A Normalform for Classes of Concatenation Hierarchies

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Abstract

Arbitrary regular languages can be expressed using Boolean operations, concatenation and the Kleene closure. If a regular language can be expressed without the Kleene closure, we call it *starfree*. A very interesting (and still open) problem is the following: Is there an algorithm computing the minimal number of alternations between Boolean operations and concatenation needed to define a given starfree language. This problem is called the *dot-depth problem*. Grouping together languages of the same concatenation complexity leads in a natural way to the definition of (infinite and strict) hierarchies that exhaust the class of starfree regular languages.

The classes $\mathcal{L}_{n+1/2}$ (resp. $\mathcal{B}_{n+1/2}$) of the Straubing-Thérien-Hierarchy (resp. Dot-Depth-Hierarchy) are defined as the polynomial closure of the classes $\mathcal{L}_n$ (resp. $\mathcal{B}_n$). We show that these classes can be defined even as the polynomial closure of the classes $\mathrm{co}\mathcal{L}_{n-1/2}$ (resp. $\mathrm{co}^{+}\mathcal{B}_{n-1/2}$, where co^{+} means that complementation is taken with respect to Σ^{+}). Since $\mathrm{co}\mathcal{L}_{n-1/2} \subsetneq \mathcal{L}_n$ (and $\mathrm{co}^{+}\mathcal{B}_{n-1/2} \subsetneq \mathcal{B}_n$), this result simplifies the conventional definition.

Notice that the only things we take for granted are a normalform for $\mathcal{L}_{3/2}$ [Arf87] and the fact that the levels are closed under finite unions and finite intersections [Arf87, PW97].

1 Introduction

In this section one can find a list of basic definitions for regular languages concerning residuals, Boolean operations and polynomial operations. Furthermore we define the Straubing-Thérien-Hierarchy and the Dot-Depth-Hierarchy. Finally we mention some known results (without proofs) which are needed to prove the normalforms.

The Straubing-Thérien-Hierarchy and the Dot-Depth-Hierarchy have been studied since 1981 and 1971, respectively. Several characterisations for these hierarchies are known (e.g. in the theory of formal languages, in the theory of finite automata, in algebra and in logic). Furthermore, these hierarchies have a close connection to complexity theory (via the leaf languages approach). It is known that these hierarchies are strict, and that they exhaust the class of starfree languages. The decidability of their levels is an open problem. For a comprehensive introduction to this area have a look at [Brz76, Pin96a, Pin96b, Tho96, Sch98].

To understand complicated objects in a better way, it is useful to develop normalforms for them. In this paper we will provide such normalforms for the levels $n + 1/2$ of both hierarchies:

- $\mathcal{L}_{n+1/2} = \mathrm{Pol}^{*}(\mathrm{co}\mathcal{L}_{n-1/2})$
- $\mathcal{B}_{n+1/2} = \mathrm{Pol}^{+}(\mathrm{co}^{+}\mathcal{B}_{n-1/2})$

Notice that $\mathrm{co}\mathcal{L}_{n-1/2} \subsetneq \mathcal{L}_n$ and $\mathrm{co}^{+}\mathcal{B}_{n-1/2} \subsetneq \mathcal{B}_n$. Therefore, our normalform simplifies the conventional definition of the levels $n + 1/2$.

Notice that we cannot derive this normalforms from the logical characterisations of $\mathcal{L}_{n+1/2}$ and $\mathcal{B}_{n+1/2}$. Perrin and Pin [PP86] showed that a language is in $\mathcal{L}_{n+1/2}$ if and only if it is FO[<]-definable by a Σ_n formula. It is easy to see that a formula $\phi \in \Sigma_n$ can be transformed into

$$\exists\exists\ldots\exists(\neg\phi_1) \quad \vee \quad \exists\exists\ldots\exists(\neg\phi_2) \quad \vee \quad \ldots \quad \vee \quad \exists\exists\ldots\exists(\neg\phi_k)$$

where $\phi_1, \ldots, \phi_k \in \Sigma_{n-1}$. Thus $\neg\phi_1, \ldots, \neg\phi_k \in \Pi_{n-1}$. Since formulas in Π_{n-1} represent languages in $\mathrm{co}\mathcal{L}_{n-1/2}$, it would be sufficient to show that the language of $\exists\exists\ldots\exists\phi'$ (with $\phi' \in \Pi_{n-1}$) is a finite union of marked concatenations of languages in $\mathrm{co}\mathcal{L}_{n-1/2}$. This would establish our normalform. Unfortunately this does not work. If we translate $\exists\exists\phi'$ into a language, then the first $\exists$-quantifier causes marked concatenations of finite intersections of residuals of languages in $\mathrm{co}\mathcal{L}_{n-1/2}$ (cf. [PP86], Lemma 3.5). Since $\mathrm{co}\mathcal{L}_{n-1/2}$ is closed under taking residuals and intersection, we obtain marked concatenations of languages in $\mathrm{co}\mathcal{L}_{n-1/2}$. Therefore, we have to apply the second $\exists$-quantifier to marked concatenations of languages in $\mathrm{co}\mathcal{L}_{n-1/2}$. This causes *intersections of marked concatenations of languages in* $\mathrm{co}\mathcal{L}_{n-1/2}$. Now we have to transform these intersections into finite unions. This is possible by Arfi's construction [Arf91]. By this construction we obtain marked concatenations of differences of languages in $\mathrm{co}\mathcal{L}_{n-1/2}$ (cf. [Arf91], Lemma 2.3.2). Since $\mathrm{co}\mathcal{L}_{n-1/2}$ *is not closed under difference*, we leave the class $\mathrm{co}\mathcal{L}_{n-1/2}$, i.e. we do not obtain marked concatenations of languages in $\mathrm{co}\mathcal{L}_{n-1/2}$.

In this paper we start from a language in $\mathcal{L}_{n+1/2}$ which can be represented as a finite union of marked products of languages in $\mathcal{L}_n$. To prove our normalform we would like to have a distributive law between concatenation and intersection of languages. Since such a law does not exist, we will provide a decomposition lemma being a kind of modified distributive law. This lemma may be of independent interest, because it holds for arbitrary languages.

In the following we provide a list of auxiliary definitions which are needed for the definitions of the Straubing-Thérien-Hierarchy and the Dot-Depth-Hierarchy. Let Σ be a finite alphabet with $|\Sigma| \geq 2$.

Definition 1.1 *Let $w \in \Sigma^*$ and $A \subseteq \Sigma^*$. We define the* ***residuals*** *of A and w as:*

$$\begin{aligned} Aw^{-1} &\stackrel{df}{=} \{v \in \Sigma^* : vw \in A\} \\ w^{-1}A &\stackrel{df}{=} \{v \in \Sigma^* : wv \in A\} \end{aligned}$$

The languages of the Straubing-Thérien-Hierarchy are subsets of Σ^*. By contrast the languages of the Dot-Depth-Hierarchy are subsets of Σ^+. Therefore, the following definitions for residuals, Booleans operations and polynomial operations distinguish between Σ^* and Σ^+.

Definition 1.2 *For arbitrary $A, B \subseteq \Sigma^*$ we define the* ***set of (positive) residuals*** *as:*

$$\begin{aligned} \mathcal{R}_r(A,B) &\stackrel{df}{=} \left\{Aw^{-1} : w \in B\right\} \\ \mathcal{R}_l(A,B) &\stackrel{df}{=} \left\{w^{-1}A : w \in B\right\} \\ \mathcal{R}_r^+(A,B) &\stackrel{df}{=} \left\{Aw^{-1} \cap \Sigma^+ : w \in B\right\} = \left\{R \cap \Sigma^+ : R \in \mathcal{R}_r(A,B)\right\} \\ \mathcal{R}_l^+(A,B) &\stackrel{df}{=} \left\{w^{-1}A \cap \Sigma^+ : w \in B\right\} = \left\{R \cap \Sigma^+ : R \in \mathcal{R}_l(A,B)\right\} \end{aligned}$$

Definition 1.3 *Let $A \subseteq \Sigma^*$. We define the* ***(positive) complement*** *as:*

$$\begin{aligned} \overline{A} &\stackrel{df}{=} \Sigma^* \setminus A \\ \overline{A}^{+} &\stackrel{df}{=} \Sigma^+ \setminus A \end{aligned}$$

Definition 1.4 *Let $\mathcal{A} \subseteq \mathcal{P}(\Sigma^*)$, we define the* ***set of (positive) complements of*** *$\mathcal{A}$:*

$$\begin{aligned} \mathrm{co}\mathcal{A} &\stackrel{df}{=} \{\overline{A} : A \in \mathcal{A}\} \\ \mathrm{co}^+\mathcal{A} &\stackrel{df}{=} \left\{\overline{A}^+ : A \in \mathcal{A}\right\} \end{aligned}$$

Definition 1.5 *Let $\mathcal{A} \subseteq \mathcal{P}(\Sigma^*)$. We define $\mathrm{BC}^*(\mathcal{A})$ —* ***the Boolean closure of*** *$\mathcal{A}$ — as the smallest subset $\mathcal{C}$ with $\mathcal{A} \subseteq \mathcal{C} \subseteq \mathcal{P}(\Sigma^*)$, such that $\mathcal{C}$ is closed under finite (possibly empty) union, finite intersection and complementation w.r.t. Σ^*.*

Definition 1.6 *Let $\mathcal{A} \subseteq \mathcal{P}(\Sigma^+)$. We define $\mathrm{BC}^+(\mathcal{A})$ —* ***the positive Boolean closure of*** *$\mathcal{A}$ — as the smallest subset $\mathcal{C}$ with $\mathcal{A} \subseteq \mathcal{C} \subseteq \mathcal{P}(\Sigma^+)$, such that $\mathcal{C}$ is closed under finite (possibly empty) union, finite intersection and positive complementation w.r.t. Σ^+.*

Definition 1.7 *Let $\mathcal{A} \subseteq \mathcal{P}(\Sigma^*)$. We define $\mathrm{Pol}^*(\mathcal{A})$ —* ***the polynomial closure of*** *$\mathcal{A}$ — as the set of all languages $L \subseteq \Sigma^*$, which can be written as a finite (possibly empty) union of languages $L_0a_1L_1 \ldots a_nL_n$, where $n \in \mathbb{N}$, $a_i \in \Sigma$ and $L_i \in \mathcal{A}$.*

Definition 1.8 *Let $\mathcal{A} \subseteq \mathcal{P}(\Sigma^+)$. We define $\mathrm{Pol}^+(\mathcal{A})$ —* ***the positive polynomial closure of*** *$\mathcal{A}$ — as the set of all languages $L \subseteq \Sigma^+$, which can be written as a finite (possibly empty) union of languages $u_0L_1u_1 \ldots u_{n-1}L_nu_n$, where $n \in \mathbb{N}$, $u_i \in \Sigma^*$ and $L_i \in \mathcal{A}$.*

If $n = 0$ in Definition 1.8 then $u_0 \neq \varepsilon$, because $L \subseteq \Sigma^+$. With the preceding definitions we are able to define the Straubing-Thérien-Hierarchy and the Dot-Depth-Hierarchy in similar ways.

