$ and $u(\sigma i)$ is undefined for $i \geq m$. For $|\sigma|$ odd, the definition is similar, but we use the set $X = \{v \in P \setminus K | u(\sigma) \leq v\}$.

From the construction it follows that $\{\sigma \in \omega^* | u(\sigma) \text{ is defined}\}$ is an infinite finitely branching tree (under the relation of being an initial segment). By König's lemma, there is an infinite path through this tree. The image of this path under u provides the desired infinite alternating chain for K. □

5 Difference Hierarchy over $\mathcal{L}_k^d$

Next we want to develop a theory of the DH over $\Sigma_1^{\tau_d}$ which generalizes the corresponding theory for signature σ developed independently in [Se01] and [GS01]. To this end, we consider some relevant partial orders on words.

Recall that in [Se01, GS01] for any $k < \omega$ the following partial order was studied: $u \leq_k v$, if $u = v \in A^{\leq k}$ or $u, v \in A^{>k}$, $p_k(u) = p_k(v)$, $s_k(u) = s_k(v)$, and there is a k-embedding $f : u \to v$. Here $p_k(u)$ ($s_k(u)$) is the prefix (resp., suffix) of u of length k, and the k-embedding f is a monotone injective function from $\{0. \ldots, |u| - 1\}$ to $\{0. \ldots, |v| - 1\}$ such that $u(i) \cdots u(i + k) = v(f(i)) \cdots v(f(i) + k)$ for all $i < |u| - k$. Note that the relation $\leq_0$ is just the inclusion of words. We say that a k-embedding f is a (k, d)-embedding, if $P_d^r(i)$ implies $P_d^r(f(i))$ for for all $i < |u|$ and $r < d$.

Definition 5.1 *For all $k < \omega$ and $d > 0$ we define a partial order $u \leq_k^d v$ in the same way as* *the relation $u \leq_k v$ above, only in the case $u, v \in A^{>k}$ we now additionally require that $|u| \equiv |v| \pmod{d}$ and f is a (k,d)-embedding.*

Note that for $d = 1$ the order $\leq_k^d$ coincides with $\leq_k$, hence our results about $\leq_k^d$ will subsume the corresponding results about $\leq_k$.

Proposition 5.2 *1. $(A^+; \leq_k^d)$ is a well partial order.*

2. If $k \leq l$ and d divides e then $u \leq_l^e v$ implies $u \leq_k^d v$.

Proof. (1) We may consider a word $u = u_0 \cdots u_n$ of length $n+1$ over the alphabeth A as a word $g_d(u) = (u_0, r_0) \cdots (u_n, r_n)$ over the alphabet $B = A \times \{0, \ldots, d-1\}$, where r_i (for any $i \leq n$) is the unique number $r < d$ satisfying $P_d^r(i)$. Then g_d is an isomorphic embedding of $(A^+; \leq_k^d)$ into $(B^+; \leq_k)$. Since $(B^+; \leq_k)$ is a wpo [Se01, GS01], so is also $(A^+; \leq_k^d)$.

(2) It suffices to show that $u \leq_{k+1}^d v$ implies $u \leq_k^d v$ and $u \leq_k^e v$ implies $u \leq_k^d v$. Let $u \leq_{k+1}^d v$. Then $g_d(u) \leq_{k+1} g_d(v)$, then $g_d(u) \leq_k g_d(v)$ [Se01, GS01], hence $u \leq_k^d v$. Now let $u \leq_k^e v$. Since $|u| \equiv |v| \pmod{e}$ implies $|u| \equiv |v| \pmod{d}$ and, as one easily checks, every (k,e)-embedding is also a (k,d)-embedding, we get $u \leq_k^d v$. □

Let $\mathcal{L}_k^d$ be the class of all upper sets in $(A^+; \leq_k^d)$. From Propositions 5.2, 4.6 and 4.7 we immediately get the following information about the DH over $\mathcal{L}_k^d$.

Theorem 5.3 *1. For all $L \subseteq A^+$ and $n < \omega$, $L \in \mathcal{L}_k^d(n)$ iff L has no 1-alternating chains with respect to $\leq_k^d$ of length n.*

2. For any $L \subseteq A^+$, $L \in BC(\mathcal{L}_k^d)$ iff L has no ω-alternating chains with respect to $\leq_k^d$.

Utilizing methods developed in [Se01, GS01] it is now straightforward to deduce the following decidability result.

Theorem 5.4 *The classes of languages $\mathcal{L}_k^d(n)$ and $BC(\mathcal{L}_k^d)$ are decidable for all $d > 0$ and $k, n < \omega$.*

We conclude this section with a result on the separation property.

Theorem 5.5 *For all $k < \omega$ and $d > 0$, the class $\mathcal{L}_k^d$ has the separation property.*

Proof. Let L, K be disjoint sets in $\mathcal{L}_k^d$. By Proposition 5.2(1), any of L, K is a finite union of 'cones' of the form $\{w \in A^+ | u \leq_k^d w\}$. By Proposition 4.3, w.l.o.g. we may assume that both L, K are such cones, i.e. $L = \{w \in A^+ | u \leq_k^d w\}$ and $K = \{w \in A^+ | v \leq_k^d w\}$ for some $u, v \in A^+$. If $|u| \leq k$, then the set $L = \{u\}$ is in $\mathcal{L}_k^d \cap co(\mathcal{L}_k^d)$ and separates L from K. The case $|v| \leq k$ is similar, hence let us assume that $u, v \in A^{>k}$.

If $p_k(u) \neq p_k(v)$ then the set $\{w \in A^+ : p_k(w) = p_k(u)\}$ is in $\mathcal{L}_k^d \cap co(\mathcal{L}_k^d)$ and separates L from K. The case $s_k(u) \neq s_k(v)$ is considered similarly. If $|u| \not\equiv |v| \pmod{d}$ then the set $\{w \in A^+ : |w| \equiv |u| \pmod{d}\}$ is in $\mathcal{L}_k^d \cap co(\mathcal{L}_k^d)$ and separates L from K.

