# Boolean Partitions and Projective Closure

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**Abstract.** When studying the complexity of partitions one often faces the situation that different partition classes have the same projection classes. The projectively closed classes are the greatest (with respect to set-inclusion) among these. In this paper we determine important partition classes that are projectively closed and we prove the rather surprising result that each projectively closed class with projection classes from the boolean hierarchy over NP is generated by labeled posets. We describe a method for calculating these posets. As a consequence for the case of set classes we obtain representations of the classes $\mathrm{NP}(m) \cap \mathrm{coNP}(m)$ by means of finite labeled posets. Moreover we explore a (probably) infinite hierarchy between the classes $\mathrm{P}^{\mathrm{NP}}[1]$ and $\mathrm{NP}(2) \cap \mathrm{coNP}(2)$.

## 1 Introduction

In many areas of life there are more than two alternatives to consider, e.g., a teacher has not only to worry about whether a student should pass the test but also which mark is appropriate, or when playing chess, the current configuration you are facing with on the board can develop into a win, loss, or tie situation depending on the next moves. In such cases evaluating the situation is rather a classification problem than simply a decision problem, or, speaking set-theoretically, rather a partition (into more than two parts) than simply a set (partition into two parts). In complexity theory classification problems have long been investigated by encodings into sets, but more recently partitions themselves have become a matter of research [KW99,KW00,Kos00].

Very important classes of classification problems originate from questions concerning relations. For a concrete example let us consider the entailment relation $\models$ for formulas of propositional logic. For propositional formulas $H$ and $H'$ it is defined as

$$H \models H' \iff_{\mathrm{def}} \text{each satisfying assignment for } H \text{ is a satisfying assignment for } H'.$$

Given two arbitrary formulas there are four possible cases to classify according to the behavior the formulas show with respect to $\models$. We translate this into the partition Entailment. The most natural way to define a partition is to fix its characteristic function. For any partition $A$ the characteristic function $c_A$ says for every $x$ to which component of $A$ this $x$ belongs. So for any pair $(H, H')$ of formulas we define

$$c_{\textsc{Entailment}}(H,H') =_{\mathrm{def}} \begin{cases} 1 & \text{if } H \not\models H' \text{ and } H' \not\models H, \\ 2 & \text{if } H \not\models H' \text{ and } H' \models H, \\ 3 & \text{if } H \models H' \text{ and } H' \not\models H, \\ 4 & \text{if } H \models H' \text{ and } H' \models H. \end{cases}$$

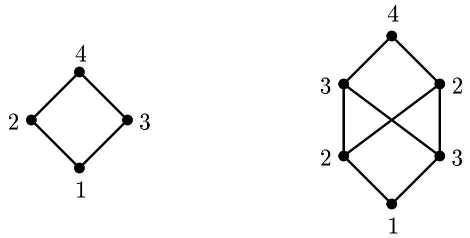


**Fig. 1.** Labeled posets being candidates for the complexity of ENTAILMENT.

Note that though the numbering of the cases to be distinguished is not essential for the classification itself yet it leads to different partitions.

Apparently there exist very close connections between ENTAILMENT and the decision problem of whether $H \models H'$ for given $H$ and $H'$. Let us explain this in more detail. For we consider two sets $A$ and $B$ that describe the decision problem formally: $A$ is the set of all pairs $(H, H')$ such that $H$ entails $H'$ and $B$ is the set of all pairs $(H, H')$ such that $H'$ entails $H$. The partition ENTAILMENT and the sets $A$ and $B$ are intimately related: Using the sets $A$ and $B$ the partition ENTAILMENT can be easily rewritten. So the first component of ENTAILMENT, denoted by $\text{ENTAILMENT}_1$, consists of all pairs of propositional formulas that do not belong to $A$ or $B$. Opposite to this the fourth component of ENTAILMENT, denoted by $\text{ENTAILMENT}_4$, is nothing else than $A \cap B$. Since obviously $A$ and $B$ are coNP-complete (note that $H$ is a tautology if and only if $H \vee \neg H \models H$) we easily observe that $\text{ENTAILMENT}_4$ is coNP-complete, whereas $\text{ENTAILMENT}_1$ is NP-complete. Equally it is not hard to verify that both the second and the third component of the entailment classification problem are complete for NP(2) where NP(2) also known as DP [PY84] is the class of all set differences of NP sets with NP sets. We can state this as ENTAILMENT is in (NP, NP(2), NP(2), coNP)—the class of all partitions having the first component in NP, the second and the third in NP(2), and the fourth in coNP. Partition from classes with components in the boolean hierarchy over NP sets are called boolean partitions.

On the other hand, ENTAILMENT and the sets $A$ and $B$ are related by the following generation principle.

$$\begin{aligned} f(1,1) = f'(0,0) = 4, \\ f(1,0) = f'(0,1) = 3, \\ f(0,1) = f'(1,0) = 2, \\ f(0,0) = f'(1,1) = 1. \end{aligned} \tag{1}$$

We immediately see that ENTAILMENT is exactly the partition being generated when $f$ is applied to the characteristic pair of the sets $A$ and $B$. That means that for all propositional formulas $H$ and $H'$ it holds that $c_{\text{ENTAILMENT}}(H, H') = f(c_A(H, H'), c_B(H, H'))$. Dually, ENTAILMENT is the partition being generated when $f'$ is applied to the complements of $A$ and $B$. Since $A$ and $B$ are coNP sets partitions similar to ENTAILMENT emerge if we release $A$ and $B$ to be arbitrary coNP sets. In this manner the function $f$ generates a whole class of partitions which we denote by coNP$(f)$. Dually we obtain a class NP$(f')$ that is easily the same as coNP$(f)$. So ENTAILMENT belongs to the class coNP$(f)$. In fact, it is one of the hardest among all partitions in this class; it is in a sense complete for $\mathrm{coNP}(f)$. The generation principle just described has been the initial point for the introduction of the boolean hierarchy of NP-partitions in [KW99,KW00]. It is known that all partition classes that are defined by functions $f : \{0,1\}^n \to \{1,\dots,k\}$ can be equivalently described by finite lattices labeled with $k$ values, e.g., the left lattice in Fig. 1 generates $\mathrm{NP}(f')$.

The natural question to ask now is whether $\mathrm{NP}(f')$ is equal to $(\mathrm{NP}, \mathrm{NP}(2), \mathrm{NP}(2), \mathrm{coNP})$. We will prove that is not the case unless $\mathrm{NP} = \mathrm{coNP}$. To do this we address the more general question of how to describe classes of boolean partitions. For that we give a method to calculate a representation of such classes by finite labeled posets. In the particular case of $(\mathrm{NP}, \mathrm{NP}(2), \mathrm{NP}(2), \mathrm{coNP})$ we obtain the right poset in Fig. 1. Since the right posets cannot be mapped to the left lattice such that order and labels are preserved, we obtain from the results in [KW99] that both generated classes are equal if and only if $\mathrm{NP} = \mathrm{coNP}$. This has the consequence that ENTAILMENT is not complete for $(\mathrm{NP}, \mathrm{NP}(2), \mathrm{NP}(2), \mathrm{coNP})$ unless $\mathrm{NP} = \mathrm{coNP}$.

Classes of boolean partitions are also important from a purely set-theoretically point of view, namely they are the largest classes with respect to set inclusion among all partition classes having the same component classes (projection classes). For instance, $\mathrm{NP}(f')$ and $(\mathrm{NP}, \mathrm{NP}(2), \mathrm{NP}(2), \mathrm{coNP})$ have the same projection classes. In fact, $(\mathrm{NP}, \mathrm{NP}(2), \mathrm{NP}(2), \mathrm{coNP})$ can be seen as the projective closure of $\mathrm{NP}(f')$. Our method for calculating posets that describe boolean partitions thus also provides a criterion for deciding whether a given labeled posets represents a projectively closed partition class (for a large class of posets).

## 2 Preliminaries

**Sets.** Let $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{N}_+ = \{1, 2, \dots\}$. The cardinality of an arbitrary finite set $A$ is denoted by $\|A\|$. For sets $A$ and $B$ we use $A \setminus B$ to denote the set-difference of $A$ with $B$, and we use $A \triangle B$ to denote the symmetric difference of $A$ and $B$. For $m \geq 1$ let $A^m$ denote the $m$-fold cartesian product of $A$ with itself.

Let $\mathcal{P}(M)$ be the power set of a fixed basic set $M$. For a set $A \subseteq M$, its complement in the basic set $M$ is denoted by $\overline{A}$, i.e., $\overline{A} = M \setminus A$. The characteristic function $c_A : M \to \{0,1\}$ is defined for all $x \in M$ as

$$c_A(x) \quad =_{\mathrm{def}} \quad \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \in \overline{A}. \end{cases}$$

Let $\mathcal{K}$ and $\mathcal{K}'$ be classes of subsets of $M$, i.e., $\mathcal{K}, \mathcal{K}' \subseteq \mathcal{P}(M)$. We define

$$\begin{aligned} \mathrm{co}\mathcal{K} &=_{\mathrm{def}} \{\ \overline{A} \mid A \in \mathcal{K}\ \}, \\ \mathcal{K} \wedge \mathcal{K}' &=_{\mathrm{def}} \{\ A \cap B \mid A \in \mathcal{K}, B \in \mathcal{K}'\ \}, \\ \mathcal{K} \vee \mathcal{K}' &=_{\mathrm{def}} \{\ A \cup B \mid A \in \mathcal{K}, B \in \mathcal{K}'\ \}, \\ \mathcal{K} \oplus \mathcal{K}' &=_{\mathrm{def}} \{\ A \triangle B \mid A \in \mathcal{K}, B \in \mathcal{K}'\ \}. \end{aligned}$$

The classes $\mathcal{K}(i)$ and $\mathrm{co}\mathcal{K}(i)$ defined by $\mathcal{K}(0) =_{\mathrm{def}} \{\emptyset\}$ and $\mathcal{K}(i+1) = \mathcal{K}(i) \oplus \mathcal{K}$ build the *boolean hierarchy over* $\mathcal{K}$ that has many equivalent definitions (see [WW85,CH86,KSW87,CGH+88]).[1] Some of them can be found in the following theorem.

[1] Usually for $\mathcal{K} = \mathrm{NP}$, a level 0 is not considered in the way we do. The zero-level there is P. However for our purposes it is more helpful to regard P not as an element of the boolean hierarchy (unless $\mathrm{P} = \mathrm{NP}$).

**Theorem 1.** *Let $\emptyset, M \in \mathcal{K}$, let $\mathcal{K}$ be closed under intersection and union, and let $m \in \mathbb{N}_+$.*

1. $\mathcal{K}(2m-1) = \{\ A_1 \cup \bigcup_{j=1}^{m-1}(A_{2j+1} \setminus A_{2j}) \mid A_1, \ldots, A_{2m-1} \in \mathcal{K}$ *and* $A_1 \subseteq \cdots \subseteq A_{2m-1}\ \}$.
2. $\mathcal{K}(2m) = \{\ \bigcup_{j=1}^{m}(A_{2j} \setminus A_{2j-1}) \mid A_1, \ldots, A_{2m} \in \mathcal{K}$ *and* $A_1 \subseteq \cdots \subseteq A_{2m}\ \}$.
3. $\mathcal{K}(2m) = \mathcal{K}(2m-1) \wedge \mathrm{co}\mathcal{K}$.
4. $\mathcal{K}(2m+1) = \mathcal{K}(2m) \vee \mathcal{K}$.
5. $\mathcal{K}(m+1) = \mathrm{co}\mathcal{K}(m) \wedge \mathcal{K}$.
6. $\mathcal{K}(m+2) = \mathcal{K}(m) \vee (\mathcal{K} \wedge \mathrm{co}\mathcal{K}) = \mathcal{K}(m) \wedge (\mathcal{K} \vee \mathrm{co}\mathcal{K})$.
7. $\mathcal{K}(m) \cup \mathrm{co}\mathcal{K}(m) \subseteq \mathcal{K}(m+1) \cap \mathrm{co}\mathcal{K}(m+1)$.

