Classes of Counting Functions and Complexity Theoretic Operators*

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1 Introduction

Most of the naturally arising classes in structural complexity theory, like P, NP, PP, all the other classes of the polynomial time hierarchy and the counting hierarchy, UP, XP [FeFoKu91], $\oplus$P [PaZa83], ..., can be characterized in terms of an underlying function class, mostly Valiant's class #P [Val79], consisting of all functions, that applied to argument x yield the number of accepting paths of some polynomial time nondeterministic Turing machine on input x, and an operator like $\exists$, $\forall$, C, $\oplus$, U, X, ... Thus we can e. g. say a set A is in $\oplus\mathrm{P} = \oplus\#\mathrm{P}$ if and only if there is a function f in #P such that $x \in A \iff f(x) \equiv 1 \pmod 2$; or a set A is in $\mathrm{NP} = \exists\mathrm{P} = \exists\#\mathrm{P}$ if and only if there is a function $f \in \#\mathrm{P}$ such that $x \in A \iff f(x) > 0$.

It now suggests itself to examine the closure properties of the class #P. It is easy to see that #P is closed under addition and multiplication, but it is not known (and suspected by most people doing research in the area) wether #P is closed under subtraction (modified or usual), division or even decrement by one. Mitsunori Ogiwara and Lane Hemachandra did a lot of work on questions of the consequences of the closure of #P under the latter operations. They gave in [OgHe91] a number of results of the following form:

#P is closed under modified subtraction if and only if $\mathrm{C}_=\mathrm{P} = \mathrm{UP}$.

The main motivation for the present paper was the question for the reasons for an equivalence like the one quoted. We answer this question in the following way:

We take an abstract approach to the examination of classes like the ones mentioned above. We start with the definition of general function classes (we call them *classes of counting functions*) and operators applied to function classes yielding either again a function class or a class of sets (i. e. a complexity class). Then we examine the underlying general relationships between operators which hold independently of the concrete function class they are applied to. We are thus able to give equations, e. g. of the form

$$\mathrm{U}(\mathcal{F} \dot{-} \mathcal{F}) = \mathrm{C}_=(\mathcal{F} \dot{-} \mathcal{F}).$$

This equation of course leads us directly to the consequence that if $\mathcal{F}$ is closed under modified subtraction then $\mathrm{U}\mathcal{F} = \mathrm{C}_=\mathcal{F}$. Since we can also, using the above equation, deduce an inverse

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relationship, we immediately get an equivalence statement which is far more general than the one of [OgHe91], which only states the case $\mathcal{F} = \#\mathrm{P}$.

We think an equation like the one above holding without any assumptions is more valuable than an equivalence $P \iff Q$, where most people think that both P and Q are false.

The main results of the paper are as follows.

1. We define the general concepts of *class of counting functions* and *operator* (taking either a class of functions or a class of sets and yielding either a class of functions or a class of sets) and give a lot of basic relationships between operators, language classes and function classes. Among the operators we examine are $\exists, \forall, \mathrm{P}^{(\cdot)}, \oplus, \mathrm{U}, \mathrm{X}, \mathrm{C}, \mathrm{C}_{=}, \mathrm{Max}, \mathrm{Min}, \mathrm{Mid} \ldots$.

2. Using this general approach to operator defined complexity classes we are able to give a set of *basic equations* holding between operators in very general contexts. Here we show among others

   (a) $\mathrm{U}(\mathcal{F} \dot{-} \mathcal{F}) = \mathrm{C}_{=}(\mathcal{F} \dot{-} \mathcal{F})$.

   (b) $\exists(\mathcal{F} - \mathcal{F}) = \text{co-}\mathrm{C}_{=}(\mathcal{F} - \mathcal{F})$.

3. From this equations we deduce in a uniform manner all of the special results of Ogiwara and Hemachandra on closure of $\#\mathrm{P}$ and span-P under modified subtraction, division, and substitution. But we are also able to state results not given on [OgHe91] for other function classes (e. g. Gap-P) and, more important, for other operations than those in [OgHe91] like usual subtraction. To express it in terms of [OgHe91], we are able to show for example that usual subtraction is hard for span-P (which is a stronger result than that of [OgHe91], which states hardness for modified subtraction).

   As corollaries we obtain among others:

   - $\#\mathrm{P}$ is closed under modified subtraction if and only if $\#\mathrm{P}$ is closed under substitution if and only if $\mathrm{UP} = \mathrm{CH}$ [OgHe91].
   - Gap-P is closed under modified subtraction if and only if Gap-P is closed under substitution if and only if $\mathrm{XP} = \mathrm{CH}$.
   - span-P ($= \#\mathrm{NP}$) is closed under *usual* subtraction if and only if span-P is closed under substitution if and only if $\mathrm{NP} = \mathrm{CH}$.
   - $\#\,$co-NP is closed under *usual* subtraction if and only if $\#\,$co-NP is closed under substitution if and only if $\mathrm{NP} = \mathrm{CH}$.

4. We introduce a general approach to optimization problems using a special maximum operator from [Ogi91] and give a general theorem on classes of differences of optimization functions, generalizing results of [Wag89, Ogi91]. We show $\mathrm{M\overline{ax}}\,\mathcal{K} - \mathrm{M\overline{ax}}\,\mathcal{K} = \mathrm{FP}^{\exists\mathcal{K}}$, i. e. a function computable in polynomial time with access to an oracle from $\exists\mathcal{K}$ can be expressed as the difference of two $\mathrm{M\overline{ax}}$ functions. In [Wag89, Ogi91] a similar result for sets decidable in polynomial time with access to an $\exists\mathcal{K}$ oracle is given.

5. We show that well studied function classes like Opt P [Kre88], Mid P [Tod90] (which we call Max FP and Mid FP) fit in our framework of general optimization classes. Thus we also obtain all the remaining results of [OgHe91] on closure of the just mentioned classes as simple consequences of our general theorem on classes of differences of optimization functions.

   We also define other natural optimization classes, which we call Med FP and Mof FP, and are able to state results for these new classes as well as for the class $\mathrm{Max}\,\#\mathrm{P}$ (in contrast to $\mathrm{Max\,FP} = \mathrm{Opt\,P}$).

   We show for example:

(a) Max FP is closed under subtraction if and only if Max FP is closed under substitution if and only if NP = co-NP [OgHe91].

(b) Mid FP is closed under subtraction if and only if Mid FP is closed under substitution if and only if NP = P^{PP} [OgHe91].

(c) Max # P is closed under subtraction if and only if Max # P is closed under substitution if and only if NP = $\mathrm{P}^{\exists\mathrm{CP}}$.

6. Again using our theorem on general optimization classes we obtain as simple corollaries hardness of function classes for certain complexity classes:

   (a) [Kre88] $\mathrm{FP}^{\mathrm{NP}} = \mathrm{FP}^{\mathrm{Max\,FP}}[1]$.

   (b) [Tod90] $\mathrm{FP}^{\mathrm{PP}} = \mathrm{FP}^{\mathrm{Mid\,FP}}[1]$.

   (c) $\mathrm{FP}^{\exists\mathrm{CP}} = \mathrm{FP}^{\mathrm{Mof\,FP}}[1]$.

The organization of the paper is as follows: In Section 2 we introduce counting function classes and operators of the four kinds mentioned before. In Section 3 we present our basic equations and show how the results of [OgHe91] and many other interesting results can be deduced. In Section 4 we present our results on general and special optimization function classes.

2 Preliminaries

2.1 Basic Definitions

We assume the reader is familiar with the basic notions of complexity theory as for example oracle Turing machine, number of queries, many one reducibility, ... [BaDiGa88]. Our notation here is standard. A concept which might possibly not be well-known is that of positive truth table reducibility ($\leq^{\mathrm{p}}_{ptt}$). A truth table reduction is called positive if the circuit computing from the answers of the oracle the result of the whole reduction consists only of AND and OR gates.

By $\dot{-}$ we denote modified subtraction, i. e. $a \dot{-} b = \max\{0, a-b\}$. For functions f and g, $f \circ g$ denotes the composition of f and g, i. e. $f \circ g(x) = f(g(x))$. $[x]$ denotes $\max\{\, n \in I\!N \mid n \leq x \,\}$. For a set M, the number of elements of M is written as $\#\, M$. For a string w the length of w is written as $|w|$. For set M and an operation o defined on M we denote by $\Gamma_o M$ the closure of M under o. By $BC(M)$ we denote the boolean closure of M.

2.2 Operators from Languages to Functions

Two of the most important ways in complexity theory to define a function f from a given language A are

$$f(x) = \#\{\, y \mid y \leq g(x) \wedge (x,y) \in A \,\}$$

and

$$f(x) = \max\{\, y \mid y \leq g(x) \wedge (x,y) \in A \,\},$$

where g is a polynomial time computable function. Particularly the first way is used to define new languages B by $x \in B \iff f(x) > 0$ ($\exists$), $x \in B \iff f(x) = g(x)$ ($\forall$), ... The latter example already shows that the function f does not completely reflect the result of the #- or max-process. The function $g \in \mathrm{FP}$ determining the range on whose base f is defined is important as well. Thus we will speak about pairs (f, g) rather than about functions f only.

Definition. A class $\mathcal{F}$ of counting functions consists of pairs (f, g) where $f: \Sigma^\star \to I\!N$ is an arbitrary function and $g: \Sigma^\star \to I\!N$ is monotonic and polynomial time computable such that $f(x) \leq g(x)$ for all x.[1] We write $f \in \mathcal{F}$ if there exists a $g \in \mathrm{FP}$ such that $(f, g) \in \mathcal{F}$. Further let co-$\mathcal{F} = \{ (g - f, g) \mid (f, g) \in \mathcal{F} \}$.

We consider the operations subtraction, modified subtraction, addition, multiplication, summation and substitution on functions. They are defined as follows:

$$\begin{array}{rcl}
\mathcal{F}_1 - \mathcal{F}_2 & = & \{ (f_1 - f_2, g) \mid f_1 \geq f_2 \wedge (f_1, g) \in \mathcal{F}_1 \wedge f_2 \in \mathcal{F}_2 \} \\
\mathcal{F}_1 \dot{-} \mathcal{F}_2 & = & \{ (f_1 \dot{-} f_2, g) \mid (f_1, g) \in \mathcal{F}_1 \wedge f_2 \in \mathcal{F}_2 \} \\
\mathcal{F}_1 + \mathcal{F}_2 & = & \{ (f_1 + f_2, g_1 + g_2) \mid (f_1, g_1) \in \mathcal{F}_1 \wedge (f_2, g_2) \in \mathcal{F}_2 \} \\
\mathcal{F}_1 \cdot \mathcal{F}_2 & = & \{ (f_1 \cdot f_2, g_1 \cdot g_2) \mid (f_1, g_1) \in \mathcal{F}_1 \wedge (f_2, g_2) \in \mathcal{F}_2 \} \\
\mathrm{Sum}\mathcal{F} & = & \{ (h, g \cdot g') \mid h(x) = \Sigma_{y \leq g'(x)} f(x, y) \text{ for some } (f, g) \in \mathcal{F} \} \\
\mathrm{SUB}\mathcal{F} & = & \{ (f(h_1, \ldots h_k), g(g'_1, \ldots, g'_k) \mid (f, g) \in \mathcal{F} \text{ is } k\text{-ary and} \\
 & & \qquad (h_i, g'_i) \in \mathcal{F} \text{ for } 1 \leq i \leq k \}
\end{array}$$

(2.1) Proposition. *Let $\mathcal{F}$ be a class of counting functions.*

1. co-$\mathcal{F} \subseteq \mathrm{FP} - \mathcal{F}$.
2. co-co-$\mathcal{F} = \mathcal{F}$.
3. $\mathcal{F} \subseteq$ co-$\mathcal{F} \iff$ co-$\mathcal{F} \subseteq \mathcal{F} \iff \mathcal{F} =$ co-$\mathcal{F}$.

The following operators will be considered subsequently.

