On NP-Partitions over Posets with an Application to Reducing the Set of Solutions of NP Problems

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Abstract. The boolean hierarchy of k-partitions over NP for $k \geq 2$ was introduced as a generalization of the well-known boolean hierarchy of sets. The classes of this hierarchy are exactly those classes of NP-partitions which are generated by finite labeled lattices.
We extend the boolean hierarchy of NP-partitions by considering partition classes which are generated by finite labeled posets. Since we cannot prove it absolutely, we collect evidence for this extended boolean hierarchy to be strict. We give an exhaustive answer to the question of which relativizable inclusions between partition classes can occur depending on the relation between their defining posets.
The study of the extended boolean hierarchy is closely related to the issue of whether one can reduce the number of solutions of NP problems. For finite cardinality types, assuming the extended boolean hierarchy of k-partitions over NP is strict, we give a complete characterization when such solution reductions are possible.

1 Introduction

Complexity theory usually investigates partitions into two parts (sets) or into infinitely many parts (functions). Starting a study of intermediate partitions into a finite number of parts (say k-partitions), Wagner and the present author [KW00] introduced the boolean hierarchy $\mathrm{BH}_k(\mathrm{NP})$ of k-partitions over NP for $k \geq 2$ as a generalization of the well-studied boolean hierarchy of NP sets. Whereas the latter hierarchy has a very clear structure with no more than two incomparable classes, for $k \geq 3$ the situation turned out be much more opaque: Already the hierarchy $\mathrm{BH}_3(\mathrm{NP})$ does not have bounded width with respect to set inclusion unless the polynomial-time hierarchy collapses.

The hierarchy $\mathrm{BH}_k(\mathrm{NP})$ is defined to consist of all classes $\mathrm{NP}(f)$ for functions $f : \{1,2\}^m \to \{1,2,\ldots,k\}$ with $m \geq 1$, that are given in the following way: Let the characteristic function of a k-partition $A = (A_1,\ldots,A_k)$ be defined as $c_A(x) = i$ if and only if $x \in A_i$. (A set B is to be identified with the 2-partition $(B,\overline{B})$). For a function f and sets $B_1,\ldots,B_m \in \mathrm{NP}$, the k-partition $A = f(B_1,\ldots,B_m)$ is defined by $c_A(x) = f(c_{B_1}(x),\ldots,c_{B_m}(x))$. Then, a class

$\mathrm{NP}(f)$ is given as $\mathrm{NP}(f) =_{\mathrm{def}} \{f(B_1, \ldots, B_m) \mid B_1, \ldots, B_m \in \mathrm{NP}\}$. The boolean hierarchy of sets appears as the case $k = 2$.

It was shown that $\mathrm{BH}_k(\mathrm{NP})$ coincides with the family of all partition classes $\mathrm{NP}(G, f)$ defined by arbitrary finite lattices G with labeling functions $f : G \to \{1, 2, \ldots, k\}$, henceforth refered to as k-lattices. This approach is very useful since frequently k-lattices are smaller descriptions of classes in $\mathrm{BH}_k(\mathrm{NP})$ than functions.

The structure inside $\mathrm{BH}_k(\mathrm{NP})$ is induced by a relation $\leq$ on k-lattices: For k-lattices (G, f) and (G', f'), let $(G, f) \leq (G', f')$ if there is a monotonic function $\varphi : G \to G'$ with $f(x) = f'(\varphi(x))$ for all $x \in G$. It holds that if $(G, f) \leq (G', f')$ then $\mathrm{NP}(G, f) \subseteq \mathrm{NP}(G', f')$. For several good reasons, it is conjectured that the converse direction is also true. More specifically, it is conjectured that $\mathrm{BH}_k(\mathrm{NP})$ is strict, i.e., $\mathrm{NP}(G, f) \not\subseteq \mathrm{NP}(G', f')$ whenever $(G, f) \not\leq (G', f')$, unless the polynomial-time hierarchy collapses.

In this paper, we extend the work presented in [KW00] to the most general case of finite labeled posets (k-posets) by a slight modification of the lattice approach. In Sect. 3 this leads us to the extended boolean hierarchy $\mathrm{EBH}_k(\mathrm{NP})$ of k-partition over NP which can be built according to the same relation $\leq$. Observe that the relation $\leq$ does not really require k-lattices. Thus, $\mathrm{BH}_k(\mathrm{NP})$ appears as a subhierarchy of $\mathrm{EBH}_k(\mathrm{NP})$.

In Sect. 4 we describe an alternative approach to $\mathrm{EBH}_k(\mathrm{NP})$ in terms of partial finite functions. Using this approach, in Sect. 5 we prove the strictness of $\mathrm{EBH}_k(\mathrm{NP})$ regarding relativizable inclusions: $(G, f) \leq (G', f')$ if and only if *relativizably* $\mathrm{NP}(G, f) \subseteq \mathrm{NP}(G', f')$, for all k-posets (G, f) and (G', f'). Certainly, this theorem provides not very strong evidence for the unrelativized strictness of $\mathrm{EBH}_k(\mathrm{NP})$ since the status of relativizations itself is questionable. But it shows that a (unrelativized) counterexample to the strictness must use the very rare non-relativizable proof techniques, hence would be very surprising.

Our main motivation for examining $\mathrm{EBH}_k(\mathrm{NP})$ is its close relation to the issue of whether one can reduce the set of solutions of NP problems. Building on previous work in [BLS84,BLS85,HNOS96,Ogi96,NRRS98], this issue has taken its ultimate shape in a work of Hemaspaandra, Ogihara, and Wechsung [HOW00].

For $A \subseteq \mathbb{N}_+$, let $\mathrm{NP}_A\mathrm{V}$ consist of all (possibly partial, possibly multivalued) functions f computed by an NP machine so that $f(x)$ is the set of all outputs ("solutions") on accepting paths and the number of outputs lies in $\{0\} \cup A$.

A (multivalued) function f' is a refinement of a (multivalued) function f iff the domains[1] of both functions are equal and $f'(x) \subseteq f(x)$ for all x. This notion induces a relation $\subseteq_c$ on function classes by saying that $\mathcal{F} \subseteq_c \mathcal{F}'$ iff each function in $\mathcal{F}$ has a refinement in $\mathcal{F}'$.

For instance, one could ask whether it is possible to construct an NP machine that always bounds, for a given satisfiable propositional formula, the cardinality of its output set of satisfying assignments to one. Such a machine reduces the set of solutions of the satisfiability problem to a singleton. The question for

[1] A multivalued function f is not defined on x iff $f(x) = \varnothing$.

this machine translates to the more formal question "$\mathrm{NP}_{\mathbb{N}_+}\mathrm{V} \subseteq_c \mathrm{NP}_{\{1\}}\mathrm{V}$?" for which it was shown by Hemaspaandra, Naik, Ogihara, and Selman [HNOS96] that the answer is negative if the polynomial-time hierarchy does not collapse.

As natural the challenge is: "Completely characterize, perhaps under some complexity-theoretic assumptions, the sets $A \subseteq \mathbb{N}_+$ and $B \subseteq \mathbb{N}_+$ such that $\mathrm{NP}_A\mathrm{V} \subseteq_c \mathrm{NP}_B\mathrm{V}$"[HOW00]. And Hemaspaandra *et al.* [HOW00] gave furthermore a sufficient condition (the "narrowing gap condition") for the case of finite sets, and they pronounced the conjecture, supported with strong theorems, that this condition is also necessary unless the polynomial-time hierarchy collapses.

In Sect. 6 we prove that the narrowing-gap condition for sets A and B is sufficient and necessary for $\mathrm{NP}_A\mathrm{V} \subseteq_c \mathrm{NP}_B\mathrm{V}$ unless for a certain k, some classes in $\mathrm{EBH}_k(\mathrm{NP})$ unexpectedly collapse.

2 Preliminaries

For classes $\mathcal{K}$ and $\mathcal{K}'$ of subsets of a set M, define $\mathrm{co}\mathcal{K} = \{\overline{L} \mid L \in \mathcal{K}\}$, $\mathcal{K}\wedge\mathcal{K}' =_{\mathrm{def}} \{A \cap B \mid A \in \mathcal{K}, B \in \mathcal{K}'\}$, and $\mathcal{K} \oplus \mathcal{K}' =_{\mathrm{def}} \{A\triangle B \mid A \in \mathcal{K}, B \in \mathcal{K}'\}$ where $A\triangle B$ denotes the symmetric difference of A and B. For a set A, $\|A\|$ denotes its cardinality.

The classes $\mathcal{K}(i)$ and $\mathrm{co}\mathcal{K}(i)$ defined by $\mathcal{K}(1) =_{\mathrm{def}} \mathcal{K}$ and $\mathcal{K}(i{+}1) =_{\mathrm{def}} \mathcal{K}(i)\oplus\mathcal{K}$ for $i \geq 1$ build the *boolean hierarchy over* $\mathcal{K}$ that has many equivalent definitions (see [WW85,CH86,KSW87,CGH+88] or the case $k = 2$ in Definition 1). The class $\mathbf{BC}(\mathcal{K})$ is the boolean closure of $\mathcal{K}$, that is the smallest class which contains $\mathcal{K}$ and which is closed under intersection, union, and complementation.

