Boolean Hierarchy of Partitions over Reducible Bases

V. L. Selivanov*
Theoretische Informatik, Universität Würzburg†

Abstract

The boolean hierarchy of partitions was introduced and studied by K. Wagner and S. Kosub, mostly over the lattice of NP-sets. We consider this hierarchy for the case of lattices having the reduction property and show that in this case the hierarchy behaves much easier and admits a deeper investigation. We completely characterize the hierarchy over some concrete important lattices, e.g. over the lattice of recursively enumerable sets.

1 Introduction

Let M be a set, $P(M)$ be the class of subsets of M, and $\mathcal{L} \subseteq P(M)$ be a class of subsets closed under $\cup, \cap$ and containing $\emptyset, M$; for the sake of brevity, we call such a class $\mathcal{L}$ *a base*. As is well-known, there is a natural classification (called the boolean hierarchy over $\mathcal{L}$) of the elements of the boolean algebra generated by $\mathcal{L}$ inside $P(M)$. The class $P(M)$ is in a natural one-one correspondence with the set 2^M of all functions $\nu : M \to 2 = \{0, 1\}$ (we identify in this paper a natural number $k \in \omega$ with the set $\{0, \ldots, k-1\}$). Is there a natural extension of the notion of the boolean hierarchy on any k in place of 2?

An answer to this natural question was suggested by K. W. Wagner some time ago (I learned his idea while visiting Würzburg in 1996). The resulting hierarchy, denoted by $BH_k(\mathcal{L})$, was called *the boolean hierarchy of k-partitions over $\mathcal{L}$*, because elements of k^M are in a natural one-one correspondence with k-partitions of M (i.e., pairwise disjoint sets $A_0, \ldots, A_{k-1}$ with $A_0 \cup \cdots \cup A_{k-1} = M$). This hierarchy was studied in [KW99, KW00], most intensively for the base $\mathcal{L} = NP$, and was extended by S. Kosub in [K00] and, in a systematized form, in [K00a] to a so called *refined boolean hierarchy of k-partitions over $\mathcal{L}$*, denoted by $RBH_k(\mathcal{L})$. It turns out that for the general case, and also for the case of NP, the structure of the hierarchies of partitions is rather complicated.

In this paper we consider the boolean hierarchies of partitions for the bases $\mathcal{L}$ having the so called reduction property (for short, we call such bases *reducible*); this means that for all $A, B \in \mathcal{L}$ there exist disjoint $A', B' \in \mathcal{L}$ such that $A' \subseteq A, B' \subseteq B$ and $A' \cup B' = A \cup B$.

*Supported by the German Research Foundation (DFG), by the Russian Foundation for Basic Research Grant 00-01-00810 and by RFBR-INTAS Grant IR-97-139.

†On a short leave from Novosibirsk Pedagogical University. Email: vseliv@nspu.ru.

Our main results show that for reducible bases the structure $(RBH_k(\mathcal{L});\subseteq)$ is much easier than for the general case (e.g., it is always a well partial ordering). This enables us to get rather complete understanding of this structure for some particular important bases, e.g. for the lattice of recursively enumerable (r.e.) sets (the reader following a new fashion in computability theory may read c.e. in place of r.e.), and for the the class of open sets of the Baire space. For the r.e. case we establish a close relationship of the boolean hierarchy of partitions with our previous work [Se82, Se89,Se89a] on complete numberings. In our opinion, these results confirm that the definitions proposed by K. Wagner and S. Kosub are the "right" ones.

In Section 2 we establish some relevant results on finite partially ordered sets (posets), in Section 3 we give the exact definition of the boolean hierarchies of k-partitions and formulate some general facts useful for the sequel. In Section 4 we prove our main result for any reducible base. In Section 5 we consider the important particular case of r.e. sets and further we give some more examples and a short discussion of a possible future work.

2 Posets and Forests

We use some standard notation and terminology on posets which may be found in any book on the subject, see e.g. [DP94]. A poset $(P;\leq)$ will be often shorter denoted just by P. Any subset of P may be considered as a poset with the induced partial ordering. In particular, this applies to the "cones" $\check{x}=\{y\in P|x\leq y\}$ and $\hat{x}=\{y\in P|y\leq x\}$ defined by any $x\in P$.

For a finite poset P, $h(P)$ will denote *the height of* P, i.e. the number of elements of the longest chain in P. For any $i, 1\leq i\leq h(P)$, let $P(i)=\{x\in P|h(\check{x})=i\}$. Then $P(1),\ldots,P(h(P))$ is a partition of P on "levels"; note that $P(1)$ is the set of all maximal elements of P. For any $x\in P$, let $x\downarrow$ denote the set of all immediate predecessors (or sons) of x in P, i.e. $x\downarrow=\{y<x|\neg\exists z(y<z<x)\}$. Note that $x\downarrow=\emptyset$ iff x is minimal in P.

By *a forest* we mean a poset in which any cone $\check{x}$ is a chain. *A tree* is a forest having the greatest element (called *the root* of the tree). Note that any finite forest is uniquely representable as a disjoint union of finite trees, the roots of the trees being the maximal elements of the forest. *A proper forest* is a forest which is not a tree.

Let us indicate a way of constructing a forest from any given finite poset which will be very useful below.

Lemma 2.1 *For any finite poset P there exist a finite forest F and a monotone function f from F onto P so that $h(F)=h(P)$, f establishes a bijection between $F(1)$ and $P(1)$, and for any $x\in F$ f establishes a bijection between $x\downarrow$ and $f(x)\downarrow$.*

Proof. We construct F level by level $F(1),F(2),\ldots$ as follows. Let $F(1)$ be any set in one-one correspondence with $P(1)$, and let $f:F(1)\to P(1)$ be any such a correspondence; all elements of $F(1)$ will be of course pairwise incomparable in F. If $h(P)=1$, the construction is over. Otherwise, for any $x\in F(1)$, for which $f(x)$ is not minimal in P, find a set Q_x which is in one-one correspondence with $f(x)\downarrow$; the sets Q_x are chosen pairwise disjoint from one another and from $F(1)$. The elements of Q_x are claimed the sons of x in F. Set $F(2) = \cup\{Q_x | x \in F(1)\}$ and extend the function f on the set $F(2)$ in such a way that it establishes a bijection between $x \downarrow = Q_x$ and $f(x) \downarrow$, for any $x \in F(1)$. Note that we have $f(F(1)) = P(1)$ and $P(2) \subseteq f(F(2)) \subseteq P(2) \cup \cdots P(h(P))$. Continuing in the same manner, we get after $h(P)$ steps the desired forest $F = F(1) \cup \cdots \cup F(h(P))$ and the monotone function f from F onto P. $\Box$

A k-poset is an object $(P; , \leq, c)$ consisting of a poset $(P; \leq)$ and a function $c : P \to k$. Often we simplify notation to (P, c). Note that usually we consider only the case $k \in \omega, k = \{0, \ldots, k-1\}$, though one could consider an arbitrary set of "labels" in place of k. E.g., any finite poset P may be considered as a P-poset (P, id), with the identity labeling $id : P \to P$. *A morphism* $f : (P; \leq, c) \to (P'; \leq', c')$ between k-posets is a monotone function $f : (P; \leq) \to (P'; \leq')$ respecting the labelings, i.e. satisfying $c = c' \circ f$. Any subset P' of a k-poset (P, c) induces in the obvious way a k-poset (P', c) which is a k-subposet of (P, c) (here and below we try to avoid cumbersome notation for restrictions of functions like $c\lceil_{P'}$). Note that if (P, c) is a k-forest then so is (P', c) for any $P' \subseteq P$.