Let $\mathcal{K}$ be a class of languages and let $r: I\!N \to I\!N$.

$$\begin{array}{rcl}
(f, g) \in \#\mathcal{K} & \iff & \text{there exists an } A \in \mathcal{K} \text{ such that for all } x \\
 & & f(x) = \#\{ y \mid y < g(x) \wedge (x, y) \in A \}. \\
(f, g) \in \mathrm{M\overline{ax}}\,\mathcal{K} & \iff & \text{there exists an } A \in \mathcal{K} \text{ such that for all } x \\
 & & f(x) = \max\{ y \mid y \leq g(x) \wedge (x, y) \in A \}. \\
(f, g) \in \mathrm{M\overline{in}}\,\mathcal{K} & \iff & \text{there exists an } A \in \mathcal{K} \text{ such that for all } x \\
 & & f(x) = \min\{ y \mid y \leq g(x) \wedge (x, y) \in A \}. \\
(f, g) \in \mathrm{FP}^{\mathcal{K}}[r] & \iff & f(x) \leq g(x) \text{ for all } x \text{ and there exists an } A \in \mathcal{K} \\
 & & \text{and a polynomial time oracle machine } M \text{ computing } f \\
 & & \text{with at most } r(|x|) \text{ queries to } A \text{ on input } x. \\
(f, g) \in \mathrm{FP}^{\mathcal{K}} & \iff & \text{there exists a polynomial } p \text{ such that } (f, g) \in \mathrm{FP}^{\mathcal{K}}[p].
\end{array}$$

(2.2) Proposition. *Let $\mathcal{K}$ be any class of languages and let $r: I\!N \to I\!N$.*

1. co- $\#\mathcal{K} = \#$ co-$\mathcal{K}$.
2. co- $\mathrm{M\overline{ax}}\,\mathcal{K} = \mathrm{M\overline{in}}\,\mathcal{K}$.
3. co- $\mathrm{M\overline{in}}\,\mathcal{K} = \mathrm{M\overline{ax}}\,\mathcal{K}$.
4. co-$\mathrm{FP}^{\mathcal{K}}[r] = \mathrm{FP}^{\mathcal{K}}[r]$.

From (2.1) and (2.2) we obtain

(2.3) Corollary. *Let $\mathcal{K}$ be any class of languages.*

1. $\#\mathcal{K} \subseteq \mathrm{FP} - \#$ co-$\mathcal{K}$.

[1] The k-ary case is defined analogously. If f is k-ary, then g has to be k-ary, too. To simplify the presentation, we will mainly concentrate on the unary case.

2. $\mathrm{M\overline{ax}}\,\mathcal{K} \subseteq \mathrm{FP} - \mathrm{M\overline{in}}\,\mathcal{K}$.

3. $\mathrm{M\overline{in}}\,\mathcal{K} \subseteq \mathrm{FP} - \mathrm{M\overline{ax}}\,\mathcal{K}$.

4. $\mathrm{FP}^{\mathcal{K}}[r] = \mathrm{FP} - \mathrm{FP}^{\mathcal{K}}[r]$.

Next we see which interesting properties of the class of functions $\#\,\mathcal{K}$ are caused by which properties of the class of languages $\mathcal{K}$.

(2.4) Proposition.

1. *If $\mathcal{K}$ is closed under $\leq_m^{\mathrm{p}}$ and union then $\#\,\mathcal{K}$ is closed under addition.*

2. *If $\mathcal{K}$ is closed under $\leq_m^{\mathrm{p}}$ and intersection then $\#\,\mathcal{K}$ is closed under multiplication.*

3. *If $\mathcal{K}$ is closed under complement then $\#\,\mathcal{K} \subseteq \mathrm{FP} - \#\,\mathcal{K}$.*

4. *If $\mathcal{K}$ is closed under $\leq_m^{\mathrm{p}}$ then $\#\,\mathcal{K} \circ \mathrm{FP} = \#\,\mathcal{K}$.*

5. *If $\mathcal{K}$ is closed under $\leq_m^{\mathrm{p}}$ then $\#\,\mathcal{K}$ is closed under summation.*

Proof. All proofs are straightforward. We show (1), (2) and (4).

Let $f_1(x) = \#\{\, y \mid |y| = p_1(|x|) \wedge (x,y) \in B_1 \,\}$ for $B_1 \in \mathcal{K}$ and $f_2(x) = \#\{\, y \mid |y| = p_2(|x|) \wedge (x,y) \in B_2 \,\}$ for $B_2 \in \mathcal{K}$.

(ad (1)) Define $B_1' = \{\, (x, y0^{p_2(|x|)}) \mid (x,y) \in B_1 \,\}$ and $B_2' = \{\, (x, y1^{p_1(|x|)}) \mid (x,y) \in B_2 \,\}$. Then $B_1' \leq_m^{\mathrm{p}} B_1$, $B_2' \leq_m^{\mathrm{p}} B_2$ and $B_1' \cap B_2' = \emptyset$ and we have

$$\#\{\, y \mid |y| = p_1(|x|) + p_2(|x|) \wedge (x,y) \in B_1' \cup B_2' \,\} = f_1(x) + f_2(x).$$

(ad (2)) Define $B_1'' = \{\, (x,y) \mid y = a_1 \ldots a_{p_1(|x|)+p_2(|x|)} \wedge (x, a_1 \ldots a_{p_1(|x|)}) \in B_1 \,\}$ and $B_2'' = \{\, (x,y) \mid y = a_1 \ldots a_{p_1(|x|)+p_2(|x|)} \wedge (x, a_{p_1(|x|)+1} \ldots a_{p_1(|x|)+p_2(|x|)}) \in B_2 \,\}$. Then $B_1'' \leq_m^{\mathrm{p}} B_1$, $B_2'' \leq_m^{\mathrm{p}} B_2$ and we have

$$\#\{\, y \mid |y| = p_1(|x|) + p_2(|x|) \wedge (x,y) \in B_1'' \cap B_2'' \,\} = f_1(x) \cdot f_2(x).$$

(ad (4)) Let $f(x) = \#\{\, y \mid |y| = p(|x|) \wedge (x,y) \in B \,\}$, where $B \in \mathcal{K}$. Let $g \in \mathrm{FP}$ such that $|g(x)| < q(|x|)$. Then define $B' = \{\, (x, y10^{p(q(|x|))-|y|-1}) \mid |y| = p(|g(x)|) \wedge (g(x), y) \in B \,\}$. Then $B' \leq_m^{\mathrm{p}} B$ and $f(g(x)) = \#\{\, y \mid |y| = p(|g(x)|) \wedge (g(x), y) \in B \,\} = \#\{\, z \mid |z| = p(q(|x|)) \wedge (x,z) \in B' \,\}$. $\square$

An immediate consequence of $\text{co-}\mathcal{K} = \mathcal{K}$ is (by (2.3)) $\#\,\mathcal{K} = \mathrm{FP} - \#\,\text{co-}\mathcal{K} = \mathrm{FP} - \#\,\mathcal{K}$. The following proposition shows that $\text{co-}\mathcal{K} = \mathcal{K}$ might be not the only reason for a class of languages to hold $\#\,\mathcal{K} \subseteq \mathrm{FP} - \#\,\mathcal{K}$.

(2.5) Proposition.

1. *If $\mathcal{K}$ is closed under $\leq_m^{\mathrm{p}}$ and intersection then $\#\,\exists\mathcal{K} \subseteq \#\,\mathcal{K} - \#\,\exists\mathcal{K}$.*

2. *For every class $\mathcal{K}$ of the polynomial hierarchy, $\#\,\mathcal{K} \subseteq \mathrm{FP} - \#\,\mathcal{K}$.*

Proof.

1. For $B \in \mathcal{K}$ and polynomials p, q we observe that

$$\#\{\, y \mid |y| = p(|x|) \wedge \exists z(|z| = q(|x|) \wedge (x,y,z) \in B)\,\} = f_1(x) - f_2(x),$$

where $f_1(x) = \#\{\,(y,z) \mid |y| = p(|x|) \wedge |z| = q(|x|) \wedge (x,y,z) \in B\,\}$ and $f_2(x) = \#\{\,(y,z) \mid |y| = p(|x|) \wedge |z| = q(|x|) \wedge (x,y,z) \in B \wedge \exists z'(|z'| = q(|x|) \wedge z' < z \wedge (x,y,z') \in B)\,\}$. The assumed closure properties of $\mathcal{K}$ ensure the existence of $B', B'' \in \mathcal{K}$ and polynomials p', p'', q'' such that $f_1(x) = \#\{\, y \mid |y| = p'(|x|) \wedge (x,y) \in B'\,\}$ and $f_2(x) = \#\{\, y \mid |y| = p''(|x|) \wedge \exists z(|z| = q''(|x|) \wedge (x,y,z) \in B'')\,\}$.

2. By induction. For $\Sigma_0^p = \Pi_0^p = \mathrm{P}$ this is true by (2.4). Let $\#\,\Sigma_k^p \subseteq \mathrm{FP} - \#\,\Sigma_k^p$ and $\#\,\Pi_k^p \subseteq \mathrm{FP} - \#\,\Pi_k^p$. We conclude by (2.3), by statement 1, and by the fact that $\#\,\exists\Pi_k^p$ and $\#\,\forall\Sigma_k^p$ are closed under addition (cf. (2.4)) $\#\,\Sigma_{k+1}^p = \#\,\exists\Pi_k^p \subseteq \#\,\Pi_k^p - \#\,\exists\Pi_k^p \subseteq \mathrm{FP} - \#\,\Pi_k^p - \#\,\exists\Pi_k^p \subseteq \mathrm{FP} - \#\,\exists\Pi_k^p = \mathrm{FP} - \#\,\Sigma_{k+1}^p$ and $\#\,\Pi_{k+1}^p = \#\,\forall\Sigma_k^p \subseteq \mathrm{FP} - \#\,\exists\Pi_k^p \subseteq \mathrm{FP} - (\mathrm{FP} - \#\,\exists\Pi_k^p) \subseteq \#\,\exists\Pi_k^p - \mathrm{FP} \subseteq (\mathrm{FP} - \#\,\forall\Sigma_k^p) - \mathrm{FP} \subseteq \mathrm{FP} - \#\,\forall\Sigma_k^p \subseteq \mathrm{FP} - \Pi_{k+1}^p$.

□

If we have the property $\#\,\mathcal{K} \subseteq \mathrm{FP} - \#\,\mathcal{K}$ then obviously several difference classes coincide.

(2.6) Proposition. *If $\#\,\mathcal{K}$ is closed under addition and $\#\,\mathcal{K} \subseteq \mathrm{FP} - \#\,\mathcal{K}$ then*

$$\#\,\mathcal{K} - \#\,\mathcal{K} = \mathrm{FP} - \#\,\mathcal{K} = \#\,\mathcal{K} - \mathrm{FP} = \#\,\text{co-}\mathcal{K} - \#\,\text{co-}\mathcal{K} = \mathrm{FP} - \#\,\text{co-}\mathcal{K} = \#\,\text{co-}\mathcal{K} - \mathrm{FP}.$$

Two other definitions of classes of counting functions can be found in the literature. Slightly modified they are as follows:

Definition.

1. [KöScTo89] A function $f\colon \Sigma^\star \to I\!N$ is in span-P if and only if there exists a nondeterministic polynomial time Turing maching M such that $f(x) = \#\{\, z \mid$ on input x and some path y the machine outputs $z\,\}$.

2. [FeFoKu91] A function $f\colon \Sigma^\star \to I\!N$ is in Gap-P if and only if there exists a nondeterministic polynomial time Turing machine M such that $f(x) = \#\{\, y \mid M$ accepts on path $y\,\} - \#\{\, y \mid M$ rejects on path $y\,\}$. Note that in [FeFoKu91] this definition was made for functions f taking also negative values. We restrict ourselves to the nonnegative case.

It turns out that these classes coincide with classes which can be expressed by $\#\,\mathcal{K}$ functions for suitable $\mathcal{K}$.

(2.7) Theorem.

1. span-P $= \#\,\mathrm{NP}$.

2. Gap-P $= \#\,\mathrm{P} - \#\,\mathrm{P}$.

Proof. Statement 2 is proved in [FeFoKu91]. We show statement 1.

For $f \in$ span-P there exists a nondeterministic polynomial time Turing machine M such that $f(x) = \#\{\, z \mid \exists y(M$ outputs z on input x and path $y)\,\}$. Hence $f \in \#\,\mathrm{NP}$.

For $f \in \#\,\mathrm{NP}$ there exist $B \in \mathrm{P}$ and polynomials p, q such that $f(x) = \#\{\, z \mid |z| = p(|x|) \wedge \exists y(|y| = q(|x|) \wedge (x,y,z) \in B)\,\}$. Defining a nondeterministic polynomial time machine M in such a way that M outputs z on input x and path y if and only if $y = uz$ for some u, $|u| = q(|x|)$, $|z| = p(|x|)$ and $(x,u,z) \in B$ we obtain $f(x) = \#\{\, z \mid \exists y(M$ outputs z on input x and path $y)\,\}$.

□

The hierarchy of counting functions over $\mathcal{K}$ is the smallest family of function classes containing $\#\,\mathcal{K}$ and being closed under the operator $\#\,\mathrm{P}^{(\cdot)}$. Let $\mathrm{FCH}\cdot\mathcal{K}$ denote the union of all classes of the hierarchy of counting functions over $\mathcal{K}$ and let $\mathrm{FCH} = \mathrm{FCH}\cdot\mathrm{P}$.

