Playing with Boolean Blocks: Constraint Satisfaction Problems[1]

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Introduction

This is the second part of a two part survey article about Boolean functions, propositional formulas and Boolean constraint satisfaction in connection with Post's lattice.

In the previous part [BCRV04] we played with Boolean functions as "building blocks" in the construction of switching circuits. In the meantime we got to know a child, having a great collection of essentially the same blocks. However, during her construction process she uses the blocks in a completely different way than we do: she uses Boolean functions not to build switching circuits but to formulate queries to a database. Nevertheless our games are closely related and there is a bridge to connect them. Therefore we will tell you more about the new game and the connection in this issue.

Every Boolean function trivially defines a Boolean relation. The relation is the set of all tuples, for which the function-value is 1. For instance the 2-ary Boolean *or*-function defines the relation $\{(0,1),(1,0),(1,1)\}$. Instead of asking whether a function maps an input-tuple to 1, we now ask whether it belongs to the relation. That means that each n-ary relation is a constraint on the set of all n-tuples. This leads us to the rich field of "constraint satisfaction", which has numerous applications in computer science (database querying, circuit design, network optimization, scheduling, logic programming), artificial intelligence (planning, belief maintenance, languages for knowledge based systems), and computational linguistics (processing of natural languages) [Kol03].

In the world of Boolean constraint satisfaction we start with a box that contains Boolean constraints. There is only a finite number of different constraint types in the box, but of each type we have an infinite number of "copies". Now we want to combine them in order to get new constraints. First fix a set of constraints. We build a new constraint as follows: Make a list of constraints which are taken out of the fixed set. Check for each suitable Boolean tuple whether it fulfills all constraints of the list. If this is the case the tuple fulfills the newly built constraint. What does that mean? This means that we simply connect the constraints using "and". In this context one important problem is to check whether a collection of constraints does not define the empty relation. Suppose we have built a constraint as explained out of a finite set of Boolean relations S (such a constraint is considered as a formula in generalized conjunctive normal form, a so called S-*formula* or $\mathrm{CNF}(S)$), then this problem is known as *generalized satisfiability problem*, denoted by $\mathrm{SAT}(S)$, and was first investigated by Thomas Schaefer in 1978. Here we are interested in the complexity of $\mathrm{SAT}(S)$. It turns out that its complexity can be characterized by the set of so called *closure properties* of S, which is the set of functions f of arity m such that if we apply f coordinate-

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wise to m vectors of any relation R from S, then the result is again in R. This correspondence is obtained through a generalization of Galois theory. Since this set of closure properties—a set of Boolean functions—is a clone, we have now a connection between the complexity of Boolean constraint satisfaction problems and the clone theory. Therefore we will show that Post's lattice can also be a very useful tool for studying the complexity of Boolean constraint satisfaction problems.

1 Closed Classes of Boolean Constraints

The study of constraint satisfaction problems (CSPs) leads to a powerful general framework in which a variety of combinatorial problems can be expressed. The aim in a constraint satisfaction problem is to find an assignment of values to the variables subject to specified constraints. This framework is used in a many research areas in computer science. Here and in the next section we will have a look at the complexity of Boolean constraint satisfaction problems and show that it depends solely on the sort of constraint relations one is allowed to use in the instance.

1.1 Boolean Constraints

Throughout the paper we use the standard correspondence between propositional formulas (predicates) and relations: a relation consists of all tuples of values for which the corresponding formula holds. We will use the same symbol for a predicate and its corresponding relation, since the meaning will always be clear from the context, and we will say that the formula *represents* the relation.

An n-ary *Boolean constraint relation* R is a Boolean relation of arity n, i.e., $R \subseteq \{0,1\}^n$. Let V be a set of variables, then a *constraint application* (or simply, a *constraint*) C is an application of R to an n-tuple of (not necessarily distinct) variables from V, i.e., $C = R(x_1, \ldots, x_n)$. An assignment $I\colon V \to \{0,1\}$ satisfies the constraint $R(x_1, \ldots, x_n)$ if and only if $(I(x_1), \ldots, I(x_n)) \in R$.

Example 1.1. For $0 \leq l \leq k$, if $R_{or_{k,l}}$ is the k-ary relation $R_{or_{k,l}} = \{0,1\}^k \setminus \{1^l 0^{k-l}\}$, then $R_{or_{k,l}}(x_1, \ldots, x_k) = (\bar{x}_1 \vee \ldots \vee \bar{x}_l \vee x_{l+1} \vee \ldots \vee x_k)$.

If R_{NAE} and $R_{\mathsf{One\text{-}Exactly}}$ are the ternary relations $R_{\mathsf{NAE}} = \{0,1\}^3 \setminus \{(0,0,0),(1,1,1)\}$ and $R_{\mathsf{One\text{-}Exactly}} = \{(1,0,0),(0,1,0),(0,0,1)\}$, then the constraint $R_{\mathsf{NAE}}(x,y,z)$ is satisfied if and only if not all the three variables are assigned the same value and the constraint $R_{\mathsf{One\text{-}Exactly}}(x,y,z)$ is satisfied if and only if exactly one of x, y and z is assigned true.

Example 1.2. The equivalence relation is the binary relation given by $\mathrm{Eq} = \{(0,0),(1,1)\}$.

Throughout the text we will refer to different types of Boolean constraint relations, following the terminology of Schaefer [Sch78].

- A Boolean relation R is *0-valid* (*1-valid* , resp.) if $(0, \ldots, 0) \in R$ ($(1, \ldots, 1) \in R$, resp.).
- A Boolean relation R is *Horn* (*anti-Horn*, resp.) if R can be represented by a conjunctive normal form (CNF) formula having at most one unnegated (negated, resp.) variable in any conjunct.
- A Boolean relation R is *bijunctive* if it can be represented by a CNF formula having at most two variables in each conjunct.
- A Boolean relation R is *affine* if it can be represented by a conjunction of linear functions, i.e., a CNF formula with $\oplus$-clauses (XOR-CNF).