From Lemma 2.1 we easily get

Lemma 2.2 *For any finite k-poset (P, c) there exist a finite k-forest (F, d) and a morphism f from (F, d) onto (P, c) with the properties specified in Lemma 2.1.*

Proof. Let F and f be defined as in the proof of Lemma 2.1. Defining d as $c \circ f$, we immediately get the desired objects. $\Box$

Define $\mathcal{P}_k, \mathcal{L}_k$ and $\mathcal{F}_k$ to be the classes of all finite k-posets, finite k-lattices and finite k-forests, respectively. Define [KW99,K00] a quasiorder $\leq$ on $\mathcal{P}_k$ as follows: $(P, c) \leq (P', c')$, if there is a morphism from (P, c) to (P', c'). By $\equiv$ we denote the equivalence relation on $\mathcal{P}_k$ induced by $\leq$.

In [KW99,K00] one can find some information on the structure of quasiorderings $(\mathcal{P}_k; \leq)$ and $(\mathcal{L}_k; \leq)$. We will establish in this section some facts on the structure $(\mathcal{F}_k; \leq)$. First we get some information about minimal k-forests, i.e. k-forests not equivalent (under $\equiv$) to a k-forest of lesser cardinality. The cardinality of a set F is denoted by $|F|$. Note that we will consider below only finite k-forests, hence from now on we usually omit the word "finite".

Lemma 2.3 *For any k-forest (F, c) the following conditions are equivalent:*
(i) (F, c) is minimal;
(ii) any endomorphism of (F, c) is injective;
(iii) $(F, c) \not\leq (F', c)$ for any proper subset $F' \subset F$.

Proof. $(i) \to (iii)$. Suppose the contrary: $(F, c) \leq (F', c)$. We trivially have $(F', c) \leq (F, c)$ (take the identity embedding). Hence, $(F, c) \equiv (F', c)$ and $|F'| < |F|$; a contradiction.

$(iii) \to (ii)$. Suppose the contrary: there is a noninjective endomorphism $f : (F, c) \to (F, c)$. Then for $F' = f(F)$ we have a contradiction.

$(ii) \to (i)$. Suppose the contrary, so $(F, c) \equiv (G, d)$ for a k-forest (G, d) with $|G| < |F|$. Choose witnessing morphisms $f : (F, c) \to (G, d)$ and $g : (G, d) \to (F, c)$. Then $g \circ f$ is a noninjective endomorphism of (F, c). A contradiction. $\Box$

The next description of the minimal k-forests is much more informative and useful than the previous one.

Theorem 2.4 *(i) Any trivial (i.e., singleton) k-forest is minimal.*

(ii) A nontrivial k-tree (T, c) is minimal iff $\forall x \in T(1) \forall y \in T(2)(c(x) \neq c(y))$ and the k-forest $(T \setminus T(1), c)$ is minimal.

(iii) A proper k-forest is minimal iff all its k-trees are minimal and pairwise incomparable under $\leq$.

Proof. Note that the assertion of the theorem is a kind of inductive description (by induction on the cardinality) of the minimal k-forests. The assertion (i), being the induction basis, is evident. Now take any k-forest with at least 2 elements and consider the alternatives when it is a k-tree (T, c) (the assertion (ii)) and when it is a proper k-forest (F, c) (the assertion (iii)).

Let us consider the first alternative. Assume that (T, c) is minimal, choose any $x \in T(1), y \in T(2)$ and check that $c(x) \neq c(y)$. Suppose the contrary, $c(x) = c(y)$, and consider the endomorphism of (T, c) sending y to x and being the identity on $T \setminus \{y\}$. This endomorphism is not injective, in a contradiction to Lemma 2.3. Next we shall check that if (T, c) is minimal then so is also $(T \setminus T(1), c)$. If it were not, then, by Lemma 2.3, there would exist a noninjective endomorphism g of $(T \setminus T(1), c)$. But then $g \cup \{(x, x)\}$, where x is the root of T, would be a noninjective endomorphism of (T, c) contradicting to Lemma 2.3.

Now assume the righthandside of (ii) and prove that (T, c) is minimal. By Lemma 2.3, it suffices to check that any endomorphism f of (T, c) is the identity, i.e. $f(x) = x$ for any $x \in T$. Let first $x \in T(1)$ (i.e., x is the root). Suppose that $f(x) \neq x$, hence $f(x) < x$ and a fortiori $f(x) \leq y \leq x$, for a unique $y \in T(2)$. We have $c(x) = c(f(x)) \neq c(y)$, hence in fact $f(x) < y < x$. By monotonicity of f, $f^2(x) \leq f(y) \leq f(x)$ and, by the condition on labels, $c(f^2(x)) = c(f(x)) \neq c(y) = c(f(y))$. Hence, $f^2(x) < f(y) < f(x)$. Continuing in this manner, we get a contradiction with the finiteness of T.

We have proved that f is identical on $T(1)$. Now let $x \in T(2)$. The case $f(x) > x$ is impossible by the condition on colors, hence $f(x) \leq y$ for a unique $y \in T(2)$. The case $y \neq x$ is impossible because otherwise the restriction $f\lceil_{\hat{x}}$ would witness that $(\hat{x}, c) \leq (\hat{y}, c)$, contradicting to (iii) and the induction hypothesis. Hence, $f(x) \leq x$ and a fortiori $f\lceil_{\hat{x}}$ is an endomorphism of $(\hat{x}, c)$. By induction, $f\lceil_{\hat{x}}$ is the identity, completing the proof of (ii).

It remains to consider the case of a proper k-forest (F, c). First assume that (F, c) is minimal; we have to check that all the k-trees $(\hat{x}, c)$, $x \in F(1)$, are minimal and pairwise incomparable. Suppose that $(\hat{x}, c)$ is not minimal. By Lemma 2.3, there is a noninjective endomorphism g of $(\hat{x}, c)$. Set $f(a) = g(a)$ for $a \leq x$ and $f(a) = a$ for $a \in F \setminus \hat{x}$. Then f is a noninjective endomorphism of (F, c); a contradiction.

Now suppose that $(\hat{x}, c) \leq (\hat{y}, c)$ for some $x, y \in F(1), x \neq y$, and let let g be a witnessing morphism. Defining f exactly as above, we again get a noninjective endomorphism of (F, c). A contradiction.

Finally, we have to check that the righthandside of (iii) implies the minimality of (F, c). As above, we easily deduce that any endomorphism of (F, c) is the identity. By Lemma 2.3, (F, c) is minimal. $\square$

By inspecting the proof of the theorem we can get some additional information on the minimal k-forests.

Corollary 2.5 *(i) A k-forest is minimal iff it has only trivial (i.e., identical) endomorphism.*

(ii) Any two minimal equivalent k-forests are isomorphic.

Proof. (i) Form right to left, the assertion follows from Lemma 2.3. In the opposite direction, it follows from the proof of Theorem 2.4.

(ii) Let (F,c) and (G,d) be minimal and equivalent k-forests, so there are mophisms $f:(F,c)\to(G,d)$ and $g:(G,d)\to(F,c)$. Then $g\circ f$ and $f\circ g$ are endomorphisms of (F,c) and (G,d), respectively. By (i), the both endomorphisms are identities. Hence, f is an isomorphism. $\Box$

Theorem 2.4 implies the following very special and useful property of k-trees.