2.3 Operators from Functions to Languages

Let $\mathcal{F}$ be any class of counting functions. Let $r\colon \mathbb{N} \to \mathbb{N}$.

$$
\begin{array}{rcl}
A \in \exists\mathcal{F} & \iff & \text{there exists } f \in \mathcal{F} \text{ such that} \\
 & & x \in A \iff f(x) > 0. \\
A \in \forall\mathcal{F} & \iff & \text{there exists } (f,g) \in \mathcal{F} \text{ such that} \\
 & & x \in A \iff f(x) = g(x). \\
A \in \mathrm{C}_{=}\mathcal{F} & \iff & \text{there exist } f \in \mathcal{F} \text{ and } g' \in \mathrm{FP} \text{ such that} \\
 & & x \in A \iff f(x) = g'(x). \\
A \in \mathrm{C}_{\neq}\mathcal{F} & \iff & \text{there exist } f \in \mathcal{F} \text{ and } g' \in \mathrm{FP} \text{ such that} \\
 & & x \in A \iff f(x) \neq g'(x). \\
A \in \mathrm{C}\mathcal{F} & \iff & \text{there exist } f \in \mathcal{F} \text{ and } g' \in \mathrm{FP} \text{ such that} \\
 & & x \in A \iff f(x) \geq g'(x). \\
A \in \mathrm{X}\mathcal{F} & \iff & \text{there exist } f \in \mathcal{F} \text{ and } g' \in \mathrm{FP} \text{ such that} \\
 & & x \in A \Rightarrow f(x) = g'(x) + 1, \\
 & & x \notin A \Rightarrow f(x) = g'(x). \\
A \in \mathrm{U}\mathcal{F} & \iff & \text{there exists } f \in \mathcal{F} \text{ such that} \\
 & & x \in A \Rightarrow f(x) = 1, \\
 & & x \notin A \Rightarrow f(x) = 0. \\
A \in \mathrm{BP}\,\mathcal{F} & \iff & \text{there exists } (f,g) \in \mathcal{F} \text{ such that} \\
 & & x \in A \Rightarrow f(x) \geq (1 - \frac{1}{|x|}) \cdot g(x), \\
 & & x \notin A \Rightarrow f(x) \leq \frac{1}{|x|} \cdot g(x). \\
A \in \oplus\,\mathcal{F} & \iff & \text{there exist } f \in \mathcal{F} \text{ and } g' \in \mathrm{FP} \text{ such that} \\
 & & x \in A \iff f(x) \equiv g'(x) \pmod 2. \\
A \in \mathrm{P}^{\mathcal{F}}[r] & \iff & \text{there exist } f \in \mathcal{F} \text{ and a polynomial time oracle machine } M \\
 & & \text{accepting } A \text{ with at most } r(|x|) \text{ queries to } f \text{ on input } x. \\
A \in \mathrm{P}^{\mathcal{F}} & \iff & \text{there exists a polynomial } p \text{ such that} \\
 & & A \in \mathrm{P}^{\mathcal{F}}[p].
\end{array}
$$

The obvious inclusional relationships are presented in Figure 1.

All statements of the following proposition are proved straightforward.

(2.8) Proposition. *Let $\mathcal{F}$ be a class of counting functions, let $r\colon \mathbb{N} \to \mathbb{N}$.*

1. co-$\exists\mathcal{F} = \forall$co-$\mathcal{F}$.
2. co-$\forall\mathcal{F} = \exists$co-$\mathcal{F}$.
3. co-$\mathrm{C}_{=}\mathcal{F} = \mathrm{C}_{\neq}\mathcal{F}$ *and* $\mathrm{C}_{=}$co-$\mathcal{F} = \mathrm{C}_{=}\mathcal{F}$.
4. co-$\mathrm{C}\mathcal{F} = \mathrm{C}$co-$\mathcal{F}$.
5. co-$\mathrm{X}\mathcal{F} = \mathrm{X}$co-$\mathcal{F}$.
6. co-$\mathrm{BP}\,\mathcal{F} = \mathrm{BP}$ co-$\mathcal{F}$.
7. co-$\oplus\,\mathcal{F} = \oplus$ co-$\mathcal{F} = \oplus\,\mathcal{F}$.
8. co-$\mathrm{P}^{\mathcal{F}}[r] = \mathrm{P}^{\text{co-}\mathcal{F}}[r] = \mathrm{P}^{\mathcal{F}}[r]$.

(2.9) Proposition. *If $\mathcal{F}$ is closed under summation then $\#\,\mathrm{U}\mathcal{F} \subseteq \mathcal{F}$.*

Proof. Let $f(x) = \#\{\, y \mid |y| = p(|x|) \wedge c_B(x,y) = 1 \,\}$ where $c_B \in \mathcal{F}$. Then $f(x) = \Sigma_{|y|=p(|x|)} c_B(x,y)$ and hence $f \in \mathcal{F}$. $\square$

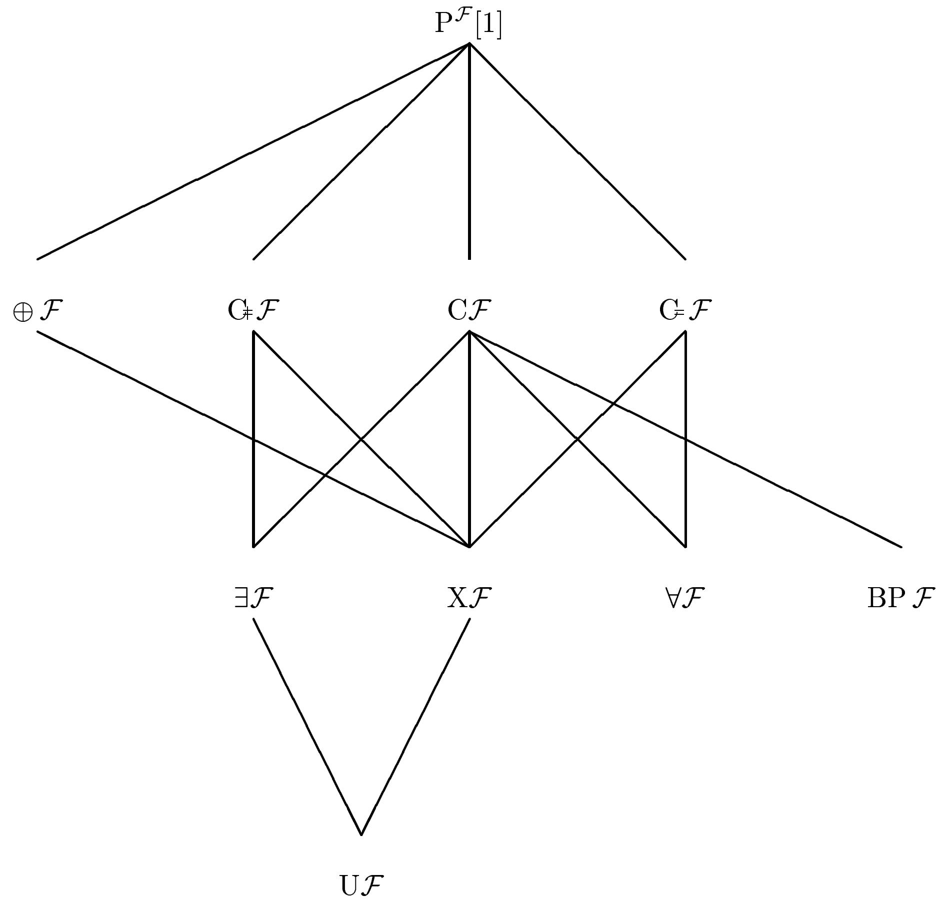

Figure 1: Inclusionships

2.4 Operators from Languages to Languages

All operators from functions to languages (cf. Section 2.3) are now redefined as operators from languages to languages.

Definition. Let $\mathcal{O}$ be one of the operators from functions to languages defined in Section 2.3, i. e. $\mathcal{O} \in \{\exists, \forall, \mathrm{C}_{=}, \mathrm{C}_{\neq}, \mathrm{C}, \mathrm{X}, \mathrm{U}, \mathrm{BP}, \oplus, \mathrm{P}^{(\cdot)}[r], \mathrm{P}^{(\cdot)}\}$. For a class $\mathcal{K}$ of languages define $\mathcal{O} \cdot \mathcal{K} = \mathcal{O}\,\#\,\mathcal{K}$.

The classes of the polynomial hierarchy over $\mathcal{K}$ are defined by $\Sigma_0^p \cdot \mathcal{K} = \Pi_0^p \cdot \mathcal{K} = \mathcal{K}$, $\Sigma_{k+1}^p \cdot \mathcal{K} = \exists\Pi_k^p \cdot \mathcal{K}$ and $\Pi_{k+1}^p \cdot \mathcal{K} = \forall\Sigma_k^p \cdot \mathcal{K}$. Furthermore define $\mathrm{PH} \cdot \mathcal{K} = \bigcup_{k\geq 0}(\Sigma_k^p \cdot \mathcal{K} \cup \Pi_k^p \cdot \mathcal{K})$, $\Sigma_k^p = \Sigma_k^p \cdot \mathrm{P}$, $\Pi_k^p = \Pi_k^p \cdot \mathrm{P}$, and $\mathrm{PH} = \mathrm{PH} \cdot \mathrm{P}$.

The counting hierarchy over $\mathcal{K}$ is the smallest family of classes containing $\mathcal{K}$ and being closed under $\exists$, $\forall$, C. Let $\mathrm{CH} \cdot \mathcal{K}$ be the union of all classes of the counting hierarchy over $\mathcal{K}$, and let $\mathrm{CH} = \mathrm{CH} \cdot \mathrm{P}$.

For later use we state the following results from the literature.

(2.10) Proposition [ToWa91]. $\mathrm{P}^{\#\mathrm{PH}} = \mathrm{P}^{\mathrm{PP}}$.

(2.11) Proposition [Tor88]. $\mathrm{P}^{\mathrm{PP}} \subseteq \exists\,\mathrm{C}_{=}\mathrm{P}$.

(2.12) Proposition [Tor88]. $\mathrm{CH} = \bigcup_{k\geq 0} \mathrm{C}_{=}^{k}\,\mathrm{P}$.

(2.13) Proposition [Wag86]. $\mathrm{FCH} = \#\,\mathrm{CH} = \mathrm{FP}^{\mathrm{CH}}$.

Every combination of a language-to-function operator with a function-to-language operator yields a language-to-language operator. Taking U as the function-to-language operator we obtain the following results which are not hard to prove.

(2.14) Proposition. *Let $\mathcal{K}$ be a class of languages.*

1. *If $\mathcal{K}$ is closed under $\leq_m^{\mathrm{p}}$, then* $\mathrm{U}\,\#\,\mathcal{K} \supset \mathcal{K}$.
2. *If $\mathcal{K}$ is closed under $\leq_m^{\mathrm{p}}$, then* $\mathrm{U}\,\mathrm{M\overline{ax}}\,\mathcal{K} = \mathcal{K}$.
3. *If $\mathcal{K}$ is closed under $\leq_m^{\mathrm{p}}$, then* $\mathrm{U}\,\mathrm{M\overline{in}}\,\mathcal{K} = \mathcal{K}$.

From (2.14), (2.9) and (2.4) we obtain

(2.15) Corollary. *If $\mathcal{K}$ is closed under $\leq_m^{\mathrm{p}}$ then* $\#\,\mathrm{U}\,\#\,\mathcal{K} = \#\,\mathcal{K}$

We will later use the following result.

(2.16) Proposition. *Let $\mathcal{F}$ be a class of counting functions such that $\mathcal{F} \circ \mathrm{FP} = \mathcal{F}$. Then*

$$\mathrm{C}_{=}\mathcal{F} \subseteq \forall\, \mathrm{C}_{\neq}\mathcal{F}$$

.