$\mathrm{BC}(\mathcal{K})$ is the boolean closure of $\mathcal{K}$, i.e., the smallest class which contains $\mathcal{K}$ and which is closed under intersection, union, and complements.

**Orders and Lattices.** We need some notions from lattice theory and order theory (see e.g., [Grä78,DP90]). A pair $(G, \leq)$ is a poset if $\leq$ is a partial order on the set $G$. Usually, we talk about the poset $G$. Where it is necessary we write $(G, \leq)$ to specify the order. A finite poset $(G, \leq)$ is a lattice if for all $x, y \in G$ there exist (a) exactly one maximal element $z \in G$ such that $z \leq x$ and $z \leq y$ (which will be denoted by $x \wedge y$), and (b) exactly one minimal element $z \in G$ such that $z \geq x$ and $z \geq y$ (which will be denoted by $x \vee y$). For a lattice $G$ we denote by $1_G$ the unique element greater than or equal to all $x \in G$ and by $0_G$ the unique element less than or equal to all $x \in G$.

**Words.** We will make no difference between $m$-tuples $(x_1, \ldots, x_m)$ over a finite set (alphabet) $M$ and words $x_1 \ldots x_m$ of length $m$ over $M$. We fix the finite alphabet $\Sigma = \{0, 1\}$ for considerations about input-output behavior of machines. More generally, let $\Delta$ be any finite alphabet. $\Delta^*$ is the set of all finite words that can be built with letters from $\Delta$. For $x, y \in \Delta^*$, $x \cdot y$ (or $xy$ for short) denotes the concatenation of $x$ and $y$. The empty word is denoted by $\varepsilon$. For $x \in \Delta^*$, $|x|$ denotes the length of $x$. For $n \in \mathbb{N}$, $\Delta^{\leq n}$ is the set of all words $x \in \Delta^*$ with $|x| \leq n$, and $\Delta^{=n}$ is the set of all words $x \in \Delta^*$ with $|x| = n$. If the alphabet $\Delta$ is ordered by $\leq$, then let $\leq_{\mathrm{lex}}$ denote the standard lexicographical order on $\Delta^*$, that is, for each $x, y \in \Delta^*$, $x \leq_{\mathrm{lex}} y$ if and only if (a) $x = y$, (b) $|x| < |y|$, or (c) $|x| = |y|$ and there is an $i$ with $x_j = y_j$ for all $j \in \{1, \ldots, i-1\}$ but $x_i < y_i$. Usually we consider words $x$ and $y$ of the same length $n$ to be partially ordered by the vector-ordering, that is, $x \leq y$ iff $x_i \leq y_i$ for all $i \in \{1, \ldots, n\}$.

**Basic Complexity Theory.** The computational model we refer to is the standard Turing machine (for a formal description see, e.g., [WW86,BDG95]). We consider nondeterministic and deterministic versions of Turing machines. We also consider Turing machines that have access to an oracle. The notions translate accordingly to such oracle Turing machines. If we consider an oracle Turing machine $M$ accessing an oracle $A$ then this is denoted by $M^A$.

Polynomial-time Turing machines are Turing machines that for a fixed polynomial $p$, make on every input $x$ at most $p(|x|)$ computation steps before reaching a final state. In case of a nondeterministic polynomial-time Turing machine $M$, the set of all words accepted by $M$, denoted by $L(M)$, is the set of all words $x \in \Sigma^*$ for which $M$, on input $x$, has at least one computation path of at most $p(|x|)$ steps of running, that ends in an accepting final state. NP (P) is the class of all sets that are accepted by nondeterministic (deterministic) polynomial-time Turing machines. $\mathrm{NP}^B$ is the class of all sets that are accepted by nondeterministic polynomial-time Turing machine accessing the set $B$. For a class $\mathcal{K}$, $\mathrm{NP}^{\mathcal{K}}$ consists of all sets that belong to $\mathrm{NP}^{B}$ for some $B \in \mathcal{K}$. The polynomial hierarchy [MS72,Sto77] is inductively defined as follows.

$$\begin{aligned} \Sigma_0^p &=_{\mathrm{def}} \mathrm{P}, \\ \Sigma_{m+1}^p &=_{\mathrm{def}} \mathrm{NP}^{\Sigma_m^p}, \\ \mathrm{PH} &=_{\mathrm{def}} \bigcup_{m \in \mathbb{N}} \Sigma_m^p. \end{aligned}$$

**Partitions.** Finally, let us make some notational conventions about partitions. For any set $M$, a $k$-tuple $A = (A_1, \ldots, A_k)$ with $A_i \subseteq M$ for each $i \in \{1, \ldots, k\}$ is said to be a *$k$-partition of $M$* if and only if $A_1 \cup A_2 \cup \cdots \cup A_k = M$ and $A_i \cap A_j = \emptyset$ for all $i, j$ with $i \neq j$. The set $A_i$ is said to be the *$i$-th component of $A$*. For two $k$-partitions $A$ and $B$ to be equal it is sufficient that $A_i \subseteq B_i$ for all $i \in \{1, \ldots, k\}$. Let $c_A : M \to \{1, \ldots, k\}$ be the characteristic function of a $k$-partition $A = (A_1, \ldots, A_k)$ of $M$, that is, $c_A(x) = i$ if and only if $x \in A_i$. For $\mathcal{C}_1, \ldots, \mathcal{C}_k \subseteq \mathcal{P}(M)$ let

$$(\mathcal{C}_1, \ldots, \mathcal{C}_k) =_{\mathrm{def}} \left\{ A \;\middle|\; A \text{ is } k\text{-partition of } M \text{ and } A_i \in \mathcal{C}_i \text{ for all } i \in \{1, \ldots, k\} \right\}.$$

We say that $(\mathcal{C}_1, \ldots, \mathcal{C}_k)$ is a bound representation of a partition class. Note that a partition class can have infinitely many bound representations. For instance, $(\mathrm{P}, \mathcal{C}) = (\mathcal{C}, \mathrm{P}) = (\mathrm{P}, \mathrm{P})$ for all $\mathrm{P} \subseteq \mathcal{C}$.

Let $\mathcal{C}$ be a class of $k$-partitions. For $i \in \{1, \ldots, k\}$ let

$$\mathcal{C}_i =_{\mathrm{def}} \left\{ A_i \;\middle|\; A \in \mathcal{C} \right\}$$

be the $i$-th projection class of $\mathcal{C}$. The partition class $(\mathcal{C}_1, \ldots, \mathcal{C}_k)$ is the *projective closure of $\mathcal{C}$*. This term is justified since the operator $\Pi$ defined as $\Pi(\mathcal{C}) = (\mathcal{C}_1, \ldots, \mathcal{C}_k)$ clearly satisfies $\mathcal{C} \subseteq \Pi(\mathcal{C})$, $\mathcal{C} \subseteq \mathcal{C}' \Rightarrow \Pi(\mathcal{C}) \subseteq \Pi(\mathcal{C}')$, and $\Pi(\Pi(\mathcal{C})) = \Pi(\mathcal{C})$, thus all conditions of a closure operator. A partition class with $\mathcal{C} = \Pi(\mathcal{C})$ is said to be *projectively closed*. Note that only projectively closed classes of $k$-partitions can have a bound representation (see Example 3).

In many cases it suffices to specify $k-1$ components of a class of $k$-partitions. This leads to free representations of partition classes. For classes $\mathcal{C}_1, \ldots, \mathcal{C}_{k-1}$ of subsets of $M$ let

$$(\mathcal{C}_1, \ldots, \mathcal{C}_{k-1}, \cdot) \quad =_{\mathrm{def}} \quad (\mathcal{C}_1, \ldots, \mathcal{C}_{k-1}, \mathcal{P}(M)).$$

Note that only for the sake of convenience we define free representations with respect to the last component. Each freely represented partition classes can be boundly represented.

**Proposition 2.** *For all classes $\mathcal{C}_1, \ldots, \mathcal{C}_{k-1}$ of subsets of $M$,*

$$(\mathcal{C}_1 \ldots, \mathcal{C}_{k-1}, \cdot) = \left( \mathcal{C}_1, \ldots, \mathcal{C}_{k-1}, \mathrm{co} \bigvee_{i=1}^{k-1} \mathcal{C}_i \right).$$

We should remark that Proposition 2 does not express that $(\mathcal{C}_1, \ldots, \mathcal{C}_{k-1}, \cdot)_k$ is equal to $\mathrm{co} \bigvee_{i=1}^{k-1} \mathcal{C}_i$.

We make the convention that a set $A$ is identified with the 2-partition $(A, \overline{A})$ and a class of set $\mathcal{C}$ is, boundly represented, identified with the class $(\mathcal{C}, \mathrm{co}\mathcal{C})$ of 2-partitions or is, freely represented, identified with the class $(\mathcal{C}, \cdot) = (\cdot, \mathrm{co}\mathcal{C})$ of 2-partitions. For instance, $\mathrm{NP} = (\mathrm{NP}, \mathrm{coNP}) = (\mathrm{NP}, \cdot)$. Classes of 2-partitions are always projectively closed since for every set $A$ its complement $\overline{A}$ is uniquely determined. In contrast, for $k \geq 3$ different class of $k$-partitions may have equal projection class as illustrated by the following example.

*Example 3.* Let $M = \{1, 2\}$. Define $\mathcal{C}$ and $\mathcal{D}$ to be the following classes of 3-partitions:

$$\begin{aligned}\mathcal{C} &=_{\mathrm{def}} \{\ (\{1\}, \emptyset, \{2\}),\ (\emptyset, \{2\}, \{1\}),\ (\{1,2\}, \emptyset, \emptyset)\ \},\\ \mathcal{D} &=_{\mathrm{def}} \mathcal{C} \cup \{\ (\{1\}, \{2\}, \emptyset)\ \}.\end{aligned}$$

Then it is easily seen that

$$\begin{aligned}\mathcal{C}_1 = \mathcal{D}_1 &= \{\emptyset, \{1\}, \{1,2\}\},\\ \mathcal{C}_2 = \mathcal{D}_2 &= \{\emptyset, \{2\}\},\\ \mathcal{C}_3 = \mathcal{D}_3 &= \{\emptyset, \{1\}, \{2\}\}\end{aligned}$$

but on the other hand $\mathcal{C} \neq \mathcal{D}$. Note that $\mathcal{D} = (\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3)$, that is $\mathcal{D}$ is projectively closed.

**Partition Classes Defined by Posets.** The following can be found in [Kos00]. Let $\mathcal{K}$ be a class with $\emptyset, M \in \mathcal{K}$ and $\mathcal{K}$ is closed under union and intersection.

**Definition 4.** *Let $G$ be a poset.*

1. *A mapping $S : G \to \mathcal{K}$ is said to be a* $\mathcal{K}$-homomorphism on $G$ *if and only if*
   (a) $\bigcup_{a \in G} S(a) = M$ *and*
   (b) $S(a) \cap S(b) = \bigcup_{c \leq a, c \leq b} S(c)$ *for all $a, b \in G$.*
2. *For any $\mathcal{K}$-homomorphism $S$ on $G$ and $a \in G$, let*

$$T_S(a) =_{\mathrm{def}} S(a) \setminus \bigcup_{b < a} S(b).$$

Note that in the case of a lattice $G$ the above notion of $\mathcal{K}$-homomorphism on $G$ coincides with $S(1_G) = M$ and $S(a) \cap S(b) = S(a \wedge b)$.