Let us make some notational conventions about partitions. For any set M, a k-tuple $A = (A_1, \ldots, A_k)$ is a *k-partition of M* iff $A_1 \cup A_2 \cup \cdots \cup A_k = M$ and $A_i \cap A_j = \varnothing$ if $i \neq j$. The set A_i is called the *i-th component of* A. Let $c_A : M \to \{1, 2, \ldots, k\}$ be the characteristic function of a k-partition $A = (A_1, \ldots, A_k)$ of M, i.e., $c_A(x) = i$ if and only if $x \in A_i$. For $\mathcal{K}_1, \ldots, \mathcal{K}_k \subseteq \mathcal{P}(M)$, we define

$$(\mathcal{K}_1, \ldots, \mathcal{K}_k) =_{\mathrm{def}} \{A \mid A \text{ is a } k\text{-partition of } M \text{ and } A_i \in \mathcal{K}_i \text{ for } 1 \leq i \leq k\}$$

and for $1 \leq i \leq k$,

$$(\mathcal{K}_1, \ldots, \mathcal{K}_{i-1}, \cdot, \mathcal{K}_{i+1}, \ldots, \mathcal{K}_k) =_{\mathrm{def}} (\mathcal{K}_1, \ldots, \mathcal{K}_{i-1}, \mathcal{P}(M), \mathcal{K}_{i+1}, \ldots, \mathcal{K}_k).$$

For a class $\mathcal{K}$ of k-partitions, let $\mathcal{K}_i =_{\mathrm{def}} \{A_i \mid A \in \mathcal{K}\}$ be the *i-th projection of* $\mathcal{K}$. Obviously, $\mathcal{K} \subseteq (\mathcal{K}_1, \ldots, \mathcal{K}_k)$. In the forthcoming we identify a set A with the 2-partition $(A, \overline{A})$, and we identify a class $\mathcal{K}$ of sets with the class $(\mathcal{K}, \mathrm{co}\mathcal{K}) = (\mathcal{K}, \cdot) = (\cdot, \mathrm{co}\mathcal{K})$ of 2-partitions.

We recall the definition of the boolean hierarchy of k-partitions over $\mathcal{K}$ for $k \geq 2$ as introduced in [KW00].

Definition 1. *Let $k \geq 2$.*

1. *For $f : \{1,2\}^m \to \{1, \ldots, k\}$ and for sets $B_1, \ldots, B_m \in \mathcal{K}$, the k-partition $f(B_1, \ldots, B_m)$ is defined by $c_{f(B_1,\ldots,B_m)}(x) = f(c_{B_1}(x), \ldots, c_{B_m}(x))$.*

2. *For $f : \{1,2\}^m \to \{1,\ldots,k\}$, the* class of k-partitions over $\mathcal{K}$ defined by f *is given by the class $\mathcal{K}(f) =_{\mathrm{def}} \{f(B_1,\ldots,B_m) \mid B_1,\ldots,B_m \in \mathcal{K}\}$.*
3. *The family* $\mathrm{BH}_k(\mathcal{K}) =_{\mathrm{def}} \{\mathcal{K}(f) \mid f : \{1,2\}^m \to \{1,2,\ldots,k\}\}$ *is the* boolean hierarchy of k-partitions over $\mathcal{K}$*.*
4. $\mathbf{BC}_k(\mathcal{K}) =_{\mathrm{def}} \bigcup \mathrm{BH}_k(\mathcal{K})$*.*

We mention some notions from lattice theory and order theory (see e.g., [Grä78]). A finite poset $(G,\leq)$ is a lattice if for all $x,y \in G$ there exist exactly one maximal element $z \in G$ such that $z \leq x$ and $z \leq y$ (which will be denoted by $x \wedge y$), and exactly one minimal element $z \in G$ such that $z \geq x$ and $z \geq y$ (which will be denoted by $x \vee y$). For a lattice G we denote by 1_G the unique element greater than or equal to all $x \in G$.

For symbols $a_1, a_2, \ldots, a_m$ from a finite alphabet Σ, we identify the m-tuple $(a_1, a_2, \ldots, a_m)$ with the word $a_1a_2\ldots a_m \in \Sigma^m$. We denote the length of a word w by $|w|$ and for a letter $a \in \Sigma$, the number of occurrence of the letter a in w by $|w|_a$. If there is an order $\leq$ on Σ, we assume Σ^m to be partially ordered by the vector-ordering, that is $a_1a_2\ldots a_m \leq b_1b_2\ldots b_m$ if and only if $a_i \leq b_i$ for all $i \in \{1,2,\ldots,m\}$. The lexicographical ordering on Σ^m is denoted by $\leq_{\mathrm{lex}}$.

The domain of any function f is denoted by D_f.

3 Partition Classes Defined by Posets

In [KW00] the boolean hierarchy of k-partitions over a set class $\mathcal{K}$ was precisely characterized by partition classes defined by finite labeled lattices. In this section we will generalize this approach to the cases of arbitrary posets rather than lattices. We follow completely the line offered in [KW00].

Let $\mathcal{K}$ be a class of subsets of M such that $\varnothing, M \in \mathcal{K}$ and $\mathcal{K}$ is closed under union and intersection.

Definition 2. *Let G be a poset.*

1. *A mapping $S : G \to \mathcal{K}$ is said to be a* $\mathcal{K}$-homomorphism on G *if and only if $\bigcup_{a\in G} S(a) = M$ and $S(a) \cap S(b) = \bigcup_{c\leq a, c\leq b} S(c)$ for all $a,b \in G$.*
2. *For any $\mathcal{K}$-homomorphism S on G, let $T_S(a) =_{\mathrm{def}} S(a) \setminus \bigcup_{b<a} S(b)$ for all $a \in G$.*

Note that in the case of a lattice G the above notion of $\mathcal{K}$-homomorphism on G coincides with the one made for lattices, i.e., $S(1_G) = M$ and $S(a) \cap S(b) = S(a \wedge b)$.

Lemma 1. *Let G be a poset, and let S be a $\mathcal{K}$-homomorphism on G.*

1. *$T_S(a) \in \mathcal{K} \wedge \mathrm{co}\mathcal{K}$ for every $a \in G$.*
2. *If $a \leq b$ then $S(a) \subseteq S(b)$ for every $a,b \in G$.*
3. *$S(a) = \bigcup_{b\leq a} T_S(b)$ for every $a \in G$.*
4. *The set of all $T_S(a)$ for $a \in G$ yields a partition of M.*

Proof. 1. Observe $T_S(a) = S(a) \cap \overline{\bigcup_{b<a} S(b)} \in \mathcal{K} \wedge \mathrm{co}\mathcal{K}$ since $\mathcal{K}$ is closed under union.

2. If $a \leq b$ then $S(b) = S(b) \cap S(b) = \bigcup_{c \leq b} S(c) \supseteq S(a)$.

3. The direction "$\supseteq$" is obvious since $T_S(b) \subseteq S(b) \subseteq S(a)$ for $b \leq a$. The converse inclusion can be verified by induction on $<$. Obviously, $S(a) = T_S(a)$ for minimal $a \in G$. For non-minimal $a \in G$ we obtain $S(a) = T_S(a) \cup \bigcup_{b<a} S(b) = T_S(a) \cup \bigcup_{b<a} \bigcup_{c \leq b} T_S(c) = T_S(a) \cup \bigcup_{c<a} T_S(c)$.

4. We have to show that every $x \in M$ is contained in exactly one $T_S(a)$. Proving the existence of such an $a \in G$, define $H_x =_{\mathrm{def}} \{a \mid x \in S(a)\}$ which is non-empty since $\bigcup_{a \in G} S(a) = M$. For $a, b \in H_x$, i.e., $x \in S(a) \cap S(b) = \bigcup_{c \leq a, c \leq b} S(c)$ there is a $c \in G$ such that $c \leq a$, $c \leq b$, and $c \in H_x$. Hence the set H_x has a least element d_x. Obviously, $x \in T_S(d_x) = S(d_x) \setminus \bigcup_{c<d_x} S(c)$.

On the other hand, assume $x \in T_S(a)$ for some $a \in G$. Then $x \in S(a)$ which implies $a \in H_x$ and $d_x \leq a$. Because of $x \in T_S(a) = S(a) \setminus \bigcup_{b<a} S(b)$ we conclude $d_x \not< a$ which means $d_x = a$. ☐

Any pair (G, f) of an arbitrary finite poset G and a function $f : G \to \{1, \ldots, k\}$ is called a *k-poset*. A k-poset which is a lattice (boolean lattice, chain, etc.) is called a *k-lattice* (*boolean k-lattice, k-chain*, etc.).

Lemma 1 provides that the following definitions are sound.

Definition 3. *Let $k \geq 2$.*

1. *For a k-poset (G, f) and a $\mathcal{K}$-homomorphism S on G let $(G, f, S) =_{\mathrm{def}} \left(\bigcup_{f(a)=1} T_S(a), \ldots, \bigcup_{f(a)=k} T_S(a)\right)$ be the k-partition defined by (G, f) and S.*
2. *Let $\mathcal{K}(G, f) =_{\mathrm{def}} \{(G, f, S) \mid S$ is $\mathcal{K}$-homomorphism on $G\}$ be the class of k-partitions defined by the k-poset (G, f).*
3. *The family* $\mathrm{EBH}_k(\mathcal{K}) =_{\mathrm{def}} \{\mathcal{K}(G, f) \mid (G, f)$ *is a k-poset$\}$ is the* extended boolean hierarchy of k-partitions over $\mathcal{K}$.