Proof. Let $(f, g) \in \mathcal{F}$ and $g' \in \mathrm{FP}$. By

$$f(x) = g'(x) \iff \forall y (y \leq g(x) \wedge y \neq g'(x) \Rightarrow f(x) \neq y)$$

the inclusion is obvious. □

2.5 Operators from Functions to Functions

Let $\mathcal{F}$ be a class of counting functions.

$$
\begin{array}{rcl}
(h, g \circ g') \in \mathrm{Max}\,\mathcal{F} & \iff & \text{there exists } (f,g) \in \mathcal{F} \text{ such that} \\
& & h(x) = \max\{\, f(x,y) \mid y \leq g'(x) \,\} \\
(h, g \circ g') \in \mathrm{Min}\,\mathcal{F} & \iff & \text{there exists } (f,g) \in \mathcal{F} \text{ such that} \\
& & h(x) = \min\{\, f(x,y) \mid y \leq g'(x) \,\} \\
(h, g \circ g') \in \mathrm{Med}\,\mathcal{F} & \iff & \text{there exists } (f,g) \in \mathcal{F} \text{ such that} \\
& & h(x) = \max\{\, z \mid \#\{\, y \mid y < g'(x) \wedge f(x,y) \geq z \,\} \\
& & \quad \geq \left\lceil \frac{1}{2} g'(x) \right\rceil \,\} \\
(h, g \circ g') \in \mathrm{Mid}\,\mathcal{F} & \iff & \text{there exists } (f,g) \in \mathcal{F} \text{ such that} \\
& & h(x) = \max\{\, z \mid \#\{\, u \mid (\exists y, y < g'(x))(f(x,y) = u) \\
& & \quad \wedge u \geq z \,\} \geq \left\lceil \frac{1}{2}\, \#\{\, u \mid \exists y (y < g'(x) \wedge f(x,y) = u) \,\} \right\rceil \,\} \\
(h, g \circ g') \in \mathrm{Mof}\,\mathcal{F} & \iff & \text{there exists } (f,g) \in \mathcal{F} \text{ such that} \\
& & h(x) = \min\{\, f(x,y) \mid y \leq g'(x) \wedge \\
& & (\forall f(x,y'), y' \leq g'(x)) \\
& & \quad (\#\{\, y'' \leq g'(x) \mid f(x,y'') = f(x,y) \,\} \\
& & \qquad \geq \#\{\, y'' \leq g'(x) \mid f(x,y'') = f(x,y') \,\}) \,\}
\end{array}
$$

Informally speaking, a Mid function yields the median of a set of function values[2], a Med function yields the median of a multiset of function values and a Mof function yields the element of a multiset which appears most often.

[2] According to our definition above Mid FP corresponds to what Ogiwara/Hemachandra call "L-MidP" in [OgHe91]. The proofs of the corresponding theorems below show that all statements hold for R-MidP as well as for L-MidP.

3 Equations on Classes of Languages

3.1 Relationships between Operators and Operations

We start with some "normal form" theorems for $\mathrm{C}_{=}$-classes.

(3.1) Corollary. *If $\mathcal{F} \subseteq \mathrm{FP} - \mathcal{F}$ and if $\mathcal{F}$ is closed under addition, then*

$$\mathrm{C}_{=}(\mathcal{F} - \mathcal{F}) = \mathrm{C}_{=}\mathcal{F}.$$

(3.2) Proposition.

1. *Let $\mathrm{FP} \subseteq \mathcal{F}$ and let $\mathcal{F}$ be closed under addition and multiplication. Then for every $A \in \mathrm{C}_{=}(\mathcal{F} - \mathcal{F})$ there exist $f_1, f_2 \in \mathcal{F}$ such that for every x*

$$\begin{array}{lcl} x \in A & \Rightarrow & f_1(x) = f_2(x) \\ x \notin A & \Rightarrow & f_1(x) < f_2(x) \end{array}$$

2. *Let $\mathrm{FP} \subseteq \mathcal{F} \subseteq \mathrm{FP} - \mathcal{F}$ and let $\mathcal{F}$ be closed under addition and multiplication. Then for every $A \in \mathrm{C}_{=}\mathcal{F}$ there exist $f \in \mathcal{F}$ and $g \in \mathrm{FP}$ such that for every x*

$$\begin{array}{lcl} x \in A & \Rightarrow & f(x) = g(x) \\ x \notin A & \Rightarrow & f(x) < g(x) \end{array}$$

Proof.

1. For $A \in \mathrm{C}_{=}(\mathcal{F}-\mathcal{F})$ there exist $h_1, h_2 \in \mathcal{F}$ and $g \in \mathrm{FP}$ such that $x \in A \iff h_1(x) - h_2(x) = g(x)$. Consequently,

$$\begin{array}{lcl} x \in A & \Rightarrow & (h_1(x) - h_2(x) - g(x))^2 = 0 \\ & \Rightarrow & 2h_1(x)(h_2(x) + g(x)) = h_1(x)^2 + h_2(x)^2 + g(x)^2 + 2h_2(x)g(x) \\ x \notin A & \Rightarrow & (h_1(x) - h_2(x) - g(x))^2 > 0 \\ & \Rightarrow & 2h_1(x)(h_2(x) + g(x)) < h_1(x)^2 + h_2(x)^2 + g(x)^2 + 2h_2(x)g(x) \end{array}$$

   Define $f_1(x) = 2h_1(x)(h_2(x) + g(x))$ and $f_2(x) = h_1(x)^2 + h_2(x)^2 + g(x)^2 + 2h_2(x)g(x)$. By Proposition 2.4 we obtain $f_1, f_2 \in \mathcal{F}$.

2. Immediate consequence of 3.1.

□

(3.3) Corollary. *Let $\mathrm{P} \subseteq \mathcal{K}$ and let $\mathcal{K}$ be closed under $\leq_m^{\mathrm{p}}$, union and intersection. Then for every $A \in \mathrm{C}_{=}(\#\mathcal{K} - \#\mathcal{K})$ there exist $f_1, f_2 \in \#\mathcal{K}$ such that for every x*

$$\begin{array}{lcl} x \in A & \Rightarrow & f_1(x) = f_2(x) \\ x \notin A & \Rightarrow & f_1(x) < f_2(x) \end{array}$$

Now we give our basic equations.

(3.4) Proposition. *Let $\mathcal{F}$ be a class of counting functions such that $\mathrm{FP} \subseteq \mathcal{F}$.*

$$\Gamma_{\dot{-}}\mathcal{F} = \Gamma_{-,\min}\mathcal{F} = \Gamma_{-,\max}\mathcal{F}$$

PROOF. The above equation is an immediate consequence of the obvious equations $f_1(x) \dot- f_2(x) = f_1(x) - \min(f_1(x), f_2(x))$, $\min(f_1(x), f_2(x)) = f_1(x) \dot- (f_1(x) \dot- f_2(x))$, $\max(f_1(x), f_2(x)) = g(x) - min(g(x) - f_1(x), g(x) - f_2(x))$ and $\min(f_1(x), f_2(x)) = g(x) - max(g(x) - f_1(x), g(x) - f_2(x))$, where $g \in \mathrm{FP}$ is chosen in such a way that $f_1(x) \le g(x)$ and $f_2(x) \le g(x)$. $\square$

(3.5) Theorem.

1. *Let* $\mathrm{FP} \subseteq \mathcal{F}$ *and let* $\mathcal{F}$ *be closed under addition.*

$$BC(\mathrm{C}(\mathcal{F} - \mathcal{F})) = \mathrm{U}\Gamma_{\dot-}\mathcal{F} = \mathrm{U}\Gamma_{-,\min}\mathcal{F} = \mathrm{U}\Gamma_{-,\max}\mathcal{F} \subseteq \mathrm{U}\Gamma_{[\div],+}\mathcal{F}$$

2. *Let* $\mathrm{FP} \subseteq \mathcal{F} \subseteq \mathrm{FP} - \mathcal{F}$ *and let* $\mathcal{F}$ *be closed under addition and multiplication. Then*

$$\mathrm{U}(\mathcal{F} \dot- \mathrm{FP}) = \mathrm{U}(\mathcal{F} \dot- \mathcal{F}) = \mathrm{C}_{=}\mathcal{F} = \mathrm{C}_{=}(\mathcal{F} \dot- \mathcal{F}).$$

PROOF.

1. $BC(\mathrm{C}(\mathcal{F} - \mathcal{F})) \subseteq \mathrm{U}\Gamma_{\dot-}\mathcal{F}$:

   By induction. For $A \in \mathrm{C}(\mathcal{F} - \mathcal{F})$ there exist $f_1, f_2 \in \mathcal{F}$ and $g \in \mathrm{FP}$ such that $f_1(x) \ge f_2(x)$ and $x \in A \iff f_1(x) - f_2(x) \ge g(x)$. Obviously, $c_A(x) = 1 \dot- (g(x) \dot- (f_1(x) \dot- f_2(x)))$. Induction step: $c_{\overline{A}}(x) = 1 \dot- c_A(x)$ and $c_{A\cup B}(x) = 1 \dot- ((1 \dot- c_A(x)) \dot- c_B(x))$.

   $\mathrm{U}\Gamma_{\dot-}\mathcal{F} \subseteq BC(\mathrm{C}(\mathcal{F} - \mathcal{F}))$:

   Using inductively the equivalences

$$z_1 + (z_2 \dot- z_3) \ge z_4 \iff (z_3 \ge z_2 \wedge z_1 \ge z_4) \vee (z_3 < z_2 \wedge z_1 + z_2 \ge z_3 + z_4)$$

   and

$$z_1 \ge (z_2 \dot- z_3) + z_4 \iff (z_3 \ge z_2 \wedge z_1 \ge z_4) \vee (z_3 < z_2 \wedge z_1 + z_3 \ge z_2 + z_4)$$

   (for natural numbers z_1, z_2, z_3, z_4) one proves that for every $f_1, f_2 \in \Gamma_{\dot-}\mathcal{F}$ the condition $f_1(x) \ge f_2(x)$ is equivalent to a boolean combination of conditions of the form $h_1(x) \ge h_2(x)$ where $h_1, h_2 \in \mathcal{F}$. The following two observations complete the proof:

   (a) For $A \in \mathrm{U}\Gamma_{\dot-}\mathcal{F}$ we have $x \in A \iff c_A(x) = 1 \iff c_A(x) \ge 1 \wedge \neg(c_A(x) \ge 2)$ where $c_A, 1 \in \Gamma_{\dot-}\mathcal{F}$.

   (b) Defining $B = \{x \mid h_1(x) \ge h_2(x)\}$ for $h_1, (h_2, g) \in \mathcal{F}$ we obtain $x \in B \iff (h_1(x) + g(x)) - h_2(x) \ge g(2)$ and hence $B \in \mathrm{C}(\mathcal{F} - \mathcal{F})$.

   $BC(\mathrm{C}(\mathcal{F} - \mathcal{F})) \subseteq \mathrm{U}\Gamma_{[\div],+}\mathcal{F}$:

   By induction. For $A \in \mathrm{C}(\mathcal{F} - \mathcal{F})$ there exist $f_1, f_2 \in \mathcal{F}$ and $g \in \mathrm{FP}$ such that $f_1(x) \ge f_2(x)$ and $x \in A \iff f_1(x) - f_2(x) \ge g(x)$. Obviously,

$$c_{\overline{A}}(x) = \left[\frac{1}{1 + \left[\frac{f_1(x)}{f_2(x)+g(x)}\right]}\right]$$

   Induction step: $c_{\overline{A}}(x) = \left[\frac{1}{1+c_A(x)}\right]$ and $c_{\overline{A\cup B}}(x) = \left[\frac{1}{1+c_A(x)+c_B(x)}\right]$.

2. For $A \in \mathrm{U}(\mathcal{F} \dot{-} \mathcal{F})$ there exist $f_1, f_2 \in \mathcal{F}$ such that $x \in A \iff f_1(x) \dot{-} f_2(x) = 1 \iff f_1(x) = f_2(x) + 1$. By $\mathcal{F} \subseteq \mathrm{FP} - \mathcal{F}$ we obtain $g \in \mathrm{FP}$ and $f_3 \in \mathcal{F}$ such that $f_2(x) = g(x) - f_3(x)$. Hence $x \in A \iff f_1(x) + f_3(x) = g(x) + 1$.

   For $A \in \mathrm{C}_{=}\mathcal{F}$ Proposition 3.2 yields $f \in \mathcal{F}$ and $g \in \mathrm{FP}$ such that

$$\begin{array}{lcl} x \in A & \Rightarrow & f(x) = g(x) \\ x \notin A & \Rightarrow & f(x) < g(x) \end{array}$$

   Consequently, $c_A(x) = f(x) \dot{-} (g(x) \dot{-} 1)$.

□

(3.6) Proposition. *1.* $\mathrm{U}(\mathcal{F} - \mathrm{FP}) = \mathrm{X}\mathcal{F}$.

2. If $\mathrm{FP} \subseteq \mathcal{F}$ *and* $\mathcal{F}$ *is closed under addition, then* $\mathrm{U}(\mathcal{F} - \mathcal{F}) = \mathrm{X}(\mathcal{F} - \mathcal{F})$.

3. Let $\mathrm{FP} \subseteq \mathcal{F}$ *and let* $\mathcal{F}$ *be closed under addition and multiplication. Then*

$$\exists(\mathcal{F} - \mathcal{F}) = \text{co-}\,\mathrm{C}_{=}(\mathcal{F} - \mathcal{F}).$$

Proof. We only show (3).