**Lemma 5.** *Let $G$ be a poset, and let $S$ be a $\mathcal{K}$-homomorphism on $G$.*

1. *$T_S(a) \in \mathcal{K} \wedge \mathrm{co}\mathcal{K}$ for every $a \in G$.*
2. *If $a \leq b$ then $S(a) \subseteq S(b)$ for every $a, b \in G$.*
3. *$S(a) = \bigcup_{b \leq a} T_S(b)$ for every $a \in G$.*
4. *The set of all $T_S(a)$ for $a \in G$ yields a partition of $M$.*

Any pair $(G, f)$ of an arbitrary finite poset $G$ and a function $f : G \to \{1, 2, \ldots, k\}$ is called a *$k$-poset.* A $k$-poset which is a lattice is called a *$k$-lattice.*

Lemma 5 provides that the following definitions are sound.

**Definition 6.** *Let $(G, f)$ be a $k$-poset. Let $k \geq 2$.*

1. *For a $\mathcal{K}$-homomorphism $S$ on $G$,* the $k$-partition defined by $(G, f)$ and $S$ *is given by*

$$(G, f, S) =_{\text{def}} \left( \bigcup_{f(a)=1} T_S(a) \;, \ldots, \; \bigcup_{f(a)=k} T_S(a) \right).$$

2. *The* class of $k$-partitions defined by the $k$-poset $(G, f)$ *is given by*

$$\mathcal{K}(G, f) =_{\text{def}} \left\{ \; (G, f, S) \;\middle|\; S \textit{ is } \mathcal{K}\textit{-homomorphism on } G \; \right\}.$$

**Definition 7.** *1. The family* $\mathrm{BH}_k(\mathcal{K}) =_{\text{def}} \{\mathcal{K}(G, f) \mid (G, f)$ *is a $k$-lattice$\}$ is the* boolean hierarchy of $k$-partitions over $\mathcal{K}$, *and* $\mathrm{BC}_k(\mathcal{K}) =_{\text{def}} \bigcup \mathrm{BH}_k(\mathcal{K})$.
*2. The family* $\mathrm{RBH}_k(\mathcal{K}) =_{\text{def}} \{\mathcal{K}(G, f) \mid (G, f)$ *is a $k$-poset$\}$ is the* refined boolean hierarchy of $k$-partitions over $\mathcal{K}$.

The classes in $\mathrm{BH}_k(\mathcal{K})$ can be alternatively characterized by means of finite functions as done in [KW99,KW00]. Let $f : \{1, 2\}^m \to \{1, \ldots, k\}$ be any function, $k \geq 2$. For $B_1, \ldots, B_m \in \mathcal{K}$ the set $f(B_1, \ldots, B_m)$ is defined by $c_{f(B_1,\ldots,B_m)}(x) = f(c_{B_1}(x), \ldots, c_{B_m}(x))$. Then the function $f$ generates a class $\mathcal{K}(f)$ by $\mathcal{K}(f) =_{\text{def}} \left\{ \; f(B_1, \ldots, B_m) \;\middle|\; B_1, \ldots, B_m \in \mathcal{K} \; \right\}$. The classes generated by functions with range $\{1, \ldots, k\}$ are exactly the classes from $\mathrm{BH}_k(\mathcal{K})$.

**Theorem 8.** [KW99,KW00] $\mathrm{BH}_k(\mathcal{K}) = \left\{ \; \mathcal{K}(f) \;\middle|\; f : \{1, 2\}^m \to \{1, \ldots, k\} \; \right\}$ *for every $k \geq 2$.*

The following proposition relate the extended boolean hierarchy to the boolean hierarchy.

**Proposition 9.** *Let $k \geq 2$.*

1. $\mathrm{BH}_k(\mathcal{K}) \subseteq \mathrm{RBH}_k(\mathcal{K})$.
2. *If $\mathcal{K}$ is not closed under complements then there exists a partition class in* $\mathrm{RBH}_k(\mathcal{K})$ *which does not belong to* $\mathrm{BH}_k(\mathcal{K})$.
3. *Let $(G, f)$ be a $k$-poset with $f : G \to \{1, 2, \ldots, k\}$ surjective.*
   *(a)* $(\mathcal{K}, \ldots, \mathcal{K}) \subseteq \mathcal{K}(G, f) \subseteq (\mathrm{BC}(\mathcal{K}), \ldots, \mathrm{BC}(\mathcal{K})) = \mathrm{BC}_k(\mathcal{K})$.
   *(b) If $\mathcal{K}$ is closed under complements then $\mathcal{K}(G, f) = (\mathcal{K}, \ldots, \mathcal{K})$.*

It would be very useful to decide whether $\mathcal{K}(G, f) \subseteq \mathcal{K}(G', f')$ for $k$-posets $(G, f)$ and $(G', f')$ by only looking at the defining posets. To this end a relation $\leq$ on finite labeled posets has been introduced.

**Definition 10.** *Let $(G, f)$ and $(G', f')$ be $k$-posets with $k \geq 2$.*

1. *$(G, f) \leq (G', f')$ if and only if there is a monotonic mapping $\varphi : G \to G'$ such that for every $x \in G$, $f(x) = f'(\varphi(x))$.*
2. *$(G, f) \equiv (G', f')$ if and only if $(G, f) \leq (G', f')$ and $(G', f') \leq (G, f)$.*

Note that among all equivalent $k$-posets there is always (up to isomorphy) a unique poset having the minimal number of elements. For the relation $\leq$ we are able to state an Embedding Lemma.

**Lemma 11. (Embedding Lemma.)** *Let $(G, f)$ and $(G', f')$ be $k$-posets. Then, if $(G, f) \leq (G', f')$, then $\mathcal{K}(G, f) \subseteq \mathcal{K}(G', f')$.*

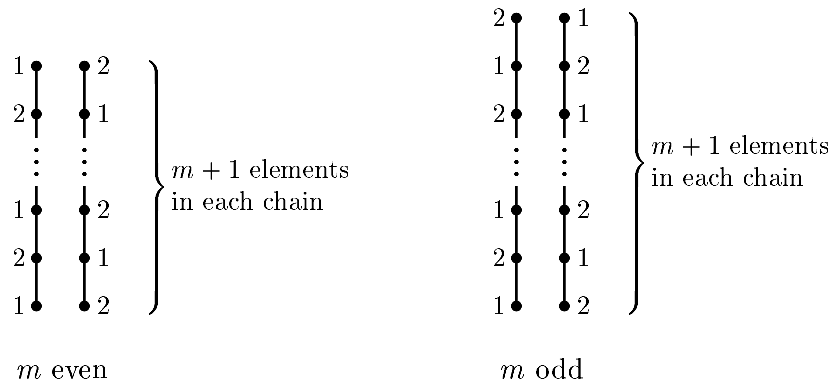


**Fig. 2.** The 2-poset $\mathfrak{D}_m$ for $m \in \mathbb{N}$

The possibility to redirect the Embedding Lemma depends on the base class $\mathcal{K}$. For NP several results has been proven. Next theorem summarizes some.

**Theorem 12.** *Let $(G, f)$ and $(G', f')$ be $k$-posets.*

1. $\mathrm{NP}^A(G, f) \subseteq \mathrm{NP}^A(G', f')$ *for all oracles $A$ if and only if $(G, f) \leq (G', f')$.*
2. *Assume the polynomial hierarchy is strict. If* $\mathrm{NP}(G, f) \subseteq \mathrm{NP}(G', f')$ *then each minimal $k$-subchain of $(G, f)$ does also occur in $(G', f')$.*

## 3 Projection Classes

From Proposition 3 we know that all components of partition classes generated by $k$-posets over a class $\mathcal{K}$ belong to the (two-valued) boolean closure $\mathrm{BC}(\mathcal{K})$ of $\mathcal{K}$. Theorem 14 tells us what exactly is the complexity of the components. We need the following lemma which is also interesting in its own. For all $m \in \mathbb{N}$ let $\mathfrak{D}_m$ be the 2-poset represented in Fig. 2. Let $\mathcal{K}$ be such that $\emptyset, M \in \mathcal{K}$ and $\mathcal{K}$ is closed under intersection and union.

**Lemma 13.** $\mathcal{K}(\mathfrak{D}_m) = (\mathcal{K} \cap \mathrm{co}\mathcal{K}) \oplus \mathcal{K}(m)$ *for all $m \in \mathbb{N}$.*

*Proof.* Let $l_0, \ldots, l_m$ be the (ascendingly ordered) elements belonging to the left chain in $\mathfrak{D}_m$ and let $r_0, \ldots, r_m$ be the (ascendingly ordered) elements belonging to the right chain in $\mathfrak{D}_m$.

$\subseteq$: Let $S$ be a $\mathcal{K}$-homomorphism on $\{l_0, \ldots, l_m, r_0, \ldots, r_m\}$. Thus, $S(l_m) \cap S(r_m) = \emptyset$ and $S(l_m) \cup S(r_m) = M$. Hence, $S(l_m) = \overline{S(r_m)}$. Consequently, $S$ maps $l_m$ and $r_m$ to sets in $\mathcal{K} \cap \mathrm{co}\mathcal{K}$. Moreover, $S(l_0) \subseteq \cdots \subseteq S(l_m)$ and $S(r_0) \subseteq \cdots \subseteq S(r_m)$. By induction the following equations can be easily verified. Let $2n \leq m$.

$$\mathop{\triangle}_{j=0}^{2n-1} (S(l_j) \cup S(r_j)) = \bigcup_{j=0}^{n-1} \left[ (S(l_{2j+1}) \backslash S(l_{2j})) \cup (S(r_{2j+1}) \backslash S(r_{2j})) \right], \tag{2}$$

$$\mathop{\triangle}_{j=0}^{2n} (S(l_j) \cup S(r_j)) = S(l_0) \cup S(r_0) \cup \\ \cup \bigcup_{j=1}^{n} \left[ (S(l_{2j}) \backslash S(l_{2j-1})) \cup (S(r_{2j}) \backslash S(r_{2j-1})) \right]. \tag{3}$$

We have two cases.

