The Boolean Hierarchy of Partitions

Sven Kosub   Klaus W. Wagner
Lehrstuhl für Theoretische Informatik
Julius-Maximilians-Universität Würzburg
{*kosub,wagner*}*@informatik.uni-wuerzburg.de*

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Abstract

We introduce the boolean hierarchy of k-partitions over NP for $k \geq 3$ as a generalization of the booelean hierarchy of sets (i.e., 2-partitions) over NP. Whereas the structure of the latter hierarchy is rather simple the structure of the boolean hierarchy of k-partitions over NP for $k \geq 3$ turns out to be much more complicated. We establish the Embedding Conjecture which enables us to get an overview on this structure. This conjecture is supported by several partial results. We also prove machine characterizations and complete problems for the classes of the boolean hierarchy of k-partitions over NP.

1 Introduction

To divide the real world into two parts like big and small, black and white, or good and bad usually oversimplifies things. In most cases a partition into many parts is more appropriate. For example, take marks in school, scores for papers submitted to a conference, salary groups, or classes of risk. In mathematics, k-valued logic is just a language for reasoning about k-valent objects.

Nevertheless, complexity theoreticians mainly investigate the complexity of sets, i.e., partitions into two parts, or the complexity of functions, i.e., partitions into usually infinitely many parts. But what about partitions into $3, 4, 5, \ldots$ parts? This paper studies, as a first step in this direction, complexity classes of k-partitions which correspond to the classes of the boolean hierarchy of sets (i.e., 2-partitions). This investigation is justified by the fact that in the cases $k \geq 3$ there are interesting new phenomena which cannot be treated appropriately when encoding k-partitions by sets.

The most general way to define the boolean hierarchy of sets over NP is as follows (see [WW85]): For a boolean function $f : \{1,2\}^m \to \{1,2\}$ and sets $B_1,\ldots,B_m$ define the set $f(B_1,\ldots,B_m)$ by $c_{f(B_1,\ldots,B_m)}(x) =_{\text{def}} f(c_{B_1}(x),\ldots,c_{B_m}(x))$. The boolean hierarchy (of sets) over NP consists of the classes $\mathrm{NP}(f) = \{\, f(B_1,\ldots,B_m) \mid B_1,\ldots,B_m \in \mathrm{NP} \,\}$. It was proved in [WW85] that every class $\mathrm{NP}(f)$ coincides with one of the classes $\mathrm{NP}(i)$ or $\mathrm{coNP}(i)$ where $\mathrm{NP}(i)$ is the class of all sets which are the symmetric difference of i NP-sets.

This approach is generalized in Section 3 to the case of k-partitions. The characteristic function of a k-partition $A = (A_1,\ldots,A_k)$ is defined by $c_A(x) = i \iff x \in A_i$. For a function $f : \{1,2\}^m \to \{1,2,\ldots,k\}$ and sets $B_1,\ldots,B_m$ define a k-partition $f(B_1,\ldots,B_m)$ by $c_{f(B_1,\ldots,B_m)} =_{\text{def}} f(c_{B_1}(x),\ldots,c_{B_m}(x))$. The boolean hierarchy of k-partitions over NP consists of the classes $\mathrm{NP}(f) =_{\text{def}} \{\, f(B_1,\ldots,B_m) \mid B_1,\ldots,B_m \in \mathrm{NP} \,\}$. The boolean hierarchy of sets over NP now appears as the special case $k = 2$.

Whereas the boolean hierarchy of sets over NP has a very simple structure (note that $\mathrm{NP}(i) \cup \mathrm{coNP}(i) \subseteq \mathrm{NP}(i+1) \cap \mathrm{coNP}(i+1)$ for all $i \geq 1$), the situation is much more complicated for the booelean hierarchy of k-partitions in the case $k \geq 3$. The main question is: Can we get an overview on the structure of this hierarchy? This question is not answered completely so far, but in the remaining sections we give partial answers, and we establish a conjecture.

A function $f : \{1,2\}^m \to \{1,2,\ldots,k\}$ which defines the class $\mathrm{NP}(f)$ of k-partitions corresponds to the finite boolean lattice $(\{1,2\}^m, \leq)$ with the labeling function f. Generalizing this idea we define for every lattice G with labeling function $f : G \to \{1,2,\ldots,k\}$ (for short: the k-lattice (G,f)) a class $\mathrm{NP}(G,f)$ of k-partitions. This does not result in more classes: In Section 5 we prove that for every k-lattice (G,f) there exists a finite function g such that $\mathrm{NP}(G,f) = \mathrm{NP}(g)$. However, the use of arbitrary lattices instead of only boolean lattices simplifies many considerations.

To get an idea of the structure of the boolean hierarchy of k-partitions over NP it is very important to have a criterion for $\mathrm{NP}(G,f) \subseteq \mathrm{NP}(G',f')$ for k-lattices (G,f) and (G',f'). In Section 5 we define a relation $\leq$ as follows: $(G,f) \leq (G',f')$ if and only if there is a monotonic mapping $\varphi : G \to G'$ such that $f(x) = f'(\varphi(x))$. We prove the Embedding Lemma which says that $(G,f) \leq (G',f')$ implies $\mathrm{NP}(G,f) \subseteq \mathrm{NP}(G',f')$, and we establish the Embedding Conjecture which says that the converse is also true unless the boolean hierarchy of sets over NP collapses.

In Section 6 we collect evidence for our Embedding Conjecture. For $k = 2$ we prove this conjecture to be true. Moreover, we prove two theorems which enable us to verify the Embedding Conjecture for $k \geq 3$ for a large class of k-lattices.

Assume the Embedding Conjecture is true. Then the set inclusion structure of the boolean hierarchy of k-partitions is isomorphic to the partial order of $\leq$-equivalence classes of k-lattices with respect to $\leq$. In Section 7 we present the partial order of all 132 equivalence classes which contain boolean 3-lattices of the form $(\{1,2\}^3, f)$. Furthermore we prove that the partial order of equivalence classes of 3-lattices does not have bounded width. This gives an impression on the complexity of the (conjectured) structure of the boolean hierarchy of 3-partitions over NP.

In Section 8 we prove that the partition classes $\mathrm{NP}(f)$ can be characterized in a natural way by nondeterministic polynomial-time machines. Using this fact we exhibite for every f a partition, which is in a sense polynomial-time many-one complete for $\mathrm{NP}(f)$.

Finally let us mention that the results in this paper besides those in Section 8 are proved even for arbitrary classes $\mathcal{K}$ of subsets of a set M such that $\varnothing, M \in \mathcal{K}$ and $\mathcal{K}$ is closed under union and intersection.

2 Preliminaries

For the classes $\mathcal{K}$ and $\mathcal{K}'$ of subsets of a set M, define $\mathrm{co}\mathcal{K} = \{\, \overline{L} \mid L \in \mathcal{K} \,\}$, $\mathcal{K} \wedge \mathcal{K}' =_{\mathrm{def}} \{\, A \cap B \mid A \in \mathcal{K}, B \in \mathcal{K}' \,\}$, $\mathcal{K} \vee \mathcal{K}' =_{\mathrm{def}} \{\, A \cup B \mid A \in \mathcal{K}, B \in \mathcal{K}' \,\}$, $\mathcal{K} \dot{\vee} \mathcal{K}' =_{\mathrm{def}} \{\, A \cup B \mid A \in \mathcal{K}, B \in \mathcal{K}', A \cap B = \varnothing \,\}$, and $\mathcal{K} \oplus \mathcal{K}' =_{\mathrm{def}} \{\, A \triangle B \mid A \in \mathcal{K}, B \in \mathcal{K}' \,\}$. The classes $\mathcal{K}(i)$ and $\mathrm{co}\mathcal{K}(i)$ defined by $\mathcal{K}(1) =_{\mathrm{def}} \mathcal{K}$ and $\mathcal{K}(i+1) =_{\mathrm{def}} \mathcal{K}(i) \oplus \mathcal{K}$ for $i \geq 1$ build the *boolean hierarchy over* $\mathcal{K}$. There are many equivalent definitions (see [WW85, Köb85, CH86, KSW87, CGH⁺88]). Some of them can be found in the following theorem.

Theorem 2.1. *Let $\varnothing, M \in \mathcal{K}$, let $\mathcal{K}$ be closed under union and intersection, and let $i \geq 1$.*

1. $\mathcal{K}(2i-1) = \{\, A_1 \cup \bigcup_{j=1}^{i-1} (A_{2j+1} \setminus A_{2j}) \mid A_1, \ldots, A_{2i-1} \in \mathcal{K} \text{ and } A_1 \subseteq \cdots \subseteq A_{2i-1} \,\}$.
2. $\mathcal{K}(2i) = \{\, \bigcup_{j=1}^{i} (A_{2j} \setminus A_{2j-1}) \mid A_1, \ldots, A_{2i} \in \mathcal{K} \text{ and } A_1 \subseteq \cdots \subseteq A_{2i} \,\}$.
3. $\mathcal{K}(2i) = \mathcal{K}(2i-1) \wedge \mathrm{co}\mathcal{K}$.
4. $\mathcal{K}(2i+1) = \mathcal{K}(2i) \vee \mathcal{K}$.
5. $\mathcal{K}(i+1) = \mathrm{co}\mathcal{K}(i) \wedge \mathcal{K}$.
6. $\mathcal{K}(i+2) = \mathcal{K}(i) \vee (\mathcal{K} \wedge \mathrm{co}\mathcal{K}) = \mathcal{K}(i) \wedge (\mathcal{K} \vee \mathrm{co}\mathcal{K})$.
7. $\mathcal{K}(i) \cup \mathrm{co}\mathcal{K}(i) \subseteq \mathcal{K}(i+1) \cap \mathrm{co}\mathcal{K}(i+1)$.