– A Boolean relation R is *complementive* if for all $(\alpha_1, \ldots, \alpha_n) \in R$ also $(\overline{\alpha_1}, \ldots, \overline{\alpha_n}) \in R$.

A set S of Boolean relations is called 0-valid (1-valid, Horn, anti-Horn, affine, bijunctive, complementive, resp.) if every relation in S is 0-valid (1-valid, Horn, anti-Horn, affine, bijunctive, complementive, resp.). A constraint set S is called *Schaefer*, if S is Horn, anti-Horn, affine, or bijunctive.

There are easy criteria to determine if a given relation is Horn, anti-Horn, bijunctive, or affine.

– A relation R is Horn, if and only if for all vectors $x, y \in R$, the vector obtained by taking coordinate-wise the logical conjunction, in symbols: $x \wedge y$, is in R [DP92], see also [CKS01, Lemma 4.8]. Similarly, the property anti-Horn is characterized by coordinate-wise disjunction.

– A relation R is bijunctive, if and only if for all vectors $x, y, z \in R$, the vector obtained by taking coordinate-wise majority (i.e., the ith coordinate is set to 1 if and only if in at least two of x, y, z the ith coordinate is 1) is in R [Sch78], see also [CKS01, Lemma 4.9].

– A relation R is affine, if and only if for all vectors $x, y, z \in R$, the vector obtained by taking coordinate-wise logical exclusive-or ($x \oplus y \oplus z$) is in R [Sch78, CH96], see also [CKS01, Lemma 4.10].

Thus, each of these criteria is given in form of a closure property of the set of all vectors in R, and each involves an operation performed on these vectors coordinate-wise.

We prove the characterization of Horn: Suppose R is Horn. Consider a clause C in the representation of R by a Horn formula, and two assignments I_1, I_2 that satisfy C. If one of I_1, I_2 satisfies one of the negative literals in C, then the same literal is satisfied in $I_1 \wedge I_2$. If both I_1, I_2 satisfy the positive literal in C, then of course this literal is also satisfied by $I_1 \wedge I_2$. Thus we see that if $x, y \in R$, then $x \wedge y \in R$.

For the converse, let us first introduce a definition. A subset of literals defined over the variables of R is a *maxterm* of R if setting each of the literals false determines the predicate to be false and it is a minimal such collection. If R is not Horn, then one can prove that R has at least one maxterm that is not Horn [CKS01, Lemma 4.7, page 29]. Suppose that this maxterm, C, contains at least two positive literals, say x_i and x_j. Let I_1 be the assignment that satisfies variable x_i plus those variables occurring negated in C and falsifies all other variables, and let I_2 be the assignment that satisfies x_j plus those variables occurring negated in C and falsifies all other variables. Then I_1, I_2 satisfy C, but the coordinate-wise conjunction $I_1 \wedge I_2$ does not satisfy C. Hence we found vectors $x, y \in R$ with the property $x \wedge y \notin R$.

The proof of the characterization of anti-Horn is analogous, and the proof of the characterization of bijunctive is very similar. The proof for affine constraints rests on a classical characterization of affine subspaces in linear algebra and can be found, e.g., in [CKS01, p. 30].

1.2 Assembling Boolean Constraints: Generalized Propositional Formulas and Conjunctive Queries

We will now consider formulas that are conjunctions of constraints, where the constraints are given by Boolean relations. Let S be a non-empty finite set of Boolean relations. Then *S-formulas* are conjunctive propositional formulas consisting of clauses built by using relations from S applied to arbitrary variables. Formally, let S be a set of Boolean relations and V be a set of variables. An *S-formula* F (over V) is a finite conjunction of clauses $F = C_1 \wedge \ldots \wedge C_p$, where each clause C_i is a constraint application of some constraint relation $R \in S$. If $F = C_1 \wedge \ldots \wedge C_p$ is such a formula over

V and I is an assignment with respect to V, then $I \models F$ if I satisfies all clauses C_i. By CNF(S) we denote the set of all S-formulas as just defined.

Schaefer in his seminal paper was interested in a classification of the complexity of satisfiability of S-formulas. In order to obtain such a result, so called *conjunctive queries* turn out to be useful. Conjunctive queries play an important role in database theory (they are equivalent to select-join-project queries in relational algebra) and thus are also of interest in their own right.

Given a set S of Boolean relations, let us denote by COQ(S) the set of all formulas of the form $\exists x_1 \exists x_2 \ldots \exists x_k \phi(x_1, \ldots, x_k, y_1, \ldots, y_l)$, where ϕ is a CNF(S); these formulas are called *conjunctive queries (over S)* [KV00]. In other words, a relation can be represented by a conjunctive query if and only if it is a projection of a relation that can be represented by an S-formula [JCG99, Definition 4, Lemma 1, and Notation 3].

Intuitively constraints using relations from COQ(S) are those which can be "simulated" by constraints using relations in S. We remark that in Schaefer's terminology, COQ(S) was denoted by Rep(S).

1.3 Closure Properties of Constraints: a Little Bit of Galois Theory

As we have already seen, there are easy criteria to determine whether a given relation is Horn, anti-Horn, bijunctive or affine. All these criteria work essentially in the same way: If you apply a special Boolean function f coordinate-wise on some of the elements of the relation R, then the result of this application must be again an element in R. Moreover we have some more subtle common properties. Recall the test for affine. A relation R is affine if and only if it is closed under coordinate-wise application of $x_1 \oplus x_2 \oplus x_3$. Since we have no particular order of the elements of R, it is clear that R is affine if and only if we can do the same test with $x_{\pi(1)} \oplus x_{\pi(2)} \oplus x_{\pi(3)}$ for any permutation $\pi \colon \{1,2,3\} \to \{1,2,3\}$. Note that this argument does not rest on the commutativity of $\oplus$ and it will also work for other tests of this kind. Now we identify variables in $x_1 \oplus x_2 \oplus x_3$ to get the 2-ary function $f'(x_1, x_2) = f(x_1, x_2, x_2)$. If the relation is affine and we apply the function f' on some elements of R we will surely obtain an element of R, which shows that f' also "preserves" the relation R. Finally take the function $f''(x_1, x_2, x_3, x_4, x_5) = x_1 \oplus x_2 \oplus (x_3 \oplus x_4 \oplus x_5)$, which is obtained by the operation of substitution from the original function $x \oplus y \oplus z$. Clearly we get another element of R if we take five elements from R and apply f'' coordinate-wise on them. We thus see that the set of functions that preserve R forms a clone. In fact, this holds for arbitrary relations R. So there is a deeper connection between clones (defined in part I) and relations.