Figure 1 shows some examples of partition classes in $\mathrm{EBH}_3(\mathcal{K})$ together with their defining 3-posets. We should mention here that, even if it seems so, not all partition classes of posets can be exactly described in their components by classes of the boolean hierarchy of sets.

The following propositions relate the extended boolean hierarchy to the boolean hierarchy.

Proposition 1. $\mathrm{BH}_k(\mathcal{K}) \subseteq \mathrm{EBH}_k(\mathcal{K})$ *for every $k \geq 2$.*

Proof. In [KW00], it was shown that a partition class $\mathcal{C}$ belongs to $\mathrm{BH}_k(\mathcal{K})$ if and only if there is a k-lattice (G, f) such that $\mathcal{C} = \mathrm{NP}(G, f)$. Since each k-lattice is in particular a k-poset, we have the inclusion. ☐

Proposition 2. *Let $\mathcal{K}$ be not closed under complementation, and let $k \geq 2$. Then there exists a partition class in* $\mathrm{EBH}_k(\mathcal{K})$ *which does not belong to* $\mathrm{BH}_k(\mathcal{K})$.

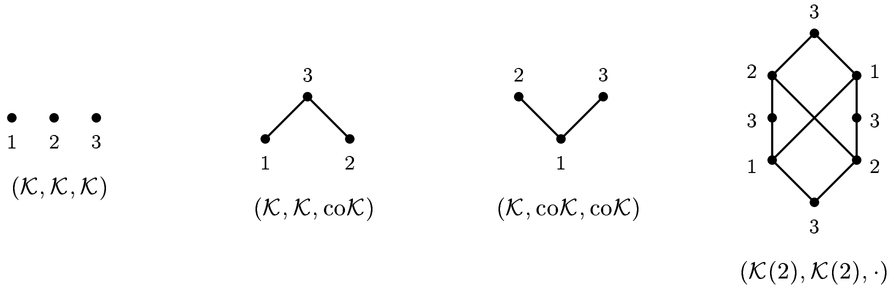


Fig. 1. Some partition classes in $\mathrm{EBH}_3(\mathcal{K})$ and their defining 3-posets.

Proof. It is easily seen that the class $(\mathcal{K}, \mathcal{K}, \dots, \mathcal{K})$ of k-partitions is defined by a k-elementary k-antichain having k different labels (see Fig. 1 for $k = 3$). Assume there is an $f : \{1,2\}^m \to \{1, 2, \dots, k\}$ with $(\mathcal{K}, \mathcal{K}, \dots, \mathcal{K}) = \mathcal{K}(f)$. Clearly, f is surjective. Let $z \in \{1,2\}^m$ be minimal with $f(1^m) \neq f(z)$. For an arbitrary set $A \in \mathcal{K}$ define the $\mathcal{K}$-homomorphism S on $(\{1,2\}^m, \leq)$ as $S(a) = A$ if $z \not\leq a$, and $S(a) = M$ otherwise. Thus, $(\{1,2\}^m, f, S)_{f(z)} = \overline{A}$. Hence, $\overline{A} \in \mathcal{K}$ and consequently, $A \in \mathrm{co}\mathcal{K}$, a contradiction. $\square$

Proposition 3. *Let (G, f) be a k-poset with $f : G \to \{1, 2, \dots, k\}$ surjective.*

1. $(\mathcal{K}, \mathcal{K}, \dots, \mathcal{K}) \subseteq \mathcal{K}(G, f) \subseteq (\mathbf{BC}(\mathcal{K}), \mathbf{BC}(\mathcal{K}), \dots, \mathbf{BC}(\mathcal{K})) = \mathbf{BC}_k(\mathcal{K})$.
2. *If $\mathcal{K}$ is closed under complementation then $\mathcal{K}(G, f) = (\mathcal{K}, \mathcal{K}, \dots, \mathcal{K})$.*

Proof. 1. $(\mathcal{K}, \mathcal{K}, \dots, \mathcal{K}) \subseteq \mathcal{K}(f)$: Let $(D_1, \dots, D_k) \in (\mathcal{K}, \mathcal{K}, \dots, \mathcal{K})$. For every $i \in \{1, \dots, k\}$ fix some $b_i \in G$ such that $f(b_i) = i$. For $a \in G$ define $S(a) =_{\mathrm{def}} \bigcup_{b_i \leq a} D_i$. The function $S : G \to \{1, 2, \dots, k\}$ is a $\mathcal{K}$-homomorphism on G because of $\bigcup_{a \in G} S(a) \supseteq \bigcup_{i=1}^{k} S(b_i) \supseteq \bigcup_{i=1}^{k} D_i = M$ and $S(a) \cap S(b) = \bigcup_{b_i \leq a} D_i \cap \bigcup_{b_i \leq a} D_i = \bigcup_{b_i \leq a,b} D_i = \bigcup_{c \leq a,b} \bigcup_{b_i \leq c} D_i = \bigcup_{c \leq a,b} S(c)$. For every $i \in \{1, \dots k\}$ we conclude $A_i = \bigcup_{f(a)=i} T_S(a) = \bigcup_{f(a)=i} (S(a) \setminus \bigcup_{b<a} S(b)) = \bigcup_{f(a)=i} (\bigcup_{b_j \leq a} D_j \setminus \bigcup_{b_j < a} D_j) = \bigcup_{f(a)=i} \bigcup_{b_j = a} D_j = D_i$. Hence $(D_1, \dots, D_k) = (G, f, S) \in \mathcal{K}(G, f)$.

$\mathcal{K}(G, f) \subseteq (\mathbf{BC}(\mathcal{K}), \mathbf{BC}(\mathcal{K}), \dots, \mathbf{BC}(\mathcal{K}))$: Let S be a $\mathcal{K}$-homomorphism on G and let $(G, f, S) = (A_1, \dots, A_k)$. For $i \in \{1, \dots, k\}$ we obtain $A_i = \bigcup_{f(a)=i} T_S(a) = \bigcup_{f(a)=i} (S(a) \setminus \bigcup_{b<a} S(b))$. Clearly, this gives $A_i \in \mathcal{K}(2 \cdot |f^{-1}(i)|)$.

$\mathbf{BC}_k(\mathcal{K}) = (\mathbf{BC}(\mathcal{K}), \dots, \mathbf{BC}(\mathcal{K})$ was proven in [KW00].

2. This statement is an immediate consequence of the first one. $\square$

Given two k-posets (G, f) and (G', f') it would be very useful to have a criterion to decide whether $\mathcal{K}(G, f) \subseteq \mathcal{K}(G', f')$. The following lemma provides a sufficient condition.

For k-posets (G, f) and (G', f') we write $(G, f) \leq (G', f')$ if and only if there is a monotonic mapping $\varphi : G \to G'$ such that $f(x) = f'(\varphi(x))$ for every $x \in G$. We write $(G, f) \equiv (G', f')$ if $(G, f) \leq (G', f')$ and $(G', f') \leq (G, f)$.

Lemma 2. (Embedding Lemma.) *Let (G, f) and (G', f') be k-posets. Then, if $(G, f) \leq (G', f')$, then $\mathcal{K}(G, f) \subseteq \mathcal{K}(G', f')$.*

Proof. Let $\varphi : G \to G'$ be a monotonic mapping such that $f(a) = f'(\varphi(a))$ for every $a \in G$. For a $\mathcal{K}$-homomorphism S on G define the mapping $S' : G' \to \mathcal{K}$ by $S'(a) =_{\text{def}} \bigcup_{\varphi(b) \leq' a} S(b)$ for all $a \in G'$. It is sufficient to prove that S' is a $\mathcal{K}$-homomorphism on G', i.e. that 1. $\bigcup_{a \in G} S'(a) = M$ and 2. $S'(a) \cap S'(b) = \bigcup_{c \leq a,b} S'(c)$ for all $a, b \in G$, and that 3. $T_S(a) \subseteq T_{S'}(\varphi(a))$ for all $a \in G$.

1. We conclude $\bigcup_{a \in G'} S'(a) = \bigcup_{a \in G'} \bigcup_{\varphi(b) \leq' a} S(x) \supseteq \bigcup_{b \in G} S(b) = M$.

2. The inclusion "$\supseteq$" is valid because of the monotonicity of S'. For the converse inclusion consider $x \in S'(a) \cap S'(b)$. There exist $c, d \in G$ such that $\varphi(c) \leq' a$, $\varphi(d) \leq' b$, $x \in S(c) \cap S(d) = \bigcup_{e \leq c,d} S(e)$. Hence there exists an $e_0 \leq c, d$ such that $x \in S(e_0)$. We obtain $x \in S'(\varphi(e_0))$ and $\varphi(e_0) \leq' \varphi(c) \leq' a$ and $\varphi(e_0) \leq' \varphi(d) \leq' b$.