The class $\mathrm{BC}(\mathcal{K})$ is the boolean closure of $\mathcal{K}$, that is the smallest class which contains $\mathcal{K}$ and which is closed under union, intersection, and complementation.

Further we need some notions from lattice theory and order theory (see e.g., [Grä78, DP90]). A finite poset $(G, \leq)$ is a lattice if for all $x, y \in G$ there exist a) exactly one maximal element $z \in G$ such that $z \leq x$ and $z \leq y$ (which will be denoted by $x \wedge y$), and b) exactly one minimal element $z \in G$ such that $z \geq x$ and $z \geq y$ (which will be denoted by $x \vee y$). For $H \subseteq G$ the unique maximal element less than or equal to all $x \in H$ is denoted by $\bigwedge H$, and the unique minimal element greater than or equal to all $x \in H$ is denoted by $\bigvee H$. For a lattice G we define $0_G =_{\text{def}} \bigwedge G$ and $1_G =_{\text{def}} \bigvee G$. An element $x \neq 1_G$ is said to be *meet-irreducible* iff $x = a \wedge b$ implies $x = a$ or $x = b$ for all $a, b \in G$.

For symbols $a_1, a_2, \ldots, a_m$ from an alphabet Σ, we identify the m-tuple $(a_1, a_2, \ldots, a_m)$ with the word $a_1 a_2 \ldots a_m \in \Sigma^m$. If there is an order $\leq$ on Σ, we assume Σ^m to be partially ordered by the vector-ordering, that is $a_1 a_2 \ldots a_m \leq b_1 b_2 \ldots b_m$ if and only if $a_i \leq b_i$ for all $i \in \{1, 2, \ldots, m\}$.

Finally let us make some notational conventions about partitions. For any set M, a k-tuple $A = (A_1, \ldots, A_k)$ with $A_i \subseteq M$ for $i = 1, \ldots, k$ is said to be a *k-partition of M* if and only if $\bigcup_{i=1}^k A_i = M$ and $A_i \cap A_j = \varnothing$ if $i \neq j$. The set A_i is said to be the *i-th component of* A. Let $c_A : M \to \{1, \ldots, k\}$ be the characteristic function of a k-partition $A = (A_1, \ldots, A_k)$ of M, i.e., $c_A(x) = i$ if and only if $x \in A_i$ for every $x \in M$ and $i \in \{1, \ldots, k\}$.

For classes $\mathcal{K}_1, \ldots, \mathcal{K}_k \subseteq \mathcal{P}(M)$ and $i = 1, \ldots, k$ define

$$(\mathcal{K}_1, \ldots, \mathcal{K}_k) =_{\text{def}} \left\{ A \,\middle|\, A \text{ is } k\text{-partition of } M \text{ and } A_i \in \mathcal{K}_i \text{ for } i = 1, \ldots, k \right\}$$

and

$$(\mathcal{K}_1, \ldots, \mathcal{K}_{i-1}, \cdot, \mathcal{K}_{i+1}, \ldots, \mathcal{K}_k) =_{\text{def}} (\mathcal{K}_1, \ldots, \mathcal{K}_{i-1}, \mathcal{P}(M), \mathcal{K}_{i+1}, \ldots, \mathcal{K}_k).$$

For a class $\mathcal{K}$ of k-partitions and $i = 1, \ldots, k$, let $\mathcal{K}_i =_{\text{def}} \left\{ A_i \,\middle|\, A \in \mathcal{K} \right\}$ be the *i-th projection of* $\mathcal{K}$. Obviously, $\mathcal{K} \subseteq (\mathcal{K}_1, \ldots, \mathcal{K}_k)$ and $\mathcal{K}_i \subseteq \mathrm{co}\dot{\bigvee}_{j \in \{1,\ldots,k\} \setminus \{i\}} \mathcal{K}_j$ for $i = 1, \ldots, k$. Whereas a class $\mathcal{K}$ of 2-partitions is uniquely determined by each of its projections, i.e., $\mathcal{K} = (\mathcal{K}_1, \mathcal{K}_2)$, it can happen that $\mathcal{K} \subset (\mathcal{K}_1, \ldots, \mathcal{K}_k)$ for $k \geq 3$ as shown by the following example. For $M = \{1, 2\}$ and $\mathcal{K} = \big\{ \big(\{1\}, \varnothing, \{2\}\big), \ \big(\varnothing, \{2\}, \{1\}\big), \ \big(\{1, 2\}, \varnothing, \varnothing\big) \big\}$ we obtain $\mathcal{K}_1 = \big\{\varnothing, \{1\}, \{1, 2\}\big\}$, $\mathcal{K}_2 = \big\{\varnothing, \{2\}\big\}$, $\mathcal{K}_3 = \big\{\varnothing, \{1\}, \{2\}\big\}$, and consequently $(\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3) = \mathcal{K} \cup \big\{ \big(\{1\}, \{2\}, \varnothing\big) \big\}$.

In what follows we identify a set A with the 2-partition $(A, \overline{A})$, and we identify a class $\mathcal{K}$ of sets with the class $(\mathcal{K}, \mathrm{co}\mathcal{K}) = (\mathcal{K}, \cdot) = (\cdot, \mathrm{co}\mathcal{K})$ of 2-partitions.

3 Partition Classes Defined by Finite Functions

Let $\mathcal{K}$ be a class of subsets of M such that $\varnothing, M \in \mathcal{K}$ and $\mathcal{K}$ is closed under union and intersection. In the literature, one way to define the classes of the boolean hierarchy of sets over $\mathcal{K}$ is as follows (see [WW85]). Let $f : \{1, 2\}^m \to \{1, 2\}$ be a boolean function. For $B_1, \ldots, B_m \in \mathcal{K}$ the set $f(B_1, \ldots, B_m)$ is defined by $c_{f(B_1,\ldots,B_m)}(x) = f(c_{B_1}(x), \ldots, c_{B_m}(x))$. Then the classes $\mathcal{K}(f) =_{\text{def}} \left\{ f(B_1, \ldots, B_m) \,\middle|\, B_1, \ldots, B_m \in \mathcal{K} \right\}$ form the boolean hierarchy over $\mathcal{K}$. Using finite functions $f : \{1,2\}^m \to \{1, 2, \ldots, k\}$ we generalize this definition (remember in which sense sets are 2-partitions) to obtain the classes of the boolean hierarchy of k-partitions over $\mathcal{K}$ as follows.

Definition 3.1.

1. For $f : \{1,2\}^m \to \{1, 2, \ldots, k\}$ and $B_1, \ldots, B_m \in \mathcal{K}$, the k-partition $f(B_1, \ldots, B_m)$ is defined by $c_{f(B_1,\ldots,B_m)}(x) = f(c_{B_1}(x), \ldots, c_{B_m}(x))$.
2. For $f : \{1,2\}^m \to \{1, 2, \ldots, k\}$, the class $\mathcal{K}(f) =_{\text{def}} \left\{ f(B_1, \ldots, B_m) \,\middle|\, B_1, \ldots, B_m \in \mathcal{K} \right\}$ is the *class of k-partitions over $\mathcal{K}$ defined by f*.
3. The family $\mathrm{BH}_k(\mathcal{K}) =_{\text{def}} \left\{ \mathcal{K}(f) \,\middle|\, f : \{1,2\}^m \to \{1, 2, \ldots, k\} \right\}$ is the *boolean hierarchy of k-partitions over $\mathcal{K}$*.
4. $\mathrm{BC}_k(\mathcal{K}) =_{\text{def}} \bigcup \left\{ \mathcal{K}(f) \,\middle|\, f : \{1,2\}^m \to \{1, 2, \ldots, k\} \right\}$.

Obviously, if $i \in \{1, \ldots, k\}$ is not a value of $f : \{1,2\}^m \to \{1, 2, \ldots, k\}$ then $\mathcal{K}(f)_i = \{\varnothing\}$, i.e., $\mathcal{K}(f)$ does not really have an i-th component. Therefore we assume in what follows that f is surjective.

The following proposition shows that every partition in $\mathcal{K}(f)$ consists of sets from the boolean hierarchy over $\mathcal{K}$. This also justifies the use of the term *boolean* in the above definition.

Proposition 3.2. *Let $f : \{1,2\}^m \to \{1, 2, \ldots, k\}$.*

1. $(\mathcal{K}, \mathcal{K}, \ldots, \mathcal{K}) \subseteq \mathcal{K}(f) \subseteq (\mathrm{BC}(\mathcal{K}), \mathrm{BC}(\mathcal{K}), \ldots, \mathrm{BC}(\mathcal{K}))$.
2. *If $\mathcal{K}$ is closed under complementation then $\mathcal{K}(f) = (\mathcal{K}, \mathcal{K}, \ldots, \mathcal{K})$.*
3. $\mathrm{BC}_k(\mathcal{K}) = (\mathrm{BC}(\mathcal{K}), \mathrm{BC}(\mathcal{K}), \ldots, \mathrm{BC}(\mathcal{K}))$.