Lemma 2.6 *For any k-tree (T,c) there are only finitely many (up to $\equiv$) k-trees (S,d) with $(S,d)\leq(T,c)$.*

Proof is by induction on $|T|$, the basis of induction being trivial (the singleton k-trees are exactly the minimal elements of $(\mathcal{F}_k,\leq)$). So assume $|T|>1$. By induction, the set of k-trees $\mathcal{U}=\{(U,e)|\exists x\in T(2)((U,e)\leq(\hat{x},c))\}$ is, up to $\equiv$, finite. Let $\mathcal{V}$ be a finite set of minimal k-trees such that any element of $\mathcal{V}$ is equivalent to an element of $\mathcal{U}$ and vice versa.

It clearly suffices to check that the following two sets of k-trees are (up to $\equiv$) finite:

$\mathcal{S}_0=\{(S,d)|(S,d)\leq(T,c)\wedge d(s)\neq c(t)\}$ and

$\mathcal{S}_1=\{(S,d)|(S,d)\leq(T,c)\wedge d(s)=c(t)\wedge(S,d)$ is minimal$\}$,

where s and t are roots of S and T, respectively. We have $\mathcal{S}_0\subseteq\mathcal{U}$ because, by $d(s)\neq c(t)$, any morphism $f:(S,d)\to(T,c)$ is in fact a morphism $f:(S,d)\to(\hat{x},c)$, for some $x\in T(2)$. Hence, $\mathcal{S}_0$ is (up to $\equiv$) finite.

Now let $(S,d)\in\mathcal{S}_1$ and $f:(S,d)\to(T,c)$. By Theorem 2.4, for any $y\in S(2)$ we have $f(y)<t$ and a fortiori $(\hat{y},d)\in\mathcal{U}$. Hence, it suffices to show that there exist (up to $\equiv$) only finitely many minimal k-trees (S,d) such that $(\hat{y},d)\in\mathcal{V}$ for all $y\in S(2)$. By Theorem 2.4.(ii), all the k-trees $(\hat{y},d)$ have to be pairwise incomparable under $\leq$. This implies the finiteness of $\mathcal{S}_1$. $\Box$

Now let us turn to the main result of this section. Recall (see e.g. [Kr72]) that a quasiorder is called *a well quasiorder (wqo)* if it has neither infinite ascending chains nor infinite antichains. Any wqo P has *a height* $h(P)$ which is by definition the greatest ordinal isomorphically embeddable into P (note that the definition agrees with the definition of height of a finite poset). With any quasiorder we associate also its *width* $w(P)$ defined as follows: if P has antichains with any finite number of elements, then $w(P)=\omega$, otherwise $w(P)$ is the greatest natural number n for which P has an antichain with n elements.

Posets considered so far were all nonempty (this is the usual assumption). Sometimes (as e.g. in the theorem below) it is technically convenient to consider also the empty k-poset assuming that $\emptyset\leq P$ for any $P\in\mathcal{P}_k$.

Theorem 2.7 *(i) For any $k\geq 2$, $(\mathcal{F}_k;\leq)$ is a wqo with $h(\mathcal{F}_k)=\omega$.*

(ii) $w(\mathcal{F}_2)=2$ and $w(\mathcal{F}_k)=\omega$ for $k>2$.

(iii) For any $k\geq 2$, (the quotient poset of) the structure $(\mathcal{F}_k\cup\{\emptyset\};\leq)$ is a distributive lattice in which any nonempty subset has an infimum.

Proof. (i) The wqo-property follows immediately from a well-known theorem of Kruskal [Kr60]. This property together with the König's lemma easily imply that $h(\mathcal{F}_k)=\omega$ iff the quotient poset $\widetilde{\mathcal{F}}_k$ of $(\mathcal{F}_k,\leq)$ is infinite and $\hat{x}$ is finite for any $x\in\widetilde{\mathcal{F}}_k$. Note that any alternating k-chain (that is a k-chain any two neighbors of which carry different labels) is a (minimal) k-tree. It is easy to see [K00a] that they already define infinitely many elements of $\widetilde{\mathcal{F}}_k$.

It remains to show that for any k-forest there are only finitely many (up to $\equiv$) k-forests below it. For this we first need to make several observations.

For any k-trees $T_0, \ldots, T_n$, let $F = T_0 \sqcup \cdots \sqcup T_n$ be their join, i.e. the disjoint union. Then F is a k-forest whose trees are exactly $T_0, \ldots, T_n$. Of course, any k-forest is (equivalent to) the join of its trees. Note that the join operation applies also to k-forests, and the join of any two k-forests is clearly their supremum under $\leq$. Hence, $\widetilde{\mathcal{F}}_k$ is an upper semilattice. Note also that any k-tree T defines a join-irreducible element of $\widetilde{\mathcal{F}}_k$ (i.e. if $T \leq G \sqcup H$ for k-forests G and H then $T \leq G$ or $T \leq H$), and, conversely, any join-irreducible element of $\widetilde{\mathcal{F}}_k$ is defined by a k-tree.

Now, let G and H be k-forests represented as joins of their k-trees, i.e. $G = G_0 \sqcup \cdots \sqcup G_m$ and $H = H_0 \sqcup \cdots \sqcup H_n$. From the remarks above follows that $G \leq H$ iff $\forall i \leq m \exists j \leq n (G_i \leq H_j)$. In other words, the lower cone $\hat{H}$ is (up to $\equiv$) the upper semilattice generated by the set of k-trees (T, c) such that $(T, c) \leq H_j$ for some $j \leq n$. From Lemma 2.6 it follows that $\hat{H}$ is (up to $\equiv$) finite.

(ii) By Theorem 2.4, minimal 2-trees are exactly the alternating 2-chains. It is clear (and known from [K00a]) that there are no 3 pairwise incomparable alternating 2-chains. Again by [K00a], for any $k > 2$ and any $n \in \omega$ there exist n pairwise incomparable alternating k-chains.

(iii) By remarks in the proof of (i), $\mathcal{F}_k \cup \{\emptyset\}$ is an upper semilattice. We have to check that any nonempty subset $\mathcal{S} \subseteq \mathcal{F}_k \cup \{\emptyset\}$ has an infimum. By the wqo-property, $\mathcal{S}$ is representable in the form $\mathcal{S} = \check{F}_0 \cup \cdots \cup \check{F}_n$ for finitely many F_i. It suffices to show that the set $\{F_0, \ldots, F_n\}$ has an infimum, because this infimum coincides with the infimum of $\mathcal{S}$. But it is clear that $inf(F_0, \ldots, F_n) = sup(\hat{F}_0 \cap \cdots \cap \hat{F}_n)$; the supremum exists because, by (i) and the presence of $\emptyset$, the set $\hat{F}_0 \cap \cdots \cap \hat{F}_n$ is nonempty and (up to $\equiv$) finite.

It remains to check the distributivity. As is well-known (see e.g. [DP94]), it suffices to show that for all $F, G, H \in \mathcal{F}_k \cup \{\emptyset\}$ with $F \leq G \sqcup H$ there exist $G' \leq G$ and $H' \leq H$ satisfying $F \equiv G' \sqcup H'$. Assume that $F \neq \emptyset$ (otherwise there is nothing to prove). Then $F \equiv F_0 \sqcup \cdots \sqcup F_l$ for some k-trees $F_0, \ldots, F_l$. By remarks in the proof of (i), $\forall i \leq l (F_i \leq G \vee F_i \leq H)$. Let

$I = \{i \leq l | F_i \leq G\}$, $J = \{i \leq l | F_i \leq H\}$, $G' = \sqcup\{F_i | i \in I\}$ and

$H' = \sqcup\{F_i | i \in J\}$ (of course, in case $I = \emptyset$ we set $G' = \emptyset$, and similarly for J). Then G' and H' have the desired properties. $\square$

Remark. The structures $(\mathcal{P}_k; \leq)$ and $(\mathcal{L}_k; \leq)$ are much more complicated than $(\mathcal{F}_k; \leq)$. E.g., in [K00a] an infinite ascending chain in $\mathcal{P}_2$ was constructed. This example is easily modified to get an infinite antichain in $\mathcal{P}_2$, as well as an infinite ascending chain and an infinite antichain in $\mathcal{L}_3$. Hence, $\mathcal{P}_k (k \geq 2)$ and $\mathcal{L}_k (k \geq 3)$ are far from being wqo's.