Definition 1.9 *For a fixed alphabet Σ we define the* ***classes of the Straubing-Thérien-Hierarchy:***

$$\begin{aligned} \mathcal{L}_0 &\stackrel{df}{=} \{\emptyset, \Sigma^*\} \\ \mathcal{L}_{n+\frac{1}{2}} &\stackrel{df}{=} \mathrm{Pol}^*(\mathcal{L}_n) \quad \textit{for } n \in \mathbb{N} \\ \mathcal{L}_{n+1} &\stackrel{df}{=} \mathrm{BC}^*(\mathcal{L}_{n+\frac{1}{2}}) \quad \textit{for } n \in \mathbb{N} \end{aligned}$$

Definition 1.10 *For a fixed alphabet Σ we define the* ***classes of the Dot-Depth-Hierarchie:***

$$\begin{aligned} \mathcal{B}_0 &\stackrel{df}{=} \{\emptyset, \Sigma^+\} \\ \mathcal{B}_{n+\frac{1}{2}} &\stackrel{df}{=} \mathrm{Pol}^+(\mathcal{B}_n) \quad \textit{for } n \in \mathbb{N} \\ \mathcal{B}_{n+1} &\stackrel{df}{=} \mathrm{BC}^+(\mathcal{B}_{n+\frac{1}{2}}) \quad \textit{for } n \in \mathbb{N} \end{aligned}$$

For our proof we need the fact that the levels of these hierarchies are closed under finite unions and finite intersections. The following theorem provides this.

Theorem 1.11 *For every $n \in \mathbb{N}$ the classes $\mathcal{L}_{\frac{n}{2}}$, $\mathrm{co}\mathcal{L}_{\frac{n}{2}}$, $\mathcal{B}_{\frac{n}{2}}$ and $\mathrm{co}^+\mathcal{B}_{\frac{n}{2}}$ are closed under finite intersections and finite unions.*

Proof In [Arf87] Arfi proved that the $\mathcal{L}$-classes are closed under finite intersections. It follows that the $\mathrm{co}\mathcal{L}$-classes are closed under finite unions. By definition the $\mathcal{L}$-classes are closed under finite unions. Therefore, the $\mathrm{co}\mathcal{L}$-classes are closed under finite intersections.

In [PW97] Pin and Weil proved that the $\mathcal{B}$-classes are closed under finite intersections. This proves the remaining facts of the theorem. $\square$

The following theorem is a special normalform for $\mathcal{L}_{3/2}$ proved by Arfi in [Arf87]. We will need this theorem for the induction base in the proof of Theorem 5.2 (the normalform for the Straubing-Thérien-Hierarchy).

Theorem 1.12 $\mathcal{L}_{\frac{3}{2}} = \mathrm{Pol}^*(\{A^* : A \subseteq \Sigma\})$

Proof Arfi proved this result in [Arf87]. $\square$

Lemma 1.13 *For arbitrary* $A, B_1, B_2, C \subseteq \Sigma^*$*:*

$$A \cdot (B_1 \cup B_2) \cdot C = A \cdot B_1 \cdot C \cup A \cdot B_2 \cdot C$$

Proof trivial $\square$

Lemma 1.14 *Let* $\mathcal{C} \subseteq \Sigma^*$ *and* $\mathcal{D} \subseteq \Sigma^+$*, then* $\mathrm{Pol}^*(\mathcal{C})$ *and* $\mathrm{Pol}^+(\mathcal{D})$ *are closed under marked concatination. This means:*

1. *If* $a \in \Sigma$ *and* $A_1, A_2 \in \mathrm{Pol}^*(\mathcal{C})$ *then* $A_1 a A_2 \in \mathrm{Pol}^*(\mathcal{C})$*.*
2. *If* $u_0, u_1 \in \Sigma^*$ *and* $B \in \mathrm{Pol}^+(\mathcal{D})$ *then* $u_0 B u_1 \in \mathrm{Pol}^+(\mathcal{D})$*.*
3. *If* $u_0, u_1, u_2 \in \Sigma^*$ *and* $B_1, B_2 \in \mathrm{Pol}^+(\mathcal{D})$ *then* $u_0 B_1 u_1 B_2 u_2 \in \mathrm{Pol}^+(\mathcal{D})$*.*

Proof This lemma follows from Lemma 1.13. $\square$

2 Weak Normalforms for $\mathcal{L}$ and $\mathcal{B}$

In this section we prove weak normalforms for both hierarchies. The proofs are quite easy, but a little bit technically. Roughly speaking, we start from the definitions of the levels $n + 1/2$ and obtain the normalforms by using DeMorgan's laws.

Lemma 2.1 *Every* $A \in \mathcal{L}_{n+\frac{1}{2}}$ *with* $n \in \mathbb{N}^+$ *can be written as*

$$A \quad = \quad \bigcup_{i=1}^{k} \left(A'_{i,0} \cap \overline{A_{i,0}}\right) a_{i,1} \left(A'_{i,1} \cap \overline{A_{i,1}}\right) \ldots a_{i,m_i} \left(A'_{i,m_i} \cap \overline{A_{i,m_i}}\right)$$

where $k, m_i \in \mathbb{N}$*,* $a_{i,j} \in \Sigma$*,* $A_{i,j} \in \mathcal{L}_{n-\frac{1}{2}}$*, and every* $A'_{i,j}$ *is of the form* $B_0 b_1 B_1 \ldots b_l B_l$ *for suitable* $l \in \mathbb{N}$*,* $b_i \in \Sigma$*,* $B_i \in \mathcal{L}_{n-1}$*. Particularly* $A'_{i,j} \in \mathcal{L}_{n-\frac{1}{2}}$*.*

Proof Let $A \in \mathcal{L}_{n+\frac{1}{2}}$ with $n \in \mathbb{N}^+$. By definition A can be written as

$$A = \bigcup_{i=1}^{k'} C_{i,0} c_{i,1} C_{i,1} \ldots c_{i,m'_i} C_{i,m'_i} \tag{1}$$

where $k', m'_i \in \mathbb{N}$, $c_{i,j} \in \Sigma$, $C_{i,j} \in \mathcal{L}_n$. Every $C_{i,j} \in \mathcal{L}_n$ is a Boolean combination of elements of $\mathcal{L}_{n-\frac{1}{2}}$ (complementation is taken w.r.t. Σ^*). We transform these Boolean combinations into a disjunctive normalform and obtain

$$C_{i,j} = \bigcup_{j'=1}^{l} \left(C'_{j',0} \cap \overline{C'_{j',1}} \cap \overline{C'_{j',2}} \cap C'_{j',3} \cap \cdots \cap C'_{j',n_{j'}} \right) \tag{2}$$

for suitable $l, n_{j'} \in \mathbb{N}$ and $C'_{i,j} \in \mathcal{L}_{n-\frac{1}{2}}$.

By Theorem 1.11 we have that $\mathcal{L}_{n-\frac{1}{2}}$ and $\mathrm{co}\mathcal{L}_{n-\frac{1}{2}}$ are closed under finite intersection. Therefore, there exist $D_{j'}, D'_{j'} \in \mathcal{L}_{n-\frac{1}{2}}$ such that:

$$C_{i,j} = \bigcup_{j'=1}^{l} \left(D'_{j'} \cap \overline{D_{j'}} \right) \tag{3}$$

We can find this representation of $C_{i,j}$ even if Equation (2) has no positive or no negative literals. In this case we use Σ^* or $\overline{\emptyset}$ for the missing literals. By definition of $\mathcal{L}_{n-\frac{1}{2}}$, every $D'_{j'}$ can be written as

$$D'_{j'} = \bigcup_{i=1}^{l'} D'_{i,0} d'_{i,1} D'_{i,1} \dots d'_{i,n'_i} D'_{i,n'_i} \tag{4}$$

for suitable $l', n'_i \in \mathbb{N}, d'_{i,j} \in \Sigma, D'_{i,j} \in \mathcal{L}_{n-1}$. If we insert this representation into Equation (3), we obtain (after transformation into an disjunctive normalform)

$$C_{i,j} = \bigcup_{j'=1}^{l''} \left(E'_{j'} \cap \overline{E_{j'}} \right) \tag{5}$$

where $l'' \in \mathbb{N}, E_{j'} \in \mathcal{L}_{n-\frac{1}{2}}$ and $E'_{j'} = B'_0 b'_1 B'_1 \dots b'_m B'_m$ for suitable $m \in \mathbb{N}$, $b'_i \in \Sigma$, $B'_i \in \mathcal{L}_{n-1}$. Now we insert this representation of $C_{i,j}$ into Equation (1) and transform it (Lemma 1.13) to

$$A = \bigcup_{i=1}^{k} \left(A'_{i,0} \cap \overline{A_{i,0}} \right) a_{i,1} \left(A'_{i,1} \cap \overline{A_{i,1}} \right) \dots a_{i,m_i} \left(A'_{i,m_i} \cap \overline{A_{i,m_i}} \right) \tag{6}$$

where $k, m_i \in \mathbb{N}$, $a_{i,j} \in \Sigma$, $A_{i,j} \in \mathcal{L}_{n-\frac{1}{2}}$, and every $A'_{i,j}$ is of the form $B_0 b_1 B_1 \dots b_l B_l$ for suitable $l \in \mathbb{N}$, $b_i \in \Sigma$, $B_i \in \mathcal{L}_{n-1}$. □