We claim that the other cases (when $|u|, |v| > k$, $p_k(u) = p_k(v)$, $p_k(u) = p_k(v)$, and $|u| \equiv |v| \pmod{d}$) are not realized because in these cases it holds $L \cap K \neq \emptyset$. To see this, let $(u_0, \ldots, u_l)$, $l > 0$, be the sequence of all $(k+1)$-factors of u, i.e. $u_i = u(i) \cdots u(i+k)$ for every $i \leq l$. Let $(v_0, \ldots, v_m)$ be the similar sequence for v. It suffices to show that, for suitable words

$x_0, \ldots, x_l, y_0, \ldots, y_m \in A^*$ the word $w = u_0 x_0 \cdots u_l x_l v_0 y_0 \cdots v_m y_m v_m$ is in $L \cap K$. Indeed, we only have to care that the lengths of the words $x_0, \ldots, x_l, y_0, \ldots, y_m$ satisfy the following equivalences modulo d:

$$|u_0x_0| \equiv 0, |u_0x_0u_1x_1| \equiv 1, \ldots, |u_0x_0 \cdots u_{l-1}x_{l-1}| \equiv l-1$$

(this guarantees the existence of a (k,d)-embedding of u into w),

$$|u_0x_0 \cdots u_lx_l| \equiv d-1, |u_0x_0 \cdots u_lx_lv_0y_0| \equiv 0, \ldots, |u_0x_0 \cdots u_lx_lv_0y_0 \cdots v_{m-1}y_{m-1}| \equiv m-1,$$

(this guarantees the existence of a (k,d)-embedding of v into w), and $|w| \equiv |u|$ (this together with the conditions above guarantees $u \leq_k^d w$ and $v \leq_k^d w$). $\square$

From the last proof one easily extracts the following description of the class $\mathcal{L}_k^d \cap co(\mathcal{L}_k^d)$ in terms of regular expressions.

Corollary 5.6 *The class $\mathcal{L}_k^d \cap co(\mathcal{L}_k^d)$ consists exactly of finite unions of languages $w_0, w_0(A^d)^*w_1$ (w_0, w_1 are nonempty words of length $\leq k$).*

6 Difference Hierarchies over $\Sigma_n^{\tau_d}$ and Σ_n^{τ}

Here we establish some results on the DH over $\Sigma_n^{\tau_d}$ and Σ_n^{τ}. In [Shu98, SS00] it was shown that DH's over any level of the Strubing-Therien hierarchy, as well as of the dot-depth hierarchy, do not collapse. The methods of those papers (using the Ehrenfeucht-Fraisse games) apply in a straightforward way to get the following.

Theorem 6.1 *For all $n, k > 0$ there holds $\Sigma_n^{\rho}(k) \not\subseteq \Pi_n^{\tau}(k)$. In particular, $\Sigma_n^{\tau_d}(k) \not\subseteq \Pi_n^{\tau_d}(k)$ and $\Sigma_n^{\tau}(k) \not\subseteq \Pi_n^{\tau}(k)$ for all $k, n, d > 0$.*

Now let $\mathcal{L}^d = \cup_k \mathcal{L}_k^d$ and $\mathcal{L} = \cup_d \mathcal{L}^d$, where $\mathcal{L}_k^d$ is the base from the previous section. The last two sets are closely related to our hierarchies.

Theorem 6.2 *There hold $\Sigma_1^{\tau_d} = \mathcal{L}^d$ and $\Sigma_1^{\tau} = \mathcal{L}$.*

Proofsketch. The second equality follows from the first one and Corollary 3.2. Towards the first equality, let $L \in \mathcal{L}^d$. Then $L \in \mathcal{L}_k^d$ for some k, hence L is a finite union of the cones $\{w \in A^+ | u \leq_k^d w\}$. It is easy to write down a Σ_1-sentence ϕ of signature τ_d such that $\{w \in A^+ | u \leq_k^d w\} = L_\phi$. Therefore, $L \in \Sigma_1^{\tau_d}$.

Conversely, let $L \in \Sigma_1^{\tau_d}$. By Theorem 3.4, L is a finite union of sets $w_0(A^d)^*w_1 \cdots (A^d)^*w_n$, where $n \geq 0$ and $w_i \in A^+$. Easy manipulations show that w.l.o.g. we may assume that all words w_i (in any fixed term $w_0(A^d)^*w_1 \cdots (A^d)^*w_n$) have the same length k. It is then straightforward to check that $w_0(A^d)^*w_1 \cdots (A^d)^*w_n \in \mathcal{L}_k^d$. Therefore, $L \in \mathcal{L}_k^d$. $\square$

The following assertion is now an immediate consequence of Theorem 5.5 and Proposition 4.5.

Theorem 6.3 *The classes $\Sigma_1^{\tau_d}$ ($d > 0$) and Σ_1^{τ} have the separation property.*

From the proof of Theorem 5.5 we get the following characterization of $\Delta_1^{\tau_d}$ in terms of regular expressions.

Corollary 6.4 *1. The class $\Delta_1^{\tau_d}$ consists exactly of finite unions of languages $w_0, w_0(A^d)^*w_1$ ($w_0, w_1 \in A^+$).*

2. The class Δ_1^{τ} consists exactly of finite unions of languages $w_0, w_0(A^d)^*w_1$ ($w_0, w_1 \in A^+$, $d > 0$).

Next we generalize some results from [GS01] about DH over Σ_1^σ to the DH over $\mathcal{L}^d$ for any $d > 1$. We start with recalling (in slightly different notation) some notions from [GS01].

Let $\mathcal{A}$ be a (deterministic) finite automaton (over A) and $c_\mathcal{A} = (n+1)^{(n+1)^{n+1}}$, where n is the number of states in $\mathcal{A}$. A word $u \in A^+$ is called an $\mathcal{A}$-idempotet, if $\delta^{uu} = \delta^u$, where δ^u is a function on states of $\mathcal{A}$ defined by $\delta^u(s) = s.u$.

Let $B = B_\mathcal{A}$ be the new alphabeth symbols of which are nonempty words over A of length $\leq c_\mathcal{A}$. Let $U = U_\mathcal{A}$ be the subset of B consisting of all $\mathcal{A}$-idempotents. By a *structured word* over B we mean words over B of the form $x_0u_1x_1 \cdots u_nx_n$, where $n \geq 0$ and all u_i are $\mathcal{A}$-idempotents. Let $S = S_\mathcal{A}$ be the set of all structured words over B. Let $f : U_\mathcal{A} \to A^+$ relates to any structured word x the word $f(x)$ over A obtained by concatinating the symbols of B written as words over A^+. In [GS01] it was shown that f is a surjection.

Let $\leq_\mathcal{A}$ be a partial order on S defined as follows: $x \leq_\mathcal{A} y$, if there exist $u_i \in U$ and $x_i, z_i \in S$ such that $x = x_0u_1x_1u_2x_2 \cdots u_mx_m$ and $y = x_0u_1z_1u_1x_1u_2z_2u_2x_2 \cdots u_mz_mu_mx_m$. This notion was the key in understanding the DH over Σ_1^σ. The following modification of this order works for the generalization of this to the DH over $\Sigma_1^{\tau_d}$.

Namely, we define a partial order $\leq_\mathcal{A}^d$ on S just in the same way as the order $\leq_\mathcal{A}$ but with the additional requirement that $|f(z_i)| \equiv 0 \pmod d$ for all i. With small modifications of proofs of the corresponding results in [GS01] we get the following.

Corollary 6.5 *Let $\mathcal{A}$ be a finite automaton.*

1. *$(S_\mathcal{A}; \leq_\mathcal{A}^d)$ is a well partial order.*
2. *If M is an upper set in $(S_\mathcal{A}; \leq_\mathcal{A}^d)$ then $f(M) \in \mathcal{L}^d$.*

Let $\mathcal{M}$ be the class of upper sets in $(S_\mathcal{A}; \leq_\mathcal{A}^d)$. Just as in the previous section, we get the following.