Proof. 1. $\mathcal{K}(f) \subseteq (\mathrm{BC}(\mathcal{K}), \mathrm{BC}(\mathcal{K}), \ldots, \mathrm{BC}(\mathcal{K}))$. Let $B_1, \ldots, B_m \in \mathcal{K}$, and consider $(A_1, \ldots, A_k) =_{\text{def}} f(B_1, \ldots, B_m)$. For $i \in \{1, \ldots, k\}$ we obtain

$$x \in A_i \iff \bigvee_{f(a_1 \ldots a_m) = i} \; \bigwedge_{j=1}^{m} c_{B_j}(x) = a_i$$

and consequently

$$A_i \quad = \bigcup_{f(a_1 \ldots a_m) = i} \Big(\bigcap_{a_i = 1} B_j \setminus \bigcup_{a_i = 2} B_j \Big). \qquad (*)$$

By Theorem 2.1.5 and 2.1.6 we get $A_i \in \mathcal{K}(2 \cdot |f^{-1}(i)|)$.

$(\mathcal{K},\mathcal{K},\ldots,\mathcal{K}) \subseteq \mathcal{K}(f)$. Let $(D_1,\ldots,D_k) \in (\mathcal{K},\mathcal{K},\ldots,\mathcal{K})$. For every $i \in \{1,\ldots,k\}$ fix some $v_i \in \{1,2\}^m$ such that $f(v_i) = i$. For $j \in \{1,\ldots,m\}$ define $B_j =_{\mathrm{def}} \bigcup_{v_i \leq 2^{j-1}12^{m-j}} D_i$. For $a_1 \ldots a_m \in \{1,2\}^m$ observe $\bigcap_{a_j=1} B_j = \bigcup_{v_l \leq a_1\ldots a_m} D_l$ and $\bigcup_{a_j=2} B_j = \bigcup_{v_l < a_1\ldots a_m} D_l$. By $(*)$ we obtain $f(B_1,\ldots,B_m) = (D_1,\ldots,D_k)$.

2. This statement is an immediate consequence of the first one.

3. The inclusion $\mathrm{BC}_k(\mathcal{K}) \subseteq (\mathrm{BC}(\mathcal{K}),\mathrm{BC}(\mathcal{K}),\ldots,\mathrm{BC}(\mathcal{K}))$ follows directly from 1. For the converse inclusion let $L \in (\mathrm{BC}(\mathcal{K}),\mathrm{BC}(\mathcal{K}),\ldots,\mathrm{BC}(\mathcal{K}))$, i.e., there exists an $r \geq 1$ such that $L_i \in \mathcal{K}(r)$ for $i = 1,2,\ldots,k$. Hence there exist sets $A_1,A_2,\ldots,A_{k\cdot r} \in \mathcal{K}$ such that $L_i = A_{(i-1)\cdot r+1} \triangle A_{(i-1)\cdot r+2} \triangle \ldots \triangle A_{i\cdot r}$ for $i = 1,2,\ldots,k$. Observe that for every $a_1a_2\ldots a_{k\cdot r}$ there exists an $i \in \{1,2,\ldots,k\}$ such that $(\bigcap_{a_j=1} A_j) \cap (\bigcap_{a_j=2} \overline{A_j}) \subseteq L_i$. So we can define a function $f : \{1,2\}^{k\cdot r} \to \{1,2,\ldots,k\}$ by $f(a_1a_2\ldots a_{k\cdot r}) = i \Leftrightarrow_{\mathrm{def}} (\bigcap_{a_j=1} A_j) \cap (\bigcap_{a_j=2} \overline{A_j}) \subseteq L_i$ for all $a_1a_2\ldots a_{k\cdot r} \in \{1,2\}^{k\cdot r}$, and we obtain $L = f(A_1,A_2,\ldots,A_{k\cdot r})$. ❑

For $k = 2$ the classes $\mathcal{K}(f)$ of the boolean hierarchy $\mathrm{BH}_2(\mathcal{K})$ of sets (2-partitions) over $\mathcal{K}$ have been completely characterized. For $f : \{1,2\}^m \to \{1,2\}$ let $\mu(f)$ be the maximum number of alternations of f-labels which can occur in a $\leq$-chain in $(\{1,2\}^m,\leq)$.

Theorem 3.3. [WW85] *For* $f : \{1,2\}^m \to \{1,2\}$,

$$\mathcal{K}(f) = \begin{cases} \mathcal{K}(\mu(f)), & \text{if } f(2^m) = 2, \\ \mathrm{co}\mathcal{K}(\mu(f)), & \text{if } f(2^m) = 1. \end{cases}$$

Consequently, $\mathrm{BH}_2(\mathcal{K}) = \{\, \mathcal{K}(k) \mid k \geq 1 \,\} \cup \{\, \mathrm{co}\mathcal{K}(k) \mid k \geq 1 \,\}$, and given a function $f : \{1,2\}^m \to \{1,2\}$ it is easy to determine the class $\mathcal{K}(k)$ or $\mathrm{co}\mathcal{K}(k)$ which coincides with $\mathcal{K}(f)$. Because of Theorem 2.1.5 the classes of $\mathrm{BH}_2(\mathcal{K})$ form a simple structure with respect to set inclusion. There do not exist three classes in $\mathrm{BH}_2(\mathcal{K})$ which are incomparable in this sense.

It is the goal of this paper to get insights into the structure of the boolean hierarchy $\mathrm{BH}_k(\mathcal{K})$ of k-partitions over $\mathcal{K}$ for $k \geq 3$. What we can say at this point is, that already for $k = 3$ the structure of $\mathrm{BH}_k(\mathcal{K})$ with respect to set inclusion is not as simple as for $k = 2$ (unless $\mathcal{K} = \mathrm{co}\mathcal{K}$). This is shown by the following example.

Example. For a,b,c such that $\{a,b,c\} = \{1,2,3\}$ define the function $f_{abc} : \{1,2\}^2 \to \{1,2,3\}$ by $f_{abc}(11) = a$, $f_{abc}(12) = f_{abc}(21) = b$, and $f_{abc}(22) = c$. Obviously, $\mathcal{K}(f_{abc})_a = \mathcal{K}$, $\mathcal{K}(f_{abc})_b = \mathcal{K}(2)$, and $\mathcal{K}(f_{abc})_c = \mathrm{co}\mathcal{K}$. Now let $abc \neq a'b'c'$. If $\mathcal{K}(f_{abc}) = \mathcal{K}(f_{a'b'c'})$ then $\mathcal{K} = \mathcal{K}(2)$ or $\mathcal{K} = \mathrm{co}\mathcal{K}$, or $\mathcal{K}(2) = \mathrm{co}\mathcal{K}$. In each of this cases we obtain $\mathcal{K} = \mathrm{co}\mathcal{K}$. Consequently, if $\mathcal{K} \neq \mathrm{co}\mathcal{K}$ then the six classes $\mathcal{K}(f_{abc})$ are pairwise incomparable with respect to set inclusion.

4 Partition Classes Defined by Lattices

It turns out that, for $f : \{1,2\}^m \to \{1,2,\dots,k\}$, a k-partition $f(B_1,\dots,B_m)$ has a very natural equivalent lattice-theoretical definition. Consider the boolean lattice $\{1,2\}^m$ with the partial vector-ordering $\leq$, and consider the function $S : \{1,2\}^m \to \mathcal{K}$ defined by $S(a_1,\dots,a_m) =_{\mathrm{def}} \bigcap_{a_i=1} B_i$, where we set $\bigcap_{\varnothing} B_i =_{\mathrm{def}} M$ (see Figure 1). Note that $S(2,\dots,2) = M$ and $S(a \wedge b) = S(a) \cap S(b)$ for all $a, b \in \{1,2\}^m$. Defining $T_S(a) =_{\mathrm{def}} S(a) \setminus \bigcup_{b<a} S(b)$ we obtain the i-th component of $f(B_1,\dots,B_m)$ as $f(B_1,\dots,B_m)_i = \bigcup_{f(a)=i} T_S(a)$, i.e., $f(B_1,\dots,B_m)$ can also be given by the function $S : \{1,2\}^m \to \mathcal{K}$.

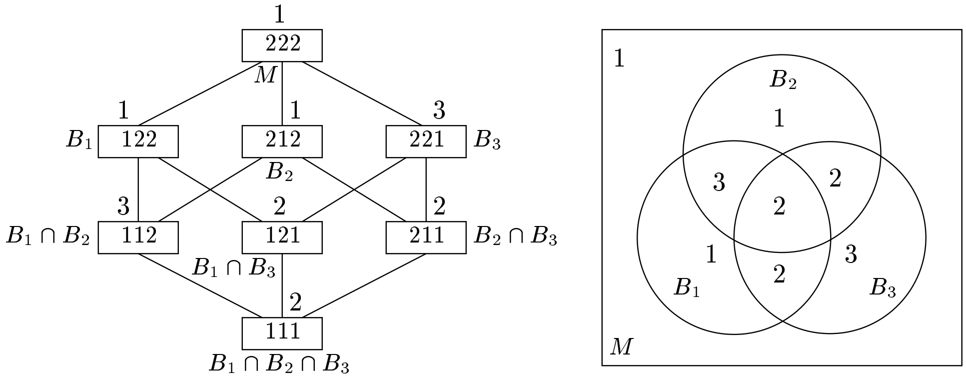


Figure 1

On the other side, if we have any function $S : \{1,2\}^m \to \mathcal{K}$ such that $S(2,\dots,2) = M$ and $S(a \wedge b) = S(a) \cap S(b)$ for all $a, b \in \{1,2\}^m$ we can define $B_j =_{\mathrm{def}} S(2^{j-1}12^{m-j})$ for $j \in \{1,2,\dots,m\}$, and we obtain $f(B_1,\dots,B_m)_i = \bigcup_{f(a)=i} T_S(i)$ for $i \in \{1,2,\dots,k\}$. In this manner the class $\mathcal{K}(f)$ of k-partitions is completely characterized by the labeled boolean lattice $\big((\{1,2\}^m, \leq), f\big)$. In this section we will see that classes of k-partitions can also be defined by weaker structures than boolean algebras.