Lemma 2.2 *Every $B \in \mathcal{B}_{n+\frac{1}{2}}$ with $n \in \mathbb{N}^+$ can be written as*

$$B = \bigcup_{i=1}^{k} u_{i,0} \left(B'_{i,1} \cap \overline{B_{i,1}}^{+} \right) u_{i,1} \dots \left(B'_{i,m_i} \cap \overline{B_{i,m_i}}^{+} \right) u_{i,m_i}$$

where $k, m_i \in \mathbb{N}$, $u_{i,j} \in \Sigma^*$ (if $m_i = 0$ then $u_{i,0} \neq \varepsilon$), $B_{i,j} \in \mathcal{B}_{n-\frac{1}{2}}$, and every $B'_{i,j}$ is of the form $v_0 B_1 v_1 \dots B_l v_l$ for suitable $l \in \mathbb{N}$, $v_i \in \Sigma^*$ (if $l = 0$ then $v_0 \neq \varepsilon$), $B_i \in \mathcal{B}_{n-1}$. Particularly $B'_{i,j} \in \mathcal{B}_{n-\frac{1}{2}}$.

Proof Analogously to the proof of Lemma 2.1. □

3 Auxiliary Results

In this section we provide further results needed to proof the normalforms. We state a distributive law between residuals and Boolean operations. Furthermore, we prove that certain co-levels are closed under concatenation with letters or words. The crucial fact is Lemma 3.4 — some kind of distributive law between intersection and concatenation.

Lemma 3.1 *For arbitrary $A, B \subseteq \Sigma^*$ and $w \in \Sigma^*$ the following holds:*

1. $w^{-1}\overline{A} = \overline{w^{-1}A}$ *and* $\overline{A}w^{-1} = \overline{Aw^{-1}}$
2. $w^{-1}\overline{A}^{+} \cap \Sigma^+ = \overline{w^{-1}A}^{+}$ *and* $\overline{A}^{+} w^{-1} \cap \Sigma^+ = \overline{Aw^{-1}}^{+}$
3. $w^{-1}(A \cap B) = w^{-1}A \cap w^{-1}B$ *and* $(A \cap B)w^{-1} = Aw^{-1} \cap Bw^{-1}$
4. $w^{-1}(A \cup B) = w^{-1}A \cup w^{-1}B$ *and* $(A \cup B)w^{-1} = Aw^{-1} \cup Bw^{-1}$

Proof clear □

Lemma 3.2 *Let $k \in \mathbb{N}^+$, $a \in \Sigma$ and $A \in \mathrm{co}\mathcal{L}_{k+\frac{1}{2}}$. Then we have $aA \in \mathrm{co}\mathcal{L}_{k+\frac{1}{2}}$.*

Proof It is easy to see that the following holds:

$$\begin{aligned}\overline{aA} &= \bigcup_{b \in \Sigma \setminus \{a\}} b\Sigma^* \quad \cup \quad a\overline{A} \quad \cup \quad \{\varepsilon\} \\ &= \bigcup_{b \in \Sigma \setminus \{a\}} \overline{\{\varepsilon\}}\, b\Sigma^* \quad \cup \quad \overline{\{\varepsilon\}}\, a\overline{A} \quad \cup \quad \{\varepsilon\}\end{aligned}$$

Now we want to show that $\overline{\{\varepsilon\}} \in \mathrm{co}\mathcal{L}_{\frac{1}{2}} \subseteq \mathcal{L}_{\frac{3}{2}}$. For this we consider the following (easy) fact:

$$\Sigma^* \setminus \{\varepsilon\} = \bigcup_{b \in \Sigma} \Sigma^* b \Sigma^* \in \mathcal{L}_{\frac{1}{2}}$$

Together with $k > 0$ we obtain

$$\overline{\{\varepsilon\}} = \overline{\Sigma^* \setminus \{\varepsilon\}} \in \mathrm{co}\mathcal{L}_{\frac{1}{2}} \subseteq \mathcal{L}_{\frac{3}{2}} \subseteq \mathcal{L}_{k+\frac{1}{2}}.$$

Hence

$$\bigcup_{b \in \Sigma \setminus \{a\}} \overline{\{\varepsilon\}}\, b\Sigma^* \in \mathcal{L}_{k+\frac{1}{2}}.$$

From $\overline{A} \in \mathcal{L}_{k+\frac{1}{2}}$ it follows that $\overline{\{\varepsilon\}}\, a\overline{A} \in \mathcal{L}_{k+\frac{1}{2}}$. Therefore, $\overline{aA} \in \mathcal{L}_{k+\frac{1}{2}}$ and $aA \in \mathrm{co}\mathcal{L}_{k+\frac{1}{2}}$. □

Lemma 3.3 *Let $n \in \mathbb{N}$, $v \in \Sigma^*$ and $B \in \mathrm{co}^{+}\mathcal{B}_{n+\frac{1}{2}}$ then we have $Bv \in \mathrm{co}^{+}\mathcal{B}_{n+\frac{1}{2}}$.*

Proof Let n, B, v as above. First of all we will show the following:

$$\overline{Bv}^{+} = \underbrace{\left(\bigcup_{\substack{w \in \Sigma^+ \\ |w| \le |v|}} w \right)}_{B_1} \cup \underbrace{\left(\bigcup_{\substack{w \in \Sigma^+ \setminus \{v\} \\ |w| = |v|}} \Sigma^+ w \right)}_{B_2} \cup \overline{B}^{+} v \tag{7}$$

„$\supseteq$“: Let $u \in B_1 \cup B_2 \cup \overline{B}^+ v$. If $u \in \overline{B}^+ v$ then $u = u_1 v$ for a suitable $u_1 \in \Sigma^+ \setminus B$. Thus we have $u \neq \varepsilon$ and $u \notin Bv$. Therefore, $u \in \overline{Bv}^+$. If $u \in B_2$ then there exist words $u_1, w \in \Sigma^+$ with $w \neq v$, $|w| = |v|$ and $u = u_1 w$. It follows that $u \neq \varepsilon$ and $u \notin Bv$. Thus $u \in \overline{Bv}^+$. Finally let $u \in B_1$, i.e. $u \neq \varepsilon$ and $|u| \leq |v|$. Thus $u \notin Bv$, because words in Bv have lengths $> |v|$. Therefore, $u \in \overline{Bv}^+$.

„$\subseteq$“: Now let $u \in \overline{Bv}^+$. If $|u| \leq |v|$ then $u \in B_1$. Now we assume $|u| > |v|$, i.e. there exist words $u_1 \in \Sigma^+, w \in \Sigma^*$ with $u = u_1 w$ and $|w| = |v|$. If $w \neq v$ then $w \neq \varepsilon$ and $u = u_1 w \in B_2$. If $w = v$, we have $u = u_1 v$ with $u_1 \in \Sigma^+$. Suppose $u_1 \in B$, then we would obtain $u \in Bv$ (a contradiction). Therefore, $u_1 \notin B$, i.e. $u_1 \in \overline{B}^+$. We conclude $u \in \overline{B}^+ v$. This proves Equation (7).

It is easy to see that $B_1, B_2 \in \mathcal{B}_{\frac{1}{2}} \subseteq \mathcal{B}_{n+\frac{1}{2}}$. From $\overline{B}^+ \in \mathcal{B}_{n+\frac{1}{2}}$ it follows that $\overline{B}^+ v \in \mathcal{B}_{n+\frac{1}{2}}$. By Equation (7) we obtain $\overline{Bv}^+ \in \mathcal{B}_{n+\frac{1}{2}}$. Since $\varepsilon \notin Bv$ we conclude:

$$Bv = \overline{\overline{Bv}^+}^+ \in \mathrm{co}^+\mathcal{B}_{n+\frac{1}{2}}$$

□

Let us consider the following example. $A \stackrel{df}{=} \{a\}^+$, $B \stackrel{df}{=} \{b\}^+$ and $C \stackrel{df}{=} A \cup B$. Then we have $(AB) \cap C = \emptyset$, but $(A \cap C)(B \cap C) = AB \neq \emptyset$. This shows that there exists no distribitive law between intersection and concatenation. However, the following lemma provides some kind of modified distributive law between intersection and concatenation. It says that we can take C into the concatenation of A and B, but we have to take residuals of C. Later we will see, that the levels of our hierarchies are closed under taking residuals. This enables us to prove the normalforms.

Lemma 3.4 *For arbitrary $A, B, C \subseteq \Sigma^*$ the following holds:*

$$AB \cap C = \bigcup_{u \in B} \left[(A \cap Cu^{-1}) \cdot \left(\bigcap_{v \in A \cap Cu^{-1}} (B \cap v^{-1}C) \right) \right]$$

Proof „$\supseteq$“: Let

$$x \in \bigcup_{u \in B} \left[(A \cap Cu^{-1}) \cdot \left(\bigcap_{v \in A \cap Cu^{-1}} (B \cap v^{-1}C) \right) \right].$$

Then there exists an $u \in B$ such that

$$x \in (A \cap Cu^{-1}) \cdot \left(\bigcap_{v \in A \cap Cu^{-1}} (B \cap v^{-1}C) \right).$$

Thus there exist $x_1 \in A \cap Cu^{-1}$ and

$$x_2 \in \bigcap_{v \in A \cap Cu^{-1}} (B \cap v^{-1}C)$$

with $x = x_1 x_2$. Since $x_1 \in A$ and $x_2 \in B$ we have $x = x_1 x_2 \in AB$. From $x_1 \in A \cap Cu^{-1}$ and

$$x_2 \in \bigcap_{v \in A \cap Cu^{-1}} (B \cap v^{-1}C)$$

we obtain $x_2 \in B \cap x_1^{-1}C$ (set $v = x_1$ in the intersection). Thus $x = x_1 x_2 \in C$.