Theorem 6.6 *1. For all $M \subseteq S_\mathcal{A}$ and $n < \omega$, $M \in \mathcal{M}(n)$ iff M has no 1-alternating chains with respect to $\leq_\mathcal{A}^d$ of length n.*

2. For any $M \subseteq S_\mathcal{A}$, $M \in BC(\mathcal{M})$ iff M has no ω-alternating chains with respect to $\leq_\mathcal{A}^d$.

The next result follows from Proposition 4.1 and manipulations with the idempotents similar to those in [GS01].

Theorem 6.7 *Let L be the regular language recognized by an automaton $\mathcal{A}$ and $d > 0$. For any n, $L \in \mathcal{L}^d(n)$ iff $f^{-1}(L) \in \mathcal{M}(n)$.*

Using the results above and adapting proofs from [GS01] (or, alternatively, the logical machinery from [Se01]), we get the following decidability result.

Theorem 6.8 *The classes of languages $\Sigma_1^{\tau_d}(n)$ and $BC(\Sigma_1^{\tau_d})$ are decidable for all $n, d > 0$.*

We do not currently know whether the classes of languages $\Sigma_1^\tau(n)$ (for $n > 1$) and $BC(\Sigma_1^\tau)$ are decidable. The decidability of Σ_1^τ follows from [Gl05] and a result in the next section.

7 plt- and ptt-Reducibilities

In this section we start our investigation of reducibilities on regular sets with discussing two reducibilities closely related to complexity theory. The first one was inroduced in [BCS92] and independently in [Ve93].

Definition 7.1
- *A language $L \subseteq A^*$ is* polylogtime reducible *to $K \subseteq B^*$, for short $L \leq_{\rm plt} K$, if there exist functions $f : A^* \times \mathbb{N} \to B$ and $g : A^* \to \mathbb{N}$, computable in polylogarithmic time (on a deterministic Turing machine which treats the input word as an oracle) such that $x \in L \leftrightarrow f(x,1)f(x,2)\dots f(x,g(x)) \in K$ for every $x \in A^*$.*
- *By a* plt-function *we mean any function of the form $x \mapsto f(x,1)f(x,2)\dots f(x,g(x))$ where f and g are computable in polylogarithmic time.*
- *A language L is plt*-decidable*, if its characteristic function is computable in polylogarithmic time.*

Examples. 1. The function $u \mapsto pu$ on A^+ which adds a fixed prefix $p \in A^*$ to a word u is a *plt*-function. The same of course applies to the operation of adding a suffix to a word.

2. Let $h : A^+ \to B^+$ be a semigroup morphism. Such functions are defined by their values $h : A \to B^+$ on the letters of A (i.e., words of length 1) because we have $h(a_0 \cdots a_l) = h(a_0) \cdots h(a_l)$, where $a_i \in A$. It is easy to see that any such function h with the property $\forall a, b \in A(|h(a)| = |h(b)|)$ is a *plt*-function.

To explain relationship of *plt*-reducibility to complexity theory, let us recall some relevant definitions. Consider a polynomial-time nondeterministic Turing machine M working on an input word x over some alphabet B and printing a letter from another alphabet A after finishing any computation path. These values are the leaves of the binary tree defined by the nondeterministic choices of M on input x. An ordering of the tuples in the program of M determines a left-to-right ordering of all the leaves. In this way, M may be considered as a deterministic transducer computing a total function $M : B^* \to A^+$. Now, relate to any language $L \subseteq A^+$ (called in this situation a leaf language) the language $M^{-1}(L) \subseteq B^*$. Denote by $\mathrm{Leaf_b}(L)$ the set of languages $M^{-1}(L)$, for all machines M specified above which have balanced computation trees, and denote by $\mathrm{Leaf_u}(L)$ the set of languages $M^{-1}(L)$, for all machines M specified above (which may have unbalanced computation trees).

Obviously, we have $\mathrm{Leaf_b}(L) \subseteq \mathrm{Leaf_u}(L)$ for every language L, and there exist languages L where $\mathrm{Leaf_b}(L) = \mathrm{Leaf_u}(L)$ is unlikely. It turns out that many inportant complexity classes have natural and useful descriptions in terms of leaf languages (see [Wa04] and references therein). The following theorem from [BCS92, Ve93] relates *plt*-reducibility to the balanced version of leaf language definability.

Theorem 7.2 *For all languages L and K, $L \leq_{\rm plt} K$ iff $\mathrm{Leaf_b}(L)^{\mathcal{O}} \subseteq \mathrm{Leaf_b}(K)^{\mathcal{O}}$ for every oracle $\mathcal{O}$.*

In [Wa04] a notion of reducibility (called *ptt*-reducibility) was introduced which is related to the unbalanced leaf language definability exactly in the same way as in Theorem 7.2. For these reasons (and because regular languages are most natural to use as leaf languages) investigation of *plt*- and *ptt*-reducibilities (especially on regular languages) seems important. For some results in this directions see [SW04, Wa04, Gl05].

We start with the following obvious fact in which we, for simplicity of notation, identify preorders with the corresponding quotient partial orders (i.e., degree structures).

Proposition 7.3 *1. $\{\emptyset\}$ and $\{A^+\}$ are two distinct minimal elements of the degree structures $(P(A^+); \leq_{\rm plt})$ and $(\mathcal{R}; \leq_{\rm plt})$ which are below any other element.*

2. *The structures $(P(A^+); \leq_{\rm plt})$ and $(\mathcal{R}; \leq_{\rm plt})$ are upper semilattices under the supremum operation $L \oplus K = aL \cup (A \setminus a)K$, where a is a fixed letter from A.*

3. *The structure $(P'(A^+); \leq_{\rm plt})$ is a distributive upper semilattice with a smallest degree consisting exactly of non-trivial plt-decidable lenguages.*

We do not know whether the structure $(\mathcal{R}'; \leq_{\rm plt})$ is a distributive upper semilattice.

Are there greatest elements in $(\mathcal{R}; \leq_{\rm plt})$ and $(\mathcal{R}; \leq_{ptt})$ (as usual, the corresponding languages are called complete)? From [HLSVW93] we get the positive answer and even a clear descripion of complete languages. Namely, for a regular language L the following statements are equivalent: L is *plt*-complete; L is *ptt*-complete; the syntactic monoid $M(L)$ is not solvable (i.e., contains a non-solvable subgroup).

We need the following characterization of $fo\tau$-definable languages from Section 2 in terms of (the graphs of) automata recognizing them.