In the following we will see that a generalized Galois theory is the connection we are looking for. The roots of Galois theory were invented by Evariste Galois to find a solution for the classical problem to solve a given polynomial equation by radicals. He was able to give a recursive criterion for solvability in this sense. Additionally his results show that polynomial equations having a degree more than 4 in general cannot be solved with radicals. The modern presentation of this beautiful result establishes a bijection between two worlds: The set of certain extension fields of the field of coefficients and the set of automorphism-groups of this extension field (see e.g. [Art98]). In the original setting, Galois asked which permutations of zeroes of a polynomial preserve all algebraic dependence relations between these zeroes. (These dependencies are given in the form of polynomial equations, for instance: The square of the first zero is equal to the second zero plus 2, that is: the square of the first zero minus the second zero minus 2 equals 0.) Let G be the set of algebraic dependence relations of the zeroes of a polynomial equation. Galois defined a group of permutations—the Galois group—that preserve these relations as follows: A permutation π belongs to the Galois group if and only if for all $h \in G$, if $h(\zeta_1, \ldots, \zeta_n) = 0$, then $h(\zeta_{\pi(1)}, \ldots \zeta_{\pi(n)}) = 0$, where

$\zeta_1, \ldots, \zeta_n$ are the zeroes of the given polynomial equation. Intuitively, if the set of dependencies is "big", then the number of permutations in the Galois group is "small", and vice versa. This matches with the abstract definition of a Galois correspondence given next.

We say that two mappings $\alpha\colon A \to B$ and $\beta\colon B \to A$ form a *Galois correspondence* (cf. [PK79, A5.1], [Pip97, Sect. 1.2.3]) between the orders $(A, \preceq^A)$ and $(B, \preceq^B)$ if the following properties hold:

1. For all $a, b \in A$ such that $a \preceq^A b$, we have $\alpha(b) \preceq^B \alpha(a)$.
2. For all $c, d \in B$ such that $c \preceq^B d$, we have $\beta(d) \preceq^A \beta(c)$.
3. For all $a \in A$, $b \in B$ we have $a \preceq^A \beta(\alpha(a))$ and $b \preceq^B \alpha(\beta(b))$.

With this definition it is not hard to show that $\lambda_1\colon A \to A$ s.t. $\lambda_1(a) = \beta(\alpha(a))$ and $\lambda_2\colon B \to B$ s.t. $\lambda_2(b) = \alpha(\beta(b))$ are closure-operators [PK79]. (Here, a closure operator in an order $(A, \preceq^A)$ is a function $f\colon A \to A$ such that $a \preceq^A f(a)$, if $a \preceq^A b$ then $f(a) \preceq^A f(b)$, and $f(a) = f(f(a))$.)

To be able to use this abstract definition in our context of Boolean functions and constraints, we introduce the central notion of a *closure property* of a Boolean relation (see, e.g., [JC95, Pip97, Dal00]). As we will see shortly this notion provides an algebraic characterization of $\mathrm{COQ}(S)$. Let R be a Boolean relation of arity n and let f be a Boolean function of arity m. For f to be a closure property of R we require that if we apply f coordinate-wise to m vectors in R, the obtained vector is again in R; see Fig. 1. More formally, we say that R is *closed* under f (or f *preserves* R, or f is a *polymorphism* of R, or R is an *invariant* of f), if for all $x_1, \ldots, x_m \in R$, where $x_i = (x_i[1], x_i[2], \ldots, x_i[n])$, we have

$$\begin{array}{ccccccc}
 & & f(& f(& f(& & f(\\
x_1 & = & 0 & 1 & 1 & \cdots & 0 \\
x_2 & = & 1 & 1 & 0 & \cdots & 1 \\
\vdots & & \vdots & \vdots & \vdots & & \vdots \\
x_m & = & 1 & 1 & 0 & \cdots & 1 \\
 & &) &) &) & &) \\
 & & \| & \| & \| & & \| \\
z & = & 0 & 1 & 0 & \cdots & 0
\end{array}$$

Figure 1: If $x_1, \ldots, x_m$ are in R, then z must be in R.

$$\Big(f\big(x_1[1], \cdots, x_m[1]\big), f\big(x_1[2], \cdots, x_m[2]\big), \ldots, f\big(x_1[n], \cdots, x_m[n]\big)\Big) \in R.$$

Example 1.3. The set of all Horn relations (of all affine relations, resp.) is the set of all relations that are preserved by the logical conjunction (by the function $x \oplus y \oplus z$, resp.).

Let us denote the set of all polymorphisms of R by $\mathrm{Pol}(R)$, and for a set S of Boolean relations, we define $\mathrm{Pol}(S)$ to be the set of Boolean functions that are polymorphisms of every relation in S. Then, as in the case of Galois' algebraic dependencies, if S is "big" then the number of closure properties of S is "small", and vice versa.