„$\subseteq$“: Now let $x \in AB \cap C$, i.e. $x = x_1x_2$ with $x_1 \in A$ and $x_2 \in B$. From $x_1x_2 \in C$ we obtain $x_1 \in Cx_2^{-1}$. Therefore, $x_1 \in A \cap Cx_2^{-1}$. Now we want to show

$$x_2 \in \bigcap_{v \in A \cap Cx_2^{-1}} \left(B \cap v^{-1}C\right).$$

Let $v \in A \cap Cx_2^{-1}$. From $v \in Cx_2^{-1} = \{w : wx_2 \in C\}$ it follows that $vx_2 \in C$. Hence $x_2 \in v^{-1}C$. Since $x_2 \in B$, we obtain $x_2 \in B \cap v^{-1}C$ for every $v \in A \cap Cx_2^{-1}$. Therefore,

$$x_2 \in \bigcap_{v \in A \cap Cx_2^{-1}} \left(B \cap v^{-1}C\right).$$

We conclude

$$x = x_1x_2 \in \left(A \cap Cx_2^{-1}\right) \cdot \left(\bigcap_{v \in A \cap Cx_2^{-1}} \left(B \cap v^{-1}C\right) \right).$$

Since $x_2 \in B$ we obtain

$$x \in \bigcup_{u \in B} \left[\left(A \cap Cu^{-1}\right) \cdot \left(\bigcap_{v \in A \cap Cu^{-1}} \left(B \cap v^{-1}C\right) \right) \right]$$

and the lemma is proved. $\square$

4 The Levels are closed under taking residuals

In this section we will show that the levels $\mathcal{L}_{\frac{n}{2}}$, co$\mathcal{L}_{\frac{n}{2}}$, $\mathcal{B}_{\frac{n}{2}}$ and co$^+\mathcal{B}_{\frac{n}{2}}$ are closed under taking left and right residuals, and that the number of residuals is finite. Notice that these results are already known from other papers (e.g. [Arf91, PW97]).

Lemma 4.1 *For $n \in \mathbb{N}$ and $D \in \mathcal{L}_{\frac{n}{2}}$ the following is true:*

1. $\mathcal{R}_r(D, \Sigma^*) \subseteq \mathcal{L}_{\frac{n}{2}}$
2. $|\mathcal{R}_r(D, \Sigma^*)| < \infty$
3. $\mathcal{R}_l(D, \Sigma^*) \subseteq \mathcal{L}_{\frac{n}{2}}$
4. $|\mathcal{R}_l(D, \Sigma^*)| < \infty$

Proof By symmetry it suffices to show the Assertions 1 and 2. Since languages in $\mathcal{L}_{\frac{n}{2}}$ are regular, the sets of its residuals are finite [Ner58]. This shows Assertion 2. We will show Assertion 1 by induction on n.

For $\mathcal{L}_0 = \{\emptyset, \Sigma^*\}$ we have $\mathcal{R}_r(\emptyset, \Sigma^*) = \{\emptyset\}$ and $\mathcal{R}_r(\Sigma^*, \Sigma^*) = \{\Sigma^*\}$. This shows the induction base ($n = 0$).

Now we assume that Assertion 1 has been shown for all $n \leq 2k$ with $k \in \mathbb{N}$ (induction hypothesis). First of all we will show the following:

$$\textit{For } a \in \Sigma \textit{ and } A \in \mathcal{L}_{\frac{2k+1}{2}} \textit{ we have } Aa^{-1} \in \mathcal{L}_{\frac{2k+1}{2}}. \tag{8}$$

Let $a \in \Sigma$ and $A \in \mathcal{L}_{\frac{2k+1}{2}}$ such that

$$A = \bigcup_{i=1}^{l} \left(A_{i,0}a_{i,1}A_{i,1} \ldots a_{i,m_i}A_{i,m_i}\right)$$

with $l, m_i \in \mathbb{N}$, $a_{i,j} \in \Sigma$, $A_{i,j} \in \mathcal{L}_{\frac{2k}{2}}$. By Lemma 3.1, it suffices to show the following:

For $m \in \mathbb{N}$, $a, a_1, \ldots, a_m \in \Sigma$, $A_0, \ldots, A_m \in \mathcal{L}_{\frac{2k}{2}}$ *we have* $(A_0 a_1 A_1 \ldots a_m A_m) a^{-1} \in \mathcal{L}_{\frac{2k+1}{2}}$. (9)

Now let $m, a, a_1, \ldots, a_m, A_0, \ldots, A_m$ as above then we consider 4 cases:

1. $A_0 a_1 A_1 \ldots a_m A_m = \emptyset$:
   Then we have $(A_0 a_1 A_1 \ldots a_m A_m) a^{-1} = \emptyset \in \mathcal{L}_{\frac{2k+1}{2}}$.

2. $A_0 a_1 A_1 \ldots a_m A_m \neq \emptyset$ and $m = 0$:
   Thus $(A_0 a_1 A_1 \ldots a_m A_m) a^{-1} = A_0 a^{-1} \in \mathcal{L}_{\frac{2k}{2}} \subseteq \mathcal{L}_{\frac{2k+1}{2}}$ by induction hypothesis.

3. $A_0 a_1 A_1 \ldots a_m A_m \neq \emptyset$, $m > 0$, $\varepsilon \in A_m$ and $a = a_m$:
   Thus $(A_0 a_1 A_1 \ldots a_m A_m) a^{-1} = (A_0 a_1 A_1 \ldots a_m (A_m a^{-1})) \cup (A_0 a_1 A_1 \ldots a_{m-1} A_{m-1})$. Since $A_m a^{-1} \in \mathcal{L}_{\frac{2k}{2}}$ (ind. hypothesis) we obtain $(A_0 a_1 A_1 \ldots a_m A_m) a^{-1} \in \mathcal{L}_{\frac{2k+1}{2}}$.

4. $A_0 a_1 A_1 \ldots a_m A_m \neq \emptyset$, $m > 0$ and ($\varepsilon \notin A_m$ or $a \neq a_m$):
   Then we have $(A_0 a_1 A_1 \ldots a_m A_m) a^{-1} = (A_0 a_1 A_1 \ldots a_m (A_m a^{-1}))$. Since $A_m a^{-1} \in \mathcal{L}_{\frac{2k}{2}}$ (ind. hypothesis) we obtain $(A_0 a_1 A_1 \ldots a_m A_m) a^{-1} \in \mathcal{L}_{\frac{2k+1}{2}}$.

Therefore, we have shown (9) and (8).

Now let $A \in \mathcal{L}_{\frac{2k+1}{2}}$ and $w \in \Sigma^*$ with $w = w_1 \ldots w_l$ (where $w_1, \ldots, w_l \in \Sigma$ and $l \in \mathbb{N}$). We have

$$A w^{-1} = (\ldots((A w_l^{-1}) w_{l-1}^{-1}) \ldots w_2^{-1}) w_1^{-1}.$$

By repeated application of (8) we obtain $A w^{-1} \in \mathcal{L}_{\frac{2k+1}{2}}$. Therefore, $\mathcal{R}_r(A, \Sigma^*) \subseteq \mathcal{L}_{\frac{2k+1}{2}}$ for every $A \in \mathcal{L}_{\frac{2k+1}{2}}$.

Every language $B \in \mathcal{L}_{\frac{2k+2}{2}}$ is a Boolean combination of languages $B_1, \ldots, B_l \in \mathcal{L}_{\frac{2k+1}{2}}$. From Lemma 3.1 we obtain: If $w \in \Sigma^*$ then $B w^{-1}$ is a Boolean combination of the languages $B_1 w^{-1}, \ldots, B_l w^{-1}$. We have already shown that these languages are in $\mathcal{L}_{\frac{2k+1}{2}}$. Therefore, $B w^{-1} \in \mathcal{L}_{\frac{2k+2}{2}}$. We have shown $\mathcal{R}_r(B, \Sigma^*) \subseteq \mathcal{L}_{\frac{2k+2}{2}}$ for every $B \in \mathcal{L}_{\frac{2k+2}{2}}$. Hence the induction step is complete. $\square$

Lemma 4.2 *For $n \in \mathbb{N}$ and $D \in \mathrm{co}\mathcal{L}_{\frac{n}{2}}$:*

1. $\mathcal{R}_r(D, \Sigma^*) \subseteq \mathrm{co}\mathcal{L}_{\frac{n}{2}}$
2. $|\mathcal{R}_r(D, \Sigma^*)| < \infty$
3. $\mathcal{R}_l(D, \Sigma^*) \subseteq \mathrm{co}\mathcal{L}_{\frac{n}{2}}$
4. $|\mathcal{R}_l(D, \Sigma^*)| < \infty$

Proof Let $D \in \mathrm{co}\mathcal{L}_{\frac{n}{2}}$ and $X \in \mathcal{R}_r(D, \Sigma^*)$, then there exists a $w \in \Sigma^*$ such that:

$$\begin{aligned} X = D w^{-1} &= \{v \in \Sigma^* : vw \in D\} \\ &= \left\{v \in \Sigma^* : vw \notin \overline{D}\right\} \\ &= \overline{\left\{v \in \Sigma^* : vw \in \overline{D}\right\}} \\ &= \overline{\overline{D} w^{-1}} \in \mathrm{co}\mathcal{R}_r(\overline{D}, \Sigma^*) \end{aligned}$$

Since $\overline{D} \in \mathcal{L}_{\frac{n}{2}}$, from Lemma 4.1 we conclude that $\mathcal{R}_r(\overline{D}, \Sigma^*)$ is a finite subset of $\mathcal{L}_{\frac{n}{2}}$. Thus $\mathrm{co}\mathcal{R}_r(\overline{D}, \Sigma^*)$ is a finite subset of $\mathrm{co}\mathcal{L}_{\frac{n}{2}}$. From $X \in \mathrm{co}\mathcal{R}_r(\overline{D}, \Sigma^*)$ we obtain $\mathcal{R}_r(D, \Sigma^*) \subseteq \mathrm{co}\mathcal{R}_r(\overline{D}, \Sigma^*)$. Therefore, $\mathcal{R}_r(D, \Sigma^*)$ is a finite subset of $\mathrm{co}\mathcal{L}_{\frac{n}{2}}$. This proves 1 and 2. Analogously one shows 3 and 4. □

Lemma 4.3 *Let $k \in \mathbb{N}$, $C \in \mathrm{Pol}^*(\mathrm{co}\mathcal{L}_{k+\frac{1}{2}})$ and $D \in \mathrm{co}\mathcal{L}_{k+\frac{1}{2}}$, then we have*

$$CD \in \mathrm{Pol}^*(\mathrm{co}\mathcal{L}_{k+\frac{1}{2}}).$$

Proof **Case 1:** D does not contain the empty word. Thus we have

$$D = \bigcup_{c \in \Sigma} c \cdot \left(c^{-1}D\right).$$

Hence

$$CD = \bigcup_{c \in \Sigma} C \cdot c \cdot \left(c^{-1}D\right).$$

Since $c^{-1}D \in \mathcal{R}_l(D, \Sigma^*) \subseteq \mathrm{co}\mathcal{L}_{k+\frac{1}{2}}$ (Lemma 4.2), from Lemma 1.14 we obtain $CD \in \mathrm{Pol}^*(\mathrm{co}\mathcal{L}_{k+\frac{1}{2}})$.