- *Case* $m = 2n$. Consider the set $B_S$ defined as

$$B_S =_{\text{def}} S(l_m) \;\triangle\; \left( \mathop{\triangle}_{j=0}^{m-1} (S(l_j) \cup S(r_j)) \right).$$

Clearly, $B_S$ is in $(\mathcal{K} \cap \text{co}\mathcal{K}) \oplus \mathcal{K}(m)$. Eliminating the first symmetric difference we get

$$B_S = \underbrace{\left[ S(l_m) \setminus \mathop{\triangle}_{j=0}^{m-1} (S(l_j) \cup S(r_j)) \right]}_{(1)} \cup \underbrace{\left[ \left( \mathop{\triangle}_{j=0}^{m-1} (S(l_j) \cup S(r_j)) \right) \setminus S(l_m) \right]}_{(2)}.$$

Using Equation (2) we further calculate for Term (1)

$$\begin{aligned}
& S(l_m) \setminus \mathop{\triangle}_{j=0}^{2n-1} (S(l_j) \cup S(r_j)) \\
&= S(l_m) \setminus \bigcup_{j=0}^{n-1} [(S(l_{2j+1} \setminus S(l_{2j})) \cup (S(r_{2j+1}) \setminus S(r_{2j}))] \\
&= S(l_m) \setminus \bigcup_{j=0}^{n-1} (S(l_{2j+1}) \setminus S(l_{2j})) \quad = \quad \bigcap_{j=0}^{n-1} [(S(l_m) \setminus S(l_{2j+1})) \cup S(l_{2j})] \\
&= S(l_0) \cup \bigcup_{j=1}^{n} (S(l_{2j}) \setminus S(l_{2j-1}))\,. \qquad (4)
\end{aligned}$$

For Term (2) we can calculate using Equation (3)

$$\begin{aligned}
& \left( \mathop{\triangle}_{j=0}^{2n-1} (S(l_j) \cup S(r_j)) \right) \setminus S(l_m) \\
&= \left( \bigcup_{j=0}^{n-1} [(S(l_{2j+1}) \setminus S(l_{2j})) \cup (S(r_{2j+1}) \setminus S(r_{2j}))] \right) \cap S(r_m) \\
&= \bigcup_{j=0}^{n-1} S(r_{2j+1}) \setminus S(r_{2j}). \qquad (5)
\end{aligned}$$

Putting together Equations (4) and (5) yield $B_S = (\mathfrak{D}_m, S)_1$.

- *Case* $m = 2n + 1$. Similar to the first case but now by defining $B_S$ as

$$B_S =_{\text{def}} S(r_m) \;\triangle\; \left( \mathop{\triangle}_{j=0}^{m-1} (S(l_j) \cup S(r_j)) \right).$$

$\supseteq$: Let $A \in (\mathcal{K} \cap \mathrm{co}\mathcal{K}) \oplus \mathcal{K}(m)$. That is $A = B \triangle C_1 \triangle \cdots \triangle C_m$ for appropriate $B \in \mathcal{K} \cap \mathrm{co}\mathcal{K}$, $C_i \in \mathcal{K}$, and $C_1 \subseteq \cdots \subseteq C_m$. Again we distinguish between odd and even numbers.

- *Case* $m = 2n$. Define a mapping $S : \{l_0, \ldots, l_m, r_0, \ldots, r_m\} \to \mathcal{K}$ as follows:

$$\begin{aligned} S(l_m) &= B, & S(r_m) &= \overline{B}, \\ S(l_j) &= B \cap C_{j+1} \text{ for } j < m, & S(r_j) &= \overline{B} \cap C_{j+1} \text{ for } j < m. \end{aligned}$$

Obviously, $S$ is a $\mathcal{K}$-homomorphism on $\{l_0, \ldots, l_m, r_0, \ldots, r_m\}$. Furthermore using Equations (2) and (3) we obtain

$$\begin{aligned} &(\mathfrak{D}_m, S)_1 \\ &= S(l_0) \cup (S(r_1) \setminus S(r_0)) \cup (S(l_2) \setminus S(l_1)) \cup \cdots \cup (S(l_m) \setminus S(l_{m-1})) \\ &= (B \cap C_1) \cup \big(\overline{B} \cap (C_2 \setminus C_1)\big) \cup (B \cap (C_3 \setminus C_2)) \cup \cdots \cup \big(B \setminus (B \cap C_m)\big) \\ &= \left[B \cap \left(\overline{C_m} \cup \bigcup_{j=0}^{n-1} (C_{2j+1} \setminus C_{2j})\right)\right] \cup \left[\overline{B} \cap \left(C_0 \cup \bigcup_{j=1}^{n-1} (C_{2j} \setminus C_{2j-1})\right)\right] \\ &= \left[B \cap \left(\overline{C_m} \cup \mathop{\triangle}_{j=0}^{m-1} Cj\right)\right] \cup \left(\overline{B} \cap \mathop{\triangle}_{j=0}^{m} C_j\right) \\ &= \left[B \cap \overline{\left(\mathop{\triangle}_{j=1}^{m} C_j\right)}\right] \cup \left[\overline{B} \cap \left(\mathop{\triangle}_{j=1}^{m} C_j\right)\right] \quad = \quad B \triangle \left(\mathop{\triangle}_{j=1}^{m} C_j\right) \quad = \quad A. \end{aligned}$$

- *Case* $m = 2n + 1$. Define a mapping $S : \{l_0, \ldots, l_m, r_0, \ldots, r_m\} \to \mathcal{K}$ as follows:

$$\begin{aligned} S(l_m) &= \overline{B}, & S(r_m) &= B, \\ S(l_j) &= \overline{B} \cap C_{j+1} \text{ for } j < m, & S(r_j) &= B \cap C_{j+1} \text{ for } j < m. \end{aligned}$$

A calculation similar to the one above gives $(\mathfrak{D}_m, S)_1 = A$.

❑

For a $k$-poset $(G, f)$ and $i \in \{1, 2, \ldots, k\}$ let $\mu_i(G, f)$ denote the maximum number of alternations between $f$-labels $i$ and $f$-labels different to $i$ in a chain of $G$ whose minimum has the label $i$, and let $\mu_{\bar{i}}(G, f)$ denote the maximum number of alternations between $f$-labels $i$ and $f$-labels different to $i$ in a chain of $G$ whose minimum has a label different to $i$.

**Theorem 14. (Projection Theorem.)** *Let $(G, f)$ be a $k$-poset with $\|f(G)\| \geq 2$ and let $i \in \{1, 2, \ldots, k\}$. Then the following holds:*

$$\mathcal{K}(G, f)_i = \begin{cases} \mathcal{K}(\mu_i(G, f)) & \text{if } \mu_i(G, f) > \mu_{\bar{i}}(G, f) \text{ and } \mu_i(G, f) \text{ is odd,} \\ \mathrm{co}\mathcal{K}(\mu_i(G, f)) & \text{if } \mu_i(G, f) > \mu_{\bar{i}}(G, f) \text{ and } \mu_i(G, f) \text{ is even,} \\ \mathcal{K}(\mu_{\bar{i}}(G, f)) & \text{if } \mu_{\bar{i}}(G, f) > \mu_i(G, f) \text{ and } \mu_{\bar{i}}(G, f) \text{ is even,} \\ \mathrm{co}\mathcal{K}(\mu_{\bar{i}}(G, f)) & \text{if } \mu_{\bar{i}}(G, f) > \mu_i(G, f) \text{ and } \mu_{\bar{i}}(G, f) \text{ is odd,} \\ (\mathcal{K} \cap \mathrm{co}\mathcal{K}) \oplus \mathcal{K}(\mu_i(G, f)) & \text{if } \mu_i(G, f) = \mu_{\bar{i}}(G, f). \end{cases}$$

*Proof.* Define a mapping $h_i : \{1, 2, \ldots, k\} \to \{1, 2\}$ for all $j \in \{1, 2, \ldots, k\}$ as

$$h_i(j) =_{\text{def}} \begin{cases} 1 \text{ if } i = j, \\ 2 \text{ if } i \neq j. \end{cases}$$

Obviously, $\mathcal{K}(G, f)_i = \mathcal{K}(G, h_i \circ f)$. If $\mu_i(G, f) \neq \mu_{\bar{i}}(G, f)$ then the minimal 2-poset that is equivalent to the 2-poset $(G, h_i \circ f)$ is a 2-chain. So in this case the first four cases of theorem can be obtained from the well-known mind-change result [WW85] by analyzing the chains. So assume that we are in case $\mu_i(G, f) = \mu_{\bar{i}}(G, f)$. Let $m = \mu_i(G, f)$. Then the minimal 2-poset which is equivalent to $(G, h_i \circ f)$ is the 2-poset $\mathfrak{D}_m$. Thus Lemma 13 completes the proof. ❑

As an example consider the 4-poset $(G, f)$ represented as described in the right part of Fig. 1. Then we easily calculate:

- $\mu_1(G, f) = 1$ and $\mu_{\bar{1}}(G, f) = 0$. Thus, $\mathcal{K}(G, f)_1 = \mathcal{K}(1) = \mathcal{K}$.
- $\mu_2(G, f) = 1$ and $\mu_{\bar{2}}(G, f) = 2$. Thus, $\mathcal{K}(G, f)_2 = \mathcal{K}(2)$.
- $\mu_3(G, f) = 1$ and $\mu_{\bar{3}}(G, f) = 2$. Thus, $\mathcal{K}(G, f)_3 = \mathcal{K}(2)$.
- $\mu_4(G, f) = 0$ and $\mu_{\bar{4}}(G, f) = 1$. Thus, $\mathcal{K}(G, f)_4 = \text{co}\mathcal{K}(1) = \text{co}\mathcal{K}$.

Hence, $\mathcal{K}(G, f) \subseteq (\mathcal{K}, \mathcal{K}(2), \mathcal{K}(2), \text{co}\mathcal{K})$. Thus the upper bound is already shown. It remains to show that in fact both classes are equal. This will be done in the next sections. Notice that for $\mathcal{K} = \text{NP}$ the class $\mathcal{K}(G, f)$ is exactly the projectively closed class for whose projection classes the components of ENTAILMENT are complete.

## 4 Partition Classes Given in Free Representations

We turn to the problem of how to determine which $k$-posets describe freely represented classes $\mathcal{C} = (\mathcal{C}_1, \ldots, \mathcal{C}_{k-1}, \cdot)$ with $\mathcal{C}_i \in \text{BH}_2(\mathcal{K})$ for $i \in \{1, \ldots, k-1\}$. Note that then the projection class $\mathcal{C}_k$ is contained in (but not necessarily equal to) classes from $\text{BH}_2(\mathcal{K})$ due to Proposition 2 and Theorem 1. Without loss of generality we suppose that none of the $\mathcal{C}_i$ with $i \leq k-1$ is $\text{co}\mathcal{K}(0)$. This is justified since $\text{co}\mathcal{K}(0) = \{\Sigma^*\}$, thus $\mathcal{C}_i = \{\Sigma^*\}$ implies $\mathcal{C}_j = \{\emptyset\}$ for all $j \neq i$, and so $\mathcal{C}$ can be considered a freely represented class with the dot in component $i$. Permuting the components yields a partition class as assumed. Admitting trivial components at all becomes extremely useful when considering classes given in bound representation.