3. For $a \in G$ and $x \in T_S(a)$ we obtain $x \in S(a) \subseteq S'(\varphi(a))$. Assume that $x \notin T_{S'}(\varphi(a))$. Then there exists a $c <' \varphi(a)$ such that $x \in S'(c)$. Consequently, there exists a $b \in G$ such that $\varphi(b) \leq' c$ and $x \in S(b)$. Hence $x \in S(a) \cap S(b) = \bigcup_{d \leq a,b} S(d)$. Hence there exists a $d_0 \leq a, b$ such that $x \in S(d_0)$. Because of $x \in T_S(a)$ we get $d_0 \not< a$ and thus $a = d_0$. We conclude $\varphi(a) = \varphi(d_0) \leq' \varphi(b) \leq' c$, a contradiction. $\square$

The next proposition indicates that there is a canonical way to represent classes in $\mathrm{EBH}_k(\mathcal{K})$.

Proposition 4. *Let (G, f) be a k-poset. Then there is a partial function $f' : \{1,2\}^m \to \{1, \ldots, k\}$ such that $(G, f) \equiv (D_{f'}, f')$ and consequently, $\mathcal{K}(G, f) = \mathcal{K}(D_{f'}, f')$.*

Proof. For all $a \in G$, consider sets $L_a = \{b \in G \mid b \leq a\}$. It holds $a \leq b$ if and only if $L_a \subseteq L_b$. Define a partial function $f' : \mathcal{P}(G) \to \{1, \ldots, k\}$ by $D_{f'} =_{\text{def}} \{L_a \mid a \in G\}$ and $f'(L_a) =_{\text{def}} f(a)$. Clearly, $(G, f) \equiv (D_{f'}, f')$ and $(\mathcal{P}(G), \subseteq)$ is isomorphic to $\left(\{1,2\}^{\|G\|}, \leq\right)$. $\square$

4 An Equivalent Approach

In Section 3 it has been described how to define classes of k-partitions by posets. This has generalized the case of lattices. In this section we will make another generalization of the lattice approach which results in the same classes of k-partitions as obtained in the poset approach.

Let $\mathcal{K}$ be a class of subsets of M such that $\emptyset, M \in \mathcal{K}$ and $\mathcal{K}$ is closed under union and intersection.

Definition 4. *Let G be a lattice and let $H \subset G$.*

1. *A mapping $S : G \to \mathcal{K}$ is said to be a* $\mathcal{K}$-homomorphism on (G, H) *if and only if $S(1_G) = M$, $S(a \wedge b) = S(a) \cap S(b)$ for all $a, b \in G$, and $S(a) = \bigcup_{b<a} S(b)$ for all $a \in H$.*
2. *For any $\mathcal{K}$-homomorphism S on (G, H), let $T_S(a) =_{\text{def}} S(a) \setminus \bigcup_{b<a} S(b)$ for all $a \in G$.*

As immediate consequences of this definition we obtain

Proposition 5. *Let S be a $\mathcal{K}$-homomorphism on (G,H) for a lattice G and $H \subset G$.*

1. *S is a $\mathcal{K}$-homomorphism on G.*
2. *$T_S(a) = \varnothing$ for all $a \in H$.*

Proposition 6. *Let S be a $\mathcal{K}$-homomorphism on a lattice G and let $H \subset G$. If $T_S(a) = \varnothing$ for all $a \in H$, then S is a $\mathcal{K}$-homomorphism on (G,H).*

Consequently we can apply all definitions made for $\mathcal{K}$-homomorphisms on G also to $\mathcal{K}$-homomorphisms on (G,H). So we are able to define classes of k-partitions.

Definition 5. *For a k-lattice (G,f) and a set $H \subset G$ define*

$$\mathcal{K}(G,H,f) =_{\mathrm{def}} \{(G,f,S) \mid S \textit{ is a } \mathcal{K}\textit{-homomorphism on } (G,H)\}.$$

It turns out that every such class $\mathcal{K}(G,H,f)$ is also of the form $\mathcal{K}(G,f)$ where (G,f) is a k-poset and vice versa. Furthermore, we can equivalently consider boolean k-lattices (G,f) and $H \subset G$.

In the following, as a shorthand we identify a function f with any restriction of f to a smaller domain, i.e., with $f|_A$ for $A \subseteq D_f$.

Lemma 3. *Let (G,f) be a k-lattice. Let S be a $\mathcal{K}$-homomorphism on (G,H) for any $H \subset G$. Then S is also $\mathcal{K}$-homomorphism on the poset $G \setminus H$. Moreover it holds $(G,f,S) = (G \setminus H, f, S)$.*

Proof. Let G' denote the poset $G \setminus H$. Due to Proposition 5 it is enough to show the first statement. That is we have to prove that 1. $\bigcup_{a\in G'} S(a) = M$ and 2. $S(a) \cap S(b) = \bigcup_{c \leq a, c\leq b, c \in G'} S(c)$ for all $a,b \in G'$.

1. We conclude $\bigcup_{a\in G'} S(w) \supseteq \bigcup_{a \in G'} T_S(w) = M$.

2. Let $a,b \in G'$. It holds $S(a) \cap S(b) = S(a \wedge b)$. If $a \wedge b \notin H$ then the statement holds obviously. Now, let $a \wedge b \in H$. Then $S(a) \cap S(b) = S(a \wedge b) = \bigcup_{c < a\wedge b} S(c) = \bigcup_{c<a\wedge b, c \in G'} S(c)$. □

For a partial function $f : \{1,2\}^m \to \{1,\dots,k\}$, let H_f denote the set of all arguments where f is not defined.

Theorem 1. *Let $f : \{1,2\}^m \to \{1,\dots,k\}$ be a partial function with non-empty domain. Then $\mathcal{K}(D_f, f) = \mathcal{K}(\{1,2\}^m, H_f, f')$ where f' is an arbitrary extension of f to $\{1,2\}^m$ with function values in $\{1,\dots,k\}$.*

Proof. The inclusion "$\supseteq$" follows from Lemma 3. For the converse inclusion let S be a $\mathcal{K}$-homomorphism on $(\{1,2\}^m, H_f)$. Then define a mapping $S' : \{1,2\}^m \to \mathcal{K}$ for all $a \in \{1,2\}^m$ as

$$S'(a) =_{\mathrm{def}} \begin{cases} S(a) & \text{if } a \in D_f, \\ \bigcup_{b<a, b \in D_f} S(b) & \text{if } a \in H_f. \end{cases}$$

Observe that S' is monotonic. First we have to make sure that S' is a $\mathcal{K}$-homorphism on $(\{1,2\}^m, H_f)$, i.e., 1. $\bigcup_{a\in\{1,2\}^m} S'(a) = M$, 2. $S'(a)\cap S'(b) = S'(a\wedge b)$ for all $a,b\in\{1,2\}^m$, and 3. $S'(a) = \bigcup_{b<a} S'(b)$ for all $a\in H_f$.

1. We conclude $\bigcup_{a\in\{1,2\}^m} S'(a) \supseteq \bigcup_{a\in D_f} S(a) = M$.

2. The inclusions "$\supseteq$" are immediate due to the monotonicity of S'. Conversely, let $x\in S'(a)\cap S'(b')$. Thus there exist $c,d\in D_f$ such that $c<a$, $d<b$, and $x\in S(c)\cap S(d)$. Hence there is an $e\in D_f$ with $e\leq c\wedge d$ and $x\in S(e)=S'(e)$. We get $e\leq a\wedge b$ and consequently $x\in S'(a\wedge b)$.

3. Again the inclusions "$\supseteq$" follow directly from the monotonicity of S'. For the other inclusion let $x\in S'(a)$ for $a\in H$. That is there exists a $b\in D_f$ with $b<a$ and $x\in S(b)=S'(b)$. Hence, $x\in\bigcup_{b<a} S'(b)$.

In order to prove $(D_f,f,S) = (\{1,2\}^m, f', S')$ it suffices to show $T_{S'}(a)\subseteq T_S(a)$ for all $a\notin H_f$. Let $x\in T_{S'}(a)$. Then $x\in S'(a) = S(a)$. Assuming $x\notin T_S(a)$ leads to a $b\in D_f$ with $b<a$ and $x\in S(b)=S'(b)$. Hence $x\notin T_S(a)$ what is a contradiction. □

Corollary 1.

$$\begin{aligned}\mathrm{EBH}_k(\mathcal{K}) &= \{\mathcal{K}(G,H,f) \mid (G,f) \textit{ is a boolean } k\textit{-lattice and } H\subset G\}\\ &= \{\mathcal{K}(G,H,f) \mid (G,f) \textit{ is a } k\textit{-lattice and } H\subset G\}.\end{aligned}$$

Proof. Follows from Proposition 4, Theorem 1, and Lemma 3. □

Corollary 2. *Let $f:\{1,2\}^m\to\{1,\dots,k\}$ be a partial function with non-empty domain. Then a k-partition L belongs to the partition class $\mathcal{K}(D_f,f)$ if and only if there are sets $B_1,\dots,B_m\in\mathcal{K}$ with $(c_{B_1}(x),\dots,c_{B_m}(x))\in D_f$ and $c_L(x) = f(c_{B_1}(x),\dots,c_{B_m}(x))$ for all $x\in M$.*

Again it is useful to have the possibility comparing structures (G,H,f) and $(G',H,'f')$. Let (G,f) and (G',f') be k-lattices, let $H\subset G$ and $H'\subset G'$. Then $(G,H,f)\leq(G',H',f')$ if and only if there is a monotonic mapping $\varphi: G\to G'$ such that $\varphi(G\setminus H)\subseteq(G'\setminus H')$ and $f(x) = f'(\varphi(x))$ for all $x\in G\setminus H$. This relation is compatible to the one made for k-posets.