Case 2: D contains the empty word. Thus we have

$$D = \bigcup_{c \in \Sigma} c \cdot \left(c^{-1}D\right) \quad \cup \quad \{\varepsilon\}.$$

Hence

$$CD = \bigcup_{c \in \Sigma} C \cdot c \cdot \left(c^{-1}D\right) \quad \cup \quad C.$$

Analogously to Case 1 we obtain $CD \in \mathrm{Pol}^*(\mathrm{co}\mathcal{L}_{k+\frac{1}{2}})$. □

Lemma 4.4 *For $n \in \mathbb{N}$ and $D \in \mathcal{B}_{\frac{n}{2}}$ the following is true:*

1. $\mathcal{R}_r^+(D, \Sigma^*) \subseteq \mathcal{B}_{\frac{n}{2}}$
2. $|\mathcal{R}_r^+(D, \Sigma^*)| \leq |\mathcal{R}_r(D, \Sigma^*)| < \infty$
3. $\mathcal{R}_l^+(D, \Sigma^*) \subseteq \mathcal{B}_{\frac{n}{2}}$
4. $|\mathcal{R}_l^+(D, \Sigma^*)| \leq |\mathcal{R}_l(D, \Sigma^*)| < \infty$

Proof By symmetry it suffices to show the Assertions 1 and 2. Since languages of $\mathcal{B}_{\frac{n}{2}}$ are regular, the sets of its residuals are finite [Ner58]. Thus $|\mathcal{R}_r(D, \Sigma^*)| < \infty$ for every $n \in \mathbb{N}$ and $D \in \mathcal{B}_{\frac{n}{2}}$. Since $\mathcal{R}_r^+(D, \Sigma^*) = \{R \cap \Sigma^+ : R \in \mathcal{R}_r(D, \Sigma^*)\}$ we have $|\mathcal{R}_r^+(D, \Sigma^*)| \leq |\mathcal{R}_r(D, \Sigma^*)|$. Thus we have shown Assertion 2. We will show Assertion 1 by induction on n.

For $\mathcal{B}_0 = \{\emptyset, \Sigma^+\}$ we obtain $\mathcal{R}_r(\emptyset, \Sigma^*) = \{\emptyset\}$ and $\mathcal{R}_r(\Sigma^+, \Sigma^*) = \{\Sigma^+, \Sigma^*\}$. Since $\emptyset \cap \Sigma^+, \Sigma^+ \cap \Sigma^+, \Sigma^* \cap \Sigma^+ \in \mathcal{B}_0$, we have shown Assertion 1 for the case $n = 0$ (induction base).

Suppose now that Assertion 1 has been shown for all $n \leq 2k$ (induction hypothesis). First of all we will show:

$$\textit{For } a \in \Sigma \textit{ and } A \in \mathcal{B}_{\frac{2k+1}{2}} \textit{ we have } Aa^{-1} \cap \Sigma^+ \in \mathcal{B}_{\frac{2k+1}{2}}. \tag{10}$$

Let $a \in \Sigma$ and $A \in \mathcal{B}_{\frac{2k+1}{2}}$. By definition,

$$A = \bigcup_{i=1}^{l} (u_{i,0} A_{i,1} u_{i,1} \ldots A_{i,m_i} u_{i,m_i})$$

where $l \in \mathbb{N}$, $m_i \in \mathbb{N}$, $u_{i,j} \in \Sigma^*$, $A_{i,j} \in \mathcal{B}_{\frac{2k}{2}}$ (if $m_i = 0$, then $u_{i,0} \neq \varepsilon$). By Lemma 3.1 we obtain

$$Aa^{-1} = \bigcup_{i=1}^{l} \left((u_{i,0} A_{i,1} u_{i,1} \ldots A_{i,m_i} u_{i,m_i})\, a^{-1} \right)$$

and

$$Aa^{-1} \cap \Sigma^+ = \bigcup_{i=1}^{l} \left((u_{i,0} A_{i,1} u_{i,1} \ldots A_{i,m_i} u_{i,m_i})\, a^{-1} \cap \Sigma^+ \right).$$

Since $\mathcal{B}_{\frac{2k+1}{2}}$ is closed under finite union, for (10) it suffices to show the following:

Let $m \in \mathbb{N}$, $a \in \Sigma$, $u_0, \ldots, u_m \in \Sigma^*$, $A_1, \ldots, A_m \in \mathcal{B}_{\frac{2k}{2}}$ (if $m = 0$, then $u_0 \neq \varepsilon$), then we have $(u_0 A_1 u_1 \ldots A_m u_m) a^{-1} \cap \Sigma^+ \in \mathcal{B}_{\frac{2k+1}{2}}$. (11)

Let $m, a, u_0, \ldots, u_m, A_1, \ldots, A_m$ be the variables defined in (11). We consider 5 cases:

1. $u_0 A_1 u_1 \ldots A_m u_m = \emptyset$:
   Thus $(u_0 A_1 u_1 \ldots A_m u_m) a^{-1} \cap \Sigma^+ = \emptyset \in \mathcal{B}_{\frac{2k+1}{2}}$.

2. $u_0 A_1 u_1 \ldots A_m u_m \neq \emptyset$ and $m = 0$:
   Thus $u_0 \neq \varepsilon$. Therefore,

   $$u_0 a^{-1} = \begin{cases} \varepsilon & : \text{ if } u_0 = a, \\ \emptyset & : \text{ if the last letter of } u_0 \text{ is no } a, \\ u \in \Sigma^+ & : \text{ otherwise.} \end{cases}$$

   Hence, either $u_0 a^{-1} \cap \Sigma^+ = \emptyset \in \mathcal{B}_{\frac{2k+1}{2}}$ or $u_0 a^{-1} \cap \Sigma^+ = \{u\} \in \mathcal{B}_{\frac{2k+1}{2}}$ for a suitable $u \in \Sigma^+$.

3. $u_0 A_1 u_1 \ldots A_m u_m \neq \emptyset$, $m \neq 0$ and $u_m \neq \varepsilon$:
   Thus $(u_0 A_1 u_1 \ldots A_m u_m) a^{-1} = (u_0 A_1 u_1 \ldots A_m)(u_m a^{-1}) \in \mathcal{B}_{\frac{2k+1}{2}}$. It follows that $(u_0 A_1 u_1 \ldots A_m u_m) a^{-1} \cap \Sigma^+ = (u_0 A_1 u_1 \ldots A_m u_m) a^{-1} \in \mathcal{B}_{\frac{2k+1}{2}}$.

4. $u_0 A_1 u_1 \ldots A_m u_m \neq \emptyset$, $m \neq 0$, $u_m = \varepsilon$ and $a \notin A_m$:
   In this case $\varepsilon \notin A_m a^{-1}$, thus $(u_0 A_1 u_1 \ldots A_m \varepsilon) a^{-1} = (u_0 A_1 u_1 \ldots A_{m-1} u_{m-1})(A_m a^{-1}) = (u_0 A_1 u_1 \ldots A_{m-1} u_{m-1})(A_m a^{-1} \cap \Sigma^+)\varepsilon$. From the induction hypothesis we conclude $A_m a^{-1} \cap \Sigma^+ \in \mathcal{B}_{\frac{2k}{2}}$. Therefore, $(u_0 A_1 u_1 \ldots A_m u_m) a^{-1} \in \mathcal{B}_{\frac{2k+1}{2}}$, and we obtain $(u_0 A_1 u_1 \ldots A_m u_m) a^{-1} \cap \Sigma^+ \in \mathcal{B}_{\frac{2k+1}{2}}$.