For labeled posets we choose a representation as sets of vectors over natural numbers. We consider an $n$-tuple $r = (r_1, \ldots, r_n)$ of integers that later on will be used to describe partition classes. For $r$ let $m = (m_1, \ldots, m_n) \in \mathbb{N}^n$ be an adjoint $n$-tuple given by

$$m_i = \begin{cases} \frac{r_i}{2} & \text{if } r_i \geq 0 \text{ and } r_i \text{ is even}, \\ \frac{|r_i|-1}{2} & \text{if } r_i \text{ is odd}, \\ \frac{-r_i-2}{2} & \text{if } r_i < 0 \text{ and } r_i \text{ is even}. \end{cases} \tag{6}$$

Define an $(n+1)$-poset $\mathfrak{B}_r = (B_r, f)$ depending on the tuple $r$ as follows:

$$\begin{aligned} B_r =_{\text{def}} & \left\{ x \in \mathbb{N}^n \mid x_\nu \in \{1, 2, \ldots, 2m_\nu + 1\} \text{ for all } \nu, \text{ and } \|\{\nu \mid x_\nu \text{ is even}\}\| \leq 1 \right\} \\ & \cup \left\{ x \in \mathbb{N}^n \mid r_\nu \geq 0, r_\nu \text{ is odd}, x_\nu = 0, \text{ and } x_\mu = 1 \text{ for } \nu \neq \mu \right\} \\ & \cup \left\{ x \in \mathbb{N}^n \mid r_\nu < 0, r_\nu \text{ is even}, x_\nu = 0, \text{ and } x_\mu = 1 \text{ for } \nu \neq \mu \right\} \\ & \cup \left\{ x \in \mathbb{N}^n \mid r_\nu < 0, x_\nu = 2m_\nu + 2, \text{ and } x_\mu = 2m_\mu + 1 \text{ for } \nu \neq \mu \right\} \end{aligned}$$

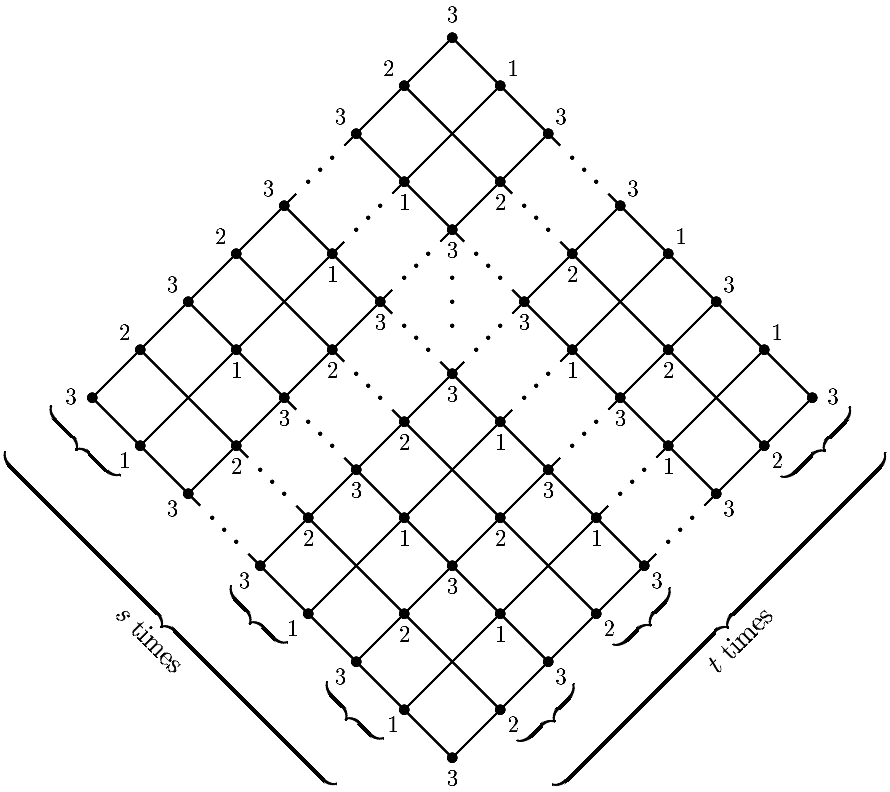


**Fig. 3.** The 3-poset $\mathfrak{B}_{2s,2t}$ for $s, t \in \mathbb{N}$

Consider $B_r$ to be partially ordered by the vector-ordering. For each $x \in B_r$ let

$$f(x) =_{\mathrm{def}} \begin{cases} \nu & \text{if } x_\nu \text{ is even and } x_\mu \text{ is odd for } \nu \neq \mu, \\ n+1 & \text{if } x_\mu \text{ is odd for all } \mu. \end{cases}$$

Although the representation seems technically involved $\mathfrak{B}_r$ is easy to handle for several cases. For instance, if $n = 1$ then $\mathfrak{B}_{2s}$ is simply a 2-chain with $2s + 1$ elements alternately labeled with 1 and 2 and the least element has label 2. For $n = 2$ the 3-poset $\mathfrak{B}_{2s,2t}$ can be drawn as the grid in Figure 3. For $n = 3$ the 4-posets $\mathfrak{B}_{2s,2t,2u}$ are depicted by 3-dimensional cuboids as done for $\mathfrak{B}_{2,2,2}$ in Figure 4.

The next proposition compares labeled posets with respect to tuples.

**Proposition 15.** *Let $r$ and $r'$ be any $n$-tuples of integers such that $|r_i| \leq |r'_i|$ for all $1 \leq i \leq n$. Then $\mathfrak{B}_r \leq \mathfrak{B}_{r'}$.*

*Proof.* In this case we have $B_r \subseteq B_{r'}$. ❑

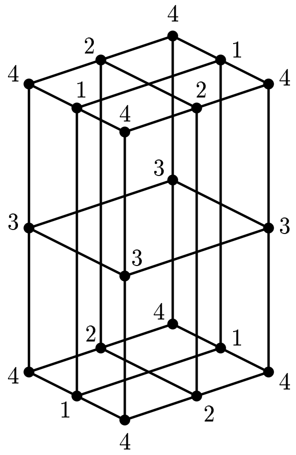


**Fig. 4.** The 4-poset $\mathfrak{B}_{2,2,2}$

The following lemma shows the connection between a labeled poset $\mathfrak{B}_r$ and its complexity in terms of label-alternations.

**Lemma 16.** *Let $r = (r_1, \ldots, r_n)$ be any $n$-tuple over integers and let $j \in \{1, \ldots, n\}$.*

1. *If $r_j = 0$ then $\mu_j(\mathfrak{B}_r) = \mu_{\bar{j}}(\mathfrak{B}_r) = 0$.*
2. *If $r_j > 0$ and $r_j$ is even then $\mu_j(\mathfrak{B}_r) < \mu_{\bar{j}}(\mathfrak{B}_r) = r_j$.*
3. *If $r_j > 0$ and $r_j$ is odd then $\mu_{\bar{j}}(\mathfrak{B}_r) < \mu_j(\mathfrak{B}_r) = r_j$.*
4. *If $r_j < 0$ and $r_j$ is even then $\mu_{\bar{j}}(\mathfrak{B}_r) < \mu_j(\mathfrak{B}_r) = -r_j$.*
5. *If $r_j < 0$ and $r_j$ is odd then $\mu_j(\mathfrak{B}_r) < \mu_{\bar{j}}(\mathfrak{B}_r) = -r_j$.*

*Proof.* The first statement is obvious. From the other statements we only prove the second one. In this case the set $B_r$ coincides with the first set in the definition. The remaining statements can be easily concluded by observing which tuples are added as new least and greatest elements via the other sets. We consider $r$ with $r_j > 0$ and $r_j$ is even. Clearly the least tuple in $B_r$ is $(1, \ldots, 1)$ and thus labeled with $n + 1$. Moreover the greatest tuple in $B_r$ is $(2m_1 + 1, \ldots, 2m_n + 1)$ and thus labeled with $n + 1$ where $(m_1, \ldots, m_n)$ is the adjoint tuple according to Equation (6). From both facts it is seen that $\mu_{\bar{j}}(\mathfrak{B}_r) > \mu_j(\mathfrak{B}_r)$. The proof of $\mu_{\bar{j}}(\mathfrak{B}_r) = r_j = 2m_j$ is by induction on $m_j$. Let the induction start at $m_j = 0$ for which the statement obviously holds. Now assume the statement is already shown for $m_j$. Consider $m_j + 1$. Then the following holds:

$$
\begin{aligned}
&B_{(r_1,\ldots,r_j+2,\ldots,r_n)} \\
&= B_{(r_1,\ldots,r_j,\ldots,r_n)} \\
&\quad \cup \left\{ (x_1, \ldots, 2m_j + 2, \ldots, x_n) \;\middle|\; 1 \le x_\nu \le 2m_\nu + 1 \text{ and } x_\nu \text{ is odd for all } \nu \neq j \right\} \\
&\quad \cup \left\{ (x_1, \ldots, 2m_j + 3, \ldots, x_n) \;\middle|\; 1 \le x_\nu \le 2m_\nu + 1 \text{ for all } \nu \neq j \text{ and} \right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left. \text{at most one } x_\nu \text{ is even} \right\}.
\end{aligned}
$$

By assumption of the induction each longest alternating chain in $\mathfrak{B}_r$ has exactly $2m_j$ alternations of $j$-labels and $l$-labels with $l \neq j$ (starting with a $j$-label). Consider a longest alternating chain that starts in $(1,\dots,1)$ and that ends in $(2m_1+1,\dots,2m_j+1,\dots,2m_n+1)$. Such a chain always exists. Now by switching in the $j$-th component from $2m_j+1$ to $2m_j+2$ to $2m_j+3$ we easily obtain

$$\mu_{\overline{j}}(\mathfrak{B}_{(r_1,\dots,r_j+2,\dots,r_n)}) = 2(m_j+1).$$

❑

The lemma is useful, e.g., in proving the following Embedding Theorem for a subclass of labeled posets.

**Theorem 17.** *Assume the boolean hierarchy of sets over $\mathcal{K}$ is infinite. Let $r$ and $r'$ any $n$-tuples of integers. Then $\mathcal{K}(\mathfrak{B}_r) \subseteq \mathcal{K}(\mathfrak{B}_{r'})$ if and only if $\mathfrak{B}_r \leq \mathfrak{B}_{r'}$.*

*Proof.* The direction "$\Leftarrow$" follows from the Embedding Lemma. For "$\Rightarrow$" suppose $\mathfrak{B}_r \not\leq \mathfrak{B}_{r'}$. Assume that $\mathcal{K}(\mathfrak{B}_r) \subseteq \mathcal{K}(\mathfrak{B}_{r'})$. By Proposition 15 there is an $i$ such that $|r_i| > |r'_i|$. By Lemma 16 and the Projection Theorem we conclude that one of the classes $\mathcal{K}(|r_i|)$ and $\mathrm{co}\mathcal{K}(|r_i|)$ coincides with $\mathcal{K}(|r'_i|)$ or $\mathrm{co}\mathcal{K}(|r'_i|)$ which implies that the boolean hierarchy collapses to level $|r'_i|$, a contradiction. ❑

Connections between posets $\mathfrak{B}_r$ and partition classes are given by boolean characteristics. Let $\mathcal{C}$ be a class of $k$-partitions that can be freely represented by $k-1$ classes from the boolean hierarchy of sets over $\mathcal{K}$. A $(k-1)$-tuple $\beta$ of integers is said to be a *boolean characteristic of $\mathcal{C}$ with respect to $\mathcal{K}$* if for all $i \in \{1,\dots,k-1\}$,

$$\mathcal{C}_i = \begin{cases} \mathcal{K}(\beta_i) & \text{if } \beta_i \geq 0, \\ \mathrm{co}\mathcal{K}(-\beta_i) & \text{if } \beta_i < 0. \end{cases}$$

Note that depending on the class $\mathcal{K}$, a freely represented partition class can only have exactly one boolean characteristic (e.g., if the boolean hierarchy of sets over $\mathcal{K}$ is infinite) or infinitely many boolean characteristics.