Proposition 7. *Let (G,f) and (G',f') be k-lattices, let $H\subset G$ and $H'\subset G'$. Then $(G,H,f)\leq(G',H',f')$ if and only if $(G\setminus H,f)\leq(G'\setminus H',f')$.*

Proof. The "$\Longrightarrow$"-direction is obvious. For "$\Longleftarrow$" let φ be a monotonic mapping witnessing $(G\setminus H,f)\leq(G'\setminus H',f')$. For $a\in G$ let $L_a = \{b\in G\setminus H \mid b\geq a\}$ and define $\psi(a) = \bigwedge_{b\in L_a}\varphi(b)$. Observe $\psi(a) = \varphi(a)$ for all $a\in G\setminus H$. Moreover, if $a\leq b$, then $L_b\subseteq L_a$, thus $\psi(a)\leq\psi(b)$. Hence, $(G,H,f)\leq(G',H',f')$. □

Corollary 3. (Embedding Lemma.) *Let (G,f) and (G',f') be k-lattices, let $H\subset G$ and $H'\subset G'$. If $(G,H,f)\leq(G',H',f')$ then $\mathcal{K}(G,H,f)\subseteq\mathcal{K}(G',H',f')$.*

Proof. If $(G,H,f)\leq(G',H',f')$ then $(G\setminus H,f)\leq(G'\setminus H',f')$. Hence by Lemma 3 and the Embedding Lemma for posets, $\mathcal{K}(G,H,f) = \mathcal{K}(G\setminus H,f)\subseteq\mathcal{K}(G'\setminus H',f') = \mathcal{K}(G',H',f')$. Note that the restricted versions of f and f' need not be surjective. □

5 On the Strictness of $\mathbf{EBH}_{k}(\mathbf{NP})$

In Sect. 3 we have defined partition classes over posets. In Sect. 4 it has been given an alternative characterization of these classes in terms of lattices with associated proper subsets. For both structures we have identified relations inducing a sufficient criterion for inclusions between partition classes.

In this section, focusing our attention to the special case $\mathcal{K} = \mathrm{NP}$, we will give some reasons of why we are convinced of the sufficient criterion to be also necessary or synonymously, that $\mathrm{EBH}_k(\mathrm{NP})$ is strict, i.e., $\mathrm{NP}(G,f) \not\subseteq \mathrm{NP}(G',f')$ whenever $(G,f) \not\leq (G',f')$.

Clearly, all results from [KW00] providing evidence for the strictness of $\mathrm{BH}_k(\mathrm{NP})$ are also arguments for the strictness of $\mathrm{EBH}_k(\mathrm{NP})$. Moreover, some proofs carry over to the case of posets.

A repetition-free k-chain is one where neighbored elements have different labels.

Theorem 2. *Assume the polynomial-time hierarchy does not collapse. Let (G,f) and (G',f') be k-posets. If $\mathrm{NP}(G,f) \subseteq \mathrm{NP}(G',f')$, then each repetition-free k-subchain of (G,f) occurs as a k-subchain of (G',f').*

Building on results like this, the Embedding Conjecture (for lattices) was posed stating that $\mathrm{BH}_k(\mathrm{NP})$ is strict unless the polynomial-time hierarchy collapses. Though we are also convinced of the strictness of $\mathrm{EBH}_k(\mathrm{NP})$, we must be warned to literally take over this conjecture for posets.

Proposition 8. *Let (G,f) and (G',f') be the 3-posets in Fig. 2. Then $(G,f) \not\leq (G',f')$ but there exists an oracle where the polynomial-time hierarchy does not collapse to any level and $\mathrm{NP}(G,f) = \mathrm{NP}(G',f')$.*

Proof. Blum and Impagliazzo [BI87] constructed an oracle with $\mathrm{NP} \cap \mathrm{coNP} = \mathrm{P}$ and the polynomial-time hierarchy is strict. In fact, their proof also shows that there is an oracle D such that the polynomial-time hierarchy is strict and for all disjoint sets $A, B \in \mathrm{NP}^D$ there is a set $C \in \mathrm{P}^D$ with $A \subseteq C \subseteq \overline{B}$. Hence, given an NP^D-homomorphism S on (G,f) we can define an NP^D-homomorphism S' on (G',f') as $S'(a') = S(a)$, $S'(b') = S(b)$, $S'(c') = C$, and $S'(d') = \overline{C}$ where C belongs to P^D with $S(a) \subseteq C \subseteq \overline{S(b)}$. This gives $(G,f,S) \in \mathrm{NP}^D(G',f')$. The other inclusion holds relativizably by the Embedding Lemma. $\square$

Does this oracle also give evidence against the Embedding Conjecture for lattices? We feel that this is not the case since the counterexample strongly depends on the weakness of structure of posets. Moreover, the oracle result affects rather the seemingly insufficient assumption of a strict polynomial-time hierarchy than the correspondence between k-posets and classes of k-partitions. For instance, under the assumption that $\mathrm{UP} \not\subseteq \mathrm{coNP}$, the counterexample fails.

Proposition 9. *Let (G,f) and (G',f') be the 3-posets in Fig. 2. If $\mathrm{NP}(G,f) = \mathrm{NP}(G',f')$ then $\mathrm{UP} \subseteq \mathrm{coNP}$.*

Fig. 2. Critical 3-posets.

Proof. Observe that if $\mathrm{NP}(G,f) \subseteq \mathrm{NP}(G',f')$ then for every pair (A,B) of disjoint NP sets there is a set $C \in \mathrm{NP} \cap \mathrm{coNP}$ with $A \subseteq C \subseteq \overline{B}$. Our proof will generalize a similar result of [GS88]. So let $L \in \mathrm{UP}$, i.e., there are a set $A \in \mathrm{P}$ and a polynomial p with $x \in L \iff \|\{y \mid |y| = p(|x|) \wedge (x,y) \in A\}\| = 1$ and $x \notin L \iff \|\{y \mid |y| = p(|x|) \wedge (x,y) \in A\}\| = 0$. Define the following sets:

$$S_A = \{(x,z) \mid (\exists y, |y| = p(|x|))[z \leq y \wedge (x,y) \in A]\}$$
$$T_A = \{(x,z) \mid (\exists y, |y| = p(|x|))[z > y \wedge (x,y) \in A]\}$$

Obviously, $S_A, T_A \in \mathrm{NP}$, $S_A \cap T_A = \varnothing$, and $S_A \cup T_A = L \times \Sigma^*$. Hence, there exists a set $C \in \mathrm{NP} \cap \mathrm{coNP}$ with $S_A \subseteq C \subseteq \overline{T_A}$. Using this set C as an oracle for binary search, one can determine for each $x \in \Sigma^*$ a value $b(x)$ such that $x \in L \iff (x, b(x)) \in A$. Hence, $L \in \mathrm{P}^{\mathrm{NP}\cap\mathrm{coNP}} \subseteq \mathrm{coNP}$. □

In the following we will show that we cannot unexpectedly collapse classes in $\mathrm{EBH}_k(\mathrm{NP})$ using relativizable proof techniques. On the one hand, certainly, this is not such a strong statement as Theorem 2 for the classes in its scope, but on the other side it is a statement about *all* classes of $\mathrm{EBH}_k(\mathrm{NP})$.

To prove the Theorem 3, we use the method of uniform characterizations of complexity classes (for a survey see [Vol99]). First we adapt some notions for the case of partition classes (formally, this is a special case of function classes [KSV98]).

Definition 6. *Let $A = (A_1, \ldots, A_k, A_{k+1})$ be a $(k+1)$-partition over alphabet Δ, $k \geq 2$. The k-partition L (over Σ) belongs to the partition class $\mathrm{Leaf}_k^{\mathrm{P}}(A)$ if and only if there are functions $f : \Sigma^* \times \mathbb{N} \to \Delta$ and $g : \Sigma^* \to \mathbb{N}$ both computable in polynomial time such that for all $x \in \Sigma^*$*

1. $f(x,0)f(x,1)\ldots f(x,g(x)) \in \Delta^* \setminus A_{k+1}$,
2. $c_L(x) = c_A(f(x,0)f(x,1)\ldots f(x,g(x)))$.

Later the component A_{k+1} of the partition A will represent in some sense a subset $H \subset G$ for the labeled lattice (G,f).

The following partitions are of particular interest.

Definition 7. *Let* $\mathrm{sgn} : \mathbb{N} \to \{0,1\}$ *be the sign function, i.e.,* $\mathrm{sgn}(x) = \min\{x,1\}$ *for all* $x \in \mathbb{N}$.

1. *Any k-partition A over $\Delta = \{a_1, \ldots, a_m\}$ is said to be* of simple counting type *if and only if* $(\mathrm{sgn}(|x|_{a_1}), \ldots, \mathrm{sgn}(|x|_{a_m})) = (\mathrm{sgn}(|y|_{a_1}), \ldots, \mathrm{sgn}(|y|_{a_m}))$ *implies $c_A(x) = c_A(y)$ for all $x, y \in \Delta^*$.*
2. *For any k-partition A of simple counting type over $\Delta = \{a_1, \ldots, a_m\}$ the function $T_A : \mathbb{N}^m \to \{1, 2, \ldots, k\}$ is defined for all $n_1, \ldots, n_m \in \mathbb{N}$ as*

$$T_A(n_1, \ldots, n_m) =_{\mathrm{def}} c_A(a_1^{n_1} a_2^{n_2} \ldots a_m^{n_m}).$$

The following lemma brings together *mutatis mutandis* results from the works of [KSV98,BCS92,Ver93,CHVW98,Her95]. A function $p : \mathbb{N}^m \to \mathbb{N}$ is a positive linear combination of multinomial coefficients iff there are $z \in \mathbb{N}^m$ and $\alpha_t \in \mathbb{N}$ such that for all $n \in \mathbb{N}^m$

$$p(n) = \sum_{t \leq z} \alpha_t \binom{n}{t}.$$

A relativization of a leaf class means that the defining functions f and g are relativized.