Definition 4.1. Let G be a lattice.

1. A mapping $S : G \to \mathcal{K}$ is said to be a *$\mathcal{K}$-homomorphism on G* if and only if $S(1_G) = M$ and $S(a \wedge b) = S(a) \cap S(b)$ for all $a, b \in G$.
2. For any $\mathcal{K}$-homomorphism S on G, let $T_S(a) =_{\mathrm{def}} S(a) \setminus \bigcup_{b<a} S(b)$ for all $a \in G$.

Lemma 4.2. *Let G be a lattice, and let S be a $\mathcal{K}$-homomorphism on G.*

1. *$T_S(a) \in \mathcal{K} \wedge \mathrm{co}\mathcal{K}$ for every $a \in G$.*
2. *$S(a) = \bigcup_{b \leq a} T_S(b)$ for every $a \in G$.*
3. *The set of all $T_S(a)$ for $a \in G$ yields a partition of M.*

4. *S is completely determined by its values for the meet-irreducible elements. That is, if S and S′ are two $\mathcal{K}$-homomorphisms on G such that $S(a) = S'(a)$ for all meet-irreducible $a \in G$ then $S(a) = S'(a)$ for all $a \in G$.*

Proof. 1. Observe $T_S(a) = S(a) \cap \overline{\bigcup_{b<a} S(b)} \in \mathcal{K} \wedge \mathrm{co}\mathcal{K}$ since $\mathcal{K}$ is closed under union.

2. The direction "$\supseteq$" is obvious since $T_S(b) \subseteq S(b) \subseteq S(a)$ for $b \leq a$. The converse inclusion can be verified by induction on $<$. Obviously, $S(0_G) = T_S(0_G)$. For $a > 0_G$ we obtain $S(a) = T_S(a) \cup \bigcup_{b<a} S(b) = T_S(a) \cup \bigcup_{b<a} \bigcup_{c \leq b} T_S(c) = T_S(a) \cup \bigcup_{c<a} T_S(c)$.

3. We have to show that every $x \in M$ is contained in exactly one $T_S(a)$. Proving the existence of such an $a \in G$, define $H =_{\mathrm{def}} \{\, a \,|\, x \in S(a) \,\}$ which is non-empty since $\bigcup_{a \in G} S(a) = M$. Since G is finite it follows that $x \in S(\bigwedge H)$. Let $b < \bigwedge H$. Then $b \notin H$, and hence $x \notin S(b)$. So, $x \in S(\bigwedge H) \setminus \bigcup_{b < \bigwedge H} S(b) = T_S(\bigwedge H)$. To show the uniqueness assume that there is an $a \neq \bigwedge H$ such that $x \in T_S(a)$. Then $x \in S(a)$ and hence $a \in H$. Consequently, $a > \bigwedge H$ and we obtain $x \notin S(a) \setminus \bigcup_{b<a} S(b) = T_S(a)$, a contradiction.

4. This is an immediate consequence of the definition of meet-irreducible elements and the condition $S(a \wedge b) = S(a) \cap S(b)$ for $\mathcal{K}$-homomorphisms. ❑

Any pair (G, f) of an arbitrary finite poset G and a function $f : G \to \{1, 2, \ldots, k\}$ is called a *k-poset*. A k-poset which is a lattice (boolean lattice) is called a *k-lattice* (*boolean k-lattice*, resp.).

Lemma 4.2 provides the soundness of the following definition.

Definition 4.3. 1. For a k-lattice (G, f) and a $\mathcal{K}$-homomorphism S on G let

$$(G, f, S) =_{\mathrm{def}} \left(\bigcup_{f(a)=1} T_S(a), \ldots, \bigcup_{f(a)=k} T_S(a)\right)$$

be the k-partition defined by (G, f) and S.

2. For a k-lattice (G, f) let

$$\mathcal{K}(G, f) =_{\mathrm{def}} \{\, (G, f, S) \,|\, S \text{ is } \mathcal{K}\text{-homomorphism on } G \,\}$$

be the class of k-partitions defined by (G, f).

Example. Consider the 3-lattice (G, f) in Figure 2. The meet-irreducible elements of G are a, b, and c. By Lemma 4.2.4 every $\mathcal{K}$-homomorphism $S : G \to \mathcal{K}$ is determined by fixing $S(a) = A$, $S(b) = B$, and $S(c) = C$. By the definition of $\mathcal{K}$-homomorphisms we get $S(1) = M$, $S(d) = S(a \wedge b) = S(a) \cap S(b) = A \cap B$, and $S(0) = S(d \wedge c) = S(d) \cap S(c) = A \cap B \cap C$. Furthermore, $C = S(c) = S(c \wedge b) = S(c) \cap S(b) = C \cap B$, i.e., $C \subseteq B$. We obtain:

$$
\begin{array}{rcl@{\quad}l}
T_S(1) & = & M \setminus (A \cup B) & = \ \overline{A} \cap \overline{B} \\
T_S(a) & = & A \setminus (A \cap B) & = \ A \cap \overline{B} \\
T_S(b) & = & B \setminus ((A \cap B) \cup C) & = \ \overline{A} \cap B \cap \overline{B} \\
T_S(c) & = & C \setminus (A \cap B \cap C) & = \ \overline{A} \cap C \\
T_S(d) & = & (A \cap B) \setminus (A \cap B \cap C) & = \ A \cap B \cap \overline{C} \\
T_S(0) & = & (A \cap B \cap C) & = \ A \cap C
\end{array}
$$

Hence

$$
\begin{array}{rcl}
(G, f, S) & = & (T_S(a) \cup T_S(0), T_S(1) \cup T_S(c), T_S(b) \cup T_S(d)) \\
 & = & (A \cap (\overline{B} \cup C), \overline{A} \cap (\overline{B} \cup C), B \cap \overline{C})
\end{array}
$$

and

$$
\begin{array}{rcl}
\mathcal{K}(G, f) & = & \left\{ (A \cap (\overline{B} \cup C), \overline{A} \cap (\overline{B} \cup C), B \cap \overline{C}) \,\middle|\, A, B, C \in \mathcal{K} \text{ and } C \subseteq B \right\} \\
 & \subseteq & (\mathcal{K}(3), \mathrm{co}\mathcal{K}(3), \mathcal{K}(2))
\end{array}
$$

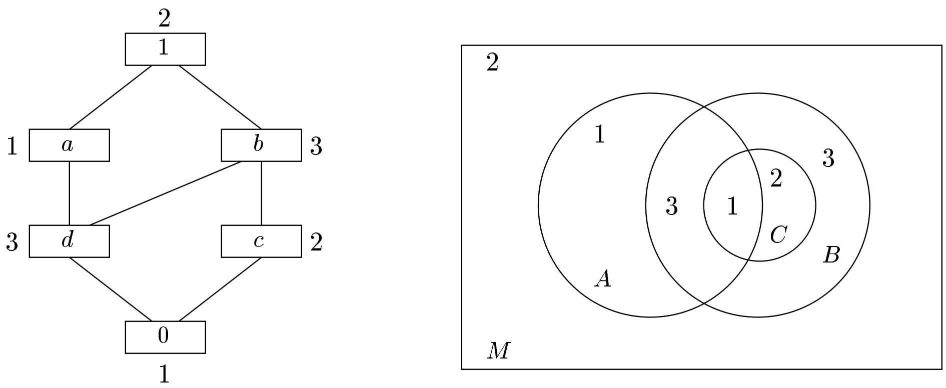


Figure 2

The discussion of the beginning of this sections yields the following proposition.

Proposition 4.4. *$\mathcal{K}(f) = \mathcal{K}((\{1,2\}^m, \leq), f)$ for every $f : \{1,2\}^m \to \{1, 2, \ldots, k\}$.*

So, if (G, f) is a boolean k-lattice then $\mathcal{K}(G, f) = \mathcal{K}(f)$. But if (G, f) is an arbitrary k-lattice, is $\mathcal{K}(G, f)$ also of the form $\mathcal{K}(g)$ for a suitable function g? The following theorem says that this is generally true. This turns out to be very important for the further study of the structure of the boolean hierarchy of k-partitions because instead of large boolean k-lattices one can handle with usually much smaller equivalent k-lattices. Notice that we will always assume $\mathcal{K}$ to be closed under intersection and finite union with $\varnothing, M \in \mathcal{K}$.

Theorem 4.5. *For every k-lattice (G, f) there exists a $g : \{1,2\}^m \to \{1, 2, \ldots, k\}$ such that $\mathcal{K}(G, f) = \mathcal{K}(g)$, where m is the number of meet-irreducible elements of G.*

We postpone the proof of this theorem to the next section where we can make use of the Embedding Lemma (Lemma 5.1).