**Theorem 18.** *Let $\mathcal{C}$ be any class of $k$-partitions that can be freely represented by $k-1$ classes from the boolean hierarchy of sets over $\mathcal{K}$. Let $\beta$ be any boolean characteristic of $\mathcal{C}$ with respect to $\mathcal{K}$. Then $\mathcal{C} = \mathcal{K}(\mathfrak{B}_\beta)$.*

*Proof.* We first prove the theorem only for classes $\mathcal{C}$ having a boolean characteristic $\beta$ with $\beta_i < 0$ and $\beta_i$ is even for all $i$. Let $m_i \in \mathbb{N}$ satisfy $\beta_i = -2m_i-2$. That means $\mathcal{C}_i = \mathrm{co}\mathcal{K}(2m_i+2)$ for all $i$. Consider the $k$-poset $\mathfrak{B}_\beta = (B_\beta, f)$. Note that the tuple $m = (m_1,\dots,m_{k-1})$ is precisely the tuple adjoint to $\beta$ in the sense of Equation (6). We have to show that $\mathcal{C} = \mathcal{K}(\mathfrak{B}_\beta)$. The inclusion "$\supseteq$" follows directly from Lemma 16 and the Projection Theorem 14. For "$\subseteq$" let $A$ be a partition in $(\mathrm{co}\mathcal{K}(2m_1+2),\dots,\mathrm{co}\mathcal{K}(2m_{k-1}+2),\cdot)$. For all $j \in \{1,2,\dots,k-1\}$ there exist sets $C^j_i \in \mathcal{K}$ with $C^j_0 \subseteq \dots \subseteq C^j_{2m_j+1}$ such that

$$A_j = C^j_0 \cup \left( \bigcup_{l=1}^{m_j} \left( C^j_{2l} \setminus C^j_{2l-1} \right) \right) \cup \overline{C^j_{2m_j+1}}.$$

Moreover we easily observe that

1. $C_0^j \subseteq C_{2m_r+1}^l$ for all $j, l$,
2. $C_{2m_j+1}^j \cup C_{2m_l+1}^l = M$ for all $j, l$ with $j \neq l$,
3. $C_{2l}^j \setminus C_{2l-1}^j \subseteq C_{2m_i+1}^i$ for all $i, j, l$.

For convenience we set $C_{-1}^j = \emptyset$ and $C_{2m_j+2}^j = M$. Define a mapping $S : B_\beta \to \mathcal{K}$ as

$$S(i_1, \ldots, i_{k-1}) =_{\text{def}} \begin{cases} C_0^j & \text{if } i_j = 0 \text{ and } i_\nu = 1 \text{ for all } \nu \neq j, \\ \left( \bigcup_{j=1}^{k-1} C_0^j \right) \cup \left( \bigcap_{j=1}^{k-1} C_{i_j}^j \right) & \text{otherwise.} \end{cases}$$

Note that $S$ is monotonic. We first have to show that $S$ is a $\mathcal{K}$-homomorphism on $B_\beta$, i.e., that the following is true:

1. $\bigcup_{a \in B_\beta} S(a) = M$,
2. $S(a) \cap S(b) = \bigcup_{c \leq a, c \leq b} S(c)$.

This is shown as follows:

1. We conclude

$$\begin{aligned} \bigcup_{a \in B_\beta} S(a) &= \left( \bigcup_{j=1}^{k-1} C_0^j \right) \cup \bigcup_{j=1}^{k-1} \bigcap_{l=1, l \neq j}^{k-1} C_{2m_l+1}^l \quad = \quad \bigcup_{j=1}^{k-1} \bigcap_{l=1, l \neq j}^{k-1} C_{2m_l+1}^l \\ &= \bigcup_{j=1}^{k-1} \left( C_{2m_j+1}^j \cap \bigcup_{l=1, l \neq j}^{k-1} C_{2m_l+1}^l \right) \quad = \quad \bigcup_{j=1}^{k_1} C_{2m_j+1}^j \quad = \quad M. \end{aligned}$$

2. It is enough to consider three cases for $a, b \in B_\beta$ with $a \neq b$.
   - *Case* $a, b > (2m_1 + 1, \ldots, 2m_{k-1} + 1)$. That means $a, b$ are maximal elements in $B_\beta$. Let $a = (2m_1 + 1, \ldots, 2m_j + 2, \ldots, 2m_{k-1} + 1)$ and let $b = (2m_1 + 1, \ldots, 2m_l + 1, \ldots, 2m_{k-1} + 1)$ with $j \neq l$. Then we have

$$\begin{aligned} S(a) \cap S(b) &= \left( \bigcap_{i=1, i \neq j}^{k-1} C_{2m_i+1}^i \right) \cap \left( \bigcap_{i=1, i \neq l}^{k-1} C_{2m_i+1}^i \right) = \bigcap_{i=1}^{k-1} C_{2m_i+1}^i \\ &= S(2m_1 + 1, \ldots, 2m_{k-1} + 1). \end{aligned}$$

     The latter is just the value of $S$ at the maximal vector less than both $a$ and $b$. So the assertion follows from the monotonicity of $S$.
   - *Case* $a, b < (1, \ldots, 1)$. That is $a, b$ are minimal elements in $B_\beta$. Let $a$ and $b$ be tuples of the form $(1, \ldots, 1, 0, 1, \ldots, 1)$ where in $a$ the zero occurs at position $j$ and in $b$ the zero occurs at position $l$ with $j \neq l$. Note that there exist no tuple in $B_\beta$ less than both $a$ and $b$. So $S(a)$ and $S(b)$ must be disjoint. Indeed, we conclude

$$S(a) \cap S(b) = C_0^j \cap C_0^l = \emptyset.$$

- *Case* $(1,\dots,1) \le a,b \le (2m_1+1,\dots,2m_{k-1}+1)$. Let $a=(i_1,\dots,i_{k-1})$ and let $b=(j_1,\dots,j_{k-1})$. Then it holds

$$S(a)\cap S(b) = \left[\left(\bigcup_{l=1}^{k-1} C_0^l\right) \cup \bigcap_{l=1}^{k-1} C_{i_l}^l\right] \;\cap\; \left[\left(\bigcup_{l=1}^{k-1} C_0^l\right) \cup \bigcap_{l=1}^{k-1} C_{j_l}^l\right]$$
$$= \left(\bigcup_{l=1}^{k-1} C_0^l\right) \cup \bigcap_{l=1}^{k-1} C_{\min\{i_l,j_l\}}^l. \tag{7}$$

There are at most two different indexes $l_1, l_2$ such that $\min\{i_{l_\nu}, j_{l_\nu}\}$ is even for $\nu \in \{1,2\}$.

If there exist strictly less than two even minima then the tuple $c=(\min\{i_1,j_1\},\dots,\min\{i_{k-1},j_{k-1}\})$ belongs to $B_\beta$. From Equation (7) we obtain $S(a)\cap S(b)=S(c)$. Since $c$ is the maximal tuple in $B_\beta$ which is less than both $a$ and $b$ the monotonicity of $S$ proves the assertion for this subcase.

If there are exactly two even minima then without loss of generality, let $i_s$ and $j_t$ be the even minima with $s<t$. Note that $i_s, j_t \ge 2$. Since the set differences $C_{i_s}^s \setminus C_{i_s-1}^s$ and $C_{i_t}^t \setminus C_{i_t-1}^t$ are disjoint we get

$$C_{i_s}^s \cap C_{j_t}^t = C_{i_s}^s \cap C_{j_t}^t \cap \left(C_{i_s-1}^s \cup C_{j_t-1}^t\right) = \left(C_{i_s-1}^s \cap C_{j_t}^t\right) \cup \left(C_{i_s}^s \cap C_{j_t-1}^t\right).$$

Using this equality and Equation (7) we easily see that

$$\begin{aligned}
&S(a)\cap S(b)\\
&= S(\min\{i_1,j_1\},\dots,\min\{i_s,j_s\}-1,\dots,\min\{i_t,j_t\},\dots,\min\{i_{k-1},j_{k-1}\})\\
&\quad \cup S(\min\{i_1,j_1\},\dots,\min\{i_s,j_s\},\dots,\min\{i_t,j_t\}-1,\dots,\min\{i_{k-1},j_{k-1}\})
\end{aligned}$$

The tuples from the last two lines have exactly one even minimum each and thus belong to $B_\beta$. Moreover, they are the only maximal tuples in $B_\beta$ less than both $a$ and $b$. Once more the monotonicity of $S$ gives the assertion.

It remains to show that $(\mathfrak{B}_\beta, S) = A$. To do that we first prove the following claim.

**Claim.** For all $s, r_2, \dots, r_{k-1} \in \mathbb{N}$ with $s \ge 1$ and $r_j \le m_j$ for $j \in \{2,3,\dots,k-1\}$ it holds

$$\bigcup_{(i_2,\dots,i_{k-1})=(0,\dots,0)}^{(r_2,\dots,r_{k-1})} T_S(2s, 2i_2+1,\dots,2i_{k-1}+1) = \left(C_{2s}^1 \setminus C_{2s-1}^1\right) \cap \bigcap_{j=2}^{k-1} C_{2r_j+1}^j.$$

*Proof of the claim.* (Induction on $(r_2,\dots,r_{k-1})$)

- Let $(r_2,\dots,r_{k-1})=(0,\dots,0)$. Then we conclude

$$
\begin{aligned}
&T_S(2s,1,\dots,1)\\
&=S(2s,1,\dots,1)\setminus S(2s-1,1,\dots,1)\\
&=\left[\left(\bigcup_{j=1}^{k-1}C_0^j\right)\cup\left(C_{2s}^1\cap\bigcap_{j=2}^{k-1}C_1^j\right)\right]\setminus\left[\left(\bigcup_{j=1}^{k-1}C_0^j\right)\cup\left(C_{2s-1}^1\cap\bigcap_{j=2}^{k-1}C_1^j\right)\right]\\
&=\overline{\left(\bigcup_{j=1}^{k-1}C_0^j\right)}\cap(C_{2s}^1\setminus C_{2s-1}^1)\cap\bigcap_{j=2}^{k-1}C_1^j \quad = \quad (C_{2s}^1\setminus C_{2s-1}^1)\cap\bigcap_{j=2}^{k-1}C_1^j
\end{aligned}
$$

This shows the base of induction.
- Assume the claim holds for $(r_2,\dots,r_t,\dots,r_{k-1})$. Consider $(r_2,\dots,r_t+1,\dots,r_{k-1})$. We only have something to show if $r_t+1\le m_t$. For convenience, we may assume that $t=2$. Set $C_{-1}^j=\emptyset$. First we calculate $T_S(2s,2r_3+3,2i_3+1,\dots,2i_{k-1}+1)$ with $(i_3,\dots,i_{k-1})\le(r_3,\dots,r_{k-1})$.

$$T_S(2s,2r_3+3,2i_3+1,\dots,2i_{k-1}+1)=D_1\cap D_2\cap D_3 \tag{8}$$

where the sets $D_1$, $D_2$, and $D_3$ are given as

$$
\begin{aligned}
D_1&=S(2s,2r_2+3,2i_3+1,\dots,2i_{k-1}+1)\\
&\qquad\qquad\qquad\setminus\ S(2s-1,2r_2+3,2i_3+1,2i_4+1,\dots,2i_{k-1}+1),\\
D_2&=S(2s,2r_2+3,2i_3+1,\dots,2i_{k-1}+1)\\
&\qquad\qquad\qquad\setminus\ S(2s,2r_2+1,2i_3+1,2i_4+1,\dots,2i_{k-1}+1),\\
D_3&=S(2s,2r_2+3,2i_3+1,\dots,2i_{k-1}+1)\\
&\qquad\setminus\big(\quad S(2s,2r_2+3,2i_3-1,2i_4+1,\dots,2i_{k-1}+1)\\
&\qquad\quad\cup\ S(2s,2r_2+3,2i_3+1,2i_4-1,\dots,2i_{k-1}+1)\\
&\qquad\quad\ \ \vdots\\
&\qquad\quad\cup\ S(2s,2r_2+3,2i_3+1,2i_4+1,\dots,2i_{k-1}-1)\ \big).
\end{aligned}
$$