The next assertion gives answer to a natural question on the relationship between $\mathcal{F}_k$ and $\mathcal{P}_k$.

Proposition 2.8 *For any $k \geq 2$, $\mathcal{F}_k$ is not an initial segment of $(\mathcal{P}_k; \leq)$.*

Proof. Let $T = (\{0, 1, 2\}; \leq, c)$ be the 2-chain where $\leq$ has its natural meaning and c is defined by $c(0) = c(2) = 0, c(1) = 1$. Let $P = (\{0, 1, 2, 3\}; \leq, d)$ be a 2-poset where $\leq$ is defined by $0 < 1, 2 < 3, 2 < 1$, and the labeling d is defined by $d(0) = d(3) = 0$, $d(1) = d(2) = 1$. One easily checks that $P < F$ and P is not equivalent to a 2-forest. $\Box$

Remark. The above-mentioned example from [K00a] of the infinite ascending chain in $\mathcal{P}_2$ is such that all members of the chain are $\leq T$, where T is from the preceding proof. Hence, $\{P \in \mathcal{P}_2 | P \leq T\}$ is not a wqo.

3 Boolean Hierarchy of Partitions

In this section we present exact definitions of hierarchies of partitions mentioned in the Introduction, and state some results which are used further.

Let $P = (P; \leq)$ be a finite poset and $\mathcal{L} \subseteq P(M)$ be a base. Let $H(P, \mathcal{L})$ denote the set of all functions $S : P \to \mathcal{L}$ such that $\cup_x S_x = M$ and $S_x \cap S_y = \cup\{S_z | z \leq x, y\}$. Note that the last condition implies that S is monotone, i.e. $S_x \subseteq S_y$ for $x \leq y$. Let $\mathcal{L}(P)$ denote the set of all P-partitions $\{T_x\}_{x \in P}$ of M such that $\cup\{T_y | y \leq x\} \in \mathcal{L}$ for any $x \in P$.

For any $S : P \to P(M)$, define $\tilde{S} : P \to P(M)$ by $\tilde{S}_x = S_x \setminus \cup\{S_y | y < x\}$. Note that our notation $\tilde{S}$ slightly differs from notation of this object from [KW99, K00] in order to stress its relevance to a similar notion from [Se95] used in the context of so called fine hierarchy.

We omit the straightforward proof of the next lemma which is implicit in [KW99, K00].

Lemma 3.1 *The map $S \mapsto \tilde{S}$ establishes a bijection between $H(P, \mathcal{L})$ and $\mathcal{L}(P)$.* $\Box$

In the next lemma, which is also known from [KW99, K00], we identify the partition $\{\tilde{S}_x\}$ with the function from M into P sending any $m \in M$ to the unique $x \in P$ satisfying $m \in \tilde{S}_x$.

Lemma 3.2 *If P, Q are finite posets and $f : P \to Q$ is a monotone function then $f \circ \tilde{S} \in \mathcal{L}(Q)$, for any $S \in H(P, \mathcal{L})$.* $\Box$

For a finite k-poset (P, c), let $\mathcal{L}(P, c) = \{c \circ \tilde{S} | S \in H(P, \mathcal{L})\}$. Note that $\mathcal{L}(P, c) \subseteq k^M$, i.e. $\mathcal{L}(P, c)$ is a class of k-partitions of M.

Lemma 3.3 *For all finite k-posets (P, c) and (Q, d), if $(P, c) \leq (Q, d)$ then $\mathcal{L}(P, c) \subseteq \mathcal{L}(Q, d)$.*

Sketch of proof. This lemma is also from [KW99, K00], we only point some details relevant to the sequel. So, let $f : (P, c) \to (Q, d)$ be a morphism, hence in particular $c = d \circ f$. Choose any element from $\mathcal{L}(P, c)$, it is of the form $c \circ \tilde{S}$ for some $S \in H(P, \mathcal{L})$. By Lemma 3.2, $f \circ \tilde{S} \in \mathcal{L}(Q)$. Hence, $c \circ \tilde{S} = d \circ (f \circ \tilde{S}) \in \mathcal{L}(Q, d)$. $\Box$

Now we are ready to present the definition of hierarchies of partitions from [KW99, K00].

Definition 3.4 *(i)* The boolean hierarchy of k-partitions over $\mathcal{L}$ *is the collection of classes $BH_k(\mathcal{L}) = \{\mathcal{L}(P, c) | (P, c) \in \mathcal{L}_k\}$.*

(ii) The refined boolean hierarchy of k-partitions over $\mathcal{L}$ *is the collection of classes $RBH_k(\mathcal{L}) = \{\mathcal{L}(P, c) | (P, c) \in \mathcal{P}_k\}$.*

Note that $BH_k(\mathcal{L})$ in this definition looks at first glance as a rather arbitrary and non-motivated restriction of $RBH_k(\mathcal{L})$. In fact, the original equivalent definition of this class in [KW99] is well motivated and is given in terms of boolean functions. We do not use the original definition here. For more discussion and motivation see [K00a].

Using the k-forests from the preceding section, we add to the defined collections the collection $FBH_k(\mathcal{L}) = \{\mathcal{L}(P,c)|(P,c) \in \mathcal{F}_k\}$, where F reminds on forests. Note that if F is a forest then the properties of functions $S \in H(F,\mathcal{L})$ may be equivalently described as follows: $\cup_x S_x = M$, $S_x \subseteq S_y$ for $x \leq y$ and $S_x \cap S_y = \emptyset$ if x, y are incomparable in F.

From Lemma 3.3 follows that the partial ordering $(RBH_k(\mathcal{L}); \subseteq)$ is a homomorphic image of $(\mathcal{P}_k; \leq)$, and similarly for the other two collections. This observation does not give two much information for BH and RBH but for the case of FBH we immediately get the important

Corollary 3.5 *For any $k \in \omega$ and any base $\mathcal{L}$, $(FBH_k(\mathcal{L}); \subseteq)$ is a well poset.*

Proof. By Lemma 3.3, the structure $(FBH_k(\mathcal{L}); \subseteq)$ is a homomorphic image of a wqo $(\mathcal{F}_k; \leq)$. As is well known [Kr72], any homomorphic image of a wqo is a wqo. $\Box$

The corollary shows that the "forest boolean hierarchy" $FBH_k(\mathcal{L})$ has more resemblance (than the hierarchies from Definition 3.4) to hierarchies considered in descriptive set theory, computability theory and some other areas of mathematics: the levels of those hierarchies form (at least) a well poset under inclusion, as does our "hierarchy" $FBH_k(\mathcal{L})$. For this reason we believe that $FBH_k(\mathcal{L})$ is a natural and useful substructure of $RBH_k(\mathcal{L})$.

Note that, according to remarks in the preceding section, there is no reason to hope that the structures $(BH_k(\mathcal{L}); \subseteq)$ and $(RBH_k(\mathcal{L}); \subseteq)$ are well posets over any base $\mathcal{L}$ (though, to the best of my knowledge, in the literature there is yet no explicit example of a base over which the structures are really not well posets). In the next section we show that for reducible bases the situation is much better.