Definition 7.4 *Let $\mathcal{A}$ be a finite automaton, q its initial state, and let $s.w$ denote the state reached by $\mathcal{A}$ when reading the word w starting from the state s. A* balanced n-counting *pattern ($n > 1$) for $\mathcal{A}$ is formed by states $s_0, \ldots, s_{n-1}$ and words x, y, u, v such that: $q.x = s_0$, $s_i.u = s_i$ for all $i < n$, $s_i.v = s_{i+1}$ for all $i < n-1$, $s_{n-1}.v = s_0$, $s_0.y$ is accepting, $s_1.y$ is rejecting, and $|u| = |v|$.*

Recall (see e.g. [St94]) that the syntactic monoid $M(L)$ may be interpreted as automaton recognizing L.

Theorem 7.5 *For every regular language L the following conditions are equivalent.*

1. *L is $fo\tau$-definable.*
2. *Every finite automaton recognizing L has no balanced n-counting patterns ($n > 1$).*
3. *The automaton $M(L)$ has no balanced n-counting patterns ($n > 1$).*

Proof. (1)→(2). Let L be $fo\tau$-definable. Toward a contradiction, let $\mathcal{A}$ be an automaton which recognizes L and has a balanced n-counting pattern for some $n > 1$. Let $K \subseteq \{0,1\}^+$ be the language of all binary words w with $\#_1(w) \equiv i \pmod{n}$ for some $i < n$ such that $s_i.y$ is an accepting state of $\mathcal{A}$ (so $0 \in K$ and $1 \notin K$). One easily checks that the *plt*-function $f : \{0,1\}^+ \to A^+$ with properties $f(w) = xh(w)y$, $h(0) = u$ and $h(1) = v$, reduces K to L. By the proof of Lemma 6 from [B95], w.l.o.g. we may assume that n is prime. Let $M_n \subseteq \{0,1\}^+$ be the set of words with $\#_1(w) \equiv 0 \pmod{n}$. Define a function g on $\{0,1\}^+$ as follows. Let $|g(w)| = |w|^{n-1}$ and for every $i \in \{1, \ldots, |w|^{n-1}\}$ the i-th letter in $g(w)$ is 1 iff $w(i_1) = \cdots = w(i_{n-1}) = 1$, where $(i_1, \ldots, i_{n-1})$ is the i-th tuple in the lexicographic ordering of $\{1, \ldots, |w|\}^{n-1}$. One easily checks that g is a *plt*-function and $\#_1(g(w)) = (\#_1(w))^{n-1}$. By Fermat's theorem, $\#_1(g(w)) \equiv 0$ (mod n) if $\#_1(w) \equiv 0 \pmod{n}$ and $\#_1(g(w)) \equiv 1 \pmod{n}$ otherwise. Hence, g reduces M_n to K. By Theorem 7.2, $Leaf_b(M_p)^O$ is contained in PH^O for all oracles O (PH is the class of sets in the polynomial-time hierarchy), which is a contradiction (see [Wa04]).

(2)→(3) is obvious.

(3)→(1). By Theorem 2.1, it suffices to assume that L is non-quasiaperiodic and find a balanced n-counting pattern in $M(L)$ for some prime n. We have that $\eta_L(A^d)$ contains a non-trivial group for some d. Then $\eta_L(A^d)$ contains an isomorphic copy $G = (\{[u_0], \ldots, [u_{n-1}]\}; \cdot)$ of the cyclic group $\mathbf{Z}_n = (\{0, \ldots, n-1\}; \oplus)$ for some prime n (so $[u_i] \cdot [u_j] = [u_{i \oplus j}]$, where $\oplus$ is the addition modulo n, $[u_0]$ is the unit of G and $[u_1]$ is a generator of G). Since $[u_0] \neq [u_1]$, there are $z, y \in A^*$ with $zu_0y \in L \leftrightarrow zu_1y \notin L$; let e.g. $zu_0y \in L$ and $zu_1y \notin L$. Let $s_i = [zu_i]$ for any $i < n$, $x = zu_0$, $u = u_0$ and $v = u_1$. One easily checks that the states $s_0, \ldots, s_{n-1}$ and the words x, y, u, v form a balanced n-counting pattern for $\mathcal{A}$ (see Definition 7.4). □

The following characterization of the relation $L \leq_{\text{plt}} 0^*1(0 \cup 1)^*$ is a "balanced" version of the corresponding result from [PW97, BKS99] for the "unbalanced" model. We need to define some more patterns.

Definition 7.6 *Let $\mathcal{A}$ be an automaton and q its initial state.*

1. *A* balanced coUP-*pattern for $\mathcal{A}$ is formed by states s_0, s_1 and words x, y, u, v such that: $q.x = s_0$, $s_i.u = s_i$ for both $i < 2$, $s_0.v = s_1$, $s_0.y$ is accepting, $s_1.y$ is rejecting, and $|u| = |v|$.*

2. *A* balanced coNP-*pattern for $\mathcal{A}$ is formed by states s_0, s_1 and words x, y, u, v such that: $q.x = s_0$, $s_i.u = s_i$ for both $i < 2$, $s_0.v = s_1 = s_1.v$, $s_0.y$ is accepting, $s_1.y$ is rejecting, and $|u| = |v|$.*

3. *A* balanced co1NP-*pattern for $\mathcal{A}$ is formed by states $s_0, s_1, \ldots, s_n$ ($n > 1$) and words x, y, u, v such that: $q.x = s_0$, $s_i.u = s_i$ for all $i \leq n$, $s_i.v = s_{i+1}$ for all $i < n$, $s_n.v = s_n$, the states $s_0.y, s_2.y, \ldots, s_n.y$ are accepting while the state $s_1.y$ is rejecting, and $|u| = |v|$.*

The next theorem follows immediately from results in [Gl05].

Theorem 7.7 *Let L be a regular language and $\mathcal{A}$ the minimal automaton recognizing L. The following conditions are equivalent:*

1. *$L \leq_{\text{plt}} 0^*1(0 \cup 1)^*$;*

2. *$\mathcal{A}$ does not have balanced coUP-patterns;*

3. *$\mathcal{A}$ does not have balanced coNP-, co1NP- and n-counting patterns ($n > 1$);*

4. *L is a finite union of languages of the form $w_0(A^d)^*w_1 \cdots (A^d)^*w_n$, where $d > 0, n \geq 0$ and $w_i \in A^*$.*

Our next result states some relationships between τ-hierarchy and $\leq_{\text{plt}}$-reducibility.

Theorem 7.8 *1. There exists $K \in \Sigma_n^\rho$ such that $L \leq_{\text{plt}} K$ for every $L \in \Sigma_n^\tau$.*

2. For any regular language L, $L \leq_{\text{plt}} 0^*1(0 \cup 1)^*$ iff $L \in \Sigma_1^\tau$.

Proof. (1) Let $K = H_n$ be the "standard" set witnessing that $\Sigma_n^\sigma \not\subseteq \Pi_n^\sigma$ from the proof of Lemma 3.1 in [SW04] (so K is in Σ_n^ρ). Let $L \in \Sigma_n^\tau$, then $L \in \Sigma_n^{\tau_d}$ for some $d > 0$. Let $f : A^+ \to A_n^+$ be the function constructed in the proof of that lemma (only this time ϕ is a quantifier-free formula of signature τ_d). One easily checks that f is a *plt*-function. Since $L = f^{-1}(K)$, $L \leq_{\text{plt}} K$.