Lemma 4. *Let A and B be $(k+1)$-partitions of simple counting type over an m-elementary alphabet $\Delta = \{a_1, \ldots, a_m\}$. Then, $\mathrm{Leaf}_k^{\mathrm{P}}(A)^C \subseteq \mathrm{Leaf}_k^{\mathrm{P}}(B)^C$ for all oracles C if and only if there are functions $p_1, \ldots, p_m : \mathbb{N}^m \to \mathbb{N}$ which are positive linear combinations of multinomial coefficients such that for all $n_1, \ldots, n_m \in \mathbb{N}$ with $a_1^{n_1} a_2^{n_2} \ldots a_m^{n_m} \in \Delta^* \setminus A_{k+1}$*

$$T_A(n_1, \ldots, n_m) = T_B(p_1(n_1, \ldots, n_m), \ldots, p_m(n_1, \ldots, n_m)).$$

Theorem 3. *Let (G, f) and (G', f') be k-posets. Then $(G, f) \leq (G', f')$ if and only if $\mathrm{NP}^C(G, f) \subseteq \mathrm{NP}^C(G', f')$ for every oracle C.*

Proof. The direction "$\Longrightarrow$" follows from the Embedding Lemma and since NP is relativizably closed under union and intersection.

For "$\Longleftarrow$", it suffices to consider k-posets (D_f, f) for partial functions with non-empty domain. We will first prove a relativizable characterization of such partition classes $\mathrm{NP}(D_f, f)$ in terms of partition classes $\mathrm{Leaf}_k^{\mathrm{P}}(A)$ with A appropriate. To do this, let Δ be the alphabet $\{0, 1, \ldots, m\}$ for a given partial function $f : \{1, 2\}^m \to \{1, 2, \ldots, k\}$. We also consider the extension $\bar{f}$ of f which is a total mapping from $\{1, 2\}^m \to \{1, \ldots, k\}$ defined for all $w \in \{1, 2\}^m$ as $\bar{f}(w) = f(w)$ for $w \in D_f$ and $\bar{f}(w) = k + 1$ for $w \notin D_f$. For a word $x \in \Delta^*$, let $\tau(x)$ denote the word $z_1 z_2 \ldots z_m$ with $z_i = 1 \iff |x|_i > 0$ and $z_i = 2 \iff |x|_i = 0$.

Define the k-partition $A(\bar{f})$ over Δ by $c_{A(\bar{f})} = \bar{f} \circ \tau$. Observe that for every f the partition $A(\bar{f})$ is of simple counting type.

Claim. $\mathrm{NP}(D_f, f) = \mathrm{Leaf}_k^{\mathrm{P}}\big(A(\bar{f})\big)$.

For "$\subseteq$", let $L \in \mathrm{NP}(D_f, f)$, i.e., there are sets $B_1, \ldots, B_m \in \mathrm{NP}$ with $(c_{B_1}(x), \ldots, c_{B_m}(x)) \in D_f$ and $c_L(x) = f(c_{B_1}(x), \ldots, c_{B_m}(x))$ for all $x \in \Sigma^*$. There are sets $C_i \in \mathrm{P}$ and without loss of generality one polynomial p such that

$x \in B_i \iff \|\{y \mid |y| = p(|x|) \wedge (x,y) \in C_i\}\| > 0$. Define functions s and t for all $x \in \Sigma^*$ as

$$t(x) = m \cdot 2^{p(|x|)} - 1$$

and for all $j \in \mathbb{N}$ with $j \leq t(|x|)$

$$s(x,j) = \begin{cases} i & \text{if } (i-1)\cdot 2^{p(|x|)} \leq j < i \cdot 2^{p(|x|)} \text{ and } \big(x, j-(i-1)\cdot 2^{p(|x|)}\big) \in C_i, \\ \\ 0 & \text{if } (i-1)\cdot 2^{p(|x|)} \leq j < i \cdot 2^{p(|x|)} \text{ and } \big(x, j-(i-1)\cdot 2^{p(|x|)}\big) \notin C_i. \end{cases}$$

Clearly, s and t are computable in polynomial time, and for all $i \in \{1,\dots,m\}$ it holds $x \in B_i \iff |s(x,0)s(x,1)\dots s(x,t(x))|_i > 0$. Hence,

$$\tau(s(x,0)s(x,1)\dots s(x,t(x))) = (c_{B_1}(x),\dots,c_{B_m}(x))$$

for all $x \in \Sigma^*$. This gives $s(x,0)s(x,1)\dots s(x,t(x)) \in \Delta^* \setminus A(\bar{f})_{k+1}$ and

$$c_L(x) = (f \circ \tau)(s(x,0)s(x,1)\dots s(x,t(x))) = c_{A(\bar{f})}(s(x,0)s(x,1)\dots s(x,t(x))).$$

Thus, $L \in \mathrm{Leaf}_k^{\mathrm{P}}\big(A(\bar{f})\big)$.

For "$\supseteq$", let $L \in \mathrm{Leaf}_k^{\mathrm{P}}\big(A(\bar{f})\big)$, and let s and t be the witnessing functions. Consider the sets $B_i =_{\mathrm{def}} \{x \in \Sigma^* \mid \|\{j \mid j \leq t(x) \wedge s(x,j) = i\}\| > 0\}$ for $i \in \{1,\dots,m\}$. Obviously, $B_1,\dots,B_m \in \mathrm{NP}$. Since $s(x,0)s(x,1)\dots s(x,t(x)) \in \Delta^* \setminus A(\bar{f})_{k+1}$, we get $(c_{B_1}(x),\dots,c_{B_m}(x)) = \tau(s(x,0)s(x,1)\dots s(x,t(x))) \in D_f$. Furthermore, it is easily seen that

$$\begin{aligned} c_{A(f)}(s(x,0)s(x,1)\dots s(x,t(x)) &= (f\circ\tau)(s(x,0)s(x,1)\dots s(x,t(x))) \\ &= f(c_{B_1}(x),\dots,c_{B_m}(x)) = c_L(x). \end{aligned}$$

Consequently, $L \in \mathrm{NP}(D_f, f)$.

The claim above enables us to apply Lemma 4. Note that the proof of the claim relativizes. Before making an explicite use of Lemma 4, we subtly have to verify that for every m-ary partial function f and for every $m' > m$ there is a m'-ary partial functions f' with $(D_f, f) \equiv (D_{f'}, f')$. Let $m' > m$. Consider the mapping $\eta : \{1,2\}^{m'} \to \{1,2\}^m$ given as $\eta(w_1 \dots w_m w_{m+1} \dots w_{m'}) =_{\mathrm{def}} w_1 \dots w_m$. Define $f' =_{\mathrm{def}} f \circ \eta$, that is $f' : \{1,2\}^{m'} \to \{1,\dots,k\}$ and $f'(w_1,\dots w_m w_{m+1} \dots w_{m'}) = f(w_1 \dots w_m)$. Then, $(D_f, f) \leq (D_{f'}, f')$ is immediately seen and $(D_f, f) \geq (D_{f'}, f')$ is witnessed by η. Hence, it is sufficient to prove the assertion of the theorem only for k-posets represented by partial functions of the same arity.

Let $f, f' : \{1,2\}^m \to \{1,2,\dots,k\}$ be partial functions with non-empty domain. Let $\mathrm{NP}^C(D_f, f) \subseteq \mathrm{NP}^C(D_{f'}, f')$ for all oracles C. We have to show that there is a monotonic mapping $\varphi : D_f \to D_{f'}$ such that $f(w) = f'(\varphi(w))$ for every $w \in D_f$. Since $A(\bar{f})$ and $A(\bar{f}')$ are k-partitions over the same alphabet $\Delta = \{0,1,\dots,m\}$ of simple counting type, we can apply Lemma 4. So, let

$p_0, p_1, \ldots, p_m$ be functions as given in Lemma 4. Since these functions are positive linear combinations of multinomial coefficients, in particular these functions are monotonic with respect to vector-ordering. For $a \in \{1,2\}$ let $a' = 1$ if $a = 1$, and $a' = 0$ if $a = 2$. Then it holds for all $w \in D_f$:

$$\begin{aligned}
&c_{A(\bar{f})}(1^{w'_1}2^{w'_2}\ldots m^{w'_m}) \\
&= T_{A(\bar{f})}(0, w'_1, \ldots, w'_m) && \big(\text{by Definition 7(2)}\big) \\
&= T_{A(\bar{f}')}\big(p_0(0, w'_1, \ldots, w'_m), p_1(0, w'_1, \ldots, w'_m), \ldots, p_m(0, w'_1, \ldots, w'_m)\big) \\
& && (\text{by Lemma 4}) \\
&= c_{A(\bar{f}')}\big(0^{p_0(0,w'_1,\ldots,w'_m)}1^{p_1(0,w'_1,\ldots,w'_m)}\ldots m^{p_m(0,w'_1,\ldots,w'_m)}\big) \\
& && \big(\text{by Definition 7(2)}\big) \\
&= c_{A(\bar{f}')}\big(1^{p_1(0,w'_1,\ldots,w'_m)}2^{p_2(0,w'_1,\ldots,w'_m)}\ldots m^{p_m(0,w'_1,\ldots,w'_m)}\big) \\
& && (A(\bar{f}') \text{ is of simple counting type})
\end{aligned}$$

Now, define φ for every $w \in D_f$ as

$$\varphi(w_1 \ldots w_m) =_{\text{def}} \tau\big(1^{p_1(0,w'_1,\ldots,w'_m)}2^{p_2(0,w'_1,\ldots,w'_m)}\ldots m^{p_m(0,w'_1,\ldots,w'_m)}\big).$$

We have to show that 1. φ maps D_f to $D_{f'}$, 2. φ is monotonic, and 3. $f(w) = f'(\varphi(w))$ for all $w \in D_f$.