4 RBH over Reducible Bases

The next theorem is one of the main results of this paper.

Theorem 4.1 *Over any reducible base $\mathcal{L}$, $RBH_k(\mathcal{L}) = FBH_k(\mathcal{L})$.*

First we prove a couple of lemmata. Let $\{A_x\}_{x\in X}$ be a finite sequence of elements of $\mathcal{L}$. By *a reduct for* $\{A_x\}_{x\in X}$ we mean a sequence $\{A'_x\}_{x\in X}$ of pairwise disjoint sets from $\mathcal{L}$ such that $A'_x \subseteq A_x$ for any $x \in X$ and $\cup_x A'_x = \cup_x A_x$. The next assertion is checked by a straightforward induction on $|X|$.

Lemma 4.2 *If $\mathcal{L}$ is reducible then any finite sequence of sets from $\mathcal{L}$ has a reduct.* $\Box$

The next lemma is an important step toward the proof of the theorem.

Lemma 4.3 *Let $\mathcal{L}$ be a reducible base, P be a finite poset, and F, f be as specified in Lemma 2.1. Then $\mathcal{L}(F,f) = \mathcal{L}(P,id)$.*

Proof. First note that (F, f) and (P, id) are considered here as P-posets, see Section 2. By Lemma 3.3, the inclusion from left to right holds true for any base $\mathcal{L}$. It remains to construct from any $S \in H(P, \mathcal{L})$ a function $T \in H(F, \mathcal{L})$ so that $f \circ \tilde{T} = \tilde{S}$. We construct $T : F \to \mathcal{L}$ by induction on the levels $F(1), F(2), \dots$ of F, see the proof of Lemma 2.1. For the level $F(1)$, choose a reduct $\{S'_p\}_{p \in P(1)}$ for the sequence $\{S_p\}_{p \in P(1)}$ and set $T_x = S'_{f(x)}$ for any $x \in F(1)$. Now let $y \in F(2)$, then $y \in x\downarrow$, for a unique $x \in F(1)$. Choose a reduct $\{S'_p\}_{p \in f(x)\downarrow}$ for $\{S_p\}_{p \in f(x)\downarrow}$, and set $T_y = T_x \cap S'_{f(y)}$ for any $y \in x\downarrow$. Continuing in this manner, we get a function $T : F \to \mathcal{L}$.

Note that $T_y \subseteq T_x$ for all $y < x$ from F (it suffices to show this only for the case $y \in x\downarrow$, and this is evident by construction). Furthermore, we have $\cup\{T_x | x \in F(1)\} = \cup\{S'_{f(x)} | x \in F(1)\} = \cup\{S_p | p \in P(1)\} = M$.

Finally, let us check that $T_y \cap T_{y_1} = \emptyset$ whenever y, y_1 are incomparable in F. First consider the case when y, y_1 belong to different trees of the forest F. Then there are unique $x \geq y$ and $x_1 \geq y_1$ with $x \neq x_1$, $x, x_1 \in F(1)$.We have $T_x = S'_{f(x)}$ and $T_{x_1} = S'_{f(x_1)}$, hence T_x and T_{x_1} are disjoint (remember that $\{S'_p\}_{p \in P(1)}$ is a reduct for $\{S_p\}_{p \in P(1)}$!). By monotonicity, $T_y \subseteq T_x$ and $T_{y_1} \subseteq T_{x_1}$, hence T_y and T_{y_1} are disjoint too.

Now consider the case when y, y_1 are in the same tree of F; then they have supremum $x = y \cup y_1$ in F. There are unique $z, z_1 \in x\downarrow$ such that $y \leq z \leq x$ and $y_1 \leq z_1 \leq x$. By construction, $T_z = T_x \cap S'_{f(z)}$ and $T_{z_1} = T_x \cap S'_{f(z_1)}$. But $S'_{f(z)}$ and $S'_{f(z_1)}$ are disjoint as different members of a reduct. Hence, T_z, T_{z_1} are disjoint and (by monotonicity) so are T_y, T_{y_1}.

Thus, we have checked that $T \in H(F, \mathcal{L})$. Now consider the partition $\{\tilde{T}_x\}_{x \in F}$. We claim that $\tilde{T}_x \subseteq \tilde{S}_{f(x)}$ for any $x \in F$.

Indeed, take any $a \in \tilde{T}_x = T_x \setminus \cup\{T_y | y < x\}$. Then $a \in S_{f(x)}$ because $T_x \subseteq S_{f(x)}$ by construction of T. It remains to show that $a \notin \cup\{S_z | z < f(x)\}$. Suppose the contrary. Then, by monotonicity of S, $a \in S_z$ for some $z \in f(x)\downarrow$. Recall that (by Lemma 2.1) f establishes a bijection between $x\downarrow$ and $f(x)\downarrow$, that $T_y = T_x \cap S'_{f(y)}$ for any $y \in x\downarrow$, and that $\cup\{S_z | z \in f(x)\downarrow\} = \cup\{S'_z | z \in f(x)\downarrow\}$. Hence, $a \in T_y$ for some $y \in x\downarrow$. A contradiction.

Now it is clear that for any $y \in P$ we have $\tilde{S}_y = \cup\{\tilde{T}_x | f(x) = y\}$. This is equivalent to $f \circ \tilde{T} = \tilde{S}$. $\square$

Now we can give a proof of the theorem.

Proof of Theorem 4.1. The inclusion $FBH_k(\mathcal{L}) \subseteq RBH_k(\mathcal{L})$ is trivial since any k-forest is a k-poset. Now take any class from $RBH_k(\mathcal{L})$, it is ot the form $\mathcal{L}(P, c)$, for a finite k-poset (P, c). Let F, f be as in Lemma 2.1, then $f : (F, c \circ f) \to (P, c)$. By Lemma 2.3, $\mathcal{L}(F, c \circ f) \subseteq \mathcal{L}(P, c)$. Now take any element of $\mathcal{L}(P, c)$, it is of the form $c \circ \tilde{S}$, for some $S \in H(P, \mathcal{L})$. By the proof of Lemma 4.3, there is a $T \in H(F, \mathcal{L})$ with $f \circ \tilde{T} = \tilde{S}$. By the proof of Lemma 3.3, $c \circ f \circ \tilde{T} = c \circ \tilde{S}$ and $(c \circ f) \circ \tilde{T} \in \mathcal{L}(F, c \circ f)$. Hence, $c \circ \tilde{S} \in \mathcal{L}(F, c \circ f)$. $\square$

From Theorem 4.1 and Corollary 3.5 we get

Corollary 4.4 *Over any reducible base $\mathcal{L}$, $(RBH_k(\mathcal{L}); \subseteq)$ is a well poset.* $\square$

From Theorem 4.1 follows that $BH_k(\mathcal{L}) \subseteq FBH_k(\mathcal{L})$, provided that $\mathcal{L}$ is reducible. The next result completely describes $BH_k(\mathcal{L})$ inside of $FBH_k(\mathcal{L})$.