(2) Let $L \in \Sigma_1^\tau$. By the proof of (1), $L \leq_{\text{plt}} K_1 = 0^*1(0 \cup 1)^*$. Conversely, let $L \leq_{\text{plt}} 0^*1(0 \cup 1)^*$. By Theorem 7.7, L is a finite union of languages of the form $w_0(A^d)^*w_1 \cdots (A^d)^*w_n$, where $n \geq 0, d > 0$ and $w_i \in A^+$. One easily writes down an existential sentence ϕ of signature τ_d such that $w_0(A^d)^*w_1 \cdots (A^d)^*w_n = L_\phi$. Therefore, $L \in \Sigma_1^\tau$. $\square$

Corollary 7.9 *1. For any $n > 0$, there is a plt-complete set in $\Sigma_n^{\tau_d}$ and Σ_n^τ.*

2. The class Σ_1^τ is a principal ideal of $(\mathcal{R}; \leq_{\text{plt}})$.

3. Σ_1^τ coincides with the class of finite unions of languages of the form $w_0(A^d)^*w_1 \cdots (A^d)^*w_n$, where $d > 0, n \geq 0$ and $w_i \in A^+$.

We do not know whether Σ_n^τ is a principal ideal of $(\mathcal{R}; \leq_{\text{plt}})$ for $n > 1$.

Now we are able to characterize (uniformly on oracles) regular languages L with the property $Leaf_b(L) \subseteq PH$. Similar characterization for the unbalanced leaf language definability is well known.

Theorem 7.10 *A regular language L is quasiaperiodic iff $Leaf_b(L)^O \subseteq PH^O$ for all oracles O.*

Proof. Let L be quasiaperiodic, then $L \in \Sigma_n^\tau$ for some $n > 0$. By Theorem 7.8(2), $L \leq_{\text{plt}} K$ for some $K \in \Sigma_n^\sigma$. By Theorem 7.2, $Leaf_b(L)^O \subseteq Leaf_b(K)^O$ for all O. It is known [Wa04] that $Leaf_b(K)^O \subseteq PH^O$ for all O, hence $Leaf_b(L)^O \subseteq PH^O$ for all O.

Conversely, let L be non-quasiaperiodic. By the proof of Theorem 7.5, $M_p \leq_{\text{plt}} L$ for some prime p. By Theorem 7.2, $Leaf_b(M_p)^O \subseteq Leaf_b(L)^O$ for all O. It is known [Wa04] that $Leaf_b(M_p)^O \not\subseteq PH^O$ for some O. Therefore, $Leaf_b(L)^O \not\subseteq PH^O$ for some O. $\square$

Corollary 7.11 *The class of quasiaperiodic languages is an ideal of $(\mathcal{R}; \leq_{\text{plt}})$.*

Next we prove a result on initial segment of the structure $(\mathcal{R}'; \leq_{\text{plt}})$. This result is implicit in [Gl05] (provided we use Theorem 7.7 and some known facts on oracle separations). Nevertheless, we present a direct proof because its ideas are also used in some proofs below. Let $(P; \leq)$ be an upper semilattice with a least element 0. Recall that an *atom* of P is a minimal non-zero element of P. The semilattice is called *atomic* if below every non-zero element there is an atom.

Theorem 7.12 *The semilattice $(\mathcal{R}'; \leq_{\text{plt}})$ is atomic with infinitely many atoms.*

Proof. Let $E \subseteq A^+$ be the language of words having at least one letter distinct from a fixed letter $a \in A$. Obviously, $E \equiv_{\text{plt}} 0^*1(0 \cup 1)^*$. For any prime p, let $M_p \subseteq A^+$ consist of all words such that the number of occurences of letters distinct from a is divided by p. We claim that the languages $E, \overline{E}, M_p$ (p prime) define exactly the atoms of $(\mathcal{R}'; \leq_{\text{plt}})$, i.e.:

1) $E, \overline{E}, M_p$ are pairwise *plt*-incomparable,

2) for any $L \in \mathcal{R}'$ of non-smallest *plt*-degree, at least one of $E, \overline{E}, M_p$ is *plt*-reducible to L.

The assertion 1) follows form Theorem 7.2 and well-known oracle separations (alternatively, it maybe easily observed from definition of *plt*-reducibility).

2) Let $L \in \mathcal{R}'$ be of non-smallest degree. By a result in [Gl05], $L \not\leq_{\rm plt} E$ or $L \not\leq_{\rm plt} \overline{E}$. We consider only the first case, the second being dual. By Theorem 7.7, the minimal automaton $\mathcal{A}$ of L contains a balanced coNP-pattern, or co1NP-pattern, or n-counting pattern.

In the case of coNP-pattern, consider the *plt*-function $f(w) = xh(w)y$, where $h : A^+ \to A^+$ is the homomorphism satisfying $h(a) = u$ and $h(b) = v$ for any $b \in A \setminus \{a\}$. The function f satisfies $\overline{E} = f^{-1}(L)$, hence $\overline{E} \leq_{\rm plt} L$.

In the case of co1NP-pattern, we similarly get $\overline{0^*10^*} = f^{-1}(L)$. Since $E \leq_{\rm plt} \overline{0^*10^*}$, we get $E \leq_{\rm plt} L$.

In the case of counting pattern, let $M'_n \subseteq \{0,1\}^+$ be the set of words with $\#_1(w) \equiv 0 \pmod n$. By the proof of Theorem 7.5, $M'_p \leq_{\rm plt} L$ for some prime p. Obviously, $M_p \equiv_{\rm plt} M'_p$. Therefore, $M_p \leq_{\rm plt} L$. □

Corollary 7.13 Δ_1^τ *coincides with the class of regular* PLT*-decidable languages.*

Proof. According to [Gl05], a regular language L is PLT-decidable iff $L \leq_{\rm plt} E$ and $L \leq_{\rm plt} \overline{E}$. This is equivalent to $L \in \Delta_1^\tau$. □

plt and *ptt*- reducibilities are not well enough related to natural hierarchies of regular sets [Gl05, Wa04]. In Section 8 we consider reducibilities which behave better in this respect.

8 Quantifier-Free Reducibilities

In [SW04] the reducibility by quantifier-free formulas of signature σ was introduced and studied. Here we generalize notions and results of [SW04] to signature τ_d for every fixed $d > 0$, and present some new results.