5 Comparing Partition Classes: The Embedding Conjecture

To study the structure of the boolean hierarchy of k-partitions over $\mathcal{K}$ it would be important to have a criterion to decide whether $\mathcal{K}(G,f) \subseteq \mathcal{K}(G',f')$ for any two k-lattices (G,f) and (G',f'). To this end we establish a relation $\leq$ between k-lattices. For k-lattices (G,f) and (G',f') we write $(G,f) \leq (G',f')$ if and only if there is a monotonic mapping $\varphi : G \to G'$ such that $f(x) = f'(\varphi(x))$ for every $x \in G$. We write $(G,f) \equiv (G',f')$ and we say that (G,f) and (G',f') are *equivalent* if $(G,f) \leq (G',f')$ and $(G',f') \leq (G,f)$.

The following lemma provides a sufficient condition for $\mathcal{K}(G,f) \subseteq \mathcal{K}(G',f')$.

Lemma 5.1. (Embedding Lemma.) *Let (G,f) and (G',f') be k-lattices. If $(G,f) \leq (G',f')$, then $\mathcal{K}(G,f) \subseteq \mathcal{K}(G',f')$.*

Proof. Let $\varphi : G \to G'$ be a monotonic mapping such that $f(a) = f'(\varphi(a))$ for every $a \in G$. For a $\mathcal{K}$-homomorphism S on G define the mapping $S' : G' \to \mathcal{K}$ by $S'(a) =_{\mathrm{def}} \bigcup_{\varphi(b) \leq' a} S(b)$ for all $a \in G'$. It is sufficient to prove that S' is a $\mathcal{K}$-homomorphism on G', i.e., that 1. $S'(1_{G'}) = M$ and 2. $S'(a \wedge' b) = S'(a) \cap S'(b)$ for all $a, b \in G$, and that 3. $T_S(a) \subseteq T_{S'}(\varphi(a))$ for all $a \in G$.

1. We conclude $S'(1_{G'}) = \bigcup_{\varphi(b) \leq' 1_{G'}} S(b) \supseteq S(1_G) = M$.

2. The inclusion "$\subseteq$" is valid because of the monotonicity of S'. For the converse inclusion consider $x \in S'(a) \cap S'(b)$. There exist $c, d \in G$ such that $\varphi(c) \leq' a$, $\varphi(d) \leq' b$, $x \in S(c)$, and $x \in S(d)$. We obtain $\varphi(c \wedge d) \leq' \varphi(c) \wedge' \varphi(d) \leq' a \wedge' b$ and $x \in S(c) \cap S(d) = S(c \wedge d)$, and consequently $x \in S'(a \wedge' b)$.

3. For $a \in G$ and $x \in T_S(a)$ we obtain $x \in S(a) \subseteq S'(\varphi(a))$. Assume that $x \notin T_{S'}(\varphi(a))$. Then there exists a $c <' \varphi(a)$ such that $x \in S'(c)$. Consequently, there exists a $b \in G$ such that $\varphi(b) \leq' c$ and $x \in S(b)$. Hence $x \in S(a) \cap S(b) = S(a \wedge b)$. Because of $x \in T_S(a)$ we get $a \wedge b \not< a$ and thus $a \leq b$. We conclude $\varphi(a) \leq' \varphi(b) \leq' c$, a contradiction. ❑

Example. The 3-lattice (G,f) shown in Figure 1 and the 3-lattice (G',f') shown in Figure 3 are equivalent.

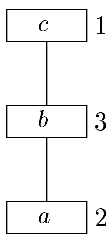


Figure 3

This can be seen as follows: Define the functions $\varphi : G \to G'$ and $\psi : G' \to G$ by $\varphi(111) = \varphi(121) = \varphi(211) = a$, $\varphi(112) = \varphi(221) = b$, $\varphi(122) = \varphi(212) = \varphi(222) = c$, $\psi(a) = 111$, $\psi(b) = 112$, and $\psi(c) = 222$. It is easy to see that φ and ψ are monotonic, $f(x) = f'(\varphi(x))$ for all $x \in G$, and $f'(x) = f(\psi(x))$ for all $x \in G'$. By the Embedding Lemma we obtain $\mathcal{K}(G, f) = \mathcal{K}(G', f')$ for all $\mathcal{K}$. Obviously, $\mathcal{K}(G', f') = \left\{ (\overline{B}, A, B \setminus A) \,\middle|\, A, B \in \mathcal{K} \text{ and } A \subseteq B \right\} = (\mathrm{co}\mathcal{K}, \mathcal{K}, \cdot) = (\mathrm{co}\mathcal{K}, \mathcal{K}, \mathcal{K}(2))$.

Now we are able to prove Theorem 4.5 from the previous section.

Proof of Theorem 4.5 Let (G, f) be an arbitrary k-lattice, let I be the set of meet-irreducible elements of G, and let $I_a =_{\mathrm{def}} \left\{ b \,\middle|\, b \geq a \text{ and } b \text{ meet-irreducible} \right\}$ for every $a \in G$. It is well-known (cf. [Grä78, DP90]) that $\bigwedge I_a = a$ for every $a \in G$. We define the boolean k-lattice $((\mathcal{P}(I), \supseteq), h)$ by $h(U) =_{\mathrm{def}} f(\bigwedge U)$ for $U \subseteq G$. The function $\varphi : G \to \mathcal{P}(G)$ defined by $\varphi(a) =_{\mathrm{def}} I_a$ is monotonic, and we get $h(\varphi(a)) = h(I_a) = f(\bigwedge I_a) = f(a)$. By the Embedding Lemma we obtain $\mathcal{K}(G, f) \subseteq \mathcal{K}((\mathcal{P}(I), \supseteq), h)$. On the other hand, the function $\psi : \mathcal{P}(I) \to G$ defined by $\psi(U) =_{\mathrm{def}} \bigwedge U$ is monotonic, and we get $f(\psi(U)) = f(\bigwedge U) = h(U)$. Again by the Embedding Lemma we obtain $\mathcal{K}((\mathcal{P}(I), \supseteq), h) \subseteq \mathcal{K}(G, f)$. So we get $\mathcal{K}(G, f) = \mathcal{K}((\mathcal{P}(I), \supseteq), h)$, but $(\mathcal{P}(I), \supseteq)$ and $(\{1, 2\}^{|I|}, \leq)$ are isomorphic. ❑

Back to the Embedding Lemma which shows that $(G, f) \leq (G', f')$ implies $\mathcal{K}(G, f) \subseteq \mathcal{K}(G', f')$. Because of Proposition 3.2.2, we cannot hope to convert this implication without an additional assumption on the strictness of the boolean hierarchy over $\mathcal{K}$. We formulate the following conjecture.

Conjecture 5.2. (Embedding Conjecture.) *Assume that the boolean hierarchy over $\mathcal{K}$ does not collapse. Let (G, f) and (G', f') be k-lattices. Then $\mathcal{K}(G, f) \subseteq \mathcal{K}(G', f')$ if and only if $(G, f) \leq (G', f')$.*

Since the collapse of the boolean hierarchy over NP implies the collapse of the polynomial-time hierarchy (cf. [Kad88]) this conjecture implies the following one.

Conjecture 5.3. (Embedding Conjecture for NP.) *Assume that the polynomial-time hierarchy does not collapse. Let (G, f) and (G', f') be k-lattices. Then* $\mathrm{NP}(G, f) \subseteq \mathrm{NP}(G', f')$ *if and only if $(G, f) \leq (G', f')$.*

In the next section we will provide the following evidence for the Embedding Conjecture:

- If $\mathcal{K}(G, f) \subseteq \mathcal{K}(G', f')$ for all $\mathcal{K}$ then $(G, f) \leq (G', f')$ (Theorem 6.1).
- The Embedding Conjecture is true for 2-lattices (Theorem 6.4).
- The Embedding Conjecture is true for a large subclass of all k-lattices (Theorem 6.6).

6 Evidence for the Embedding Conjecture

We start with a theorem which states a modification of the Embedding Conjecture.

Theorem 6.1. *Let (G, f) and (G', f') be k-lattices. If $\mathcal{K}(G, f) \subseteq \mathcal{K}(G', f')$ for every $\mathcal{K}$ with $\varnothing, M \in \mathcal{K}$ which is closed under finite union and intersection, then $(G, f) \leq (G', f')$.*

Proof. Define $\mathcal{K} =_{\text{def}} \{ D(S) \mid S \subseteq G \}$ where $D(S) =_{\text{def}} \{ a \in G \mid (\exists b \in S)[a \leq b] \}$. Clearly, $\varnothing, G \in \mathcal{K}$ and $\mathcal{K}$ is closed under finite union and intersection. Let S be the $\mathcal{K}$-homomorphism on G defined as $S(a) =_{\text{def}} D(\{a\})$ for every $a \in G$. Obviously, $T_S(a) = \{a\}$ and consequently $\big(f^{-1}(1), f^{-1}(2), \ldots, f^{-1}(k)\big) \in \mathcal{K}(G, f) \subseteq \mathcal{K}(G', f')$. Hence, a $\mathcal{K}$-homomorphism $S' : G' \to \mathcal{K}$ on G' exists such that $\bigcup_{f'(d)=i} T_{S'}(d) = f^{-1}(i)$ for every $1 \leq i \leq k$. Define $h : G \to G'$ to be the function which assigns to each $a \in G$ the uniquely determined $d \in G'$ such that $a \in T_{S'}(d)$, i.e., $h^{-1}(d) = T_{S'}(d)$. Obviously, $a \in T_{S'}(h(a))$ and $f'(h(a)) = f(a)$. It remains to show that h is monotonic. Let $a, b \in G$ with $a \leq b$. Then $b \in T_{S'}(h(b)) \subseteq S'(h(b))$, so $a \in S'(h(b))$. From Lemma 4.2.2 there follows the existence of $c \in G'$ with $c \leq h(b)$ and $a \in T_{S'}(c)$. Thus $c = h(a)$, hence $h(a) \leq h(b)$. ❑

Now we prove that the Embedding Conjecture is true for 2-lattices. To this end we first prove an analogue to Theorem 3.3 for 2-lattices. For a 2-lattice (G, f) let $\mu(G, f)$ be the maximum number of alternations of f-labels which can occur in a $\leq$-chain in the lattice G.