Theorem 4.5 *If $\mathcal{L}$ is reducible then $BH_k(\mathcal{L})$ is the collection of classes $\mathcal{L}(T, d)$, where (T, d) is a finite k-tree, all leafs of which carry the same label.*

Proof. First let us check that if (T, d) is as specified above then $\mathcal{L}(T, d) \in BH_k(\mathcal{L})$. If T is a chain then it is also a lattice and the assertion is trivial. Otherwise, adjoin to T an extra least element and assign to this element the label carried by the leafs of T; denote the resulting k-poset (P, c). Note that P is indeed a lattice, hence $\mathcal{L}(P, c) \in BH_k(\mathcal{L})$. Now let $(F, c \circ f)$ be the k-forest constructed from (P, c) as in the proof of Lemma 2.2. Then $(F, c \circ f)$ is a k-tree obtained from (T, d) by adding a unique son to any leaf of T and by assigning to the son the label carried by the leafs. Evidently, $(F, c \circ f) \equiv (T, d)$. Hence, $\mathcal{L}(T, d) = \mathcal{L}(F, c \circ f) = \mathcal{L}(P, c) \in BH_k(\mathcal{L})$.

Conversely, take any class from $BH_k(\mathcal{L})$, it is of the form $\mathcal{L}(P, c)$ for a finite k-lattice (P, c). Consider again the k-forest $(F, c \circ f)$ from Lemma 2.2. The poset P has the greatest element, hence F is a tree. The poset P has also the least element b carrying the label $c(b)$. Then any leaf x of F carries the label $c(b)$ (otherwise, we would have $f(x) > b$ and, by construction of F, we have to add a son of x to F; a contradiction). $\square$

From the last proof we immediately get

Corollary 4.6 *Let $\mathcal{L}$ be a reducible base and let (P, c) be a k-poset with a least and a greatest element. Then $\mathcal{L}(P, c) \in BH_k(\mathcal{L})$.* $\square$

5 RBH over Recursively Enumerable Sets

In this section we consider the particular case when $\mathcal{L}$ is the lattice $\mathcal{E}$ of all r.e. subsets of ω and $M = \omega$. As is well known (see e.g. [R67]), $\mathcal{E}$ is a reducible base. In this case the k-partitions of ω are naturally identified with elements of the set k^ω, i.e. with numberings of the form $\nu : \omega \to k = \{0.\ldots, k-1\}$. We will show that in this case there are deep connections of the objects studied above with some well developed parts of the theory of numberings [E73, E75], namely with computable numberings and complete numberings.

Let us start with recalling some relevant definitions. By *a numbering* we mean any function with domain ω. A numbering $\nu : \omega \to \mathcal{E}$ is *computable* if the set $\{(n, x) | x \in \nu_n\}$ is r.e. For any family $\mathcal{S} \subseteq \mathcal{E}$ of r.e. sets and for any $c : \mathcal{S} \to k$, let $C(\mathcal{S})$ be the set of all computable numberings of the form $\nu : \omega \to \mathcal{S}$, and let $C(\mathcal{S}, c) = \{c \circ \nu | \nu \in C(\mathcal{S})\}$. Any family $\mathcal{S}$ of r.e. sets is partially ordered under inclusion; let $\sqsubseteq$ denote the dual partial order on $\mathcal{S}$, i.e. $(\mathcal{S}; \sqsubseteq) = (\mathcal{S}; \supseteq)$.

Let us relate introduced notions to notions from Section 3.

Lemma 5.1 *For any finite family $\mathcal{S}$ of r.e. sets and for any labeling $c : \mathcal{S} \to k$ we have $\mathcal{E}(\mathcal{S}; \sqsubseteq) = C(\mathcal{S})$ and $\mathcal{E}(\mathcal{S}; \sqsubseteq, c) = C(\mathcal{S}, c)$.*

Proof. As is well known [E73, E75], a numbering $\nu : \omega \to \mathcal{S}$ is computable iff the index set $\nu^{-1}(\{X \in \mathcal{S} | A \subseteq X\})$ is r.e. for any $A \in \mathcal{S}$. Comparing this with the definition of $\mathcal{L}(P; \leq)$ in Section 3 we immediately get the first equality. The second equality follows from the first one. $\square$

The next assertion describes the boolean hierarchies of partitions over r.e. sets in terms of computable numberings.

Theorem 5.2 *(i) $RBH_k(\mathcal{E})$ is the collection of all classes $C(\mathcal{S}, c)$, where $\mathcal{S}$ is a finite family of r.e. sets and $(\mathcal{S}; \sqsubseteq, c) \in \mathcal{P}_k$.*

(ii) $BH_k(\mathcal{E})$ is the collection of all classes $C(\mathcal{S}, c)$, where $\mathcal{S}$ is a finite family of r.e. sets and $(\mathcal{S}; \sqsubseteq, c) \in \mathcal{L}_k$.

(iii) $FBH_k(\mathcal{E})$ is the collection of all classes $C(\mathcal{S}, c)$, where $\mathcal{S}$ is a finite family of r.e. sets and $(\mathcal{S}; \sqsubseteq, c) \in \mathcal{F}_k$.

Proof. (i) By Lemma 5.1, any class $C(\mathcal{S}, c)$ is in $RBH_k(\mathcal{E})$. It remains to show that any class of $RBH_k(\mathcal{E})$, that by Definition 3.4 is of the form $\mathcal{E}(P; \leq, d)$ for some $(P; \leq, d) \in \mathcal{P}_k$, coincides with $C(\mathcal{S}, c)$ for a finite $\mathcal{S} \subseteq \mathcal{E}$. Without loss of generality we may assume that $P \subseteq \omega$, because any finite k-poset is isomorphic to a k-poset with this property. Now let $\mathcal{S} = \{\check{x} | x \in P\}$. Then $\mathcal{S}$ is a finite family of (finite) r.e. sets and $(\mathcal{S}; \sqsubseteq)$ is isomorphic to $(P; \leq)$. Hence, $(\mathcal{S}; \sqsubseteq, c)$ is isomorphic to $(P; \leq, d)$ for some $c : \mathcal{S} \to k$. By Lemmas 5.1 and 3.3, $\mathcal{E}(P; \leq, d) = \mathcal{E}(\mathcal{S}; \sqsubseteq, c) = C(\mathcal{S}, c)$ completing the proof of (i).

The same proof applies to (ii) and (iii). $\square$

From Theorems 5.2 and 4.1 we immediately get

Corollary 5.3 *$RBH_k(\mathcal{E})$ coincides with the collection of all classes $C(\mathcal{S}, c)$, where $\mathcal{S}$ is a finite family of r.e. sets and $(\mathcal{S}; \sqsubseteq, c) \in \mathcal{F}_k$.*

Let us recall a couple of other definitions from the numbering theory. A numbering μ is called *reducible* to a numbering ν (in symbols, $\mu \leq \nu$) if $\mu = \nu \circ f$ for a recursive function f. Numberings μ and ν are called *equivalent* ($\mu \equiv \nu$) if $\mu \leq \nu$ and $\nu \leq \mu$. A class $\mathcal{C}$ of numberings is called *a principal ideal* if it is downward closed and has a biggest numbering under $\leq$. In other words, $\mathcal{C} = \{\mu | \mu \leq \nu\}$ for some numbering ν. $\square$

Corollary 5.4 *Any class of $RBH_k(\mathcal{E})$ is a principal ideal of $(k^\omega; \leq)$.*

Proof. It is well known [E73, E75] that for any finite family $\mathcal{S}$ of r.e. sets the class $C(\mathcal{S})$ is a principal ideal. For any $c : \mathcal{S} \to k$, the class $C(\mathcal{S}, c)$ will be a principal ideal too, because $\mu \leq \nu$ implies $c \circ \mu \leq c \circ \nu$. It remains to apply Theorem 5.2. $\square$

Let us present another, and the last, portion of relevant notions from the numbering theory. Define a binary operation $\oplus$ on numberings by

$(\mu \oplus \nu)(2n) = \mu(n)$ and $(\mu \oplus \nu)(2n+1) = \nu(n)$.