Generalizing the observations we made for affine relations R in the first paragraph of Sect. 1.3, we see that for every set S, $\mathrm{Pol}(S)$ is a clone. (This holds since immediately by definition, $\mathrm{Pol}(S)$ is closed under introduction of fictive variables, permutation and identification of variables and substitution, and moreover, contains the projection functions.) Conversely, if B is a set of Boolean functions, then $\mathrm{Inv}(B)$ is defined to be the set of all relations which are preserved by the functions in B, i.e., the set of relations that are invariants of all functions in B. It turns out that sets $\mathrm{Inv}(B)$ have particular common properties: They contain the equivalence relation, and each such set is closed under Cartesian product, projection and identification of variables (see, e.g., [JCG99, Lemma 1]). More generally, for a set S of Boolean relations let $\langle S\rangle$ denote the closure of S plus the equivalence relation under the mentioned operations. We say that $\langle S\rangle$ is the *co-clone* generated by S (see [Dal00, page 81], [Pip97, Sect. 1.7]). Hence, for every set of Boolean functions B, $\mathrm{Inv}(B)$ is a co-clone.

Example 1.4. Horn relations, as well as affine relations, form a co-clone.

Hence Inv and Pol play the role of the functions α and β, resp., in the abstract definition of a Galois correspondence, when we identify the lattice of sets Boolean functions with A and the lattice of sets of Boolean relations with B. A basic introduction to this correspondence can be found in [Pip97, Pös01] and a comprehensive study in [PK79].

Interestingly, the closure operators λ_1 and λ_2 from the general setting of a Galois correspondence, i.e., in our case the functions $\mathrm{Inv}(\mathrm{Pol}(\cdot))$ and $\mathrm{Pol}(\mathrm{Inv}(\cdot))$, turn out to coincide with the co-clone- and clone-closure:

Theorem 1.5 ([Gei68, BKKR69],[PK79, Folgerung 1.2.4]). *Let S be a set of Boolean relations and B be a set of Boolean functions.*

- $\mathrm{Inv}(\mathrm{Pol}(S)) = \langle S\rangle$
- $\mathrm{Pol}(\mathrm{Inv}(B)) = [B]$

Example 1.6. Consider the constraint $R_{\textsf{One-Exactly}}$ defined in Example 1.1. The set of polymorphisms $\mathrm{Pol}(\{R_{\textsf{One-Exactly}}\})$ is a clone. It contains neither any constant operation nor the negation operation nor the disjunction nor the conjunction nor the majority operation nor the ternary exclusive-or operation. Therefore, looking at Post's lattice and at the basis of the classes, we see that $\mathrm{Pol}(\{R_{\textsf{One-Exactly}}\}) = \mathrm{I}_2$, i.e., consists only of all projections. Thus, $\mathrm{Inv}(\mathrm{Pol}(\{R_{\textsf{One-Exactly}}\})$ consists of all Boolean relations, and then so does $\langle\{R_{\textsf{One-Exactly}}\}\rangle$ by Theorem 1.5.

In the first part we have characterized the set of all functions that can be computed by a B-circuit as the closure of B under superposition: $\mathrm{CIRC}(B) = [B]$. Analogously, if we want to know what Boolean relations can be expressed by conjunctive queries over $S \cup \{\mathrm{Eq}\}$ it suffices to look at the closure of the Boolean relations (together with the equivalence relation), $S \cup \{\mathrm{Eq}\}$, under Cartesian product, projection and identification of variables. More formally (see [JCG99, Theorem 2], [Dal00, Theorem 20, p. 98]):

Proposition 1.7. *For every set S of Boolean relations,*

$$\mathrm{COQ}(S \cup \{\mathrm{Eq}\}) = \langle S\rangle.$$

Proof. The relation associated with a conjunctive query Φ over $S\cup\{\mathrm{Eq}\}$ is the projection (existential quantification) of the conjunction of relations occurring in Φ. Thus it is easy to see that $\mathrm{COQ}(S \cup \{\mathrm{Eq}\}) \subseteq \langle S\rangle$. Conversely, $S \subseteq \mathrm{COQ}(S \cup \{\mathrm{Eq}\})$. Now it is easy to check that $\mathrm{COQ}(S \cup \{\mathrm{Eq}\})$ is a co-clone, therefore we have also $\langle S\rangle \subseteq \mathrm{COQ}(S \cup \{\mathrm{Eq}\})$, thus proving the equality. ☐

1.4 The Lattice of Co-clones

As already mentioned in the first part, the set of all clones form a lattice. Because we want to connect clones and co-clones, the following definition is helpful: Let $(A, \preceq^A, \sqcup^A, \sqcap^A)$ and $(B, \preceq^B, \sqcup^B, \sqcap^B)$ be ordered lattices. A bijective lattice-homomorphism $\lambda\colon (A, \sqcup^A, \sqcap^A) \to (B, \sqcup^B, \sqcap^B)$ that reverses the order in the sense that if $a \preceq^A b$ then $\lambda(b) \preceq^B \lambda(a)$ is called *lattice-anti-isomorphism.*

Theorem 1.8 [PK79, Satz 3.1.2]. *The lattices of Boolean clones and Boolean co-clones are anti-isomorphic. The function* Inv *is a lattice-anti-homomorphism from the set of Boolean clones to the set of Boolean co-clones, and the function* Pol *for the opposite direction.*

Just as a convenient shorthand let us denote the co-clone that is invariant under B by $\mathrm{I}B$, hence $\mathrm{I}B = \mathrm{Inv}(B)$. Hence, for example, IE_2 is the co-clone of Horn relations and IL_2 is the co-clone of affine relations. The co-clone of all relations is denoted by BR. The complete structure of the lattice of co-clones is given by Fig. 2. A look at this graph shows that below the class BR there exist seven maximal closed classes of Boolean relations (circled bold in Fig. 2). Hence if we want to know whether all Boolean relations can be "simulated" by conjunctive queries built with constraints from a given set S, then we have to check that S is not included in one of these seven classes. This is the same idea as in part 1, where we used the five Post classes to obtain a completeness criterion for classes of Boolean functions and used it to prove that the *nand*-function is a base of BF.