Theorem 6.2. *For every 2-lattice (G, f),*

$$\mathcal{K}(G, f) = \begin{cases} \mathcal{K}(\mu(G, f)), & \text{if } f(1_G) = 2, \\ \text{co}\mathcal{K}(\mu(G, f)), & \text{if } f(1_G) = 1. \end{cases}$$

Proof. In the proof of Theorem 4.5 we defined a function $h : \{1, 2\}^{|I|} \to \{1, 2\}$ (remember that I is the set of meet-irreducible elements of G and that $(\mathcal{P}(I), \supseteq)$ and $(\{1, 2\}^{|I|}, \leq)$ are isomorphic) such that $(G, f) \equiv (\{1, 2\}^{|I|}, h)$. Consequently, $\mathcal{K}(G, f) = \mathcal{K}(\{1, 2\}^{|I|}, h) = \mathcal{K}(h)$, $\mu(G, f) = \mu(\{1, 2\}^{|I|}, h) = \mu(h)$, and $f(1_G) = h(2^{|I|})$. By Theorem 3.3 we obtain the statement. ❑

Corollary 6.3. 1. *The unique minimal 2-lattice (G, f) such that $\mathcal{K}(G, f) = \mathcal{K}(i)$ is a chain with $i + 1$ elements with alternating labels 1 and 2 such that the maximum of the chain has label 2.*

2. *The unique minimal 2-lattice (G, f) such that $\mathcal{K}(G, f) = \text{co}\mathcal{K}(i)$ is a chain with $i+1$ elements with alternating labels 1 and 2 such that the maximum of the chain has label 1.*

As a consequence of Theorem 6.2 we get the validity of the Embedding Conjecture for 2-lattices.

Theorem 6.4. *Assume that* $\mathrm{BH}_2(\mathcal{K})$ *does not collapse. For* 2*-lattices* (G, f) *and* (G', f') *the following are equivalent:*

1. $\mathcal{K}(G, f) \subseteq \mathcal{K}(G', f')$,
2. $\mu(G, f) < \mu(G', f')$ *or* $\big(\mu(G, f) = \mu(G', f')$ *and* $f(1_G) = f'(1_{G'})\big)$,
3. $(G, f) \leq (G', f')$.

Proof. $1 \Rightarrow 2$ is a consequence of Theorem 6.2, and $3 \Rightarrow 1$ follows from the Embedding Lemma. For $2 \Rightarrow 3$ take a $\leq$-chain $(c_0, c_1, \ldots, c_r)$ in G' with maximum number of alternations between f'-labels, i.e., $r = \mu(G', f')$ and $f'(c_{i-1}) \neq f'(c_i)$ for $i = 1, \ldots, r$. For $a \in G$ define $\varphi(a)$ as follows: If $f(1_G) = f'(1_{G'})$ $(f(1_G) \neq f'(1_{G'}))$ then $\varphi(a) =_{\mathrm{def}} c_i$ ($\varphi(a) =_{\mathrm{def}} c_{i+1}$, resp.) where i is the maximum number of alternations between f-labels in a chain from a to 1_G. Obviously, φ is monotonic and $f'(\varphi(a)) = f(a)$. ❑

We now establish a theorem which shows that the Embedding Conjecture is true for a large subclass of k-lattices. At this, we make use of the following simple principle.

Proposition 6.5. *Let* (G, f) *and* (G', f') *be* k*-lattices, and let* π *be a function mapping* $\{1, 2, \ldots, k\}$ *to* $\{1, 2, \ldots, m\}$*. If* $\mathcal{K}(G, f) \subseteq \mathcal{K}(G', f')$*, then* $\mathcal{K}(G, \pi \circ f) \subseteq \mathcal{K}(G', \pi \circ f')$*.* [1] *Moreover, if* π *is injective, then the equivalence holds.*

Let (G, f) be a k-lattice. For $I, J \subseteq \{1, 2, \ldots, k\}$ with $I \cap J = \varnothing$, define $\mu_{I,J}(G, f)$ to be the maximum number of alternations between f-labels from I and f-labels from J in a chain of G whose minimum has an f-label from I.

Theorem 6.6. *Assume that* $\mathrm{BH}_2(\mathcal{K})$ *does not collapse. For* k*-lattices* (G, f) *and* (G', f')*, if* $\mathcal{K}(G, f) \subseteq \mathcal{K}(G', f')$*, then* $\mu_{I,J}(G, f) \leq \mu_{I,J}(G', f')$ *for all* $I, J \subseteq \{1, 2, \ldots, k\}$ *with* $I \cap J = \varnothing$*.*

Proof. If $I = \varnothing$ or $J = \varnothing$, then $\mu_{I,J}(G, f) = 0$ for all (G, f). So, suppose I and J to be non-empty and $I \cap J = \varnothing$. Consider the function π mapping elements from I to $\min I$, elements from J to $\min J$, and elements not in I or J to themselves. Then, for all k-lattices (G, f), it holds $\mu_{I,J}(G, f) = \mu_{\pi(I),\pi(J)}(G, \pi \circ f)$. Therefore and because of Proposition 6.5, without loss of generality, we can assume that I and J are singletons; $I = \{i\}$, $J = \{j\}$, and $i \neq j$. For convenience, we write $\mu_{ij}(G, f)$ instead of $\mu_{\{i\},\{j\}}(G, f)$.

[1] Our use of $\circ$ is $(g \circ h)(x) = g(h(x))$.

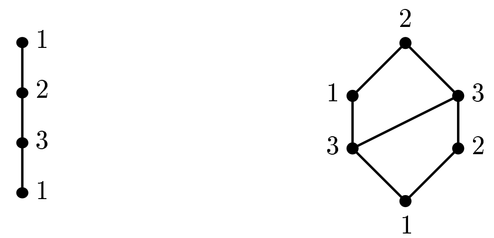
Let C be a maximal chain in G such that $\mu_{ij}(C, f|_C) = \mu_{ij}(G, f)$. Hence, $\mathcal{K}(C, f|_C) \subseteq \mathcal{K}(G', f')$. Since $f|_C : C \to \{i, j\}$ we have also $\mathcal{K}(C, f|_C) \subseteq \mathcal{K}(G', h)$ for all $h : G' \to \{1, 2, \ldots, k\}$ such that $h(a) = f'(a)$ if $f'(a) \in \{i, j\}$.

If there is no $b \in G'$ with $f'(b) \notin \{i, j\}$, then the claim is just the same already proven in Theorem 6.4. So, fix some $b \in G'$ such that $f'(b) \notin \{i, j\}$. Define for $a \in G'$

$$h(a) =_{\text{def}} \begin{cases} f'(a), & \text{if } a \neq b \\ i, & \text{if } a = b \text{ and } \mu_{ij}(G'_b, f'|_{G'_b}) \text{ is even} \\ j, & \text{if } a = b \text{ and } \mu_{ij}(G'_b, f'|_{G'_b}) \text{ is odd} \end{cases}$$

where $G'_b =_{\text{def}} \left\{ c \in G' \,\middle|\, c \leq b \right\}$. Hence, $\mathcal{K}(C, f|_C) \subseteq \mathcal{K}(G', h)$ and $\mu_{ij}(G', f') \leq \mu_{ij}(G', h)$. Consider a maximal chain $a_0 < a_1 < \cdots < a_r$ in G' such that $r = \mu_{ij}(G', h)$, $h(a_s) \in \{i, j\}$, $h(a_0) = i$, and $h(a_{s-1}) \neq h(a_s)$ for $s = 1, \ldots, r$. If $b \notin \{a_0, a_1, \ldots, a_r\}$ then $h(a_s) = f'(a_s)$ for all $s = 0, 1, \ldots, r$ and hence $\mu_{ij}(G', f') \geq \mu_{ij}(G', h)$, thus $\mu_{ij}(G', f') = \mu_{ij}(G', h)$. Now let $b = a_s$ for some $s \in \{0, 1, \ldots, r\}$. Since $f'(a_{s-1}) = h(a_{s-1}) \neq h(a_s)$ and, by definition, $h(b) = h(a_s)$, the chain $a_0 < a_1 < \cdots < a_{s-1}$ cannot be a maximum chain in G'_b with alternating f'-labels starting with f'-label i. Hence there exists such a chain $b_0 < b_1 < \cdots < b_{s-1} < b_s$ in $(G'_b, f'|_{G'_b})$ and consequently such a chain $b_0 < b_1 < \cdots < b_{s-1} < b_s < a_{s+1} < \cdots < a_r$ in (G', f'). This means $\mu_{ij}(G', f') \geq r = \mu_{ij}(G', h)$ and hence, $\mu_{ij}(G', f') = \mu_{ij}(G', h)$.