1. Let $w \in D_f$, i.e., $1^{w'_1}2^{w'_2}\ldots m^{w'_m} \in \Delta^* \setminus A(\bar{f})_{k+1}$. Thus by the arguments above the word $1^{p_1(0,w'_1,\ldots,w'_m)}2^{p_2(0,w'_1,\ldots,w'_m)}\ldots m^{p_m(0,w'_1,\ldots,w'_m)}$ belongs to $\Delta^* \setminus A(\bar{f}')_{k+1}$. Hence, $\varphi(w) = \tau\big(1^{p_1(0,w'_1,\ldots,w'_m)}2^{p_2(0,w'_1,\ldots,w'_m)}\ldots m^{p_m(0,w'_1,\ldots,w'_m)}\big) \in D_{f'}$ by definition of $A(\bar{f}')$.

2. Let $v \leq w$. It is enough to show that $\varphi(w)_j = 2$ whenever $\varphi(v)_j = 2$ for all $j \in \{1, \ldots, m\}$. But this is immediate from the following facts:

- $\varphi(v)_j = 2 \iff p_j(0, v'_1, \ldots, v'_m) = 0$ by construction of τ,
- $p_j(0, w'_1, \ldots, w'_m) \leq p_j(0, v'_1, \ldots, v'_m)$, since $v_j \leq w_j$ implies $w'_j \leq v'_j$ and p_j is monotonic with respect to vector-ordering.

Thus, we have shown $\varphi(v) \leq \varphi(w)$.

3. Let $w \in D_f$. Then we can conclude as follows:

$$\begin{aligned}
f(w) &= c_{A(\bar{f})}(1^{w'_1}2^{w'_2}\ldots m^{w'_m}) \\
&= c_{A(\bar{f}')}\big(1^{p_1(0,w'_1,\ldots,w'_m)}2^{p_2(0,w'_1,\ldots,w'_m)}\ldots m^{p_m(0,w'_1,\ldots,w'_m)}\big) \\
&= (f' \circ \tau)\big(1^{p_1(0,w'_1,\ldots,w'_m)}2^{p_2(0,w'_1,\ldots,w'_m)}\ldots m^{p_m(0,w'_1,\ldots,w'_m)}\big) \\
&= f'(\varphi(w))
\end{aligned}$$

Thus, the proof of Theorem 3 is complete. $\square$

6 Application

In Sect. 3 and Sect. 4 we established the extended boolean hierarchy of k-partitions and gave alternative characterizations of the classes in it. In the previous section we presented arguments supporting the conjecture that $\mathrm{EBH}_k(\mathrm{NP})$ is strict. In this section we will show that $\mathrm{EBH}_k(\mathrm{NP})$ is closely related to other issues in complexity theory such as the issue of whether one can reduce the set of solutions of NP problems.

The class NPMV introduced by Book, Long, and Selman [BLS84,BLS85] contains all possibly partial, possibly multivalued functions (set functions) f for which there exists a nondeterministic polynomial-time Turing transducer such that $f(x)$ is exactly the set of all outputs ("solutions") made by the transducer on x on accepting paths. (A function f is not defined on x iff $f(x) = \varnothing$). For any $A \subseteq \mathbb{N}_+$, let $\mathrm{NP}_A\mathrm{V}$ denote the class of all NPMV functions f satisfying for all $x \in \Sigma^*$ that the number of solutions of $f(x)$ is an element of $\{0\} \cup A$.

Solution reduction (or output reduction) is captured by the notion of refinements. Let $\mathcal{F}$ and $\mathcal{F}'$ be classes of (possibly partial, possibly multivalued) functions. Then we say that $\mathcal{F}$ functions always have $\mathcal{F}'$ refinements, in symbols $\mathcal{F} \subseteq_c \mathcal{F}'$, iff for all $f \in \mathcal{F}$ there is a $f' \in \mathcal{F}'$ such that $D_f = D_{f'}$ and $f'(x) \subseteq f(x)$ for all $x \in \Sigma^*$.

We focus on the problem of determining for which finite sets $A \subseteq \mathbb{N}_+$ and $B \subseteq \mathbb{N}_+$ it holds $\mathrm{NP}_A\mathrm{V} \subseteq_c \mathrm{NP}_B\mathrm{V}$. To cope this challenge, the following property has been detected as a promising one.

Definition 8. [HOW00] *Let $A, B \subseteq \mathbb{N}_+$ be finite, $A = \{a_1, \ldots, a_m\}$ with $a_1 < a_2 < \cdots < a_m$. We say that the pair (A, B) satisfies the* narrowing-gap condition *if and only if $\|A\| = 0$ or there exist $b_1, \ldots, b_m \in B$ such that $a_1 - b_1 \geq a_2 - b_2 \geq \cdots \geq a_m - b_m \geq 0$.*

Narrowing gaps are sufficient for refinements.

Theorem 4. [HOW00] *Let $A, B \subseteq \mathbb{N}_+$ be finite. If (A, B) satisfies the narrowing-gap condition, then $\mathrm{NP}_A\mathrm{V} \subseteq_c \mathrm{NP}_B\mathrm{V}$.*

And very sophisticated necessary conditions for refinements have been proven under the assumption that the polynomial-time hierarchy does not collapse (cf. [HOW00,HNOS96,Ogi96,NRRS98]). However, the theorems do not fully match the narrowing-gap condition.

Adopting $\mathrm{EBH}_k(\mathrm{NP})$ to be strict for all $k \geq 2$, we can prove that in fact, the narrowing-gap condition is necessary for refinements. To do that, we define particular posets representing a pair of finite sets. Note that if $\min A < \min B$ then $\mathrm{NP}_A\mathrm{V} \not\subseteq_c \mathrm{NP}_B\mathrm{V}$ unrelativized (and in all relativizations).

Definition 9. *Let $A, B \subseteq \mathbb{N}_+$ be finite with $\min B \leq \min A$.*

1. *Then, $\pi(A)$ denotes the finite labeled poset $((G, \leq), f)$ with*
   - $G = \{x \in \{1,2\}^m \mid |x|_1 \in \{0\} \cup A\}$,
   - $f(x) = \|\{z \in G \mid x \leq_{\mathrm{lex}} z\}\|$ *for all* $x \in G$.

2. Then, $\pi(A,B)$ denotes the finite labeled poset $((G,\leq),f)$ with
  - $G = \{(x,y) \mid x,y \in \{1,2\}^m, |x|_1 \in A, |y|_1 \in B, x \leq y\} \cup \{(2^m,2^m)\}$,
  - $f(x,y) = \|\{z \mid x \leq_{\text{lex}} z \wedge (\exists t)[(z,t) \in G]\}\|$ *for all* $(x,y) \in G$.

In Fig. 3, posets representing the question of whether $\mathrm{NP}_{\{1,3\}}\mathrm{V} \subseteq_c \mathrm{NP}_{\{1,2\}}\mathrm{V}$ (which is known to imply a collapse of the polynomial-time hierarchy) are drawn. Observe that $\pi(A) \not\leq \pi(A,B)$.

Theorem 5. *Let $A, B \subseteq \mathbb{N}_+$ be finite with $\min B \leq \min A$. Then the following statements are equivalent.*

1. $\mathrm{NP}_A\mathrm{V}^D \subseteq_c \mathrm{NP}_B\mathrm{V}^D$ *for all oracles D.*
2. $\mathrm{NP}^D(\pi(A)) \subseteq \mathrm{NP}^D(\pi(A,B))$ *for all oracles D.*
3. $\pi(A) \leq \pi(A,B)$.
4. *(A,B) satisfies the narrowing-gap condition.*

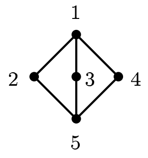
Proof. (1) $\Longrightarrow$ (2): Note that the proof we present relativizes. Let $(G,f) = \pi(A)$ and $(G',f') = \pi(A,B)$. Let $(G,f,S) \in \mathrm{NP}(G,f)$. Define a multivalued function φ as

$$\varphi(x) =_{\text{def}} \left\{ i \;\middle|\; \text{there is a } w \in G \text{ with } w_i = 1 \text{ and } x \in S(w) \right\}$$

for all $x \in \Sigma^*$. We have to show that $\varphi \in \mathrm{NP}_A\mathrm{V}$.