For any set S, we get a structure $(S^\omega; \leq, \oplus)$ which is a popular object of the numbering theory. We enrich this structure by unary operations p_s, for any $s \in S$, as follows (for more details see [Se82]). Let υ be a universal partial recursive function, see [Ro67] (a concrete example of such a function is defined by $\upsilon\langle n, x\rangle = \phi_n(x)$), where $\langle , \rangle$ is a recursive pairing function and $\{\phi_n\}$ is a standard numbering of all partial recursive functions. For any ν, we define $p_s(\nu)$ by

$[p_s(\nu)](n) = s$, is $\upsilon(n)$ is undefined, and $[p_s(\nu)](n) = \nu\upsilon(n)$, otherwise.

Let us state without proof some known properties of the resulting structure $(S^\omega; \leq, \oplus, p_s)$. Proofs way be found in [E73, E75, Se82]. Note that some proofs are nontrivial and use a version of the Kleene Recursion Theorem for complete numberings.

Lemma 5.5 *1)* $\leq$ *is a quasiorder and* $\oplus$ *is a supremum operation for* $\leq$.

2) $\nu \leq p_s(\nu)$, $\mu \leq \nu \to p_s(\mu) \leq p_s(\nu)$, $p_s(p_s(\nu)) \leq p_s(\nu)$.

3) $p_s(\mu) \leq p_u(\nu) \wedge s \neq u \to p_s(\mu) \leq \nu$.

4) $p_s(\mu) \leq \nu \oplus \xi \to p_s(\mu) \leq \nu \vee p_s(\mu) \leq \xi$.

5) If $f : S \to T$ *then* $f \circ (\mu \oplus \nu) = (f \circ \mu) \oplus (f \circ \nu)$ *and* $f \circ p_s(\nu) = p_{f(s)}(f \circ \nu)$.

□

With any finite family $\mathcal{S}$ of r.e. sets such that $(\mathcal{S}; \sqsubseteq)$ is a forest we associate a numbering $\nu_{\mathcal{S}}$ by induction on $|\mathcal{S}|$ as follows:

if $\mathcal{S} = \{A\}$ is a singleton then $\nu_{\mathcal{S}}(n) = A$ for any $n \in \omega$;

if $\mathcal{S}$ is a proper forest then $\nu_{\mathcal{S}} = \nu_{\mathcal{S}_0} \oplus \cdots \oplus \nu_{\mathcal{S}_n}$, where $\mathcal{S}_0, \ldots, \mathcal{S}_n$ are the trees of $\mathcal{S}$;

if $\mathcal{S}$ is a nontrivial tree then $\nu_{\mathcal{S}} = p_A(\nu_{\mathcal{S}_0} \oplus \cdots \oplus \nu_{\mathcal{S}_n})$, where A is the root of $\mathcal{S}$ and $\mathcal{S}_i = \hat{A}_i$ for the sons $A_0, \ldots, A_n$ of A (of course, in case $n = 0$ the sum is identified with $\nu_{\mathcal{S}_0}$).

Note that the numbering $\nu_{\mathcal{S}}$ is defined only up to $\equiv$ (e.g., the definition uses the associativity and commutativity of $\oplus$ modulo $\equiv$ following from Lemma 5.5). The next result provides some additional information to Corollary 5.4 because it gives an explicit description of the greatest numbering in $C(\mathcal{S})$.

Theorem 5.6 *Let* $\mathcal{S}$ *be a finite family of r.e. sets such that* $(\mathcal{S}; \sqsubseteq)$ *is a forest. Then* $\nu_{\mathcal{S}}$ *is a greatest element of* $C(\mathcal{S})$.

Proof sketch. The proof is a routine exercise on the numbering theory so we give only a brief sketch. First we show, by induction on $|\mathcal{S}|$, that $\nu_{\mathcal{S}} \in C(\mathcal{S})$, i.e. $\nu_{\mathcal{S}}$ is a computable numbering. Next, again by induction on $|\mathcal{S}|$, we show that $\mu \leq \nu_{\mathcal{S}}$ for any $\mu \in C(\mathcal{S})$. The proof actually does not use Lemma 5.5, it depends only on definitions of $p_s(\nu), \oplus$ and on the well known properties of the numbering ϕ. □

Now let us consider the structure $(k^\omega; \leq, \oplus, p_0, \ldots, p_{k-1})$, where again $k = \{0, \ldots, k-1\}$. With any finite k-forest $F = (F; \leq, c)$ we associate a numbering $\nu_F \in k^\omega$ by induction on $|F|$ as follows:

if $F = \{x\}$ is a singleton then $\nu_F(n) = c(x)$ for any $n \in \omega$;

if F is a proper forest then $\nu_F = \nu_{F_0} \oplus \cdots \oplus \nu_{F_n}$, where $F_0, \ldots, F_n$ are the trees of F;

if F is a nontrivial tree then $\nu_F = p_{c(x)}(\nu_{F_0} \oplus \cdots \oplus \nu_{F_n})$, where x is the root of F and $F_i = \hat{x}_i$ for the sons $x_0, \ldots, x_n$ of x (in case $n = 0$ the sum is identified with ν_{F_0}).

The numbering ν_F is closely related to the numbering $\nu_{\mathcal{S}}$ for a suitable $\mathcal{S}$. Namely, choose a family $\mathcal{S}$ of r.e. sets so that $(\mathcal{S}; \sqsubseteq)$ is isomorphic to $(F; \leq)$, and let f be an isomorphism (see the proof of Theorem 5.2). From Property 5) in Lemma 5.5 it follows that $\nu_F = c \circ f \circ \nu_{\mathcal{S}}$. From this remark and Theorem 5.6 one easily gets

Corollary 5.7 *The class of numberings* $\mathcal{E}(F, c) \subseteq k^\omega$ *is a principal ideal and* ν_F *is a greatest element in* $\mathcal{E}(F, c)$. □

Now we prove the main result of this section.

Theorem 5.8 *For all finite k-forests $F = (F, c)$ and $G = (G, d)$, $(F, c) \leq (G, d)$ iff $\mathcal{E}(F, c) \subseteq \mathcal{E}(G, d)$.*

Proof. From left to right, the assertion follows from Lemma 3.3. It remains to prove the converse. By Corollary 5.7, it suffices to show that if $F \not\leq G$ then $\nu_F \not\leq \nu_G$. This is by induction on $|F| + |G|$, the induction basis (i.e. the case when F and G are singletons) being trivial. For $|F| + |G| > 2$, consider the following 3 cases.

Case 1. F is a proper forest with the trees $F_0, \ldots, F_m$.
By the proof of Theorem 2.7, $F_i \not\leq G$ for some $i \leq m$. By induction, $\nu_{F_i} \not\leq \nu_G$ for some $i \leq m$. By Property 1), $\nu_{F_i} \leq \nu_F = \nu_{F_0} \oplus \cdots \oplus \nu_{F_m}$, Hence, $\nu_F \not\leq \nu_G$.

Case 2. F is a tree and G is a proper forest with the trees $G_0, \ldots, G_n$.
By the proof of Theorem 2.7, $F \not\leq G_j$ for all $j \leq n$. By induction, $\nu_F \not\leq \nu_{G_j}$ for all $j \leq n$. By Property 4), $\nu_F \not\leq \nu_G = \nu_{G_0} \oplus \cdots \oplus \nu_{G_n}$.