For these sets we obtain

$$D_1=\overline{\left(\bigcup_{j=1}^{k-1}C_0^j\right)}\cap\left(C_{2s}^1\setminus C_{2s-1}^1\right)\cap C_{2r_2+3}^2\cap\bigcap_{j=3}^{k-1}C_{2i_j+1}^j,$$

$$D_2=\overline{\left(\bigcup_{j=1}^{k-1}C_0^j\right)}\cap C_{2s}^1\cap\left(C_{2r_2+3}^2\setminus C_{2r_2+1}^2\right)\cap\bigcap_{j=3}^{k-1}C_{2i_j+1}^j,$$

$$D_3=\overline{\left(\bigcup_{j=1}^{k-1}C_0^j\right)}\cap C_{2s}^1\cap C_{2r_2+3}^2\cap\bigcap_{j=3}^{k-1}\left[(C_{2i_j+1}^j\setminus C_{2i_j-1}^j)\cap\bigcap_{l=3,l\neq j}^{k-1}C_{2i_l+1}^l\right].$$

Putting these identities in Equation (8) and simplifying the terms it is seen that

$$
\begin{aligned}
&T_S(2s, 2r_3+3, 2i_3+1, \ldots, 2i_{k-1}+1) \\
&= \left(C^1_{2s} \setminus C^1_{2s-1}\right) \cap \left(C^2_{2r_2+3} \setminus C^2_{2r_2+1}\right) \cap \bigcap_{j=3}^{k-1} \left(C^j_{2i_j+1} \setminus C^j_{2i_j-1}\right).
\end{aligned}
$$

Using this equality and the assumption of induction we finally calculate

$$
\begin{aligned}
&\bigcup_{(i_2,i_3,\ldots,i_{k-1})=(0,0,\ldots,0)}^{(r_2+1,r_3\ldots,r_{k-1})} T_S(2s, 2i_2+1, \ldots, 2i_{k-1}+1) \\
&= \left( \bigcup_{(i_3,\ldots,i_{k-1})=(0,\ldots,0)}^{(r_3,\ldots,r_{k-1})} T_S(2s, 2r_2+3, 2i_3+1, \ldots, 2i_{k-1}+1) \right) \\
&\qquad \cup \left( \bigcup_{(i_2,i_3\ldots,i_{k-1})=(0,0,\ldots,0)}^{(r_2,r_3,\ldots,r_{k-1})} T_S(2s, 2i_2+1, 2i_3+1, \ldots, 2i_{k-1}+1) \right) \\
&= \left( \left(C^1_{2s} \setminus C^1_{2s-1}\right) \cap \left(C^2_{2r_2+3} \setminus C^2_{2r_2+1}\right) \cap \bigcup_{(i_3,\ldots,i_{k-1})=(0,\ldots,0)}^{(r_3,\ldots,r_{k-1})} \bigcap_{j=3}^{k-1} \left(C^j_{2i_j+1} \setminus C^j_{2i_j-1}\right) \right) \\
&\qquad \cup \left( \left(C^1_{2s} \setminus C^1_{2s-1}\right) \cap \bigcap_{j=2}^{k-1} C^j_{2r_j+1} \right) \\
&= \left( \left(C^1_{2s} \setminus C^1_{2s-1}\right) \cap \left(C^2_{2r_2+3} \setminus C^2_{2r_2+1}\right) \cap \bigcap_{j=3}^{k-1} C^j_{2r_j+1} \right) \\
&\qquad \cup \left( \left(C^1_{2s} \setminus C^1_{2s-1}\right) \cap C^2_{2r_2+1} \cap \bigcap_{j=3}^{k-1} C^j_{2r_j+1} \right) \\
&= \left(C^1_{2s} \setminus C^1_{2s-1}\right) \cap C^2_{2r_2+3} \cap \bigcap_{j=3}^{k-1} C^j_{2r_j+1}.
\end{aligned}
$$

Hence, the claim is proven.

As an immediate consequence of the claim we get that for $s \in \mathbb{N}_+$ the following is true:

$$
\bigcup_{(i_2,\ldots,i_{k-1})=(0,\ldots,0)}^{(m_2,\ldots,m_{k-1})} T_S(2s, 2i_2+1, \ldots, 2i_{k-1}+1) = C^1_{2s} \setminus C^1_{2s-1}. \tag{9}
$$

In fact, Equation (9) has been the intention why we have proven the claim at all.

Now we determine the first component of the partition $(\mathfrak{B}_\beta, S)$. Consider the set of all tuples $(j_1, j_2, \ldots, j_{k-1}) \in B_\beta$ satisfying that $j_1 > 0$ is even and $j_2, \ldots, j_{k-1}$ are odd, i.e., all tuples $(j_1, j_2, \ldots, j_{k-1})$ such that there exist $i_1 \in \mathbb{N}_+$, $i_2, \ldots, i_{k-1} \in \mathbb{N}$ with $i_l \le m_l$ for all $l \in \{1, 2, \ldots, k-1\}$, $j_1 = 2i_1$, and $j_l = 2i_l + 1$ for all $l \in \{2, 3, \ldots, k-1\}$. With Equation (9) in mind we then conclude

$$
\begin{aligned}
&(\mathfrak{B}_\beta, S)_1 \\
&= S(0,1,\ldots,1) \cup \left( \bigcup_{i=1}^{m_1} \bigcup_{(j_2,\ldots,j_{k-1})=(0,\ldots,0)}^{(m_2,\ldots,m_{k-1})} T_S(2i, 2j_2+1, \ldots, 2j_{k-1}+1) \right) \\
&\qquad \cup \ \ (S(2m_1+3, 2m_2+1, \ldots, 2m_{k-1}+1) \setminus S(2m_1+1, 2m_2+1, \ldots, 2m_{k-1}+1)) \\
&= C_0^j \cup \left( \bigcup_{i=1}^{m_1} (C_{2i}^1 \setminus C_{2i-1}^1) \right) \cup \left[ \left( \bigcap_{i=2}^{k-1} C_{2m_i-1}^i \right) \setminus \bigcap_{i=1}^{k-1} C_{2m_i+1}^i \right] \\
&= C_0^j \cup \left( \bigcup_{i=1}^{m_1} (C_{2i}^1 \setminus C_{2i-1}^1) \right) \cup \left( \overline{C_{2m_1+1}^1} \cap \bigcap_{i=2}^{k-1} C_{2m_i+1}^i \right) \\
&= C_0^j \cup \left( \bigcup_{i=1}^{m_1} (C_{2i}^1 \setminus C_{2i-1}^1) \right) \cup \overline{C_{2m_1+1}^1} \quad = \quad A_1.
\end{aligned}
$$

Completely similar to this (also including the claim) one can show the equality for all other components $j \in \{2, 3, \ldots, k-1\}$. Hence, $(\mathfrak{B}_\beta, S) = A$. Consequently, $A \in \mathcal{K}(\mathfrak{B}_\beta)$.

Thus, the first step of the proof of our theorem is completely proven, i.e., the statement of the theorem is true for partition classes $\mathcal{C}$ and boolean characteristics $\beta$ with $\beta_i < 0$ and $\beta$ is even. Validity of the theorem for other partition classes result from slight modifications. For instance, if $\beta_i > 0$ and $\beta_i$ is even then by Lemma 16 and the Projection Theorem $\mathcal{K}(\mathfrak{B}_\beta)_i \subseteq \mathcal{K}(\beta_i)$. Conversely, it is enough to observe that for $A$ with $A_i \in \mathcal{K}(2m_i)$ with $\beta_i = 2m_i$ we can simply set $C_0^i = \emptyset$ and $C_{2m_i+2}^i = M$ in the proof of "$\supseteq$" above. All other cases can be shown analogously. ❑

As a consequence of this theorem we conclude:

**Corollary 19.** *$\mathcal{K}(\mathfrak{B}_r)$ is projectively closed for all $n$-tuples $r$ of integers.*

Moreover, Theorem 18 give us a method to figure out labeled posets for each projectively closed class. As an example, the partition class $(\mathrm{co}\mathcal{K}(4), \mathrm{co}\mathcal{K}(5), \cdot)$ can be described by the 3-poset pictured in Figure 5.

## 5 Partition Classes Given in Bound Representations

In the previous subsection we have completely solved the problem of characterizing the freely represented $k$-partition classes that are explicitly given by $k-1$ components of the boolean hierarchy of sets over $\mathcal{K}$. Now we are going to solve the same problem for boundly represented partition classes. The key for this is already given by Theorem 18 because we clearly have the following bridge between freely and boundly represented partition classes.

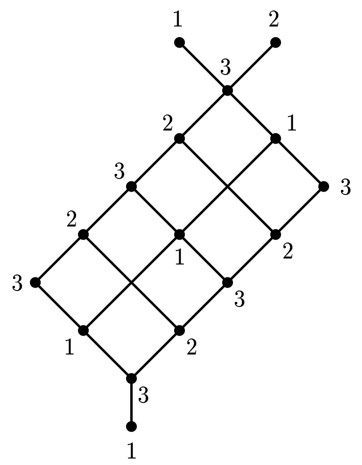


**Fig. 5.** The 3-poset generating $(\mathrm{co}\mathcal{K}(4), \mathrm{co}\mathcal{K}(5), \cdot)$

**Proposition 20.** *Let $(\mathcal{C}_1, \ldots, \mathcal{C}_{k-1}, \cdot)$ be any partition class and let $(G, f)$ be any $k$-poset such that $\mathcal{K}(G, f) = (\mathcal{C}_1, \ldots, \mathcal{C}_{k-1}, \cdot)$. Then it holds that $(\mathcal{C}_1, \ldots, \mathcal{C}_{k-1}, \{\emptyset\}) = \mathcal{K}(G', f')$ with $G' = G \setminus \{a \in G \mid f(a) = k\}$.*

From this proposition we easily conclude

**Theorem 21.** *Let $\mathcal{C} = (\mathcal{C}_1, \ldots, \mathcal{C}_k)$ be any class of $k$-partitions that can be boundly represented by classes from the boolean hierarchy of sets over $\mathcal{K}$. Let $\beta$ be any boolean characteristic of the corresponding freely represented class $(\mathcal{C}_1, \ldots, \mathcal{C}_k, \cdot)$ with respect to $\mathcal{K}$. Then $\mathcal{C} = \mathcal{K}(\mathfrak{B}'_\beta)$ where $\mathfrak{B}'_\beta$ is the $k$-poset that emerges from the $(k+1)$-poset $\mathfrak{B}_\beta$ by eliminating all elements having label $k+1$.*

*Proof.* Follows from Theorem 18 and Proposition 20 via

$$(A_1, \ldots, A_k) \in (\mathcal{C}_1, \ldots, \mathcal{C}_k) \Longleftrightarrow (A_1, \ldots, A_k, \emptyset) \in (\mathcal{C}_1, \ldots, \mathcal{C}_k, \cdot)$$

❑

Applying this theorem leads directly to the right 4-poset in Fig. 1 as a representation of the class $(\mathcal{K}, \mathcal{K}(2), \mathcal{K}(2), \mathrm{co}\mathcal{K})$.

We should note that the $k$-posets we obtain from Theorem 21 are, in general, not minimal $k$-posets opposite to those constructed in Theorem 18. So after calculating the $k$-poset for a given boundly represented classes of $k$-partitions we have to minimize the labeled poset. Remind that there exist a up to isomorphy unique minimal equivalent $k$-poset. As an example, the $k$-poset in Figure 6 is the minimal $k$-poset which is equivalent to the $k$-poset we obtain from the theorem above to represent $(\mathcal{K}(2), \mathcal{K}(2), \mathcal{K}(2))$. Note that $(\mathcal{K}(2), \mathcal{K}(2), \mathcal{K}(2), \cdot) = \mathcal{K}(\mathfrak{B}_{2,2,2})$ (see Figure 4).