By Figure 1 we have "$\subseteq$". For "$\supseteq$" we have by 3.3 for every $A \in \text{co-}\,\mathrm{C}_{=}(\mathcal{F} - \mathcal{F})$ functions $f_1, f_2 \in \mathcal{F}$ such that $x \in A \Rightarrow f_1(x) < f_2(x)$ and $x \notin A \Rightarrow f_1(x) = f_2(x)$. Consequently, $f_2(x) \geq f_1(x)$ and $x \in A \iff f_2(x) - f_1(x) > 0$, i. e. $A \in \exists(\mathcal{F} - \mathcal{F})$. □

(3.7) Corollary. *Let* $\mathrm{P} \in \mathcal{K}$ *and let* $\mathcal{K}$ *be closed under* $\leq_m^{\mathrm{p}}$*, union and intersection. Then*

$$\exists(\#\mathcal{K} - \#\mathcal{K}) = \text{co-}\,\mathrm{C}_{=}(\#\mathcal{K} - \#\mathcal{K}).$$

3.2 Consequences

We now give consequences of our basic equations.

First we use 3.5 to deduce consequences of closure of counting function classes under modified subtraction.

(3.8) Theorem. *Let* $\mathrm{FP} \subseteq \mathcal{F} \subseteq \mathrm{FCH}$ *and let* $\mathcal{F}$ *be closed under addition, multiplication and summation. Then the following statements are equivalent:*

1. $\mathcal{F}$ *is closed under substitution.*

2. $\mathcal{F}$ *is closed under modified subtraction.*

3. $\mathcal{F}$ *is closed under integer division.*

4. $\mathcal{F}$ *is closed under maximum and subtraction.*

5. $\mathrm{U}\mathcal{F} = \mathrm{C}_{=}\mathcal{F}$.

6. $\mathrm{U}\mathcal{F} = \mathrm{CH}$.

7. $\mathcal{F} = \mathrm{FCH}$.

Proof.

(1) $\Rightarrow$ (2), (1) $\Rightarrow$ (3) and (1) $\Rightarrow$ (4) are obvious.

(2) $\Rightarrow$ (5), (3) $\Rightarrow$ (5) and (4) $\Rightarrow$ (5) are immediate consequences of 3.5, since $\mathrm{U}\mathcal{F} \subseteq \mathrm{C}_{=}\mathcal{F} \subseteq BC(\mathrm{C}\mathcal{F})$.

(5) $\Rightarrow$ (6): Because of $\mathrm{CH} = \bigcup_{k\geq 1} \mathrm{C}_{=}^{k}\,\mathrm{P}$ it suffices to prove $\mathrm{C}_{=}^{k}\,\mathrm{P} \subseteq \mathrm{U}\mathcal{F}$ for $k \geq 1$. For $k = 1$ this is our assumption. Let $\mathrm{C}_{=}^{k}\,\mathrm{P} \subseteq \mathrm{U}\mathcal{F}$ for some $k \geq 1$. Then $\mathrm{C}_{=}^{k+1}\,\mathrm{P} = \mathrm{C}_{=}\mathrm{C}_{=}^{k}\,\mathrm{P} \subseteq \mathrm{C}_{=}\mathrm{U}\mathcal{F} = \mathrm{C}_{=}\#\,\mathrm{U}\mathcal{F} \subseteq \mathrm{C}_{=}\mathcal{F}$ (the latter inclusion is by 2.9).

(6) $\Rightarrow$ (7): By 2.9 and (6) we conclude $\#\,\mathrm{U}\mathcal{F} \subseteq \mathcal{F} \subseteq \mathrm{FCH} \subseteq \#\,\mathrm{CH} \subseteq \#\,\mathrm{U}\mathcal{F}$.

(7) $\Rightarrow$ (1): We have $\mathcal{F} = \mathrm{FCH}$, but by 2.13 $\mathrm{FCH} = \mathrm{FP}^{\mathrm{CH}}$ is closed under substitution. □

Observe that from Theorem 3.5 we could get even more simply the following corollary for $\mathcal{F} - \mathcal{F}$:

(3.9) Corollary. *Let* $\mathrm{FP} \subseteq \mathcal{F} \subseteq \mathrm{FCH}$ *and let* $\mathcal{F}$ *be closed under addition, multiplication and summation. Then the following statements are equivalent:*

1. $\mathcal{F} - \mathcal{F}$ *is closed under substitution.*
2. $\mathcal{F} - \mathcal{F}$ *is closed under max.*
3. $\mathcal{F} - \mathcal{F}$ *is closed under min.*
4. $\mathrm{U}(\mathcal{F} - \mathcal{F}) = \mathrm{C}(\mathcal{F} - \mathcal{F})$.
5. $\mathrm{U}(\mathcal{F} - \mathcal{F}) = \mathrm{CH}$.
6. $\mathcal{F} - \mathcal{F} = \mathrm{FCH}$.

(3.10) Theorem. *Let* $\mathrm{P} \subseteq \mathcal{K} \subseteq \mathrm{CH}$ *and let* $\mathcal{K}$ *be closed under* $\leq_m^{\mathrm{p}}$*, union and intersection. Then the following statements are equivalent:*

1. $\#\,\mathcal{K}$ *is closed under substitution.*
2. $\#\,\mathcal{K}$ *is closed under modified subtraction.*
3. $\#\,\mathcal{K}$ *is closed under integer division.*
4. $\#\,\mathcal{K}$ *is closed under maximum and subtraction.*
5. $\mathrm{U}\mathcal{K} = \mathrm{C}_{=}\mathcal{K}$.
6. $\mathrm{U}\mathcal{K} = \mathrm{CH}$.
7. $\#\,\mathcal{K} = \mathrm{FCH}$.

Proof. Put $\mathcal{F} = \#\,\mathcal{K}$ in Theorem 3.8 and use 2.4. □

(3.11) Corollary.

$$\begin{array}{llll}
(1) & \#\,\mathrm{P} = \Gamma_{\mathrm{SUB}}\,\#\,\mathrm{P} & \iff \#\,\mathrm{P} = \Gamma_{\dot{-}}\,\#\,\mathrm{P} & \iff \#\,\mathrm{P} = \Gamma_{[\div]}\,\#\,\mathrm{P} \\
 & & \iff \#\,\mathrm{P} = \Gamma_{\min,-}\,\#\,\mathrm{P} & \iff \mathrm{UP} = \mathrm{C}_{=}\mathrm{P} \\
 & & \iff \mathrm{UP} = \mathrm{CH} & \iff \#\,\mathrm{P} = \mathrm{FCH} \\
(2) & \text{span-P} = \Gamma_{\mathrm{SUB}}\text{span-P} & \iff \text{span-P} = \Gamma_{\dot{-}}\text{span-P} & \iff \text{span-P} = \Gamma_{[\div]}\text{span-P} \\
 & & \iff \text{span-P} = \Gamma_{\min,-}\text{span-P} & \iff \mathrm{NP} = \mathrm{C}_{=}\mathrm{P} \\
 & & \iff \mathrm{NP} = \mathrm{CH} & \iff \text{span-P} = \mathrm{FCH} \\
(3) & \text{Gap-P} = \Gamma_{\mathrm{SUB}}\text{Gap-P} & \iff \text{Gap-P} = \Gamma_{\dot{-}}\text{Gap-P} & \iff \text{Gap-P} = \Gamma_{[\div]}\text{Gap-P} \\
 & & \iff \text{Gap-P} = \Gamma_{\min,-}\text{Gap-P} & \iff \mathrm{XP} = \mathrm{C}_{=}\mathrm{P} \\
 & & \iff \mathrm{XP} = \mathrm{CH} & \iff \text{Gap-P} = \mathrm{FCH}
\end{array}$$

Proof. This is a corollary of Theorem 3.10

For (2) note that span-P $= \#\,\mathrm{NP}$ (2.7) and that $\mathrm{NP} = \mathrm{C}_=\exists\mathrm{P} \iff \mathrm{NP} = \mathrm{C}_=\mathrm{P}$. The latter is seen as follows: Let $\mathrm{NP} = \mathrm{C}_=\exists\mathrm{P}$. Then $\mathrm{C}_=\mathrm{P} \subseteq \mathrm{NP}$. Because of co-NP $\subseteq \mathrm{C}_=\mathrm{P}$ this means co-NP $= \mathrm{NP} = \mathrm{C}_=\mathrm{P}$. Now let $\mathrm{NP} = \mathrm{C}_=\mathrm{P}$. Because of $\mathrm{P}^{\#\,\mathrm{PH}} = \mathrm{P}^{\mathrm{PP}}$ (2.10) and $\mathrm{P}^{\mathrm{PP}} \subseteq \exists\,\mathrm{C}_=\mathrm{P}$ (2.11) we conclude $\mathrm{C}_=\exists\mathrm{P} \subseteq \mathrm{P}^{\#\,\exists\mathrm{P}} \subseteq \mathrm{P}^{\mathrm{PP}} \subseteq \exists\,\mathrm{C}_=\mathrm{P} \subseteq \exists\mathrm{NP} \subseteq \mathrm{NP}$.

For (3) note that Gap-P $= \#\,\mathrm{P} - \#\,\mathrm{P}$ (2.7) and $\mathrm{XP} = \mathrm{U}(\#\,\mathrm{P} - \#\,\mathrm{P})$ (3.6). □

Next we use 3.7 to deduce consequences of closure of counting function classes under usual subtraction.

(3.12) Theorem. *Let $\mathcal{K} = \exists\mathcal{K}'$ or $\mathcal{K} = \forall\mathcal{K}'$. Let $\mathrm{P} \subseteq \mathcal{K}' \subseteq \mathrm{CH}$ and let $\mathcal{K}'$ be closed under $\leq^{\mathrm{p}}_m$, union and intersection. Then the following statements are equivalent:*

1. *$\#\,\mathcal{K}$ is closed under substitution.*
2. *$\#\,\mathcal{K}$ is closed under subtraction.*
3. *$\mathcal{K} = \mathrm{C}_=\mathcal{K}$.*
4. *$\mathcal{K} = \mathrm{CH}$.*
5. *$\#\,\mathcal{K} = \mathrm{FCH}$.*

Proof. Let $\mathcal{K} = \exists\mathcal{K}'$.

(1) $\Rightarrow$ (2): Obviously, subtraction is polynomial time computable. But $\mathrm{FP} \subseteq \#\,\mathrm{P} \subseteq \#\,\mathcal{K}' \subseteq \#\,\exists\mathcal{K}'$.

(2) $\Rightarrow$ (3): Because of (2.3) we have $\#\,\text{co-}\exists\mathcal{K}' \subseteq 2^{\mathrm{Pol}} - \#\,\exists\mathcal{K}' \subseteq \#\,\exists\mathcal{K}' - \#\,\exists\mathcal{K}'$. If $\#\,\exists\mathcal{K}'$ is closed under subtraction then $\#\,\text{co-}\exists\mathcal{K}' \subseteq \#\,\exists\mathcal{K}'$! Consequently, $\text{co-}\exists\mathcal{K}' \subseteq \text{co-}\forall\exists\mathcal{K}' \subseteq \exists\text{co-}\exists\mathcal{K}' \subseteq \exists\exists\mathcal{K}' \subseteq \exists\mathcal{K}'$ and hence $\exists\mathcal{K}' = \text{co-}\exists\mathcal{K}'$. On the other hand, (3.7) implies $\text{co-}\,\mathrm{C}_=\exists\mathcal{K}' = \text{co-}\,\mathrm{C}_=(\#\,\exists\mathcal{K}') \subseteq \text{co-}\,\mathrm{C}_=(\#\,\exists\mathcal{K}' - \#\,\exists\mathcal{K}') \subseteq \exists(\#\,\exists\mathcal{K}' - \#\,\exists\mathcal{K}') \subseteq \exists(\#\,\exists\mathcal{K}') \subseteq \exists\mathcal{K}'$ which results in $\exists\mathcal{K}' = \mathrm{C}_=\exists\mathcal{K}'$.

(3) $\Rightarrow$ (4): By (2.12) we have $\mathrm{CH} = \bigcup_{k\geq 0} \mathrm{C}_=^k\,\mathrm{P} = \bigcup_{k\geq 0} \mathrm{C}_=^k\,\exists\mathrm{P}$. Thus it is sufficient to prove $\mathrm{C}_=^k\,\exists\mathrm{P} \subseteq \exists\mathcal{K}'$ for every $k \geq 0$. For $k = 0$ this is obvious. Assume that this is true for k. We conclude $\mathrm{C}_=^{k+1}\,\exists\mathrm{P} \subseteq \mathrm{C}_=\mathrm{C}_=^k\,\exists\mathrm{P} \subseteq \mathrm{C}_=\exists\mathcal{K}' \subseteq \exists\mathcal{K}'$.

(4) $\Rightarrow$ (5): is an immediate consequence of (2.13).