Example 1.9. Let be $S = \{R_{or_{3,0}}, R_{or_{3,1}}, R_{or_{3,2}}, R_{or_{3,3}}\}$. Clearly $R_{or_{3,0}}$ is not 0-valid and $R_{or_{3,3}}$ is not 1-valid. Using our tests we can show that $R_{or_{3,0}}$ is not Horn, because $(1,0,0) \in R_{or_{3,0}}$ and $(0,0,1) \in R_{or_{3,0}}$ but $(0,0,0) \notin R_{or_{3,0}}$. Similarly we have that $R_{or_{3,3}}$ is not anti-Horn, because $(1,0,0) \in R_{or_{3,3}}$ and $(0,1,1) \in R_{or_{3,3}}$ but $(1,1,1) \notin R_{or_{3,3}}$. Additionally $R_{or_{3,0}}$ is not bijunctive, because $(1,0,0) \in R_{or_{3,0}}$, $(0,1,0) \in R_{or_{3,0}}$ and $(0,0,1) \in R_{or_{3,0}}$ but $(0,0,0) \notin R_{or_{3,0}}$. Next note that $(1,0,0) \in R_{or_{3,3}}$, $(0,1,0) \in R_{or_{3,3}}$ and $(0,0,1) \in R_{or_{3,3}}$ but $(1,1,1) \notin R_{or_{3,3}}$, which shows that $R_{or_{3,3}}$ is not affine. Finally we have that $(0,0,0) \notin R_{or_{3,0}}$ but $(1,1,1) \in R_{or_{3,0}}$ showing that $R_{or_{3,0}}$ is not complementive. Hence we have that the co-clone of S is $\mathrm{II}_2 = \mathrm{BR}$, which gives that $\mathrm{COQ}(S)$ is the set of all Boolean relations. In fact, our argument above shows that already $\mathrm{COQ}(\{R_{or_{3,0}}, R_{or_{3,3}}\}) = \mathrm{BR}$, the set of all relations.

2 The Complexity of Constraint Satisfaction Problems

The main topic in this section is the complexity of the satisfiability problem for S-formulas, denoted by $\mathrm{SAT}(S)$. Comparing conjunctive queries with S-formulas, the only difference is that in the former some of the variables are existentially quantified, but certainly this does not lead to a different complexity of the satisfiability problem. Formally, if $\mathrm{SAT}(\mathrm{COQ}(S))$ denotes the satisfiability problem for conjunctive queries over S, then:

Proposition 2.1. $\mathrm{SAT}(S)$ *is polynomial-time equivalent to* $\mathrm{SAT}(\mathrm{COQ}(S))$.

Thus it is clear that the algebraic correspondence described above can be of use to determine the complexity of constraint satisfaction problems. For instance, let us consider the relation $R_{\textsf{One-Exactly}}$. We have seen above that $\langle\{R_{\textsf{One-Exactly}}\}\rangle$ is the set of all Boolean relations, hence so is $\mathrm{COQ}(\{R_{\textsf{One-Exactly}}, \mathrm{Eq}\})$ by Proposition 1.7. Therefore, the satisfiability problem $\mathrm{SAT}(\{R_{\textsf{One-Exactly}}, \mathrm{Eq}\})$ is as hard as the general propositional satisfiability problem, i.e., it is NP-complete.