5. $u_0 A_1 u_1 \ldots A_m u_m \neq \emptyset$, $m \neq 0$, $u_m = \varepsilon$ and $a \in A_m$:
   In this case $\varepsilon \in A_m a^{-1}$, thus we obtain:

$$\begin{aligned} (u_0 A_1 u_1 \ldots A_m \varepsilon) a^{-1} \cap \Sigma^+ &= [(u_0 A_1 u_1 \ldots A_{m-1} u_{m-1})(A_m a^{-1})] \cap \Sigma^+ && (12) \\ &= [(u_0 A_1 u_1 \ldots A_{m-1} u_{m-1})(A_m a^{-1} \cap \Sigma^+)\varepsilon \cup && (13) \\ &\quad\ (u_0 A_1 u_1 \ldots A_{m-1} u_{m-1})] \cap \Sigma^+ && (14) \\ &= [((u_0 A_1 u_1 \ldots A_{m-1} u_{m-1})(A_m a^{-1} \cap \Sigma^+)\varepsilon) \cap \Sigma^+] \cup && (15) \\ &\quad\ [(u_0 A_1 u_1 \ldots A_{m-1} u_{m-1}) \cap \Sigma^+] && (16) \end{aligned}$$

By induction hypothesis, $A_m a^{-1} \cap \Sigma^+ \in \mathcal{B}_{\frac{2k}{2}}$. Therefore,

$$(u_0 A_1 u_1 \dots A_{m-1} u_{m-1})(A_m a^{-1} \cap \Sigma^+)\varepsilon \in \mathcal{B}_{\frac{2k+1}{2}}$$

and

$$((u_0 A_1 u_1 \dots A_{m-1} u_{m-1})(A_m a^{-1} \cap \Sigma^+)\varepsilon) \cap \Sigma^+ \in \mathcal{B}_{\frac{2k+1}{2}}. \tag{17}$$

Let us consider the language $[(u_0 A_1 u_1 \dots A_{m-1} u_{m-1}) \cap \Sigma^+]$, we distinguish 3 cases:

(a) We have $m = 1$ and $u_0 = \varepsilon$. Thus $(u_0 A_1 u_1 \dots A_{m-1} u_{m-1}) \cap \Sigma^+ = \varepsilon \cap \Sigma^+ = \emptyset \in \mathcal{B}_{\frac{2k+1}{2}}$.

(b) We have $m = 1$ and $u_0 \neq \varepsilon$. Thus $(u_0 A_1 u_1 \dots A_{m-1} u_{m-1}) \cap \Sigma^+ = u_0 \in \mathcal{B}_{\frac{2k+1}{2}}$.

(c) We have $m > 1$. Thus $(u_0 A_1 u_1 \dots A_{m-1} u_{m-1}) \in \mathcal{B}_{\frac{2k+1}{2}}$. It follows that $(u_0 A_1 u_1 \dots A_{m-1} u_{m-1}) \cap \Sigma^+ \in \mathcal{B}_{\frac{2k+1}{2}}$.

Hence in all 3 cases we obtain:

$$(u_0 A_1 u_1 \dots A_{m-1} u_{m-1}) \cap \Sigma^+ \in \mathcal{B}_{\frac{2k+1}{2}} \tag{18}$$

From the Equations (12) to (18) we conclude:

$$(u_0 A_1 u_1 \dots A_m \varepsilon) a^{-1} \cap \Sigma^+ \in \mathcal{B}_{\frac{2k+1}{2}}$$

We have shown (11) and (10).

Now let $w \in \Sigma^*$ and $A \in \mathcal{B}_{\frac{2k+1}{2}}$. Furthermore, $w = w_1 \dots w_l$ with $w_1, \dots, w_l \in \Sigma$ and $l \in \mathbb{N}$. It is easy to see that for every $B \subseteq \Sigma^*$ and $b \in \Sigma$ the following holds: $Bb^{-1} = (B \cap \Sigma^+)b^{-1}$. Thus we obtain:

$$\begin{aligned} Aw^{-1} \cap \Sigma^+ &= (\dots((Aw_l^{-1})w_{l-1}^{-1}) \dots w_2^{-1})w_1^{-1} \cap \Sigma^+ \\ &= (\dots((Aw_l^{-1} \cap \Sigma^+)w_{l-1}^{-1} \cap \Sigma^+) \dots w_2^{-1} \cap \Sigma^+)w_1^{-1} \cap \Sigma^+ \end{aligned}$$

By repeated application of (10) we obtain $Aw^{-1} \cap \Sigma^+ \in \mathcal{B}_{\frac{2k+1}{2}}$. We have shown $\mathcal{R}_r^+(A, \Sigma^*) \subseteq \mathcal{B}_{\frac{2k+1}{2}}$ for every $A \in \mathcal{B}_{\frac{2k+1}{2}}$.

Every language $B \in \mathcal{B}_{\frac{2k+2}{2}}$ can be written in (a kind of) disjunktive normalform. This means

$$B = \bigcup_{i=1}^{l} \left(B_{i,1} \cap \overline{B_{i,2}}^{+} \cap \overline{B_{i,3}}^{+} \cap B_{i,4} \cap \dots \cap B_{i,m_i} \right)$$

where $l \in \mathbb{N}$, $m_i \in \mathbb{N}^+$ and $B_{i,j} \in \mathcal{B}_{\frac{2k+1}{2}}$. Notice that complementation is taken w.r.t. Σ^+. For this complementation we have $\overline{D}^{+} = \overline{D \cap \Sigma^+}^{+}$ for every language $D \subseteq \Sigma^*$. From Lemma 3.1

we conclude, that for every $w \in \Sigma^*$ the following holds:

$$\begin{aligned}
Bw^{-1} \cap \Sigma^+ &= \left[\bigcup_{i=1}^{l} \left(B_{i,1} \cap \overline{B_{i,2}}^+ \cap \overline{B_{i,3}}^+ \cap B_{i,4} \cap \cdots \cap B_{i,m_i}\right)\right] w^{-1} \cap \Sigma^+ \\
&= \left[\bigcup_{i=1}^{l} \left(B_{i,1}w^{-1} \cap \overline{B_{i,2}}^+ w^{-1} \cap \overline{B_{i,3}}^+ w^{-1} \cap B_{i,4}w^{-1} \cap \cdots \cap B_{i,m_i}w^{-1}\right)\right] \cap \Sigma^+ \\
&= \bigcup_{i=1}^{l} \left(B_{i,1}w^{-1} \cap \overline{B_{i,2}}^+ w^{-1} \cap \overline{B_{i,3}}^+ w^{-1} \cap B_{i,4}w^{-1} \cap \cdots \cap B_{i,m_i}w^{-1} \cap \Sigma^+\right) \\
&= \bigcup_{i=1}^{l} \left(B_{i,1}w^{-1} \cap \overline{B_{i,2}w^{-1}}^+ \cap \overline{B_{i,3}w^{-1}}^+ \cap B_{i,4}w^{-1} \cap \cdots \cap B_{i,m_i}w^{-1} \cap \Sigma^+\right) \\
&= \bigcup_{i=1}^{l} \begin{pmatrix} B_{i,1}w^{-1} \cap \overline{B_{i,2}w^{-1} \cap \Sigma^+}^+ \cap \overline{B_{i,3}w^{-1} \cap \Sigma^+}^+ \cap \\ \cap B_{i,4}w^{-1} \cap \cdots \cap B_{i,m_i}w^{-1} \cap \Sigma^+ \end{pmatrix} \\
&= \bigcup_{i=1}^{l} \begin{pmatrix} [B_{i,1}w^{-1} \cap \Sigma^+] \cap \overline{[B_{i,2}w^{-1} \cap \Sigma^+]}^+ \cap \overline{[B_{i,3}w^{-1} \cap \Sigma^+]}^+ \cap \\ \cap [B_{i,4}w^{-1} \cap \Sigma^+] \cap \cdots \cap [B_{i,m_i}w^{-1} \cap \Sigma^+] \end{pmatrix}
\end{aligned}$$

We have already shown that the languages in the square brackets are in $\mathcal{B}_{\frac{2k+1}{2}}$. Thus $Bw^{-1} \cap \Sigma^+ \in \mathcal{B}_{\frac{2k+2}{2}}$. Hence $\mathcal{R}_r^+(B, \Sigma^*) \subseteq \mathcal{B}_{\frac{2k+2}{2}}$ for every $B \in \mathcal{B}_{\frac{2k+2}{2}}$. Therefore, the induction step is complete. □

Lemma 4.5 *For $n \in \mathbb{N}$ and $D \in \mathrm{co}^+\mathcal{B}_{\frac{n}{2}}$:*

1. $\mathcal{R}_r^+(D, \Sigma^*) \subseteq \mathrm{co}^+\mathcal{B}_{\frac{n}{2}}$
2. $|\mathcal{R}_r^+(D, \Sigma^*)| \leq |\mathcal{R}_r(D, \Sigma^*)| < \infty$
3. $\mathcal{R}_l^+(D, \Sigma^*) \subseteq \mathrm{co}^+\mathcal{B}_{\frac{n}{2}}$
4. $|\mathcal{R}_l^+(D, \Sigma^*)| \leq |\mathcal{R}_l(D, \Sigma^*)| < \infty$

Proof Let $D \in \mathrm{co}^+\mathcal{B}_{\frac{n}{2}}$ and $X \in \mathcal{R}_r^+(D, \Sigma^*)$, then there exists a $w \in \Sigma^*$ such that:

$$\begin{aligned}
X = Dw^{-1} \cap \Sigma^+ &= \{v \in \Sigma^* : vw \in D\} \cap \Sigma^+ = \left\{v \in \Sigma^+ : vw \in D\right\} \\
&= \{v \in \Sigma^+ : vw \notin \overline{D}\} = \left\{v \in \Sigma^+ : vw \notin \overline{D}^+\right\} \\
&= \overline{\left\{v \in \Sigma^+ : vw \in \overline{D}^+\right\}}^+ = \overline{\left\{v \in \Sigma^* : vw \in \overline{D}^+\right\} \cap \Sigma^+}^+ \\
&= \overline{\overline{D}^+ w^{-1} \cap \Sigma^+}^+ \in \mathrm{co}^+\mathcal{R}_r^+(\overline{D}^+, \Sigma^*)
\end{aligned}$$

Since $\overline{D}^+ \in \mathcal{B}_{\frac{n}{2}}$, from Lemma 4.4 we conclude that $\mathcal{R}_r^+(\overline{D}^+, \Sigma^*)$ is a finite subset of $\mathcal{B}_{\frac{n}{2}}$. Thus $\mathrm{co}^+\mathcal{R}_r^+(\overline{D}^+, \Sigma^*)$ is a finite subset of $\mathrm{co}^+\mathcal{B}_{\frac{n}{2}}$. Since $X \in \mathrm{co}^+\mathcal{R}_r^+(\overline{D}^+, \Sigma^*)$ we have $\mathcal{R}_r^+(D, \Sigma^*) \subseteq \mathrm{co}^+\mathcal{R}_r^+(\overline{D}^+, \Sigma^*)$. Therefore, $\mathcal{R}_r^+(D, \Sigma^*)$ is a finite subset of $\mathrm{co}^+\mathcal{B}_{\frac{n}{2}}$. This proves 1 and 2. Analogously one shows 3 and 4. □

5 A Normalform for $\mathcal{L}_{n+\frac{1}{2}}$

This section provides a main result of the paper, i.e. the normalform for the levels $n+\frac{1}{2}$ of the Straubing-Thérien-Hierarchy. Lemma 5.1 combines our modified distributive law (Lemma 3.4) and the fact that the levels are closed under taking residuals (Lemma 4.1 and 4.2). Together with the weak normalform for the Straubing-Thérien-Hierarchy (Lemma 2.1), we obtain the promised result (Theorem 5.2).