(5) $\Rightarrow$ (1): We have $\#\,\exists\mathcal{K}' = \mathrm{FCH}$, but by (2.13) $\mathrm{FCH} = \mathrm{FP}^{\mathrm{CH}}$ is closed under substitution.

For $K = \forall\mathcal{K}'$ we prove only (2) $\Rightarrow$ (3), because the other implications are as for $\mathcal{K} = \exists\mathcal{K}'$. Let $\#\,\forall\mathcal{K}'$ be closed under subtraction. Then $\#\,\text{co-}\forall\mathcal{K}' \subseteq \mathrm{FP} - \#\,\forall\mathcal{K}' \subseteq \#\,\forall\mathcal{K}'$ and hence $\text{co-}\forall\mathcal{K}' \subseteq \forall\text{co-}\forall\mathcal{K}' = \forall(\#\,\text{co-}\forall\mathcal{K}') \subseteq \forall(\#\,\forall\mathcal{K}') = \forall\mathcal{K}'$. Consequently, $\#\,\forall\mathcal{K}' = \#\,\text{co-}\forall\mathcal{K}' = \#\,\exists\text{co-}\mathcal{K}'$, and thus $\#\,\exists\text{co-}\mathcal{K}'$ is closed under subtraction. By the first part of the proof we obtain $\text{co-}\forall\mathcal{K}' = \exists\text{co-}\mathcal{K}' = \mathrm{C}_=\exists\text{co-}\mathcal{K}'$ and thus $\forall\mathcal{K}' = co\,\mathrm{C}_=\exists\text{co-}\mathcal{K}'$. By (2.16) we conclude $\mathrm{C}_=\forall\mathcal{K}' \subseteq \forall\text{co-}\,\mathrm{C}_=\forall\mathcal{K}' \subseteq \forall\forall\mathcal{K}' \subseteq \forall\mathcal{K}$. □

(3.13) Corollary.

$$
\begin{array}{llllll}
(1) & \text{span-P} = \Gamma_{\mathrm{SUB}}\text{span-P} & \iff & \text{span-P} = \Gamma_{-}\text{span-P} & \iff & \text{span-P} = \mathrm{FCH} \\
 & & \iff & \mathrm{NP} = \mathrm{CH} & \iff & \mathrm{NP} = \mathrm{C}_=\exists\mathrm{P} \\
 & & \iff & \mathrm{NP} = \mathrm{C}_=\mathrm{P} & & \\
(2) & \#\,\text{co-NP} = \Gamma_{\mathrm{SUB}}\,\#\,\text{co-NP} & \iff & \#\,\text{co-NP} = \Gamma_{-}\,\#\,\text{co-NP} & \iff & \#\,\text{co-NP} = \mathrm{FCH} \\
 & & \iff & \mathrm{NP} = \mathrm{CH} & \iff & \text{co-NP} = \mathrm{C}_=\forall\mathrm{P}
\end{array}
$$

Proof. This is a corollary to Theorem 3.12. For 1 see also proof of Corollary 3.11 □

4 Equations on Classes of Functions

In this section we first give a general theorem on classes of differences of functions defined via the $\mathrm{M\overline{ax}}$ operator introduced in Section 2. Then we show how we can exploit this theorem to get results about different function classes which all have some optimization process implicit in their definition, like $\mathrm{Opt\,P} = \mathrm{Max\,FP}$, $\mathrm{Mid\,FP}$ (here we are interested in the *maximal* element m of a set M, sucht that the number of elements of M less then m is less than the number of elements of M greater than or equal to m; this is an optimization problem), or $\mathrm{Mof\,FP}$ (here we are interested in the element m with the *maximal* number of paths on which m is printed; again an optimization problem).

4.1 Relationships between Operations and Operators

We start with the definition of a technical operator taking a class of sets and filtering out the sets which are *monotonic* in the second argument concerning membership in the set.

Definition. Let $\mathcal{K}$ be any class of languages. Then we say that A belongs to $\mathrm{M}\mathcal{K}$ ($A \in \mathrm{M}\mathcal{K}$) if and only if $A \in \mathcal{K}$ and $(x, y+1) \in A$ implies $(x, y) \in A$ for all x, y.

Next we observe some relationships between the M-operator and the $\mathrm{M\overline{ax}}$- operator.

(4.1) Lemma. *If $\mathcal{K}$ is closed under intersection with* L*-sets, then*

1. $\mathrm{M\overline{ax}}\,\mathrm{M}\exists\mathcal{K} = \mathrm{M\overline{ax}}\,\exists\mathcal{K}$.
2. $\mathrm{M\overline{ax}}\,\exists\mathcal{K} \subseteq \mathrm{M\overline{ax}}\,\mathcal{K} - \mathrm{M\overline{ax}}\,\mathcal{K}$.
3. $\mathrm{M\overline{ax}}\,\exists\mathcal{K} - \mathrm{M\overline{ax}}\,\exists\mathcal{K} = \mathrm{M\overline{ax}}\,\mathcal{K} - \mathrm{M\overline{ax}}\,\mathcal{K}$.

PROOF. 1. Let $A \in \exists\mathcal{K}$, $(x, y) \in A \iff \exists z, |z| = p(|xy|)\colon (x, y, z) \in B$ for some $B \in \mathcal{K}$ and polynomial p. Then define

$$\begin{aligned} A' &= \{\, (x, y) \mid \exists y' > y, |y'| = p(|x|)\colon (x, y') \in A \,\} \\ &= \{\, (x, y) \mid \exists y', |y'| = p(|x|) \exists z\colon y' > y \wedge (x, y', z) \in B \,\} \end{aligned}$$

Clearly $\max\{\, y \mid |y| = p'(|x|) \wedge (x, y) \in A \,\} = \max\{\, y \mid |y| = p'(|x|) \wedge (x, y) \in A' \,\}$ for every x and $(x, y+1) \in A' \Rightarrow (x, y) \in A'$.

2. Let $f \in \mathrm{M\overline{ax}}\,\exists\mathcal{K}$, $f(x) = \max\{\, y \mid |y| = p(|x|) \wedge (x, y) \in A \,\}$, where $A \in \exists\mathcal{K}$, $(x, y) \in A \iff \exists z, |z| = q(|x, y|)\colon (x, y, z) \in B$ for some $B \in \mathcal{K}$. Now define

$$\begin{aligned} A_1 &= \{\, (x, yzy) \mid |y| = p(|x|), |z| = q(|xy|)\colon (x, y, z) \in B \,\} \\ A_2 &= \{\, (x, yzw) \mid |y| = p(|x|), |z| = q(|xy|), w = 0^{p(|x|)}\colon (x, y, z) \in B \,\} \end{aligned}$$

Then

$$\begin{aligned} f(x) &= \max\{\, u \mid |u| = 2p(|x|) + q(|x| + p(|x|))\colon (x, u) \in A_1 \,\} \\ &- \max\{\, u \mid |u| = 2p(|x|) + q(|x| + p(|x|))\colon (x, u) \in A_2 \,\}. \end{aligned}$$

3. Follows from part 2 with the additional observation that

$$(\mathrm{M\overline{ax}}\,\mathcal{K} + \mathrm{M\overline{ax}}\,\mathcal{K}) - (\mathrm{M\overline{ax}}\,\mathcal{K} + \mathrm{M\overline{ax}}\,\mathcal{K}) = \mathrm{M\overline{ax}}\,\mathcal{K} - \mathrm{M\overline{ax}}\,\mathcal{K}.$$

To see this, now encode in the last component of elements of the set A_1 not y but values of arithmetical expressions of the above form.

Let f_i, $1 \le i \le 4$, be functions from $\mathrm{M\overline{ax}}\,\mathcal{K}$, defined via B_i, $1 \le i \le 4$, $B_i \in \mathcal{K}$. Now define

$$\begin{aligned} A_1 &= \{\, (x, y_1y_2y_3y_4z) \mid && |y_i| = p(|x|), |z| = p(|x|)+1 \text{ for suitable } p \text{ and} \\ &&& (x, y_i) \in B,\ 1 \le i \le 4, \text{ and } z = (y_1+y_2) - (y_3+y_4) \,\} \\ A_2 &= \{\, (x, y_1y_2y_3y_4z) \mid && \text{as above, but now } z = 0^{p(|x|)+2} \,\} \end{aligned}$$

Then obtain the desired f as in the last proof.

□

Definition. We say a function f is in $\mathrm{FP}_{\mathit{suff}}$, if $f(x)$ is equal to a suffix of x of a certain length which is computable in polynomial time on input $1^{|x|}$.

Now we state and prove our main theorem in this section.

(4.2) Theorem. *If $\mathcal{K}$ is closed under $\le^{\mathrm{p}}_{ptt}$, then*

$$\mathrm{FP}^{\mathcal{K}} = \mathrm{FP}^{\mathrm{M\overline{ax}\,M}\mathcal{K}} = \mathrm{FP}^{\mathrm{M\overline{ax}\,M}\mathcal{K}}[1] = \mathrm{M\overline{ax}\,M}\mathcal{K} - \mathrm{M\overline{ax}\,M}\mathcal{K} = \mathrm{FP}_{\mathit{suff}} \circ \mathrm{M\overline{ax}\,M}\mathcal{K}.$$

Proof. $\mathrm{FP}_{\mathit{suff}} \circ \mathrm{M\overline{ax}\,M}\mathcal{K} \subseteq \mathrm{M\overline{ax}\,M}\mathcal{K} - \mathrm{M\overline{ax}\,M}\mathcal{K}$:

Given a function whose value is the suffix of length ℓ of a given $\mathrm{M\overline{ax}\,M}\mathcal{K}$ function, we define a second $\mathrm{M\overline{ax}\,M}\mathcal{K}$ function which yields the value of the first function but with the last ℓ bits all set to zero.

To be more precise let $f \in \mathrm{M\overline{ax}\,M}\mathcal{K}$, $f(x) = \max\{\, y \mid |y| = p(|x|) \wedge (x,y) \in B \,\}$ for some polynomial p and some set $B \in \mathrm{M}\mathcal{K}$. Then define

$$\begin{aligned} B' = \{\, (x,w) \mid \quad & |w| = p(|x|), w = y_1y_2, |y_2| = \ell \text{ and} \\ & (y_2 = 0^\ell \wedge (x,w) \in B) \text{ or } (x, y'0^\ell) \in B \text{ where } y' = y_1 + 1 \,\}; \end{aligned}$$

and

$$f'(x) = \max\{\, w \mid |w| = p(|x|) \wedge (x,w) \in B' \,\}.$$

Then $f' \in \mathrm{M\overline{ax}\,M}\mathcal{K}$ and the last ℓ bits of $f(x)$ are equal to $f(x) - f'(x)$.

$\mathrm{M\overline{ax}\,M}\mathcal{K} - \mathrm{M\overline{ax}\,M}\mathcal{K} \subseteq \mathrm{FP}^{\mathrm{M\overline{ax}\,M}\mathcal{K}}[1]$:

Let $f_i(x) = \max\{\, y \mid |y| = p_i(|x|) \wedge (x,y) \in B_i \,\}$ for $i \in \{1,2\}$, polynomials p_i and sets $B_i \in \mathrm{M}\mathcal{K}$. Then define

$$\begin{aligned} B = \{\, (x, y_1y_2) \mid \quad & |y_1| = p_1(|x|), |y_2| = p(|x|) \text{ and} \\ & ((x,y_1) \in B_1 \wedge (x,y_2) \in B_2) \text{ or } ((x,y_1) \in B_1 \wedge (x, y_1+1) \in B_1) \,\}; \end{aligned}$$

and

$$f(x) = \max\{\, y \mid |y| = p(|x|) \wedge (x,y) \in B \,\}$$

for suitable p.

Then clearly a polynomial time transducer given x can compute $f_1(x) - f_2(x)$ by asking only one question to f.

$\mathrm{FP}^{\mathrm{M\overline{ax}\,M}\mathcal{K}}[1] \subseteq \mathrm{FP}^{\mathrm{M\overline{ax}\,M}\mathcal{K}}$: trivial.

$\mathrm{FP}^{\mathrm{M\overline{ax}\,M}\mathcal{K}} \subseteq \mathrm{FP}^{\mathcal{K}}$: Obvious. Use binary search to compute the maximum.