- $\varphi \in \mathrm{NPMV}$: Since S is an NP-homomorphism on a finite poset G there is for each $w \in G$ a nondeterministic polynomial-time Turing machine M_w accepting $S(w)$. Define M to be a transducer that on input x, (i) guesses nondeterministically a $w \in G$, (ii) simulates M_w on x, and (iii) if simulation accepts along a computation path, then outputs all i with $w_i = 1$ on different accepting path prolongations. Clearly, M runs in polynomial time and the set of outputs of $M(x)$ along accepting paths is equal to $\varphi(x)$.
- $\|\varphi(x)\| \in \{0\} \cup A$ for all $x \in \Sigma^*$: Let $x \in \Sigma^*$. There exists exactly one $w \in G$ with $x \in T_S(w)$. Further let R_w denote the set $\{i \mid w_i = 1\}$. Obviously, $R_w \subseteq \varphi(x)$. It suffices to show $\varphi(x) = R_w$. Assume $R_w \neq \varphi(x)$, i.e., there is a $j \in \varphi(x)$ which is not in R_w. Thus there is a $v \in G$ with $v \not\geq w$, $v_j = 1$, and $x \in S(v)$. Hence, $x \in S(v) \cap S(w) = \bigcup_{z<w, z\leq v} S(z)$. So there must exist a $z \in G$ with $z < w$, $z_i = 1$, and $x \in S(z)$. Consequently, $x \notin T_S(w)$, a contradiction.

From the assumption that $\mathrm{NP}_A\mathrm{V} \subseteq_c \mathrm{NP}_B\mathrm{V}$ we get a multivalued function $\varphi' \in \mathrm{NP}_B\mathrm{V}$ with $D_\varphi = D_{\varphi'}$ and $\varphi'(x) \subseteq \varphi(x)$ for all $x \in \Sigma^*$. Consider the following mapping T for all $b \in \{1,2\}^m$ given as $T(b) = \{x \mid \{i \mid b_i = 1\} \subseteq \varphi'(x)\}$. Clearly, $T(b) \in \mathrm{NP}$ and $T(b) \cap T(c) = T(b \wedge c)$. Define a mapping $S' : G' \to \mathrm{NP}$ by $S'(a,b) = S(a) \cap T(b)$ for all $(a,b) \in G'$. We will show that S' is an NP-homomorphism on G'. For that, first observe $S'(2^m,2^m) = \Sigma^*$. So it remains to verify $S'(a,b) \cap S'(a',b') = \bigcup_{(c,d)\in G', (c,d)\leq(a,b),(a',b')} S'(c,d)$. The inclusion "$\supseteq$" holds because S' is monotonic. For "$\subseteq$" let $x \in S'(a,b) \cap S'(a',b') = S(a) \cap S(a') \cap T(b \wedge b')$. Since S is an NP-homomorphism on G and $a, a' \in G$, there are



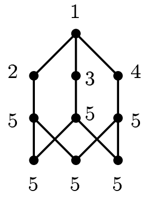


Fig. 3. The 5-posets for the pair (A, B) where $A = \{1, 3\}$ and $B = \{1, 2\}$.

$c \in G$ with $c \leq a$, $c \leq a'$, and $x \in S(c)$, among them the word c with $x \in T_S(c)$. Considering this word we obtain $x \in S(c) \cap T(b \wedge b')$ and $\{i \mid c_i = 1\} = \varphi(x)$ by construction of φ. Let c' be the word in $\{1, 2\}^m$ satisfying $\{i \mid c'_i = 1\} = \varphi'(x)$. Thus $|c'|_1 \in \{0\} \cup B$. Since $x \in T(b \wedge b')$, i.e., $\{i \mid (b \wedge b')_i = 1\} \subseteq \varphi'(x)$, we get $c' \leq b \wedge b'$. Moreover, $\varphi'(x) \subseteq \varphi(x)$ implies $c \leq c'$. Hence, $(c, c') \in G'$, $(c, c') \leq (a, b)$, $(c, c') \leq (a', b')$, and $x \in S(c) \cap T(c') = S'(c, c')$.

To complete the proof of this direction, we will show that $T_{S'}(a, b) \subseteq T_S(a)$ for all $(a, b) \in G'$. The case for $(a, b) = (2^m, 2^m)$ is trivial. So let $a \neq 2^m$ and $b \neq 2^m$. Let $x \in T_{S'}(a, b)$. Then $x \in S'(a, b) = S(a) \cap T(b) \subseteq S(a)$. Assume $x \notin T_S(a)$. Then there is a $c < a$ with $x \in S(c)$, say that c with $x \in T_S(c)$. Thus $\{i \mid c_i = 1\} = \varphi(x)$. Since $D_\varphi = D'_\varphi$ and $\{i \mid b_i = 1\} \subseteq \varphi'(x) \subseteq \varphi(x)$ there is a $d \leq b$ with $c \leq d$, $|d|_1 \in B$, and $x \in T(d)$. Hence, $(c, d) < (a, b)$, $(c, d) \in G'$ and $x \in S(c) \cap T(d) = S'(c, d)$. Thus, $x \notin T_{S'}(a, b)$ what is a contradiction.

(2) $\Longrightarrow$ (3): Application of Theorem 3.

(3) $\Longrightarrow$ (4): Let $(G, f) = \pi(A)$ and $(G', f') = \pi(A, B)$. Let $\varphi : G \to G'$ be a monotonic mapping which witnesses $(G, f) \leq (G', f')$. Let $\varphi_A(w)$ denote the first component of $\varphi(w)$ and $\varphi_B(w)$ denote the second component of $\varphi(w)$, i.e., $\varphi(w) = (\varphi_A(w), \varphi_B(w))$. Since f is injective we have that φ is injective. Thus, it holds $\varphi_A(w) = w$ and $w \leq \varphi_B(w)$ for all $w \in G$. The latter implies $|w|_1 \geq |\varphi_B(w)|_1$. Let $A = \{a_1, \ldots, a_m\}$ with $a_1 < \cdots < a_m$. We will find a sequence $b_1, \ldots, b_m \in B$ with $a_1 - b_1 \geq \cdots \geq a_m - b_m \geq 0$. For that, consider sets $G_j = \{w \in G \mid |w|_1 = a_j\}$. Assume for the moment the claim below would already be proven. If we iterate this claim we get a sequence $1^{a_m}, w_{m-1}, \ldots, w_1$ with

$$a_1 - |\varphi_B(w_1)|_1 \geq \cdots \geq a_{m-1} - |\varphi_B(w_{m-1})|_1 \geq a_m - |\varphi_B(1^{a_m})|_1 \geq 0.$$

Hence, setting $b_j = |\varphi_B(w_j)|_1$ proves that (A, B) satisfies the narrowing-gap condition. So it remains to show the following claim:

Claim. Let $w \in G_j$ for $j \geq 2$. There exists a $v \in G_{j-1}$ with $v \geq w$ and $a_{j-1} - |\varphi_B(v)|_1 \geq a_j - |\varphi_B(w)|_1$.

To prove it, without loss of generality, let w have the prefix consisting of a_j times the letter 1 and, for convenience, let $\varphi_B(w)$ have the prefix consisting of $|\varphi_B(w)|_1$ times the letter 1. Assume to the contrary that for all $v \in G_{j-1}$ with

$v > w$ it holds $a_{j-1} - |\varphi_B(v)|_1 < a_j - |\varphi_B(w)|_1$. Let $b_j = |\varphi_B(w)|_1$ and $b_{j-1} = \min\{|\varphi_B(v)|_1 \mid v \in G_{j-1}, v > w\}$. The assumption implies $b_j - b_{j-1} < a_j - a_{j-1}$. Consider the word v' having the prefix starting with $a_j - a_{j-1}$ times the letter 2 and then having a_{j-1} times the letter 1 before coming 2's. Clearly, $v' > w$ and $v' \in G_{j-1}$. Since φ is monotonic, we have (i) $\varphi_B(v') \geq \varphi_B(w)$ and furthermore (ii) $\varphi_B(v') \geq v'$. From (ii) it follows that $\varphi_B(v')$ must be a word starting with at least $a_j - a_{j-1}$ times the letter 2 and from (i) it follows that $\varphi_B(v')$ can have at most $b_j - b_{j-1}$ times the letter 2 in any prefix before coming a letter 1. Hence, $\varphi_B(v') \not\geq v'$, a contradiction.

(4) $\Longrightarrow$ (1): Relativization of Theorem 4. □

Corollary 4. *Assume the hierarchy* $\mathrm{EBH}_k(\mathrm{NP})$ *is strict for all* $k \geq 2$. *Let* $A, B \subseteq \mathbb{N}_+$ *be finite. Then* (A, B) *satisfies the narrowing-gap condition if and only if* $\mathrm{NP}_A\mathrm{V} \subseteq_c \mathrm{NP}_B\mathrm{V}$.

Corollary 5. *Let* $A, B \subseteq \mathbb{N}_+$ *be finite. Then* (A, B) *satisfies the narrowing-gap condition if and only if* $\mathrm{NP}_A\mathrm{V}^D \subseteq_c \mathrm{NP}_B\mathrm{V}^D$ *for all oracles* D.

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