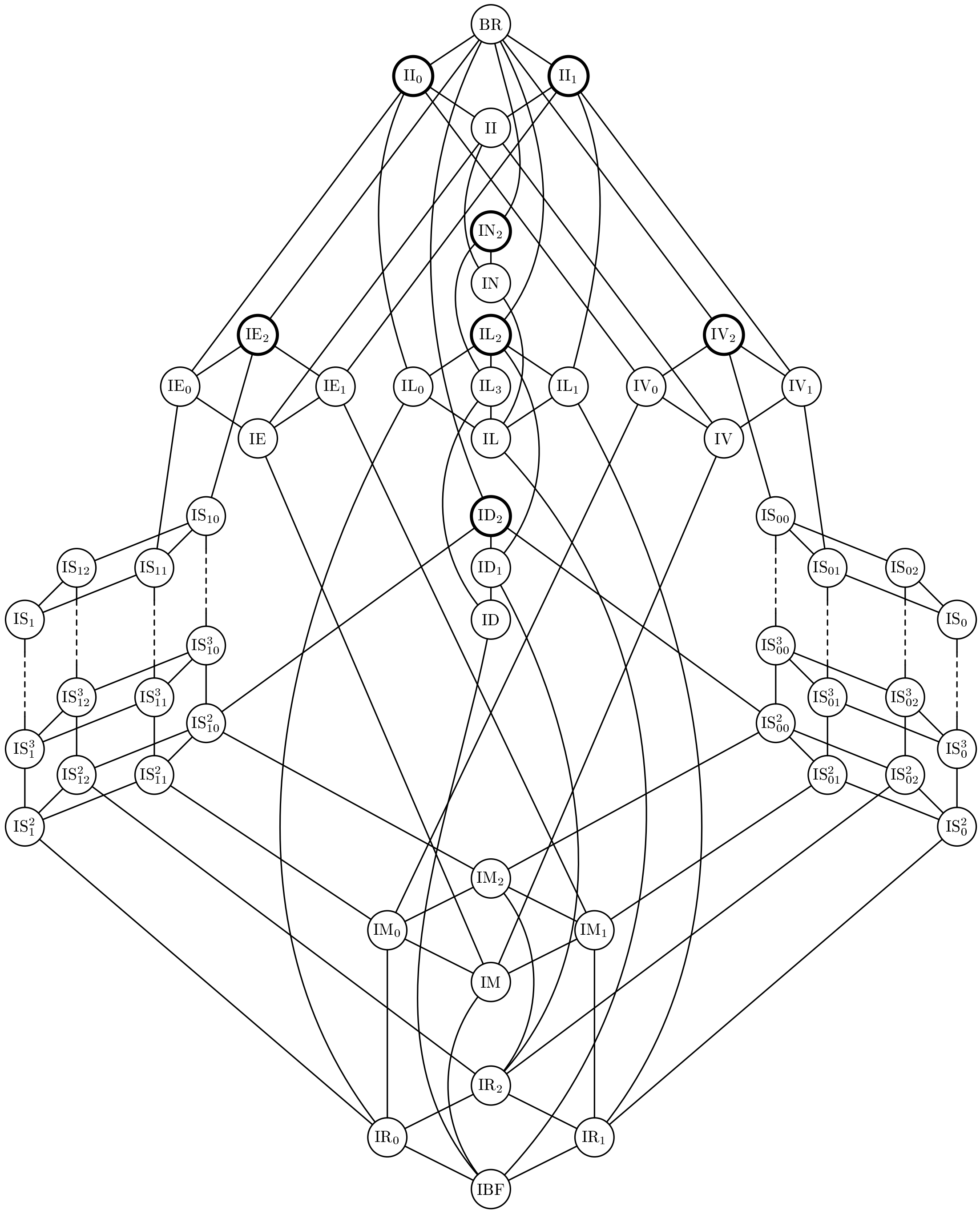

Figure 2: Graph of all co-clones

As a consequence we have that the complexity of $\mathrm{SAT}(S \cup \{\mathrm{Eq}\})$ is characterized by the co-clone generated by S, $\langle S \rangle$, or equivalently by its clone of polymorphisms $\mathrm{Pol}(S)$. We observe next that the equality relation actually is of no importance in the context of satisfiability; the reason is simply that in a $\mathrm{CNF}(S \cup \{\mathrm{Eq}\})$ we can delete a constraint $\mathrm{Eq}(x, y)$ and identify the variables x, y in the rest of the formula [Jea98]. Thus we obtain:

Proposition 2.2. *Let S be a non-empty set of Boolean relations, then $\mathrm{SAT}(S \cup \{\mathrm{Eq}\})$ is polynomial-time reducible to $\mathrm{SAT}(S)$. Hence, $\mathrm{SAT}(S)$ is polynomial-time equivalent to $\mathrm{SAT}(\langle S \rangle)$.*

The following two easy reductions provide the central tools to obtain a complexity classification of the satisfiability problem. The first one is obtained, if all constraints in one formula can be expressed by conjunctive queries over another constraint set:

Proposition 2.3. *Let S_1 and S_2 be finite non-empty sets of Boolean relations. If $\mathrm{COQ}(S_1) \subseteq \mathrm{COQ}(S_2)$, then $\mathrm{SAT}(S_1)$ is polynomial-time reducible to $\mathrm{SAT}(S_2)$.*

Proof. If $\mathrm{COQ}(S_1) \subseteq \mathrm{COQ}(S_2)$, then in particular every relation in S_1 can be expressed by a conjunctive query over S_2. Thus, given an S_1-formula, one can construct an S_2-formula in locally replacing (in each conjunct) every relation from S_1 by its equivalent conjunctive query in $\mathrm{COQ}(S_2)$ and simply deleting the existential quantifiers. This local replacement clearly provides a polynomial-time (and even logarithmic-space) reduction from $\mathrm{SAT}(S_1)$ to $\mathrm{SAT}(S_2)$. □

Combining this with Proposition 1.7 and Theorem 1.5 we obtain the following reduction:

Theorem 2.4 [Jea98]. *Let S_1 and S_2 be finite non-empty sets of Boolean relations. If $\mathrm{Pol}(S_2) \subseteq \mathrm{Pol}(S_1)$, then $\mathrm{SAT}(S_1)$ is polynomial-time reducible to $\mathrm{SAT}(S_2)$.*

Proof. By Theorem 1.8, if $\mathrm{Pol}(S_2) \subseteq \mathrm{Pol}(S_1)$, then $\mathrm{Inv}(\mathrm{Pol}(S_1)) \subseteq \mathrm{Inv}(\mathrm{Pol}(S_2))$. Hence, following Theorem 1.5, $\langle S_1 \rangle \subseteq \langle S_2 \rangle$. Thus, according to Proposition 1.7, $\mathrm{COQ}(S_1 \cup \{\mathrm{Eq}\}) \subseteq \mathrm{COQ}(S_2 \cup \{\mathrm{Eq}\})$ and hence $\mathrm{COQ}(S_1) \subseteq \mathrm{COQ}(S_2 \cup \{\mathrm{Eq}\})$. Therefore, according to Proposition 2.3, $\mathrm{SAT}(S_1)$ is polynomial-time reducible to $\mathrm{SAT}(S_2 \cup \{\mathrm{Eq}\})$, and thus to $\mathrm{SAT}(S_2)$ by Proposition 2.2. □

We are now in a position to prove Schaefer's main result, a dichotomy theorem for satisfiability of constraint satisfaction problems which can be stated as follows:

Theorem 2.5 [Sch78]. *Let S be a set of Boolean relations. If S is 0-valid or 1-valid or Schaefer then $\mathrm{SAT}(S)$ is polynomial-time decidable, in all other cases $\mathrm{SAT}(S)$ is NP-complete.*

We now give a proof of Theorem 2.5 making use of the algebraic correspondence described above and of the lattice of co-clones. Our presentation follows an argument sent to the authors in an e-mail correspondence by Phokion Kolaitis, but is implicit in the papers by Jeavons and his group (see, e.g., [JCG97, JCG99]) and in Dalmau's Ph. D. thesis [Dal00].

Let S be a set of constraints. We make a case distinction, making use of the graph of co-clones in Fig. 2.

1. $S \subseteq \mathrm{II}_0$, i.e., $\mathrm{Pol}(S) \supseteq \mathrm{I}_0$. Then we see that S must contain the all-zero vector, i.e., S is 0-valid. Hence every instance of $\mathrm{CSP}(S)$ is satisfiable via the all-0 vector.