$\mathrm{FP}^{\mathcal{K}} \subseteq \mathrm{FP}_{\mathit{suff}} \circ \mathrm{M\overline{ax}\,M}\mathcal{K}$:

Let $f \in \mathrm{FP}^B$ for some $B \in \mathcal{K}$ be computed by the deterministic polynomial time oracle transducer M. We may assume that M is normalized in the sense that on inputs of length n

M asks exactly $p(n)$ oracle queries and outputs at most $q(n)$ bits for some polynomials p and q. Then define

$$\begin{array}{rcl} D & = & \{ (x,\, a_1 \dots a_{p(|x|)} y) \mid a_i \in \{0,1\} \text{ for } 1 \le i \le p(|x|), |y| = q(|x|); \\ & & \quad \text{if we interpret } a_1 \dots a_{p(|x|)} \text{ as answers from the oracle in a} \\ & & \quad \text{computation path of } M \text{, then } z_1, \dots, z_{p(|x|)} \text{ are the questions in this} \\ & & \quad \text{computation, } y' \text{ is the output in this computation, and we have that} \\ & & \quad a_i = 1 \Rightarrow z_i \in B \text{ for } 1 \le i \le p(|x|) \text{ and} \\ & & \quad y \le 0^{q(|x|)-|y'|} y' \,\} \\ & \cup & \{ (x,\, a_1 \dots a_m y) \mid a_i \in \{0,1\} \text{ for } 1 \le i \le m, |y| = p(|x|) + q(|x|) - m - 1; \\ & & \quad \text{if we interpret } a_1 \dots a_m \text{ as answers from the oracle in a} \\ & & \quad \text{computation path of } M \text{, then } z_1, \dots, z_{m+1} \text{ are the first questions in} \\ & & \quad \text{this computation; } a_i = 1 \Rightarrow z_i \in B \text{ for } 1 \le i \le m \\ & & \quad \text{and } z_{i+1} \in B . \} \end{array}$$

Obviously $D \le_{ptt}^{\mathrm{p}} B$ and therefore $D \in \mathrm{M}\mathcal{K}$. Define

$$h(x) = \max\{\, z \mid |z| = p(|x|) + q(|x|) \wedge (x,z) \in D \,\}.$$

Then $h \in \mathrm{M\overline{ax}}\,\mathrm{M}\mathcal{K}$ and $h(x) \in \{0,1\}^{p(|x|)} 0^{q(|x|)-|f(x)|} f(x)$ which shows $f \in \mathrm{FP}_{\mathit{suff}} \circ \mathrm{M\overline{ax}}\,\mathrm{M}\mathcal{K}$. □

Remark. The last theorem can be seen as an analogue of a result about sets (instead of functions), shown by Ogiwara [Ogi91] and earlier in a restricted context by Wagner [Wag89]. They showed that $\mathrm{P}^{\mathcal{K}} = \mathcal{K}(2^{\mathrm{Pol}})$, if $\mathcal{K}$ fulfills certain conditions, where

$$A \in \mathcal{K}(f) \iff \begin{array}{l} \text{there exists } B \in \mathcal{K} \text{ such that} \\ x \in A \text{ iff } \max\{\, y \mid y \le f(|x|) \wedge (x,y) \in B \,\} \text{ is odd.} \end{array}$$

Using our notation, this can be written as $\mathrm{P}^{\mathcal{K}} = \oplus\,\mathrm{M\overline{ax}}\,\mathrm{M}\mathcal{K}$.

So they proved that, given a set $A \in \mathrm{P}^{\mathcal{K}}$, there is an associated $\mathrm{M\overline{ax}}\,\mathrm{M}\mathcal{K}$ function f such that the last bit of the value of $f(x)$ corresponds directly to the value $c_A(x)$ of the characteristic function of A. We showed that $\mathrm{FP}^{\mathcal{K}} = \mathrm{FP}_{\mathit{suff}} \circ \mathcal{K}$, in other words we proved that, given a function $h \in \mathrm{FP}^{\mathcal{K}}$, there exists an associated $\mathrm{M\overline{ax}}\,\mathrm{M}\mathcal{K}$ function f such that the suffix of a certain length of $f(x)$ is exactly the value $h(x)$.

This is a direct generalization of Ogiwara's and Wagner's result.

(4.3) Corollary. *If $\mathcal{K}$ is closed under $\le_{ptt}^{\mathrm{p}}$, then*

$$\begin{array}{rcccl} \mathrm{FP}^{\mathcal{K}} &= \mathrm{M\overline{ax}}\,\mathrm{M}\mathcal{K} - \mathrm{M\overline{ax}}\,\mathrm{M}\mathcal{K} &= \mathrm{FP}^{\mathrm{M\overline{ax}}\,\mathrm{M}\mathcal{K}} &= \Gamma_{\mathrm{SUB}}\,\mathrm{M\overline{ax}}\,\mathrm{M}\mathcal{K} &= \Gamma_{-}\,\mathrm{M\overline{ax}}\,\mathrm{M}\mathcal{K} \\ \mathrm{FP}^{\exists\mathcal{K}} &= \mathrm{M\overline{ax}}\,\mathcal{K} - \mathrm{M\overline{ax}}\,\mathcal{K} &= \mathrm{FP}^{\mathrm{M\overline{ax}}\,\mathcal{K}} &= \Gamma_{\mathrm{SUB}}\,\mathrm{M\overline{ax}}\,\mathcal{K} &= \Gamma_{-}\,\mathrm{M\overline{ax}}\,\mathcal{K} \end{array}$$

Proof. Lemma 4.1 and Theorem 4.2 □

4.2 Consequences

(4.4) Theorem. *The following statements are equivalent:*

1. $\mathrm{M\overline{ax}}\,\mathrm{M}\mathcal{K}$ *is closed under substitution.*
2. $\mathrm{M\overline{ax}}\,\mathrm{M}\mathcal{K}$ *is closed under subtraction.*
3. $\mathrm{FP}^{\mathrm{M\overline{ax}}\,\mathrm{M}\mathcal{K}} = \mathrm{M\overline{ax}}\,\mathrm{M}\mathcal{K}$.

But we can show even more:

(4.5) Theorem. *The following statements are equivalent:*

1. $\mathrm{M\overline{ax}}\,\mathcal{K}$ *is closed under substitution.*

2. $\mathrm{M\overline{ax}}\,\mathcal{K}$ *is closed under subtraction.*

3. $\mathrm{FP}^{\mathrm{M\overline{ax}}\,\mathcal{K}} = \mathrm{M\overline{ax}}\,\mathcal{K}$.

4. $\mathcal{K} = \exists\mathcal{K} = \forall\mathcal{K}$.

5. $\mathcal{K} = \mathrm{PH}\cdot\mathcal{K}$.

Proof. $1 \Rightarrow 2$: trivial.

$2 \Rightarrow 3$: by Corollary 4.3.

$3 \Rightarrow 4$: since $\mathrm{U}(\mathrm{FP}^{\mathrm{M\overline{ax}}\,\mathcal{K}}) = \mathrm{U}(\mathrm{FP}^{\exists\mathcal{K}}) = \mathrm{P}^{\exists\mathcal{K}}$ and $\mathrm{U}\,\mathrm{M\overline{ax}}\,\mathcal{K} = \mathcal{K}$ we directly get $\mathcal{K} = \exists\mathcal{K}$ and $\mathcal{K} = \text{co-}\mathcal{K}$ and therefore also $\mathcal{K} = \forall\mathcal{K}$.

$4 \Rightarrow 5$: trivial.

$5 \Rightarrow 1$: $\Gamma_{\mathrm{SUB}}\,\mathrm{M\overline{ax}}\,\mathcal{K} \subseteq \mathrm{FP}^{\mathrm{M\overline{ax}}\,\mathcal{K}} \subseteq \mathrm{FP}^{\exists\mathcal{K}} \subseteq \mathrm{M\overline{ax}}\,\mathrm{P}^{\exists\mathcal{K}} \subseteq \mathrm{M\overline{ax}}\,\mathrm{PH}\cdot\mathcal{K} \subseteq \mathrm{M\overline{ax}}\,\mathcal{K}$. □

(4.6) Corollary.

1. $\mathrm{M\overline{ax}}\,\mathrm{P}$ *is closed under subtraction if and only if* $\mathrm{M\overline{ax}}\,\mathrm{P}$ *is closed under substitution if and only if* $\mathrm{P} = \mathrm{PH}$.

2. $\mathrm{M\overline{ax}}\,\mathrm{PP}$ *is closed under subtraction if and only if* $\mathrm{M\overline{ax}}\,\mathrm{PP}$ *is closed under substitution if and only if* $\mathrm{PP} = \mathrm{PH}\cdot\mathrm{PP}$.

Now we show how we can exploit Theorem 4.2 to get results on other optimization function classes.

We first give some definitions.

Definition. For any function $f\colon\Sigma^\star \to I\!N$, we set $\mathrm{graph}_f = \{\, (x, f(x)) \mid x \in \Sigma^\star \,\}$

Definition. We say that $\mathcal{F}$ is closed under restricted multiplication, if given any two place 0/1-valued function $f \in \mathcal{F}$, we also have $\lambda xy.f(x,y)\cdot y \in \mathcal{F}$.

(4.7) Proposition. *For* $\mathcal{O} \in \{\mathrm{Max}, \mathrm{Mid}, \mathrm{Med}, \mathrm{Mof}\}$*, if* $\mathcal{F}$ *is closed under multiplication with* FP *functions, then* $\mathcal{O}\mathcal{F}$ *is closed under restricted multiplication.*

Proof. Let $f \in \mathcal{O}\mathcal{F}$ be defined via $g \in \mathcal{F}$. Then $\lambda xy.f(x,y)\cdot y \in \mathcal{F}$ via $g' = \lambda xyz.g(x,y,z)\cdot y$, which is by assumption a function from $\mathcal{F}$. □

Definition. $\mathcal{F}$ is closed under restricted summation, if given a two place function $f \in \mathcal{F}$ such that $|\{\, y \mid f(x,y) \neq 0 \,\}| = 1$, then also $\lambda x.\Sigma_y f(x,y) \in \mathcal{F}$.

(4.8) Proposition. $\mathrm{Max}\,\mathcal{F}$, $\mathrm{Mid}\,\mathcal{F}$ *are closed under restricted summation.*

Proof. Given f, where $f(x,y)$ for $|y| = q(|x|)$ is defined as the maximum/median of $\{\, g(x,y,z) \mid |z| = p(|xy|) \,\}$, simply define $f'(x)$ as the maximum/median of $\{\, g(x,y,z) \mid |y| = q(|x|), |z| = p(|xy|) \,\}$. That f' has the desired property is a consequence of the observation that if the maximum or median of a set M is 0, then we must have $M = \emptyset$ or $M = \{0\}$. □

To get relationships for a function class $\mathcal{F}$ as consequences of Theorem 4.2 we have to show that $\mathcal{F}$ has a characterization in terms of the $\mathrm{M\overline{ax}}$-operator. This is made precise in the next theorem.

(4.9) Theorem. *If* $\mathrm{FP} \subseteq \mathcal{F}$, $\mathrm{M\overline{ax}\,M}\mathcal{K} \subseteq \mathrm{FP}^{\mathcal{F}}[1] \cap \mathcal{F} - \mathcal{F}$ *and* $\mathcal{F} \subseteq \mathrm{FP}^{\mathrm{M\overline{ax}\,M}\mathcal{K}}$ *for some class of functions* $\mathcal{F}$*, then:*

1. $\mathcal{F} - \mathcal{F} = \mathrm{FP}^{\mathcal{F}} = \Gamma_{\mathrm{SUB}}\mathcal{F} = \Gamma_{-}\mathcal{F}$.
2. $\mathcal{F}$ *is closed under substitution if and only if* $\mathcal{F}$ *is closed under substraction if and only if* $\mathcal{F} = \mathrm{FP}^{\mathcal{F}}$.
3. *If* $\mathcal{F}$ *is closed under subtraction then* $\mathrm{P}^{\mathcal{F}} = \mathrm{P}^{\mathcal{K}} = \mathrm{U}\mathcal{F}$.
4. *If* $\mathcal{F}$ *is closed under restricted multiplication and restricted summation, then:* $\mathcal{F}$ *is closed under subtraction if and only if* $\mathcal{F}$ *is closed under substitution if and only if* $\mathrm{P}^{\mathcal{K}} = \mathrm{P}^{\mathcal{F}} = \mathrm{U}\mathcal{F}$.

Proof.