Lemma 5.1 *For $n \in \mathbb{N}^+$, $m \in \mathbb{N}$, $c_1, \ldots, c_m \in \Sigma$ and $D, C_0, \ldots, C_m \in \mathrm{co}\mathcal{L}_{n+\frac{1}{2}}$ we have $C_0c_1C_1 \ldots c_mC_m \cap D \in \mathrm{Pol}^*(\mathrm{co}\mathcal{L}_{n+\frac{1}{2}})$.*

Proof We will show this by induction on m. For $m = 0$ we have $C_0, D \in \mathrm{co}\mathcal{L}_{n+\frac{1}{2}}$. Since $\mathrm{co}\mathcal{L}_{n+\frac{1}{2}}$ is closed under finite intersection (Theorem 1.11) we obtain $C_0 \cap D \in \mathrm{co}\mathcal{L}_{n+\frac{1}{2}} \subseteq \mathrm{Pol}^*(\mathrm{co}\mathcal{L}_{n+\frac{1}{2}})$.

Now we assume that the assertion has been shown for $m = k \in \mathbb{N}$ (induction hypothesis), and we want to show it for $m = k+1$ (induction step). Let $n, D, c_1, \ldots, c_{k+1}, C_0, \ldots, C_{k+1}$ as above. By Lemma 3.4, for arbitrary $A, B, C \subseteq \Sigma^*$:

$$AB \cap C = \bigcup_{u \in B} \left[\left(A \cap Cu^{-1}\right) \cdot \left(\bigcap_{v \in A \cap Cu^{-1}} \left(B \cap v^{-1}C\right) \right) \right]$$

In our case ($A \stackrel{df}{=} C_0c_1C_1 \ldots c_kC_k$, $B \stackrel{df}{=} c_{k+1}C_{k+1}$, $C \stackrel{df}{=} D$) this means:

$$C_0c_1C_1 \ldots c_{k+1}C_{k+1} \cap D = \bigcup_{u \in c_{k+1}C_{k+1}} \left[\overbrace{\left(C_0c_1C_1 \ldots c_kC_k \cap Du^{-1}\right)}^{D_u} \cdot \underbrace{\left(\bigcap_{v \in \underbrace{C_0c_1C_1 \ldots c_kC_k \cap Du^{-1}}_{D_u}} \left(c_{k+1}C_{k+1} \cap v^{-1}D\right) \right)}_{E_u} \right] \tag{19}$$

By Lemma 4.2, $\mathcal{R}_r(D, c_{k+1}C_{k+1}) = \left\{Du^{-1} : u \in c_{k+1}C_{k+1}\right\}$ is a finite subset of $\mathrm{co}\mathcal{L}_{n+\frac{1}{2}}$. Therefore, the number of different D_u is finite. Thus the union in Equation (19) is finite, say it ranges over the finite set $U \subseteq c_{k+1}C_{k+1}$. By Lemma 4.2, $\mathcal{R}_l(D, D_u) = \left\{v^{-1}D : v \in D_u\right\}$ is a finite subset of $\mathrm{co}\mathcal{L}_{n+\frac{1}{2}}$ for every $u \in U$. Therefore, the intersection E_u in Equation (19) is finite.

Since $C_{k+1} \in \mathrm{co}\mathcal{L}_{n+\frac{1}{2}}$, from Lemma 3.2 we conclude $c_{k+1}C_{k+1} \in \mathrm{co}\mathcal{L}_{n+\frac{1}{2}}$. By Lemma 4.2, $v^{-1}D \in \mathrm{co}\mathcal{L}_{n+\frac{1}{2}}$ for every $v \in \Sigma^*$. Therefore, the intersection E_u in Equation (19) is a finite intersection of languages in $\mathrm{co}\mathcal{L}_{n+\frac{1}{2}}$. Since $\mathrm{co}\mathcal{L}_{n+\frac{1}{2}}$ is closed under finite intersection (Theorem 1.11), we conclude $E_u \in \mathrm{co}\mathcal{L}_{n+\frac{1}{2}}$ for every $u \in U$. We obtain the following simplification of Equation (19):

$$C_0c_1C_1 \ldots c_{k+1}C_{k+1} \cap D = \bigcup_{u \in U} \left[\overbrace{\left(C_0c_1C_1 \ldots c_kC_k \cap Du^{-1}\right)}^{D_u} \cdot E_u \right] \tag{20}$$

with $Du^{-1} \in \mathrm{co}\mathcal{L}_{n+\frac{1}{2}}$ for every $u \in U$ by Lemma 4.2. Therefore, from the induction hypothesis we conclude $D_u \in \mathrm{Pol}^*(\mathrm{co}\mathcal{L}_{n+\frac{1}{2}})$. By Lemma 4.3 we obtain $D_uE_u \in \mathrm{Pol}^*(\mathrm{co}\mathcal{L}_{n+\frac{1}{2}})$ for every $u \in U$. Hence $C_0c_1C_1 \ldots c_{k+1}C_{k+1} \cap D \in \mathrm{Pol}^*(\mathrm{co}\mathcal{L}_{n+\frac{1}{2}})$. $\square$

Theorem 5.2 *For $n \in \mathbb{N}^+$ we have $\mathcal{L}_{n+\frac{1}{2}} = \mathrm{Pol}^*(\mathrm{co}\mathcal{L}_{n-\frac{1}{2}})$.*

Proof Since $\mathcal{L}_{n+\frac{1}{2}} \supseteq \mathrm{Pol}^*(\mathrm{co}\mathcal{L}_{n-\frac{1}{2}})$ by definition, we have to show $\mathcal{L}_{n+\frac{1}{2}} \subseteq \mathrm{Pol}^*(\mathrm{co}\mathcal{L}_{n-\frac{1}{2}})$ for all $n \in \mathbb{N}^+$. Let $n = 1$. By Theorem 1.12, $\mathcal{L}_{3/2} = \mathrm{Pol}^*(\{A^* : A \subseteq \Sigma\})$. For $A \subseteq \Sigma$ we have $A^* = \overline{\bigcup_{a \notin A} \Sigma^* a \Sigma^*}$. Thus A^* is in $\mathrm{co}\mathcal{L}_{1/2}$. This proves the assertion for $n = 1$ (induction base). Now let $n \in \mathbb{N}$ with $n > 1$ and $A \in \mathcal{L}_{n+\frac{1}{2}}$. By Lemma 2.1, we can represent A as

$$A \quad = \quad \bigcup_{i=1}^{k} \overbrace{\left(A'_{i,0} \cap \overline{A_{i,0}}\right)}^{C_{i,1}} a_{i,1} \overbrace{\left(A'_{i,1} \cap \overline{A_{i,1}}\right)}^{C_{i,m_i}} \ldots a_{i,m_i} \left(A'_{i,m_i} \cap \overline{A_{i,m_i}}\right)$$

where $k, m_i \in \mathbb{N}$, $a_{i,j} \in \Sigma$, $A_{i,j} \in \mathcal{L}_{n-\frac{1}{2}}$, and every $A'_{i,j}$ is of the form $B_0 b_1 B_1 \ldots b_l B_l$ for suitable $l \in \mathbb{N}$, $b_i \in \Sigma$, $B_i \in \mathcal{L}_{n-1}$. Thus every $C_{i,j}$ is of the form

$$C_{i,j} = B_0 b_1 B_1 \ldots b_l B_l \cap B'$$

for suitable $l \in \mathbb{N}$, $b_i \in \Sigma$, $B_i \in \mathcal{L}_{n-1} \subseteq \mathrm{co}\mathcal{L}_{n-\frac{1}{2}}$ and $B' \in \mathrm{co}\mathcal{L}_{n-\frac{1}{2}}$. By Lemma 5.1 we obtain $C_{i,j} \in \mathrm{Pol}^*(\mathrm{co}\mathcal{L}_{n-\frac{1}{2}})$ (notice $n > 1$). Since $\mathrm{Pol}^*(\mathrm{co}\mathcal{L}_{n-\frac{1}{2}})$ is closed under finite union and marked concatenation (Lemma 1.14), we conclude $A \in \mathrm{Pol}^*(\mathrm{co}\mathcal{L}_{n-\frac{1}{2}})$. □

6 A NormalForm for $\mathcal{B}_{n+\frac{1}{2}}$

Similar to Section 5 we provide the normalform for the levels $n + \frac{1}{2}$ of the Dot-Depth-Hierarchy. Lemma 6.1 combines our modified distributive law (Lemma 3.4) and the fact that the levels are closed under taking residuals (Lemma 4.4 and 4.5). Together with the weak normalform for the Dot-Depth-Hierarchy (Lemma 2.2), we obtain the promised result (Theorem 6.2).