2. $S \subseteq \mathrm{II}_1$, i.e., $\mathrm{Pol}(S) \supseteq \mathrm{I}_1$. In this case, S must be 1-valid, and every instance of $\mathrm{CSP}(S)$ is satisfiable via the all-1 vector.

The maximal co-clones not included in II_1 or II_0 are BR, IN_2, IE_2, IL_2, IV_2, and ID_2. We consider these in turn.

3. $S \subseteq \mathrm{IE}_2$, i.e., $\mathrm{Pol}(S) \supseteq \mathrm{E}_2$ and contains the function $x \wedge y$. This means that every relation $R \in S$ is closed under coordinate-wise $\wedge$, hence R is Horn. Satisfiability for Horn formulas is known to be polynomial-time solvable by the so called "Horn algorithm" (HORNSAT $\in$ P, [Pap94, p. 78-79]).

4. $S \subseteq \mathrm{IV}_2$, i.e., $\mathrm{Pol}(S) \supseteq \mathrm{V}_2$ and contains the function $x \vee y$. Analogous to the above, we obtain that R can be represented by an anti-Horn-formula. Again, satisfiability is in P by a variation of the Horn algorithm.

5. $S \subseteq \mathrm{IL}_2$, i.e., $\mathrm{Pol}(S) \supseteq \mathrm{L}_2$ and contains in particular its basis, the function $x \oplus y \oplus z$. This means that every constraint in R must be affine. A CSP with affine constraints can be looked at as an equation system over GF[2], hence satisfiability can be tested in polynomial time with the Gaussian elimination algorithm.

6. $S \subseteq \mathrm{ID}_2$, i.e., $\mathrm{Pol}(S) \supseteq \mathrm{D}_2$ and contains in particular its basis, the function $xz \wedge yz \wedge xy$, i.e., the majority function of arity 3. This means that every constraint in R must be bijunctive. Satisfiability for 2-CNF formulas is polynomial-time solvable (2SAT is even in NL, [Pap94, p. 184-185]).

The remaining cases are now those of the classes $\mathrm{BR} = \mathrm{II}_2$ and IN_2. We will prove that if $\langle S \rangle = \mathrm{IN}_2$, then $\mathrm{SAT}(S)$ is NP-complete. According to Theorem 2.4 this will be sufficient since $\mathrm{II}_2 \supseteq \mathrm{IN}_2$, thus concluding the proof.

7. $\langle S \rangle = \mathrm{IN}_2$, i.e., $\mathrm{Pol}(S) = \mathrm{N}_2$. In this case, $\langle S \rangle$ is the set of all Boolean relations R that are closed under all projection functions and their negations. In particular, $\langle S \rangle$ contains the ternary relation R_{NAE} defined in Example 1.1. (It can be shown that in fact, $\langle S \rangle$ is equal to the set of all complementive relations.) Looking at CNF formulas with clauses that are applications of the constraint R_{NAE}, we see that we deal with a particular satisfiability problem for 3-CNF formulas where we ask if there is an assignment that in every clause makes one variable false and one variable true (stated otherwise: not all three variables in a clause receive the same truth assignment). This is the so called NOT-ALL-EQUAL-3SAT problem which is known to be NP-complete, see, e.g., [Pap94]. Hence, also in this case, $\mathrm{SAT}(S)$ is NP-complete, and the proof of Schaefer's dichotomy theorem is finished.

The only cases for $\mathrm{Pol}(S)$ that lead to an NP-complete satisfiability problem are those of I_2 and N_2. A common property of these classes is the following: Say that an n-ary function f is *essentially unary* if it is not constant and depends on only one of its arguments, i.e., there is some i, $1 \leq i \leq n$ and some unary non-constant Boolean function f' such that $f(x_1, \ldots, x_i, \ldots, x_n) = f'(x_i)$. Of course, the only possibilities for f' are the unary identity or the unary negation. Thus we obtain the following corollary:

Corollary 2.6 [JC95, JCG99]. *Let S be a set of Boolean constraints. If* $\mathrm{Pol}(S)$ *consists of only essentially unary functions, then* $\mathrm{SAT}(S)$ *is* NP*-complete, otherwise* $\mathrm{SAT}(S)$ *is polynomial-time solvable.*

Further Complexity Results

In the context of Boolean constraint satisfaction, not only the satisfiability problem was looked at, but many more problems were classified w.r.t. their computational complexity. Considering different versions of satisfiability, equivalence, optimization and counting problems, dichotomy theorems for classes as NP, US, MaxSNP, OptP, and #P were obtained [Cre95, CH96, CH97, KS98, Jub99, BHRV02, RV03, KK03, BHRV04], see also the monograph [CKS01]. Satisfiability and learnability of generalized quantified Boolean formulas were studied by Dalmau [Dal00]. Also, the study of Schaefer's formulas lead to remarkable results about approximability of optimization problems in the constraint satisfaction context [KST97, KSW97, Zwi98].

Connections between algebraic theory and the computational complexity of constraint satisfaction problems were originally developed for studying decision problems where the question is to decide the existence of a solution. Until now, only Schaefer's original dichotomy theorem for satisfiability and the metaproblem in the upcoming section have been re-proven making use of Post's lattice. Only in a very recent paper, Krokhin and Jonsson [KJ03] have shown that this approach can lead to general results for a wider range of problems with different computational properties. They have studied the complexity of recognizing frozen variables (i.e., variables that take the same values in all possible solutions) in an S-formula and completely classified the complexity of this problem. We consider it interesting to give an alternative proof along the same lines of, e.g., the dichotomy theorem for counting the number of satisfying assignments from [CH96].

3 The Metaproblem: The Complexity of the Co-clones

We have given a definition of co-clones as objects that are useful for the complexity study of problems for Boolean constraints. But how complicated are the co-clones themselves, i.e., how difficult is it to decide whether a relation given in a reasonably compact form, e.g. a propositional formula over $\{and, or, not\}$, is element of a specified co-clone?