A $qf\tau_d$*-interpretation* I *over alphabets* $A = \{a, \ldots\}$ *and* $B = \{b, \ldots\}$ is given by a tuple

$$(\phi_U(\bar{x}), \phi_<(\bar{x}, \bar{y}), \phi_\bot(\bar{x}), \phi_\top(\bar{x}), \phi_S(\bar{x}, \bar{y}), \phi_b(\bar{x}), \ldots, \phi_d^r(\bar{x}))$$

where $\bar{x} = (x_1, \ldots, x_n)$ and $\bar{y} = (y_1, \ldots, y_n)$ are sequences of different variables of the same length $n > 0$ (n is fixed in advance) and $\phi_U(\bar{x}), \ldots, \phi_d^r(\bar{x})$ are quantifier-free formulas of τ_d^A with the following properties. Let $u = u_0 \cdots u_l$ be any word over A of length $|u| = l + 1$. Then the set $T = \{\bar{x} \in \{0, \ldots, l\}^n | \mathbf{u} \models \phi_U(\bar{x})\}$ should be non-empty and the formulas $\phi_<(\bar{x}, \bar{y}), \phi_\bot(\bar{x}), \phi_\top(\bar{x}), \phi_S(\bar{x}, \bar{y})), \phi_b(\bar{x}), ..., \phi_d^r(\bar{x})$ interpreted in $\mathbf{u}$ should define a model of $\mathrm{CLO}^{\tau_d}{}_B$ with the universe T (the formulas $\phi_\bot(\bar{x}), \phi_\top(\bar{x})$ should be true exactly on the first and the last element, respectively). Since finite models of $\mathrm{CLO}^{\tau_d}{}_B$ are in a bijective correspondence with elements of B^+, any $qf\tau_d$-interpretation I induces a function $u \mapsto u^I$ from A^+ into B^+.

Examples. 1. Let $\phi_U(\bar{x})$ be a valid formula, let $\phi_<(\bar{x}, \bar{y})$ define the lexicographic ordering between $\bar{x}$ and $\bar{y}$, let $\phi_\bot(\bar{x}), \phi_\top(\bar{x}), \phi_S(\bar{x}, \bar{y})$ be defined in the obvious way according to their intended interpretations, and let $\phi_b(\bar{x}, \ldots), \phi_d^r(\bar{x})$ be choosen in a way to get a model of $\mathrm{CLO}^{\tau_d}{}_B$ on this ordering. Then u^I is of length $|u|^n$, and letters of the word u^I are easily computed from the interpretation. Note that $u \mapsto u^I$ is a *plt*-function.

2. Let $\phi_U(x)$ be a valid formula of one variable x, let $\phi_<(x,y)$ be $x > y$, let $\phi_a(x)$ be $Q_a(x)$ for any $a \in A$, let $\phi_\perp(\bar{x}), \phi_\top(\bar{x})$ be $x = \top$, $x = \perp$ respectively, and let $\phi_S(x,y)$ be $S(y,x)$. Let $\phi^r_d(\bar{x})$ be defined in the obvious way to satisfy the axioms of CLO^{τ_d}. Then we get a $qf\tau_d$-interpretation I over A and A such that u^I is the reverse of the word $u \in A^+$. Note that $u \mapsto u^I$ is a *plt*-function.

3. Let $u \mapsto pu$ be the function on A^+ which adds a fixed prefix $p \in A^*$ to a word u. Is there a $qf\tau_d$-interpretation I over A and A such that $u^I = pu$ for any u? For $p = \varepsilon$ the answer is of course positive, otherwise it is negative (since any $qf\tau_d$-interpretation sends words of length 1 to words of length 1). But it is easy to see that there is a $qf\tau_d$-interpretation I such that $u^I = pu$ for any u of length > 1. The same of course applies to the operation of adding a suffix to a word.

4. For any semigroup morphism $h : A^+ \to B^+$ with the property $\forall a, b \in A(|h(a)| \equiv |h(b)| \pmod{d})$ there is a $qf\tau_d$-interpretation I over A and B such that $u^I = h(u)$ for almost all $u \in A^+$ (i.e. for all but finitely many words).

Definition 8.1

- *A function $f : A^+ \to B^+$ is called a $qf\tau_d$-function if there is a $qf\tau_d$-interpretation I over A and B such that $u^I = f(u)$ for almost all $u \in A^+$.*
- *We say that $L \subseteq A^+$ is $qf\tau_d$-reducible to $K \subseteq B^+$ (in symbols $L \leq_{\mathrm{qf}\tau_\mathrm{d}} K$) if $L = f^{-1}(K)$ for some $qf\tau_d$-function $f : A^+ \to B^+$.*

The next theorem generalizes corresponding facts from [SW04] obtained there for the case of $qf\sigma$-reducibility. In the proof of the item (6) below one has only to note that the function from the proof of Theorem 7.8(1) is a $qf\tau_d$ function. The proof of the item (7) makes use of Theorem 6.3 and Proposition 4.4.

Theorem 8.2

1. *The relation $\leq_{\mathrm{qf}\tau_\mathrm{d}}$ is reflexive and transitive.*
2. *$\{\emptyset\}$ and $\{A^+\}$ are two distinct minimal elements of the degree structure $(P(A^+); \leq_{\mathrm{qf}\tau_\mathrm{d}})$ which are below any other element.*
3. *The structure $(P(A^+); \leq_{\mathrm{qf}\tau_\mathrm{d}})$ is an upper semilattice.*
4. *The structure $(P'(A^+); \leq_{\mathrm{qf}\tau_\mathrm{d}})$ is a distributive upper semilattice with a least element which consists exactly of non-trivial $\Delta_1^{\tau_\mathrm{d}}$-languages.*
5. *The classes $\cup_n \Sigma_n^{\tau_\mathrm{d}}$, $\mathcal{R}$, $FO + MOD(\mathcal{P})$ ($\mathcal{P}$ is any set of positive integers containing d) are ideals of $(P(A^+); \leq_{\mathrm{qf}\tau_\mathrm{d}})$.*
6. *The classes $\Sigma_n^{\tau_\mathrm{d}}, \Pi_n^{\tau_\mathrm{d}}$, as well as all levels of the difference hierarchy over $\Sigma_n^{\tau_\mathrm{d}}$, are principal ideals of $(P(A^+); \leq_{\mathrm{qf}\tau_\mathrm{d}})$.*
7. *Let D_n be a $qf\tau_d$-complete set in the n-th level of the difference hierarchy over $\Sigma_1^{\tau_\mathrm{d}}$. Then $D_n \oplus \overline{D}_n$ is the infimum of sets $D_{n+1}, \overline{D}_{n+1}$ under $\leq_{\mathrm{qf}\tau_\mathrm{d}}$.*

Next we state a 'parametrized' version of Theorem 7.5. For any fixed $d > 0$, the notion of a d-balanced n-counting pattern is obtained from Definition 7.4 by replacing the equality $|u| = |v|$ on the equivalence $|u| \equiv |v| \equiv 0 \pmod{d}$. Note that for $d = 1$ the last condition becomes trivial, and we speak simply about n-counting pattern.