Repeating this construction we obtain finally a function $g : G' \to \{i, j\}$ such that $\mathcal{K}(C, f|_C) \subseteq \mathcal{K}(G', g)$, $\mu_{ij}(C, f|_C) = \mu_{ij}(G, f)$, and $\mu_{ij}(G', g) = \mu_{ij}(G', f')$. In fact, $\mathcal{K}(C, f|_C)$ and $\mathcal{K}(G', g)$ are classes of 2-partitions. By Theorem 6.4, we obtain $\mu(C, f|_C) < \mu(G', g)$ or, $\mu(C, f|_C) = \mu(G', g)$ and $f(1_C) = g(1_{G'})$, from which we can conclude $\mu_{ij}(C, f|_C) \leq \mu_{ij}(G', g)$. ❑



(G, f) (G', f')

Figure 4.

Example. Consider the 3-lattices (G, f) and (G', f') in Figure 4. To show $\mathcal{K}(G, f) \not\subseteq \mathcal{K}(G', f')$ if $\mathrm{BH}_2(\mathcal{K})$ is strict, let $I =_{\text{def}} \{1\}$ and $J =_{\text{def}} \{2\}$. Then we have $\mu_{I,J}(G, f) = 2$ and $\mu_{I,J}(G', f') = 1$. Hence, by Theorem 6.6, $\mathcal{K}(G, f) \not\subseteq \mathcal{K}(G', f')$ unless $\mathrm{BH}_2(\mathcal{K})$ collapses. Reversely, let $I =_{\text{def}} \{1\}$ and $J =_{\text{def}} \{2, 3\}$. Then, $\mu_{I,J}(G', f') = 3$ and $\mu_{I,J}(G, f) = 2$. Thus, again by Theorem 6.6, $\mathcal{K}(G', f') \not\subseteq \mathcal{K}(G, f)$ unless $\mathrm{BH}_2(\mathcal{K})$ collapses.

7 On the Structure of the Boolean Hierarchy of k-partitions for $k \geq 3$

Assume the Embedding Conjecture is true. Then the structure of the boolean hierarchy of k-partitions with respect to set inclusion is identical with the partial order of $\leq$-equivalence classes of k-lattices with respect to $\leq$. To get an idea of the richness of the latter structure we will now present (without proof of its completeness) the partial order of all equivalence classes of 3-lattices which include a boolean 3-lattice of the form $(\{1,2\}^3, f)$ with surjective f (for non-surjective f these k-lattices do not really define 3-partitions). The 5796 different boolean 3-lattices of the form $(\{1,2\}^3, f)$ with surjective f are in 132 different equivalence classes. Figure 5 shows the partial order of the 44 equivalence classes which contain boolean 3-lattices of the form $(\{1,2\}^3, f)$ such that $f(1,1,1) = 1$. The cases $f(1,1,1) = 2$ and $f(1,1,1) = 3$ yield isomorphic partial orders. A line from equivalence class $\mathcal{G}$ up to equivalence class $\mathcal{G}'$ means that $(G, f) \leq (G', f')$ for every $(G, f) \in \mathcal{G}$ and $(G', f') \in \mathcal{G}'$. We emphasize that such a study would be intractable without the possibility to present boolean k-lattices by equivalent k-lattices.

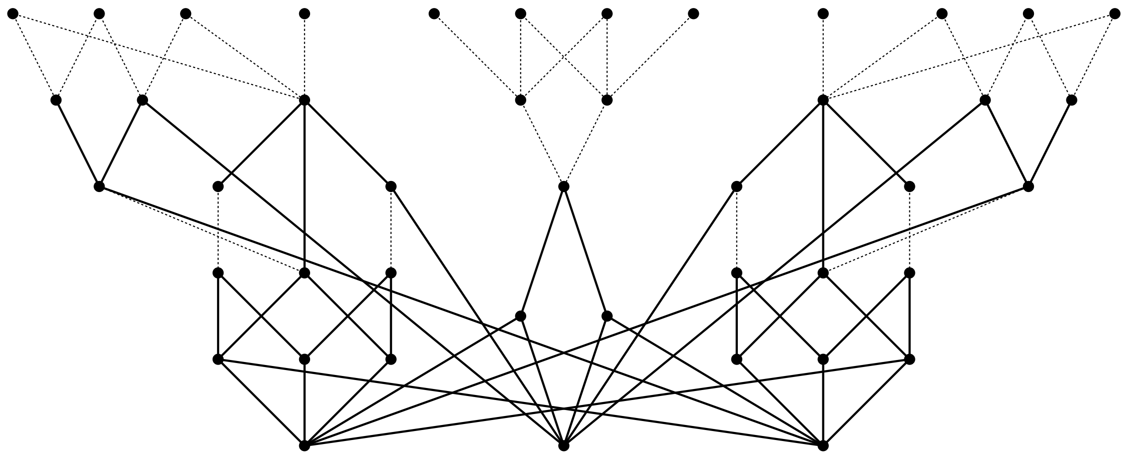


Figure 5.

Figure 6 shows the middle part of the partial order in Figure 5 where each equivalence class is represented by a minimal equivalent 3-lattice. Figure 7 shows the right part of the partial order in Figure 5 where each equivalence class is represented by a minimal equivalent 3-lattice. The left part of the partial order in Figure 2 is symmetric to the right part where the labels 2 and 3 change their role.

Theorem 7.1. *Assume that* $\mathrm{BH}_2(\mathcal{K})$ *does not collapse. If there is a solid line from class* $\mathcal{G}$ *up to class* $\mathcal{G}'$ *in Figure 6 or Figure 7 then* $\mathcal{K}(G, f) \subset \mathcal{K}(G', f')$ *for every* $(G, f) \in \mathcal{G}$ *and* $(G', f') \in \mathcal{G}'$.

Proof. This is just a lot of simple applications of Theorem 6.6 like in the example after Theorem 6.6. ❑

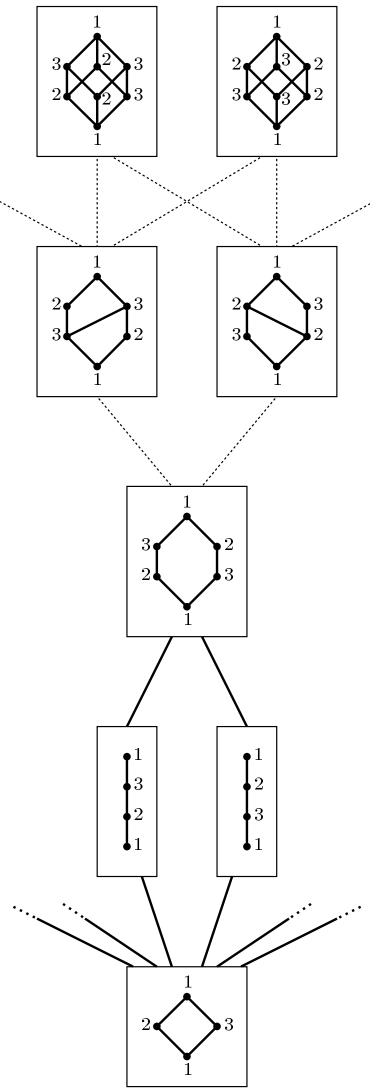


Figure 6.

At the end of this section we prove that the partial order of equivalence classes of 3-lattices does not have "bounded width".

Proposition 7.2. *For every $m \in \mathbb{N}$ there exist at least m 3-lattices that are incomparable with respect to $\leq$.*

Proof. Let $G_m = (\{1, 2, \ldots, m\}, \leq)$ be the chain with the natural order on $\{1, 2, \ldots, m\}$. For every $i \in \{1, 2, \ldots, m\}$ let $f_m^i : G_m \to \{1, 2, 3\}$ be the function defined as

$$f_m^i(j) = \begin{cases} 1, & \text{if } (j < i \text{ and } j \text{ is odd}) \text{ or } (j > i \text{ and } j \text{ is even}), \\ 2, & \text{if } (j < i \text{ and } j \text{ is even}) \text{ or } (j > i \text{ and } j \text{ is odd}), \\ 3, & \text{if } j = i. \end{cases}$$

It is easy to see that for all $i, j \in G_m$ with $i \neq j$ the 3-lattices (G_m, f_m^i) and (G_m, f_m^j) are incomparable with respect to $\leq$. ❑