Case 3. F and G are k-trees with the roots x and y, respectively.
We have $\nu_F = p_{c(x)}(\nu_{F_0} \oplus \cdots \oplus \nu_{F_m})$ and $\nu_G = p_{d(y)}(\nu_{G_0} \oplus \cdots \oplus \nu_{G_n})$ for F_i, G_j specified in the definition above. Suppose, for a contradiction, that $\nu_F \leq \nu_G$. If $c(x) \neq d(y)$ then, by Property 3), $\nu_F \leq \nu_{G \setminus \{y\}}$. By induction, $F \leq G \setminus \{y\}$. Hence, $F \leq G$; a contradiction.

Finally, let $c(x) = d(y)$. By Property 2), $\nu_{F \setminus \{x\}} \leq \nu_G$. By induction, $F \setminus \{x\} \leq G$. Let f be a morphism from $(F \setminus \{x\}; c)$ to (G, d). Then $f \cup \{(x, y)\}$ is a morphism from (F, c) to (G, d). Hence, $F \leq G$. A contradiction. $\Box$

We see that the poset $(FBH_k(\mathcal{E}); \subseteq) = (RBH_k(\mathcal{E}); \subseteq)$ is isomorphic to the quotient structure of $(\mathcal{F}_k; \leq)$ from Section 2. This gives a lot of information about the boolean hierarchy of partitions over r.e. sets. E.g., $(RBH_2(\mathcal{E}); \subseteq)$ is a well poset of width 2 and height ω. In fact, from the properties of Ershov's difference hierarchy over $\mathcal{E}$ [E70, Se82, Se89a] it follows that this poset coincides with the poset formed by the finite levels of the difference hierarchy (one has only to add also the Δ-levels).

For $k > 2$, $(RBH_k(\mathcal{E}); \subseteq)$ is a well poset of height ω and of unbounded width which is (if augmented with an extra least element) even a distributive lattice. By Theorem 5.6, all classes of this hierarchy are principal ideals the greatest elements of which have a very natural description. Namely, they are exactly the elements of the subalgebra of $(k^\omega; \oplus, p_0, \ldots, p_{k-1})$ generated by the constant numberings $\nu_0, \ldots, \nu_{k-1}$ defined by $\nu_i(n) = i$, for all $n \in \omega$. Note that this subalgebra was considered earlier [Se82, Se89, Se89a] but here we get much new information about this subalgebra.

6 Conclusion

Here we briefly discuss some more examples of reducible bases and some open questions. Results in this section are formulated without proofs which are rather similar to the proofs in Section 5 and will appear either in the journal version of this preprint or in a separate publication.

First of all, all notions and results of Section 5 are generalizable in a straightforward way (using relativization) to any level of the arithmetical hierarchy in place of $\mathcal{E}$.

The notions, results and proofs from Section 5 are also not hard to mimic for the class $\mathbf{\Sigma}_1$ of open sets of the Cantor space $m^\omega (m \geq 2)$, as well as of the Baire space, in place of $\mathcal{E}$ (note that, as is well known (see e.g. [M80]), the class $\mathbf{\Sigma}_1$, as well as any level $\mathbf{\Sigma}_\alpha (\alpha < \omega_1)$) of the Borel hierarchy, is a reducible base). In particular, the following result holds true.

Theorem 6.1 *For any $k \geq 2$, the structure $(RBH_k(\mathbf{\Sigma}_1); \subseteq)$ is isomorphic to the quotient structure of $(\mathcal{F}_k; \leq)$, and, consequently, also to the structure $(RBH_k(\mathcal{E}); \subseteq)$.* $\Box$

Other interesting examples of reducible bases are the classes $\mathcal{R}\mathbf{\Sigma}_n$, $n = 1, 2$, of regular $\mathbf{\Sigma}_n$-subsets of the Cantor space m^ω. Analogs of notions and results from Section 5 can be again without big problems mimicked at least for the case of regular open sets. In particular, we have the following result.

Theorem 6.2 *For any $k \geq 2$, the structure $(RBH_k(\mathcal{R}\mathbf{\Sigma}_1); \subseteq)$ is isomorphic to the quotient structure of $(\mathcal{F}_k; \leq)$, and, consequently, to $(RBH_k(\mathcal{E}); \subseteq)$.* $\Box$

For the higher levels of the Borel hierarchy the situation becomes more complicated though we have a definite feeling that analogs of results of Section 5 are provable also in this situation.

This paper gives of course no information for the non-reducible bases, in particular for NP, hence the interesting problems about the behavior of the boolean hierarchies of partitions in this case remain open. Other natural and interesting objects of investigation are e.g. the structures $FBH_k(co(\mathcal{E}); \subseteq)$ and $(FBH_k(\mathbf{\Pi}_1); \subseteq))$.

7 Acknowledgment

I am grateful to Klaus Wagner for inviting me to the University of Würzburg for a research stay. Additional thanks are to him and to Sven Kosub for acquainting me with their results on the boolean hierarchy of partitions and for many helpful discussions.

References

[DP94] B. A. Davey and H. A. Pristley. *Introduction to Lattices and Order.* Cambridge, 1994.

[E73] Y. L. Ershov. Theorie der Numerierungen I. *Zeitschr. math. Logik Grundl. Math.*, 19 (1973), 289—388.

[E75] Y. L. Ershov. Theorie der Numerierungen II. *Zeitschr. math. Logik Grundl. Math.*, 21 (1975), 473—584.

[E70] Y. L. Ershov. On a hierarchy of sets III. *Algebra and Logic*, 9 (1970) (Russian).

[K00] S. Kosub. On NP-partitions over posets with an application of reducing the set of solutions of NP problems. In: *Proc. 25th Symp. on Math. Foundations of Comp. Sci., Lecture Notes of Computer Science*, 1893 (2000), 467—476, Berlin, Springer.

[K00a] S. Kosub. *Complexity and Partitions.* PhD Thesis, Würzburg, 2000.

[Kr60] J. B. Kruskal. Well-quasi-ordering, the tree theorem, and Varzsonyi's conjecture. *Trans. Amer. Math. Soc.*, 95 (1960), 210—225.

[Kr72] J. B. Kruskal. The theory of well-quasi-ordering: a frequently discovered concept. *J. Combinatorics Th.(A)*, 13 (1972), 297—305.

[KW99] S. Kosub and K. Wagner. The boolean hierarchy of partitions. Technical Report 233, Institut für Informatik, Universität Würzburg, 1999.

[KW00] S. Kosub and K. Wagner. The boolean hierarchy of NP-partitions. In: *Proc. 17th Symp. on Theor. Aspects of Comp. Sci., Lecture Notes of Computer Science*, 1770 (2000), 157—168, Berlin, Springer.

[M80] Y. N. Moschovakis. *Descriptive Set Theory.* Amsterdam, North Holland, 1980.

[R67] H. Rogers. *Theory of recursive functions and effective computability.* New York, McGrow Hill, 1967.

[Se82] V. L. Selivanov. On the structure of degrees of generalized index sets. *Algebra and Logic*, 21, N 4 (1982), 472-491 (Russian, there is an English translation).

[Se89] V. L. Selivanov. Fine hierarchies of arithmetical sets and definable index sets. *Trans. Inst. Math., Novosibirsk,* 12 (1989), 165-185 (Russian).

[Se89a] V. L. Selivanov. *Hierarchical classification of arithmetical sets and index sets.* Doctor dissertation (Habilitation Thesis), Institute of Mathematics SD RAS, Novosibirsk, 1989 (Russian).

[Se95] V. L. Selivanov. Fine hierarchies and Boolean terms. *Journal of Symbolic Logic*, 60 (1995), p. 289—317.

[Se98] V. L. Selivanov. Fine hierarchy of regular ω-languages. *Theor. Comp. Sci.*, 191 (1998), 37—59.