Theorem 8.3 *For every regular language L the following conditions are equivalent.*

1. *L is $fo\tau_d$-definable.*

2. *Every finite automaton recognizing L has no d-balanced n-counting patterns ($n > 1$).*

3. *The automaton $M(L)$ has no d-balanced n-counting patterns ($n > 1$).*

The proof is almost the same as that of Theorem 7.5, only now, in the proof of (3) $\to$ (1), we have $|u_i| \equiv 0 \pmod{d}$ for every $i < n$. This guaranties that the pattern will be d-balanced.

In [SW04] some relationships between *plt*- and $qf\sigma$-reducibilities were established. These results also generalize to signatures τ_d.

Theorem 8.4 *1. If $L \leq_{\mathrm{qf}\tau_{\mathrm{d}}} M$ and M is of counting type then $L \leq_{\mathrm{plt}} M$.*

2. *If both L and M are of finite counting type then $L \leq_{\mathrm{qf}\tau_{\mathrm{d}}} M$ iff $L \leq_{\mathrm{plt}} M$.*

3. *Within the class of differences of $\Sigma_1^{\tau_{\mathrm{d}}}$ sets there are infinitely many languages modulo $\equiv_{\mathrm{qf}\tau_{\mathrm{d}}}$.*

Next we want to state 'parametrized' versions of Theorems 7.7 and 7.12. The notions of d-balanced coUP-, coNP- and co1NP-patterns are obtained from Definition 7.6 by replacing the equality $|u| = |v|$ on the equivalence $|u| \equiv |v| \equiv 0 \pmod{d}$.

Theorem 8.5 *Let L be a regular language and $\mathcal{A}$ the minimal automaton recognizing L. The following conditions are equivalent:*

1. *$L \leq_{\mathrm{qf}\tau_{\mathrm{d}}} 0^*1(0 \cup 1)^*$;*

2. *$\mathcal{A}$ does not have d-balanced coUP-patterns;*

3. *$\mathcal{A}$ does not have d-balanced coNP-, co1NP- and n-counting patterns ($n > 1$);*

4. *L is a finite union of languages of the form $w_0(A^d)^*w_1 \cdots (A^d)^*w_n$, where $n \geq 0$ and $w_i \in A^*$.*

Note that for $d = 1$ we get the corresponding result from [BKS99] (and that other forbidden-pattern results from [BKS99, Gl05] are also parametrizable in the same way). The proof of the last theorem is obtained by easy modifications of the proofs of Claim 1 and Corollary 2 in [Gl05]).

Corollary 8.6 *$\Sigma_1^{\tau_{\mathrm{d}}}$ is the class of finite unions of languages $w_0(A^d)^*w_1 \cdots (A^d)^*w_n$, where $n \geq 0$ and $w_i \in A^+$.*

Proof. The inclusion from right to left is obvious. The opposite inclusion follows from the last theorem and the fact that $0^*1(0 \cup 1)^*$ is $qf\tau_d$-complete in $\Sigma_1^{\tau_{\mathrm{d}}}$. $\square$

Next we prove an analog of Theorem 7.12.

Theorem 8.7 *The semilattice $(\mathcal{R}'; \leq_{\mathrm{qf}\tau_{\mathrm{d}}})$ is atomic with infinitely many atoms.*

Proof. We claim that the languages $E, \overline{E}, M_p$ (p prime) from the proof of Theorem 7.12 define exactly the atoms of $(\mathcal{R}'; \leq_{\mathrm{qf}\tau_{\mathrm{d}}})$, i.e.:

1) $E, \overline{E}, M_p$ are pairwise $qf\tau_d$-incomparable,

2) for any $L \in \mathcal{R} \setminus \Delta_1^{\tau_d}$, at least one of sets $E, \overline{E}, M_p$ is $qf\tau_d$-reducible to L.

1) The incomparability of E and $\overline{E}$ follows from the fact that E is $qf\tau_d$-complete in $\Sigma_1^{\tau_d}$ and $\Sigma_1^{\tau_d} \neq \Pi_1^{\tau_d}$. Since M_p is not a quasiaperiodic language (see [St94]), we have $M_p \not\leq_{\mathrm{qf}\tau_d} E$. Suppose that $E \leq_{\mathrm{qf}\tau_d} M_p$. Since M_p is of counting type, $E \leq_{plt} M_p$, contradicting to the proof of Theorem 7.12. Therefore, E and M_p are incomparable. A similar proof shows that $\overline{E}$ and M_p are incomparable. It remains to check that $M_p \not\leq_{\mathrm{qf}\tau_d} M_q$ for $p \neq q$. Suppose $M_p \leq_{\mathrm{qf}\tau_d} M_q$. This implies $M_p \leq_{\mathrm{plt}} M_q$, a contradiction.

With Theorem 8.5 at hand, the assertion 2) is proved in the same way as the corresponding assertion in the proof of Theorem 7.12. One has only to observe that the function g from the proof of Theorem 7.5 is a $qf\tau_d$-function for all $d > 0$. $\square$

From distributivity of the semilattice $(\mathcal{R}'; \leq_{\mathrm{qf}\tau_d})$ we immediately get the following.

Corollary 8.8 *The lattice $(Fin; \subseteq)$ of all finite subsets of ω is isomorphic to an ideal of $(\mathcal{R}'; \leq_{\mathrm{qf}\tau_d})$.*

We note that the result from [SW04] on the pincipal ideal generated by the set $U \oplus \overline{U}$, where $U = 0^*10^*$, is also true for the structure $(P(A^+); \leq_{\mathrm{qf}\tau_d})$. This principal ideal is shown in Figure 1, where $\mathbf{u}$ is the degree of U, $\mathbf{d}_0$ is the degree of $\emptyset$ and $\mathbf{d}_1$ is the degree of $0^*1(0 \cup 1)^*$.

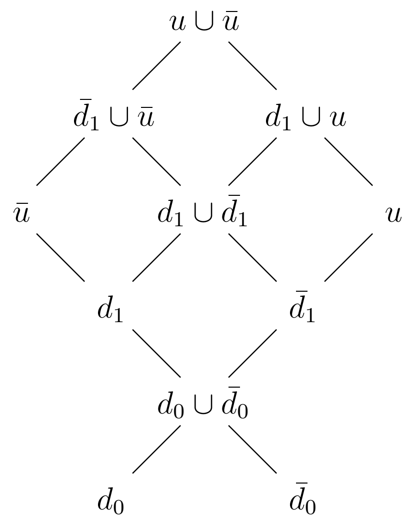

Figure 1. The principal ideal generated by $\mathbf{u} \cup \bar{\mathbf{u}}$.

One may wonder about relationships between reducibilities studied above. In [Wa04] it is shown that the *plt*-, *ptt*- and $qf\sigma$-reducibiliites are pairwise incomparable (with only one exception: it is still open whether $L \leq_{qf\sigma} K$ implies $L \leq_{ptt} K$). What about relationships between $qf\tau_d$-reducibilities for different d? We guess that all these reducibilities are pairwise incomparable. Here is one particular case.