1. What remains to show is $\mathrm{FP}^{\mathcal{F}} \subseteq \mathcal{F} - \mathcal{F}$. But if $\mathcal{F} \in \mathrm{M\overline{ax}\,M}\mathcal{K} - \mathrm{M\overline{ax}\,M}\mathcal{K} = \mathrm{FP}^{\mathcal{K}} = \mathrm{FP}^{\mathrm{M\overline{ax}\,M}\mathcal{K}}[1]$, then $\mathrm{FP}^{\mathcal{F}} \subseteq \mathrm{FP}^{\mathcal{K}} \subseteq \mathrm{FP}^{\mathrm{M\overline{ax}\,M}\mathcal{K}}[1] \subseteq \mathrm{M\overline{ax}\,M}\mathcal{K} - \mathrm{M\overline{ax}\,M}\mathcal{K} \subseteq \mathcal{F} - \mathcal{F}$.
2. Immediate consequence of statement (1).
3. Obvious, since under the assumptions of the proposition $\mathrm{U}(\mathcal{F} - \mathcal{F}) = \mathrm{U}\mathcal{F}$ and $\mathrm{U}(\mathrm{FP}^{\mathcal{F}}) = \mathrm{P}^{\mathcal{F}}$ and $\mathrm{U}(\mathrm{FP}^{\mathcal{F}}) = \mathrm{P}^{\mathcal{F}}$.
4. For any $f, g \in \mathrm{FP}^{\mathcal{K}}$ we have $\mathrm{graph}_{f-g} \in \mathrm{P}^{\mathcal{K}}$. Therefore by the assumption there is a function $h \in \mathcal{F}$ such that
$$h(x,y) = 1 \iff f(x) - g(x) = y.$$
Now define $h'(x,y) = h(x,y) \cdot y$. If $\mathcal{F}$ is closed under restricted multiplication, then $h' \in \mathcal{F}$. But now
$$\Sigma_y h'(x,y) = f(x) - g(x),$$
and since we know that there is only y for every x such that $h'(x,y) \neq 0$, this summation can be carried out in $\mathcal{F}$.

$\square$

We now give some results on concrete classes of optimization functions.

(4.10) Lemma.

1. $\mathrm{M\overline{ax}\,NP} \subseteq \mathrm{FP}^{\mathrm{Max\,FP}}[1] \cap \mathrm{Max\,FP} - \mathrm{Max\,FP}$ *and* $\mathrm{Max\,FP} \subseteq \mathrm{FP}^{\mathrm{M\overline{ax}\,NP}}$.
2. $\mathrm{M\overline{ax}\,MCP} = \mathrm{Med\,FP}$.
3. $\mathrm{M\overline{ax}\,MCP} \subseteq \mathrm{FP}^{\mathrm{Mid\,FP}}[1] \cap \mathrm{Mid\,FP} - \mathrm{Mid\,FP}$ *and* $\mathrm{Mid\,FP} \subseteq \mathrm{FP}^{\mathrm{M\overline{ax}\,MCP}}$.
4. $\mathrm{M\overline{ax}\,\exists CP} \subseteq \mathrm{FP}^{\mathrm{Mof\,FP}}[1] \cap \mathrm{Mof\,FP} - \mathrm{Mof\,FP}$ *and* $\mathrm{Mof\,FP} \subseteq \mathrm{FP}^{\mathrm{M\overline{ax}\,\exists CP}}$.
5. $\mathrm{M\overline{ax}\,\exists CP} \subseteq \mathrm{FP}^{\mathrm{Max\,\#P}}[1] \cap \mathrm{Max\,\#P} - \mathrm{Max\,\#P}$ *and* $\mathrm{Max\,\#P} \subseteq \mathrm{FP}^{\mathrm{M\overline{ax}\,\exists CP}}$.

PROOF. We briefly sketch in some points, that the function classes fulfill the inclusionships.

$\mathrm{Max\,FP} \subseteq \mathrm{M\overline{ax}\,P} - \mathrm{M\overline{ax}\,P}$: Let $g \in \mathrm{FP}$, $f(x) = \max\{\, g(x,y) \mid |y| = p(|x|) \,\}$ for some polynomial p. Then using a reasonable pairing function, it is clear that

$$f(x) = \max\{\, \langle z, y, z\rangle \mid g(x,y) = z \,\} - \max\{\, \langle z, y, 0\rangle \mid g(x,y) = z \,\}$$

and therefore $f \in \mathrm{M\overline{ax}\,P} - \mathrm{M\overline{ax}\,P}$.

$\mathrm{M\overline{ax}\,NP} \subseteq \mathrm{Max\,FP} - \mathrm{Max\,FP} \cap \mathrm{FP}^{\mathrm{Max\,FP}}[1]$: similar to the above. Informally speaking, we now code the value we are interested in in the last bits of the output-strings of an NP-machine.

$\mathrm{Med\,FP} \subseteq \mathrm{M\overline{ax}\,MCP}$: z is the median of a multiset MS if and only if the number of elements in MS less than z is less than the number of elements in MS greater or equal to z. This can be directly expressed as a $\mathrm{M\overline{ax}\,MCP}$ condition.

$\mathrm{M\overline{ax}\,MCP} \subseteq \mathrm{Med\,FP}$: Let $f \in \mathrm{M\overline{ax}\,MCP}$, $f(x) = \max\{\, y \mid y < e(|x|) \wedge (x,y) \in A \,\}$, where $e \in 2^{\mathrm{Pol}}$, $A \in \mathrm{MCP}$, witnessed by the nondeterministic machine M.

Now we compute the maximal y bit for bit, using some kind of binary search:

First look at the computation tree of M with input $(x, y_0$, where $y_0 = \frac{e(|x|)}{2}$. Because of the monotonicity of A, the median of the leaves of M in this tree will indicate the first bit of y. Depending on wether it is 0 or 1 look next at M on input (x, y_1), where $y_1 = \frac{e(|x|)}{4}$ or $y_1 = \frac{3e(|x|)}{4}$. Now the median of the leaves determines the second bit of y. Continuing this procedure, build a polynomial time machine, such that on the middle path the maximal y is printed.

The equation from the theorem is now obtained as in the case of Max FP.

The case of Mid FP proceeds similar to the Med-case. When we formulate a Mid condition we need in comparison to Med one extra existential quantifier. So we get $\mathrm{Mid\,FP} \subseteq \mathrm{M\overline{ax}\,MC\exists P}$, but we know that $\mathrm{FP}^{\mathrm{M\overline{ax}\,MC\exists P}} = \mathrm{FP}^{\mathrm{C\exists P}} = \mathrm{FP}^{\mathrm{CP}} = \mathrm{FP}^{\mathrm{M\overline{ax}\,MCP}}$. On the other hand we show $\mathrm{M\overline{ax}\,MCP} \subseteq \mathrm{FP}^{\mathrm{Mid\,FP}}[1] \cap \mathrm{Mid\,FP} - \mathrm{Mid\,FP}$ exactly as in the case of Med.

$\mathrm{Mof\,FP} \subseteq \mathrm{M\overline{ax}\,PP} - \mathrm{M\overline{ax}\,PP}$: Let $g \in \mathrm{FP}$, $f(x)$ is the smallest of the most often appearing values in $G_x = \{\, g(x,y) \mid |y| = p(|x|) \,\}$ for some polynomial p. Let $\overline{z}$ for some z, $|z| = \ell$, denote the binary representation of $10^\ell - z$.

Now define

$$\begin{array}{rcl} A & = & \{\, (x, \langle k, \overline{z}\rangle) \mid z \text{ appears } \geq k \text{ times in } G_x \,\} \in \mathrm{PP} \\ A_1 & = & \{\, (x, \langle k, \overline{z}, z\rangle) \mid (x, \langle k, z\rangle) \in A \,\} \in \mathrm{PP} \\ A_2 & = & \{\, (x, \langle k, \overline{z}, 0\rangle) \mid (x, \langle k, z\rangle) \in A \,\} \in \mathrm{PP} \end{array}$$

Then it is clear that $f \in \mathrm{M\overline{ax}\,PP} - \mathrm{M\overline{ax}\,PP}$.

$\mathrm{M\overline{ax}\,\exists CP} \subseteq \mathrm{Mof\,FP} - \mathrm{Mof\,FP} \cap \mathrm{FP}^{\mathrm{Mof\,FP}}[1]$: Again we use a kind of modified binary search as in the proof of Med. On the middle path determine the first bit of the maximum and print all strings starting with that bit as often as necessary such that all other paths cannot destroy this choice. Then on the paths at one quarter and three quarters determine the second bit and again print all strings with known first and second bit, and so on.

For the case of $\mathrm{Max\,\#P}$, we can show using simple arguments as above that $\mathrm{Mof\,FP} \subseteq \mathrm{Max\,\#P} - \mathrm{Max\,\#P} \cap \mathrm{FP}^{\mathrm{Max\,\#P}}[1]$ and $\mathrm{Max\,\#P} \subseteq \mathrm{M\overline{ax}\,\exists CP} - \mathrm{M\overline{ax}\,\exists CP}$. □

(4.11) Corollary.

$$\begin{array}{llllll} \mathrm{Max\,FP} & - \mathrm{Max\,FP} & = \mathrm{FP}^{\mathrm{Max\,FP}}[1] & = \mathrm{FP}^{\mathrm{Max\,FP}} & = \mathrm{FP}^{\mathrm{NP}} \\ \mathrm{Min\,FP} & - \mathrm{Min\,FP} & = \mathrm{FP}^{\mathrm{Min\,FP}}[1] & = \mathrm{FP}^{\mathrm{Min\,FP}} & = \mathrm{FP}^{\mathrm{NP}} \\ \mathrm{Med\,FP} & - \mathrm{Med\,FP} & = \mathrm{FP}^{\mathrm{Med\,FP}}[1] & = \mathrm{FP}^{\mathrm{Med\,FP}} & = \mathrm{FP}^{\mathrm{PP}} \\ \mathrm{Mid\,FP} & - \mathrm{Mid\,FP} & = \mathrm{FP}^{\mathrm{Mid\,FP}}[1] & = \mathrm{FP}^{\mathrm{Mid\,FP}} & = \mathrm{FP}^{\mathrm{PP}} \\ \mathrm{Mof\,FP} & - \mathrm{Mof\,FP} & = \mathrm{FP}^{\mathrm{Mof\,FP}}[1] & = \mathrm{FP}^{\mathrm{Mof\,FP}} & = \mathrm{FP}^{\exists\mathrm{CP}} \\ \mathrm{Max\,\#P} & - \mathrm{Max\,\#P} & = \mathrm{FP}^{\mathrm{Max\,\#P}}[1] & = \mathrm{FP}^{\mathrm{Max\,\#P}} & = \mathrm{FP}^{\exists\mathrm{CP}} \end{array}$$

The 4. statement of this corollary was proved independently by Diana Emme and Krzysztof Loryś.

PROOF. All proofs have the same structure. We give the proof for the first statement.

Use Lemma 4.10, Theorem 4.9, and Theorem 4.3 as follows: $\mathrm{Max\,FP} \subseteq \mathrm{FP}^{\mathrm{NP}} = \mathrm{FP}^{\overline{\mathrm{Max}}\,\mathrm{NP}}[1]$. Therefore $\mathrm{FP}^{\mathrm{Max\,FP}} \subseteq \mathrm{FP}^{\overline{\mathrm{Max}}\,\mathrm{NP}}[1] \subseteq \mathrm{FP}^{\mathrm{Max\,FP}}[1] \subseteq \mathrm{Max\,FP} - \mathrm{Max\,FP}$. □

(4.12) Corollary.

1. Max FP *is closed under subtraction if and only if* Max FP *is closed under substitution if and only if* NP = co-NP.
2. Mid FP *is closed under subtraction if and only if* Mid FP *is closed under substitution if and only if* $\mathrm{NP} = \mathrm{P}^{\mathrm{PP}}$.
3. *If* Med FP *is closed under subtraction, then* $\mathrm{PP} = \mathrm{P}^{\mathrm{PP}}$.
4. *If* Mof FP *is closed under subtraction, then* $\mathrm{PP} = \mathrm{P}^{\exists\mathrm{CP}}$ *and therefore* $\mathrm{PP} = \mathrm{PH} \cdot \mathrm{PP}$.
5. Max # P *is closed under subtraction if and only if* Max # P *is closed under substitution if and only if* $\mathrm{NP} = \mathrm{P}^{\exists\mathrm{CP}}$ *if and only if* $\mathrm{NP} = \mathrm{P}^{\mathrm{PP}}$.

PROOF. Theorem 4.9, Corollary 4.11, together with the obervations U Max FP = NP, U Mid FP = NP, U Med FP = PP and U Max # P = NP. □

Finally, observe that corollary 4.11 also yields several results about hardness of several function classes for certain complexity classes (for some of which very complicated proofs appeared in the literature), e. g. $\mathrm{FP}^{\mathrm{NP}} = \mathrm{FP}^{\mathrm{Max\,FP}}[1]$ [Kre88], $\mathrm{FP}^{\mathrm{PP}} = \mathrm{FP}^{\mathrm{Mid\,FP}}[1]$ [Tod90], but even for the in a sense weaker operator Med we get$\mathrm{FP}^{\mathrm{PP}} = \mathrm{FP}^{\mathrm{Med\,FP}}[1]$; and $\mathrm{FP}^{\exists\mathrm{CP}} = \mathrm{FP}^{\mathrm{Mof\,FP}}[1]$.

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