Definition 3.1. For a co-clone S, we define

$$\mathrm{coMEM}(S) = \{\, F \mid F \text{ is propositional formula and the relation represented by } F \text{ is in } S \,\}.$$

Analogously to duality for Boolean functions, we define the notion of *duality* for Boolean relations.

Definition 3.2. Let R be an n-ary Boolean relation. The *dual relation* of R is defined by $\mathrm{dual}(R) = \{\, (\overline{a_1}, \ldots, \overline{a_n}) \mid (a_1, \ldots, a_n) \in R \,\}$. For a set of Boolean relations S let $\mathrm{dual}(S) = \{\, \mathrm{dual}(R) \mid R \in S \,\}$. Note that the dual-operator establishes a bijection between the elements of S and the elements of $\mathrm{dual}(S)$.

We get the following easy connection between duality of Boolean functions and Boolean relations:

Proposition 3.3. *Let f be a Boolean function and R be a Boolean relation. Then f preserves R if and only if* $\mathrm{dual}(f)$ *preserves* $\mathrm{dual}(R)$.

The following result about the complexity of the co-clones appeared in [Cre98]. We now present a proof relying on the Galois correspondence between Boolean constraints and function.

Theorem 3.4. *If S is the co-clone of all Boolean relations, the co-clone of all 0-valid relations, the co-clone of all 1-valid relations, or the co-clone of all relations that are both 0- and 1-valid (i.e., $S \in \{\mathrm{II}_2, \mathrm{II}_0, \mathrm{II}_1, \mathrm{II}\}$), then the problem* $\mathrm{coMEM}(S)$ *is in* P. *For every other co-clone S, the problem* $\mathrm{coMEM}(S)$ *is* coNP*-complete.*

Proof. As before, we will not distinguish between formulas and the relations they represent and we will speak, e.g., of tuples in a relation R as well as tuples in a formula F.

The claim is obvious for the easy cases, since to check if a relation R is 0-valid or 1-valid one just has to check if one tuple is in R.

For a Boolean function $f\colon \{0,1\}^m \to \{0,1\}$ and tuples $\alpha_1, \ldots, \alpha_m \in \{0,1\}^n$, let $f(\alpha_1, \ldots, \alpha_m)$ be the n-tuple obtained by coordinate-wise applying f to the α_i's.

First of all, observe, that coMEM(S) is in coNP for all co-clones S: For a given n-ary propositional formula F, guess any function f from a fixed basis of Pol(S). Suppose, that f is m-ary. Guess m tuples $\alpha_1, \ldots, \alpha_m \in \{0,1\}^n$. Test, whether these tuples are in F. If this is not the case, accept. If they are all in F, then accept if and only if $f(\alpha_1, \ldots, \alpha_m) \in F$. Then the relation represented by F is in S if and only if all guesses are successful.

Let TAUT be the set of tautological propositional formulas, which is known to be coNP-complete. We will reduce TAUT to coMEM(S) for all co-clones S that do not belong to the easy cases. Take a n-ary propositional formula H, let $g(H)(x_1, \ldots, x_n) =_{\text{def}} H(x_1, \ldots, x_n) \wedge H(\overline{x_1}, \ldots, \overline{x_n})$ and let

$$h(H) = (x_1 = x_2) \wedge (g(H)(y_1, \ldots, y_n) \vee \overline{z_1}) \wedge (H(u_1, \ldots, u_n) \vee z_2 \vee z_3 \vee z_4).$$

If $H \in$ TAUT, then $h(H) \equiv x_1 = x_2$ and therefore $h(H) \in S$. On the other hand, if $H \notin$ TAUT, there is an $\alpha \in \{0,1\}^n$ such that $H(\alpha) = 0$.

- Observe that $h(H)(0,0,\alpha,0,\alpha,1,0,0) = 1 \neq h(H)(1,1,\overline{\alpha},1,\overline{\alpha},0,1,1)$. Thus $h(H)$ is not complementive.
- Observe that the tuples $(0,0,\alpha,0,\alpha,1,0,0)$ and $(0,0,\alpha,0,\alpha,0,1,0)$ are in $h(H)$, but on the other hand, $\mathit{and}((0,0,\alpha,0,\alpha,1,0,0),(0,0,\alpha,0,\alpha,0,1,0)) = (0,0,\alpha,0,\alpha,0,0,0)$ is not in $h(H)$. Thus $h(H)$ is not closed under *and* and therefore it is not equivalent to a Horn-formula.
- Let $f(x,y,z) = xy \vee xz \vee yz$, i.e., f is a basis for clone D_2. Observe that the tuples $(0,0,\alpha,0,\alpha,1,0,0)$, $(0,0,\alpha,0,\alpha,0,1,0)$, $(0,0,\alpha,0,\alpha,0,0,1)$ are in $h(H)$, but on the other hand, $f((0,0,\alpha,0,\alpha,1,0,0),(0,0,\alpha,0,\alpha,0,1,0),(0,0,\alpha,0,\alpha,0,0,1))) = (0,0,\alpha,0,\alpha,0,0,0)$ is not in $h(H)$. Thus $h(H)$ is not bijunctive.
- Let $f(x,y,z) = x \oplus y \oplus z$, i.e., f is a basis for clone L_2. Observe that the tuples $(0,0,\alpha,0,\alpha,1,1,0)$, $(0,0,\alpha,0,\alpha,0,1,0)$, $(0,0,\alpha,0,\alpha,1,0,0)$ are in $h(H)$, but on the other hand, $f((0,0,\alpha,0,\alpha,1,1,0),(0,0,\alpha,0,\alpha,0,1,0),(0,0,\alpha,0,\alpha,1,0,0)) = (0,0,\alpha,0,\alpha,0,0,0)$ is not in $h(H)$. Thus $h(H)$ is not affine.

Thus, membership for co-clones that are complementive, Horn, bijunctive, or affine is coNP-complete. Finally, Proposition 3.3 yields coNP-completeness for co-clones that are anti-Horn. □

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