Proposition 8.9 *For any $d > 1$, $qf\tau_d$-reducibility is incomparable with $qf\tau_1$-reducibility.*

Proof. Since $(A^d)^+ \in \Delta_1^{\tau_d}$, $(A^d)^+ \leq_{\mathrm{qf}\tau_d} 0^*1(0 \cup 1)^*$. Since $(A^d)^+$ is not aperiodic, $(A^d)^+ \not\leq_{qf\tau_1} 0^*1(0 \cup 1)^*$. Therefore, $qf\tau_d$-reducibility does not imply $qf\tau_1$-reducibility.

Conversely, let $h : \{0,1\}^+ \to \{0,1\}^+$ be the homomorphism satisfying $h(0) = 0^d$ and $h(1) = 1$. Then $\#_1(h(w)) = \#_1(w)$ and $|h(w)| \equiv \#_1(w) \pmod{d}$. Thus,

$$w \in M_d \leftrightarrow \#_1(w) \equiv 0 \pmod{d} \leftrightarrow |h(w)| \equiv 0 \pmod{d}.$$

Since h is a $qf\tau_1$-function, $M_d \leq_{qf\tau_1} (A^d)^+$. Since M_d is not quasiaperiodic, $M_d \not\leq_{\mathrm{qf}\tau_\mathrm{d}} (A^d)^+$. Therefore, $qf\tau_1$-reducibility does not imply $qf\tau_d$-reducibility. $\square$

We have seen that $df\tau_d$-reducibility fits the τ_d-hierarchy in the sense that any level $\Sigma_n^{\tau_\mathrm{d}}$ is downward closed and has a complete set under $\leq_{\mathrm{qf}\tau_\mathrm{d}}$. Is there a reducibility which fits the τ-hierarchy? A positive solution to this problem is given by the following reducibility: $L \leq_\tau K$, if there exists $c > 0$ such that $L \leq_{\mathrm{qf}\tau_\mathrm{d}} K$ for all multiples d of c. Analogs of many results of this sections hold true for the τ-reducibility. E.g., τ-reducibility fits the τ-hierarchy, the degree structure $(\mathcal{R}'; \tau)$ is a semilattice with a smallest element consisting exactly of nontrivial languages in Δ_1^τ, the structure $(\mathcal{R}'; \tau)$ is atomic with infinitely many atoms (defined by the same sets as in the proof of Theorem 7.12).

9 First Order Reducibilities

Here we consider weaker logical reducibilities, namely reducibilities $\leq_{\mathrm{fo}\tau_\mathrm{d}}$ by first order formulas of signature τ_d. Definition of $\leq_{\mathrm{fo}\tau_\mathrm{d}}$ is the same as that of $\leq_{\mathrm{qf}\tau_\mathrm{d}}$, only now the interpretation I consists of arbitrary first order formulas of τ_d. The following result is straightforward.

Theorem 9.1 *1. The relation $\leq_{\mathrm{fo}\tau_\mathrm{d}}$ is reflexive and transitive.*

2. $\{\emptyset\}$ and $\{A^+\}$ are two distinct minimal elements of $(P(A^+); \leq_{\mathrm{fo}\tau_\mathrm{d}})$ which are below any other element.

3. The structure $(P(A^+); \leq_{\mathrm{fo}\tau_\mathrm{d}})$ is an upper semilattice.

4. The classes $\mathcal{R}$, $FO + MOD(\mathcal{P})$ ($\mathcal{P}$ is any set of positive integers containing d) are ideals of $(P(A^+); \leq_{\mathrm{fo}\tau_\mathrm{d}})$.

5. The structures $(P'(A^+); \leq_{\mathrm{fo}\tau_\mathrm{d}})$ and $(\mathcal{R}'; \leq_{\mathrm{fo}\tau_\mathrm{d}})$ are distributive upper semilattices with a least element consisting exactly of non-trivial $fo\tau_d$-definable languages.

The next theorem is an analog of Theorems 8.7 and 7.12.

Theorem 9.2 *The semilattice $(\mathcal{R}'; \leq_{\mathrm{fo}\tau_\mathrm{d}})$ is atomic with infinitely many atoms.*

Proof. We claim that the sets M_p (p prime) from the proof of Theorem 7.12 define exactly the atoms of $(\mathcal{R}'; \leq_{\mathrm{fo}\tau_\mathrm{d}})$, i.e.:

1) M_p are pairwise $fo\tau_d$-incomparable,

2) for any regular non-$fo\tau_d$-definable language L, at least one of M_p is $fo\tau_d$-reducible to L.

To prove 1), suppose that $M_q \leq_{\mathrm{fo}\tau_\mathrm{d}} M_p$ for some $q \neq p$. Since $M_p \in (FO + MOD)(p)$ and $(FO + MOD)(p)$ is an ideal of $(P(A^+); \leq_{\mathrm{fo}\tau_\mathrm{d}})$, $M_q \in (FO + MOD)(p)$. This contradicts to Theorem VII.2.1 in [St94].

2) Let L be regular and non-$fo\tau_d$-definable. By Theorem 8.5 and the proof of Theorem 8.7, $M_p \leq_{\mathrm{qf}\tau_\mathrm{d}} L$ for some prime p. Therefore, $M_p \leq_{\mathrm{fo}\tau_\mathrm{d}} L$. $\square$

Corollary 9.3 *The lattice* $(Fin; \subseteq)$ *is isomorphic to an ideal of* $(\mathcal{R}'; \leq_{\mathrm{fo}\tau_{\mathrm{d}}})$.

The reducibility $\leq_{\mathrm{fo}\sigma}$ seems to be related to a well-known open problem of automata theory, namely the problem of generalized star-height. We believe that better understanding of the structure $(\mathcal{R}; \leq_{\mathrm{fo}\sigma})$ may shed some light on this problem.

10 Other Reducibilities and Open Questions

There are also other natural reducibilities on regular sets. E.g., let $\leq_{fom}$ be defined in the same way as $\leq_{\mathrm{fo}\tau_{\mathrm{d}}}$ but this time the interpretation I consists of arbitrary $(FO{+}MOD)$-formulas. One can easily establish for this reducibility an analog of Theorem 9.1. In particular, the smallest degree in $(\mathcal{R}'; \leq_{fom})$ consists exactly of non-trivial regular languages with solvable syntactic monoid. Therefore, there exist two distinct (modulo $\equiv_{fom}$) non-trivial regular languages. We do not know whether there exist three non-trivial regular languages which are pairwise distinct modulo $\equiv_{fom}$.

We do not also currently know whether there exist regular languages which are complete under 'logical' reducibilities considered above.

Analogs of the open questions from [SW04] for $qf\tau_d$-reducibilities seem also interesting.

One could consider also reductions by functions computable by natural classes of finite transducers. Such reducibilities were successfully applied for classification of regulat ω-languages (see [Wa79]).

Acknowledgement. I am grateful to Klaus Wagner for hosting my stay at the University of Würzburg in the pre-Christmas days of 2004, and also to him and Christian Glaßer for helpful discussions.

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