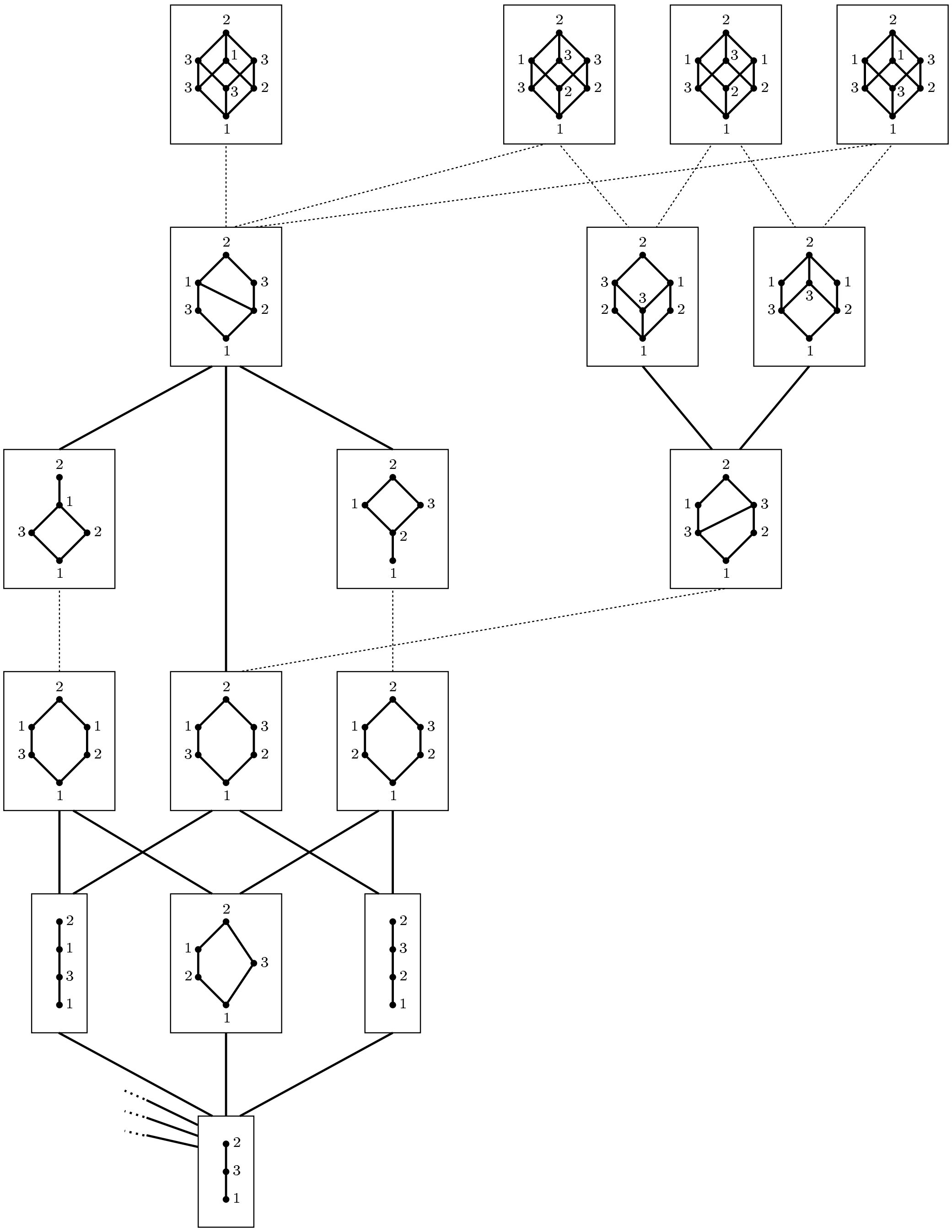


Figure 7.

8 Machines That Accept Partitions

In this section we will see how the partitions of a class $\mathrm{NP}(f)$ can be accepted in a natural way by nondeterministic polynomial-time machines with a notion of acceptance which depends on the function f.

Definition 8.1. For $m \geq 1$, a *polynomial-time m-machine* M is a nondeterministic polynomial-time machine producing on every computation path an element from the set $\{0, 1, \ldots, m\}$. For an input x let

$$M(x) =_{\mathrm{def}} \left\{ i \neq 0 \,\middle|\, \text{there exists a path of } M \text{ on } x \text{ with result } i \right\}.$$

Obviously, a polynomial-time 1-machine is an ordinary nondeterministic polynomial-time machine. For $i \in \{1, \ldots, m\}$ the sets

$$L_i(M) =_{\mathrm{def}} \left\{ x \,\middle|\, \text{there exists a path of } M \text{ on } x \text{ with result } i \right\}$$

are in NP and we obtain $M(x) = \left\{ i \,\middle|\, x \in L_i(M) \right\}$ and $c_{L_i(M)}(x) = c_{M(x)}(i)$ for all x.

Definition 8.2. For a function $f : \mathcal{P}(\{1, \ldots, m\}) \to \{1, 2, \ldots, k\}$ and a polynomial-time m-machine M let (M, f) be the k-partition defined by $c_{(M,f)}(x) = f(M(x))$ for all x.

Note that every function $f : \mathcal{P}(\{1, \ldots, m\}) \to \{1, 2, \ldots, k\}$ can also be considered to be the function $f : \{1, 2\}^m \to \{1, 2, \ldots, k\}$ and vice versa by the relationships $f(a_1, \ldots, a_m) = f(\left\{ i \,\middle|\, a_i = 1 \right\})$ for $a_1, \ldots, a_m \in \{1, 2\}$ and $f(A) = f(c_A(1), \ldots, c_A(m))$ for $A \subseteq \{1, \ldots, m\}$.

Theorem 8.3. *Let $m \geq 1$ and $f : \{1, 2\}^m \to \{1, 2, \ldots, k\}$.*

$$\mathrm{NP}(f) = \left\{ (M, f) \,\middle|\, M \textit{ is a polynomial-time } m\textit{-machine} \right\}.$$

Proof. For the inclusion "$\subseteq$" consider $B_1, \ldots, B_m \in \mathrm{NP}$. There exist nondeterministic polynomial-time machines $M_1, \ldots, M_m$ such that M_i accepts B_i for $i = 1, \ldots, m$. Define M to be a nondeterministic polynomial-time machine which simulates $M_1, \ldots, M_m$ in parallel but when simulating M_i it outputs i rather than 1. Obviously, $L_i(M) = B_i$ for $i = 1, \ldots, m$, and we conclude

$$\begin{aligned} c_{f(B_1,\ldots,B_m)}(x) &= f(c_{B_1}(x), \ldots, c_{B_m}(x)) = f(c_{L_1(M)}(x), \ldots, c_{L_m(M)}(x)) \\ &= f(c_{M(x)}(1), \ldots, c_{M(x)}(m)) = f(M(x)) = c_{(M,f)}(x). \end{aligned}$$

For the inclusion "$\supseteq$" consider a polynomial-time m-machine M and conclude

$$\begin{aligned} c_{(M,f)}(x) &= f(M(x)) = f(c_{M(x)}(1), \ldots, c_{M(x)}(m)) \\ &= f(c_{L_1(M)}(x), \ldots, c_{L_m(M)}(x)) = c_{f(L_1(M),\ldots,L_m(M))}(x). \end{aligned}$$

❑

Finally, we will discuss completeness for the partition classes $\mathrm{NP}(f)$. We will see that it is easy to construct from an arbitrary NP-complete problem a problem which is complete for $\mathrm{NP}(f)$. We say that the k-partition A is polynomial-time many-one reducible to the k-partition B (for short $A \leq_m^p B$) if and only if there exists a polynomial-time computable function g such that $c_A(w) = c_B(g(w))$ for all w. Note that in the case $k = 2$ this yields exactly the classical notion of polynomial-time many-one reducibility for sets. A k-partition A is said to be $\leq_m^p$*-complete* for a class $\mathcal{C}$ if and only if $A \in \mathcal{C}$ and $B \leq_m^p A$ for every k-partition $B \in \mathcal{C}$.

Let $h : (\Sigma^*)^m \to \Sigma^*$ be any polynomial-time computable surjective function such that the inverse functions $h_1, h_2, \ldots, h_m : \Sigma^* \to \Sigma^*$ are also polynomial-time computable. For a set $A \subseteq \Sigma^*$ and a function $f : \{1,2\}^m \to \{1,2,\ldots,k\}$ define the k-partition $A(f)$ by $c_{A(f)}(w) =_{\mathrm{def}} f((c_A \circ h_1)(w) \ldots (c_A \circ h_m)(w))$ for all $w \in \Sigma^*$.

Theorem 8.4. *For $f : \{1,2\}^m \to \{1,2,\ldots k\}$, if A is $\leq_m^p$-complete for* NP *then $A(f)$ is $\leq_m^p$-complete for* $\mathrm{NP}(f)$*.*

Proof. Defining $A_i =_{\mathrm{def}} \{ w \mid h_i(w) \in A \}$ for $i = 1, \ldots, m$ we obtain $A_i \in \mathrm{NP}$. For every $w \in \Sigma^*$ we conclude $c_{A(f)}(w) = f((c_A \circ h_1)(w) \ldots (c_A \circ h_m)(w)) = f(c_{A_1}(w) \ldots c_{A_m}(w)) = c_f(A_1, \ldots, A_m)$. Consequently, $A(f) = f(A_1, \ldots, A_m) \in \mathrm{NP}(f)$.

Now take any $B_1, B_2, \ldots, B_m \in \mathrm{NP}$. Since A is $\leq_m^p$-complete for NP there exist polynomial-time computable functions $g_1, g_2, \ldots, g_m$ such that $w \in B_i \leftrightarrow g_i(w) \in A$ for every $i = 1, \ldots, m$. Defining $g(w) =_{\mathrm{def}} h(g_1(w), \ldots g_m(w))$ for every $w \in \Sigma^*$, we can conclude $c_{f(B_1,B_2,\ldots,B_m)}(w) = f(c_{B_1}(w) \ldots c_{B_m}(w)) = f((c_A \circ g_1)(w) \ldots (c_A \circ g_m)(w))$ $= f((c_A \circ h_1 \circ g)(w) \ldots (c_A \circ h_m \circ g)(w)) = c_{A(f)}(g(w))$. Hence $f(B_1, \ldots, B_m) \leq_m^p A(f)$. ❑

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