Lemma 6.1 *For $n, m \in \mathbb{N}$, $v_0, \ldots, v_m \in \Sigma^*$ and $B, B_1, \ldots, B_m \in \mathrm{co}^+\mathcal{B}_{n+\frac{1}{2}}$ we have $v_0 B_1 v_1 \ldots B_m v_m \cap B \in \mathrm{Pol}^+(\mathrm{co}^+\mathcal{B}_{n+\frac{1}{2}})$.*

Proof We will show this by induction on m. Since $B \in \mathrm{co}^+\mathcal{B}_{n+\frac{1}{2}}$ we have $\varepsilon \notin B$. Therefore, in the case $m = 0$ either $v_0 \cap B = \{v_0\} \neq \{\varepsilon\}$ or $v_0 \cap B = \emptyset$. Since both languages are in $\mathrm{Pol}^+(\emptyset) \subseteq \mathrm{Pol}^+(\mathrm{co}^+\mathcal{B}_{n+\frac{1}{2}})$, the assertion is proved for the case $m = 0$ (induction base).
Now we assume that the assertion has been proved for $m = k \in \mathbb{N}$ (induction hypothesis), and we want to show it for $m = k + 1$ (induction step). Now let $v_0, \ldots, v_{k+1} \in \Sigma^*$ and $B, B_1, \ldots, B_{k+1} \in \mathrm{co}^+\mathcal{B}_{n+\frac{1}{2}}$.

Case 1: $v_0 = \varepsilon$ and $k = 0$: Since $B_1 \in \mathrm{co}^+\mathcal{B}_{n+\frac{1}{2}}$, by Lemma 3.3 we obtain $v_0 B_1 v_1 = B_1 v_1 \in \mathrm{co}^+\mathcal{B}_{n+\frac{1}{2}}$. Since $\mathrm{co}^+\mathcal{B}_{n+\frac{1}{2}}$ is closed under finite intersection (Theorem 1.11) we conclude $v_0 B_1 v_1 \cap B \in \mathrm{co}^+\mathcal{B}_{n+\frac{1}{2}} \subseteq \mathrm{Pol}^+(\mathrm{co}^+\mathcal{B}_{n+\frac{1}{2}})$.

Case 2: $v_0 \neq \varepsilon$ or $k > 0$: If we set $A \overset{df}{=} v_0 B_1 v_1 \ldots B_k v_k$, $B \overset{df}{=} B_{k+1} v_{k+1}$ and $C \overset{df}{=} B$ then by Lemma 3.4:

$$(v_0 B_1 v_1 \ldots B_{k+1} v_{k+1}) \cap B = \bigcup_{u \in B_{k+1} v_{k+1}} \left[\overbrace{\left((v_0 B_1 v_1 \ldots B_k v_k) \cap B u^{-1}\right)}^{D_u} \cdot \bigcap_{\underbrace{v \in (v_0 B_1 v_1 \ldots B_k v_k) \cap B u^{-1}}_{D_u}} \overbrace{\left(B_{k+1} v_{k+1} \cap v^{-1} B\right)}^{E_u} \right] \tag{21}$$

In this case $\varepsilon \notin v_0B_1v_1\dots B_kv_k$, hence $v_0B_1v_1\dots B_kv_k = (v_0B_1v_1\dots B_kv_k)\cap\Sigma^+$. Therefore, we can rewrite the terms D_u of Equation (21):

$$D_u = (v_0B_1v_1\dots B_kv_k)\cap Bu^{-1} = (v_0B_1v_1\dots B_kv_k)\cap Bu^{-1}\cap\Sigma^+$$

By Lemma 4.5, $\mathcal{R}_r^+(B,\Sigma^*) = \{Bu^{-1}\cap\Sigma^+ : u\in\Sigma^*\}$ is a finite subset of $\mathrm{co}^+\mathcal{B}_{n+\frac{1}{2}}$. Thus the number of different D_u is finite. Therefore, the union in Equation (21) is finite too, say it ranges over the finite set $U\subseteq B_{k+1}v_{k+1}$. Since $\varepsilon\notin B_{k+1}v_{k+1}$ we can rewrite the terms E_u of Equation (21):

$$E_u = B_{k+1}v_{k+1}\cap v^{-1}B = B_{k+1}v_{k+1}\cap v^{-1}B\cap\Sigma^+$$

By Lemma 4.5, $\mathcal{R}_l^+(B,\Sigma^*) = \{v^{-1}B\cap\Sigma^+ : v\in\Sigma^*\}$ is a finite subset of $\mathrm{co}^+\mathcal{B}_{n+\frac{1}{2}}$. Therefore, for a fixed $u\in U$ the intersection in Equation (21) is finite. Since $B_{k+1}\in\mathrm{co}^+\mathcal{B}_{n+\frac{1}{2}}$ we have $B_{k+1}v_{k+1}\in\mathrm{co}^+\mathcal{B}_{n+\frac{1}{2}}$ (Lemma 3.3). By Lemma 4.5 it holds that $v^{-1}B\cap\Sigma^+\in\mathrm{co}^+\mathcal{B}_{n+\frac{1}{2}}$ for every $v\in\Sigma^*$. Therefore, the intersection in Equation (21) is a finite intersection of languages in $\mathrm{co}^+\mathcal{B}_{n+\frac{1}{2}}$. By Theorem 1.11, this intersection is in $\mathrm{co}^+\mathcal{B}_{n+\frac{1}{2}}$. Thus we can rewrite Equation (21) as

$$(v_0B_1v_1\dots B_{k+1}v_{k+1})\cap B = \bigcup_{u\in U}\left[\overbrace{\left((v_0B_1v_1\dots B_kv_k)\cap Bu^{-1}\cap\Sigma^+\right)}^{D_u}\cdot F_u\right] \tag{22}$$

where $U\subseteq\Sigma^*$ is a finite set, and $F_u\in\mathrm{co}^+\mathcal{B}_{n+\frac{1}{2}}$ for every $u\in U$. Since $Bu^{-1}\cap\Sigma^+\in\mathrm{co}^+\mathcal{B}_{n+\frac{1}{2}}$ for every $u\in U$ (Lemma 4.5), from the induction hypothesis we conclude that $D_u\in\mathrm{Pol}^+(\mathrm{co}^+\mathcal{B}_{n+\frac{1}{2}})$. From Lemma 1.14 we obtain $D_uF_u = \varepsilon D_u\varepsilon F_u\varepsilon\in\mathrm{Pol}^+(\mathrm{co}^+\mathcal{B}_{n+\frac{1}{2}})$ for every $u\in U$. Thus $(v_0B_1v_1\dots B_{k+1}v_{k+1})\cap B\in\mathrm{Pol}^+(\mathrm{co}^+\mathcal{B}_{n+\frac{1}{2}})$. $\square$

Theorem 6.2 *For $n\in\mathbb{N}^+$ we have $\mathcal{B}_{n+\frac{1}{2}} = \mathrm{Pol}^+(\mathrm{co}^+\mathcal{B}_{n-\frac{1}{2}})$.*

Proof Let $n\in\mathbb{N}^+$. By definition we have $\mathcal{B}_{n+\frac{1}{2}}\supseteq\mathrm{Pol}^+(\mathrm{co}^+\mathcal{B}_{n-\frac{1}{2}})$. Now let $B\in\mathcal{B}_{n+\frac{1}{2}}$. By Lemma 2.2, we can represent B as

$$B = \bigcup_{i=1}^{k} u_{i,0}\overbrace{\left(B'_{i,1}\cap\overline{B_{i,1}}^{+}\right)}^{C_{i,1}}u_{i,1}\dots\overbrace{\left(B'_{i,m_i}\cap\overline{B_{i,m_i}}^{+}\right)}^{C_{i,m_i}}u_{i,m_i}$$

where $k,m_i\in\mathbb{N}$, $u_{i,j}\in\Sigma^*$ (if $m_i=0$, then $u_{i,0}\neq\varepsilon$), $B_{i,j}\in\mathcal{B}_{n-\frac{1}{2}}$, and every $B'_{i,j}$ is of the form $v_0B_1v_1\dots B_lv_l$ for suitable $l\in\mathbb{N}$, $v_i\in\Sigma^*$ (if $l=0$, then $v_0\neq\varepsilon$), $B_i\in\mathcal{B}_{n-1}$. Thus every $C_{i,j}$ is of the form

$$C_{i,j} = v_0B_1v_1\dots B_lv_l\cap B'$$

for suitable $l\in\mathbb{N}$, $v_i\in\Sigma^*$ (if $l=0$, then $v_0\neq\varepsilon$), $B_i\in\mathcal{B}_{n-1}\subseteq\mathrm{co}^+\mathcal{B}_{n-\frac{1}{2}}$ and $B'\in\mathrm{co}^+\mathcal{B}_{n-\frac{1}{2}}$. By Lemma 6.1 we obtain $C_{i,j}\in\mathrm{Pol}^+(\mathrm{co}^+\mathcal{B}_{n-\frac{1}{2}})$. Since $\mathrm{Pol}^+(\mathrm{co}^+\mathcal{B}_{n-\frac{1}{2}})$ is closed under finite union and marked concatenation (Lemma 1.14), we conclude $B\in\mathrm{Pol}^+(\mathrm{co}^+\mathcal{B}_{n-\frac{1}{2}})$. $\square$

7 Summary and Open Questions

We have proved a normalform for the classes $n+\frac{1}{2}$ of the Straubing-Thérien-Hierarchy and the Dot-Depth-Hierarchy. More precisely, for $n \in \mathbb{N}^+$ we have shown:

$$\begin{aligned} \mathcal{L}_{n+\frac{1}{2}} &= \mathrm{Pol}^*(\mathrm{co}\mathcal{L}_{n-\frac{1}{2}}) \\ \mathcal{B}_{n+\frac{1}{2}} &= \mathrm{Pol}^+(\mathrm{co}^+\mathcal{B}_{n-\frac{1}{2}}) \end{aligned}$$

These results show that the essential operations needed to define the hierarchies are the *polynomial closure* and the *complementation*. We do not need the full power of Boolean operations for our purpose.

Whether these normalforms can help us to understand the hierachies better is of course an open question. Another one is the question on the existence of normalforms for the so called Δ_k-classes. These classes are defined as follows:

$$\begin{aligned} \Delta_k^{\mathcal{L}} &\stackrel{df}{=} \mathcal{L}_{k+\frac{1}{2}} \cap \mathrm{co}\mathcal{L}_{k+\frac{1}{2}} \\ \Delta_k^{\mathcal{B}} &\stackrel{df}{=} \mathcal{B}_{k+\frac{1}{2}} \cap \mathrm{co}^+\mathcal{B}_{k+\frac{1}{2}} \end{aligned}$$

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