Theorem 21 has an interesting consequence for the boolean hierarchy of sets. We can describe each class $\mathcal{K}(m) \cap \mathrm{co}\mathcal{K}(m)$ as a class generated by labeled posets. For $m \in \mathbb{N}_+$ let $\mathfrak{H}_m$ be the 2-poset presented in Figure 7.

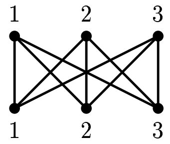

**Fig. 6.** The 3-poset generating $(\mathcal{K}(2), \mathcal{K}(2), \mathcal{K}(2))$

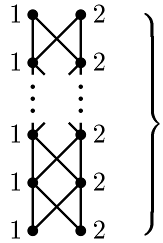


**Fig. 7.** The 2-poset $\mathfrak{H}_m$ for $m \in \mathbb{N}_+$

**Corollary 22.** *$\mathcal{K}(m) \cap \mathrm{co}\mathcal{K}(m) = \mathcal{K}(\mathfrak{H}_m)$ for all $m \in \mathbb{N}_+$.*

*Proof.* Note that $\mathcal{K}(m) \cap \mathrm{co}\mathcal{K}(m) = (\mathcal{K}(m), \mathcal{K}(m))_1$. According to Theorem 21 we first have to calculate the 3-poset that describes $(\mathcal{K}(m), \mathcal{K}(m), \cdot)$. This is the $k$-poset $\hat{\mathfrak{G}}_{m,m}$ from Theorem 18. Then we eliminate all elements labeled with 3. This yields the set

$$G = \begin{cases} \{\ (i,j) \mid i,j \in \{0,1,\dots,2m+1\} \text{ and either } i \text{ or } j \text{ is odd }\} & \text{if } m \text{ is odd,} \\ \{\ (i,j) \mid i,j \in \{1,\dots,2m+1\} \text{ and either } i \text{ or } j \text{ is odd }\} & \text{if } m \text{ is even.} \end{cases}$$

Thus $(G, f_m)$ generates $(\mathcal{K}(m), \mathcal{K}(m))$. Now consider the set $G'$ defined as

$$G' =_{\mathrm{def}} \begin{cases} \{\ (i,j) \mid i,j \in \{0,1,\dots,2m+1\} \wedge (i = j+1 \vee j = i+1)\ \} & \text{if } m \text{ is odd,} \\ \{\ (i,j) \mid i,j \in \{1,\dots,2m+1\} \wedge (i = j+1 \vee j = i+1)\ \} & \text{if } m \text{ is even.} \end{cases}$$

Note that $G'$ consists only of pairs $(i,j)$ with $i \neq j$. It holds that $(G, f_m) \equiv (G', f_m)$ as can be seen as follows: $(G', f_m) \leq (G, f_m)$ is witnessed by the identity function and $(G, f_m) \leq (G', f_m)$ is witnessed by the following monotonic function $\varphi : G \to G'$

$$\varphi(i,j) =_{\mathrm{def}} \begin{cases} (i, i-1) & \text{if } i > j, \\ (j-1, j) & \text{if } i < j. \end{cases}$$

But $(G', f_m)$ is isomorphic to $\mathfrak{H}_m$. By the Embedding Lemma, $(\mathcal{K}(m), \mathcal{K}(m)) = \mathcal{K}(\mathfrak{H}_m)$. ❑

## 6 Fine Hierarchies inside $\mathbf{BH_2(NP)}$

The approach to investigate partition classes generated by $k$-posets leads in the case that $k = 2$ to a much shrewder structuring of the boolean hierarchy of NP sets. For instance,

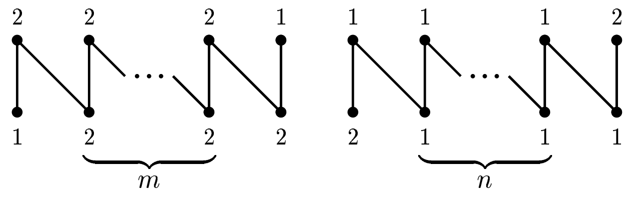


**Fig. 8.** The 2-poset $\mathfrak{F}(m,n)$

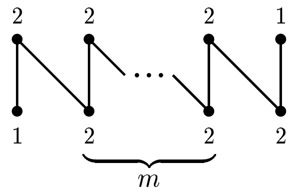


**Fig. 9.** The 2-poset $\mathfrak{F}(m,\infty)$

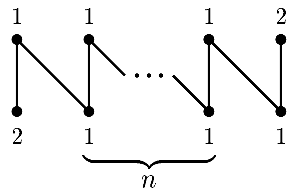


**Fig. 10.** The 2-poset $\mathfrak{F}(\infty,n)$

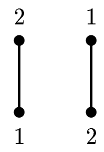


**Fig. 11.** The 2-poset $\mathfrak{F}(\infty,\infty)$

consider the following well-known inclusion chain of the lowest non-trivial levels of the boolean hierarchy $\mathrm{BH}_2(\mathrm{NP})$

$$\mathrm{NP} \cup \mathrm{coNP} \;\subseteq\; \mathrm{P}_{\|}^{\mathrm{NP}}[1] \;\overset{*}{\subseteq}\; \mathrm{NP}(2) \cap \mathrm{coNP}(2) \;\subseteq\; \mathrm{NP}(2) \cup \mathrm{coNP}(2) \;\subseteq\; \mathrm{P}_{\|}^{\mathrm{NP}}[2].$$

For each of the inclusions except the one marked by $*$ it holds that the inclusion is strict unless the polynomial-time hierarchy collapses. More specifically, it holds that

1. $\mathrm{NP} \cup \mathrm{coNP} = \mathrm{P}_{\|}^{\mathrm{NP}}[1]$ implies $\mathrm{NP} = \mathrm{coNP}$,
2. $\mathrm{NP}(2) \cap \mathrm{coNP}(2) = \mathrm{NP}(2) \cup \mathrm{coNP}(2)$ implies $\mathrm{NP}(2) = \mathrm{coNP}(2)$,
3. $\mathrm{NP}(2) \cup \mathrm{coNP}(2) = \mathrm{P}_{\|}^{\mathrm{NP}}[2]$ implies $\mathrm{NP}(2) = \mathrm{coNP}(2)$.

The second statement is obvious. The first statement and third statement are easily seen since for each $r \in \mathbb{N}$ the class $\mathrm{P}_{\|}^{\mathrm{NP}}[r]$ has a $\leq_m^p$-complete problem.

Only for the inclusion $*$ it is not known whether equality collapses the polynomial-time hierarchy. The difference between these two classes seems apparently too fine to exploit them

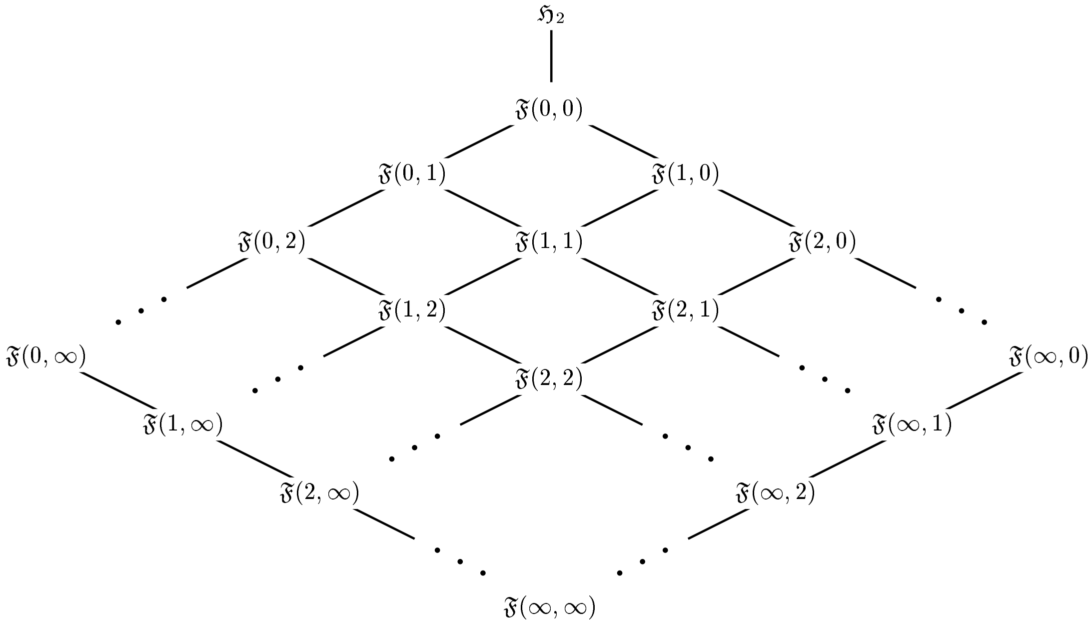


**Fig. 12.** The 2-posets generating classes between $(\mathrm{NP} \cap \mathrm{coNP}) \oplus \mathrm{NP}$ and $\mathrm{NP}(2) \cap \mathrm{coNP}(2)$

regarding such collapse consequences. However, the difference is large enough to contain probably infinite subhierarchies.

The following theorem is easy to conclude from the results we mentioned in the preliminary section. Note that $\mathrm{P}_{||}^{\mathrm{NP}}[1] = \mathrm{P} \oplus \mathrm{NP}$ [Wag98] and that $\mathfrak{F}(\infty,\infty) = \mathfrak{D}_2$ (see Figure 2).

**Theorem 23.** *Let $m, n \in \mathbb{N} \cup \{\infty\}$ and let $\mathfrak{F}(m,n)$ be the 2-posets presented in Figures 8-11.*

1. $\mathrm{NP}(\mathfrak{F}(0,0)) \subseteq \mathrm{NP}(\mathfrak{H}_2) = \mathrm{NP}(2) \cap \mathrm{coNP}(2)$.
2. $\mathrm{NP}^C(\mathfrak{F}(m,n)) \subseteq \mathrm{NP}^C(\mathfrak{F}(m',n'))$ *for all oracles $C$ if and only if $m \leq m'$ and $n \leq n'$.*
3. $\mathrm{P}_{||}^{\mathrm{NP}}[1] \subseteq \mathrm{NP}(\mathfrak{F}(\infty,\infty)) = (\mathrm{NP} \cap \mathrm{coNP}) \oplus \mathrm{NP}$.

The class $(\mathrm{NP} \cap \mathrm{coNP}) \oplus \mathrm{NP}$ is astonishingly robust in a sense that it has many different characterizations (see [Wag98,HHH98]). It can be considered as:

1. The class generated by $\mathfrak{D}_2 = \mathfrak{F}(\infty,\infty)$ over NP.
2. The class of all sets that are 1-truth-table strong nondeterministic reducible to some NP set.
3. The class of all sets that can be decided by deterministic polynomial-time machine making one query to an oracle from $\mathrm{NP} \cap \mathrm{coNP}$ followed by one query to some NP set (and the other way round).
4. The class of all sets that can be decided by deterministic polynomial-time machine making, simultaneously, one query to an oracle from $\mathrm{NP} \cap \mathrm{coNP}$ and one query to some NP.

Figure 12 shows the structure of all complexity classes generated by posets between $\mathrm{P}_{||}^{\mathrm{NP}}[1]$ and $\mathrm{NP}(2) \cap \mathrm{coNP}(2)$. Each of the classes described by their defining posets is different to another class in at least one relativized world.

Similar and still more complicated subhierarchies as those we considered here can be observed in higher levels of the boolean hierarchy of sets over NP.

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