# Location & Information

Maarten Löffler
*Utrecht University*

*Based on joint work with*

Kevin Buchin
Ivor van der Hoog
Irina Kostitsyna
Marc van Kreveld
Joe Mitchell
Pat Morin
Wolfgang Mulzer
Jack Snoeyink
Bettina Speckmann

# Chapter 1
*PROXIMITY STRUCTURES*

Point sets and proximity

Computational questions

State of the art in 1D

State of the art in 2D

# Chapter 2
*PARTIAL INFORMATION*

Uncertain locations

Quantifying information

State of the art in 1D

State of the art in 2D

# Chapter 3
*FINE-GRAINED MEASURES*

Ambiguity of regions

Information theory

State of the art in 1D

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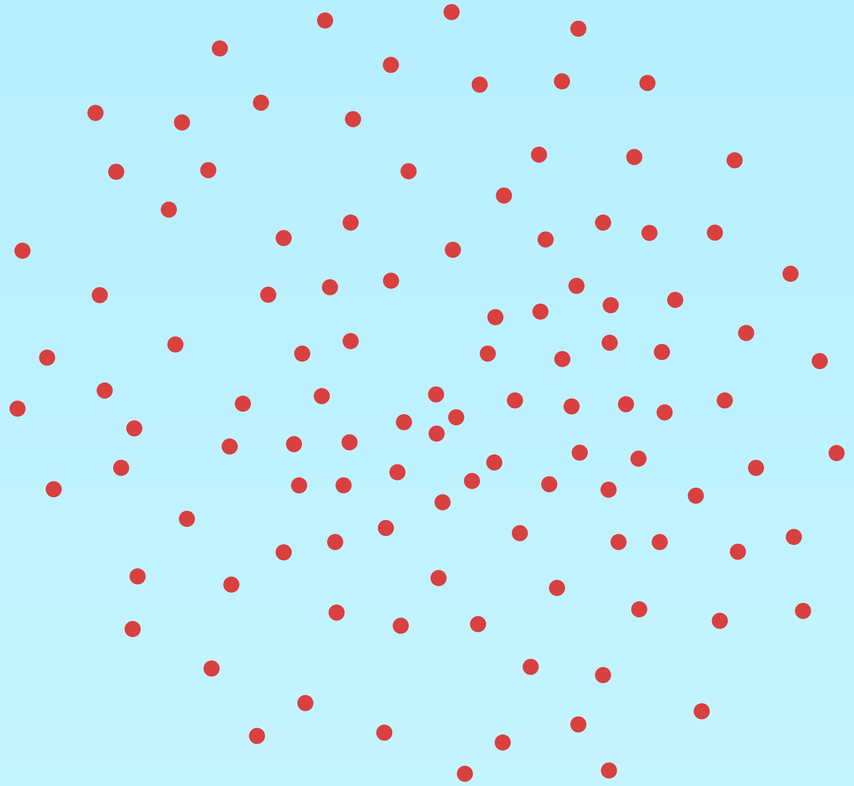

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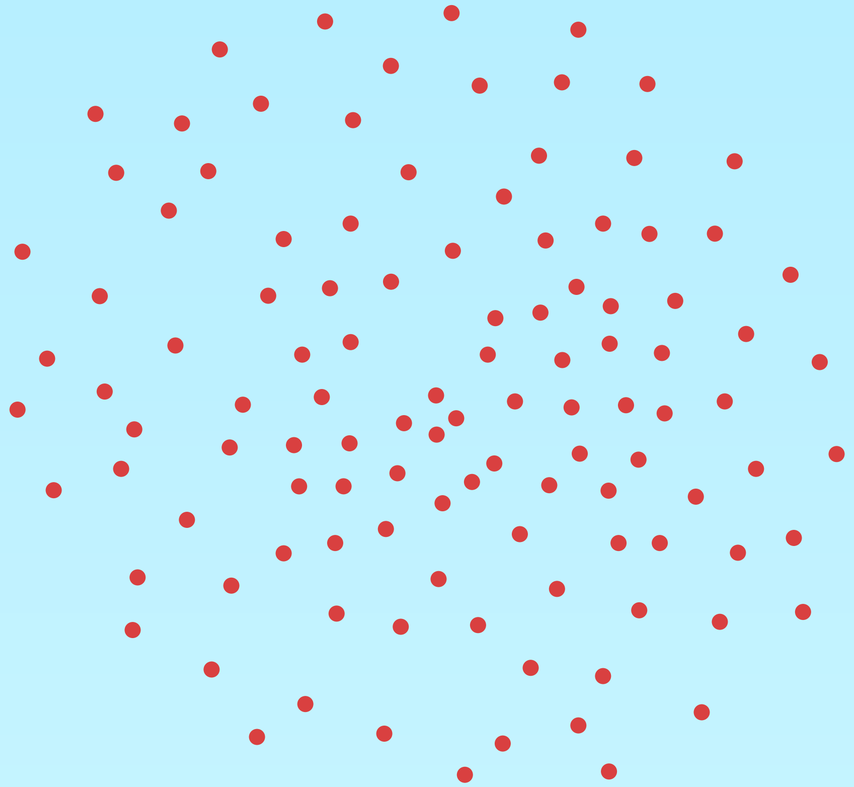

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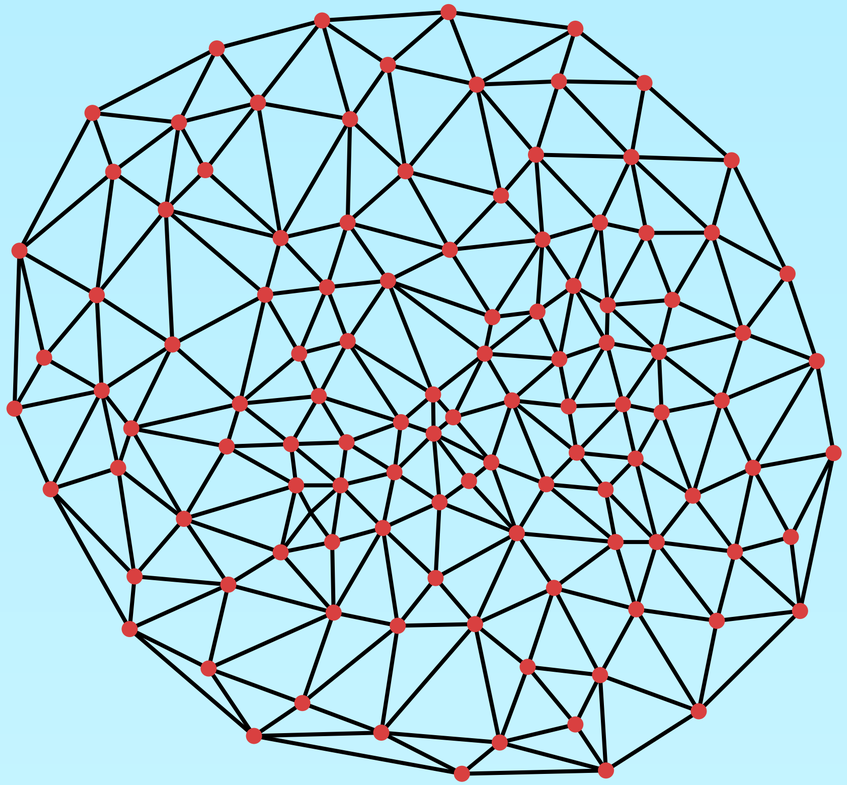

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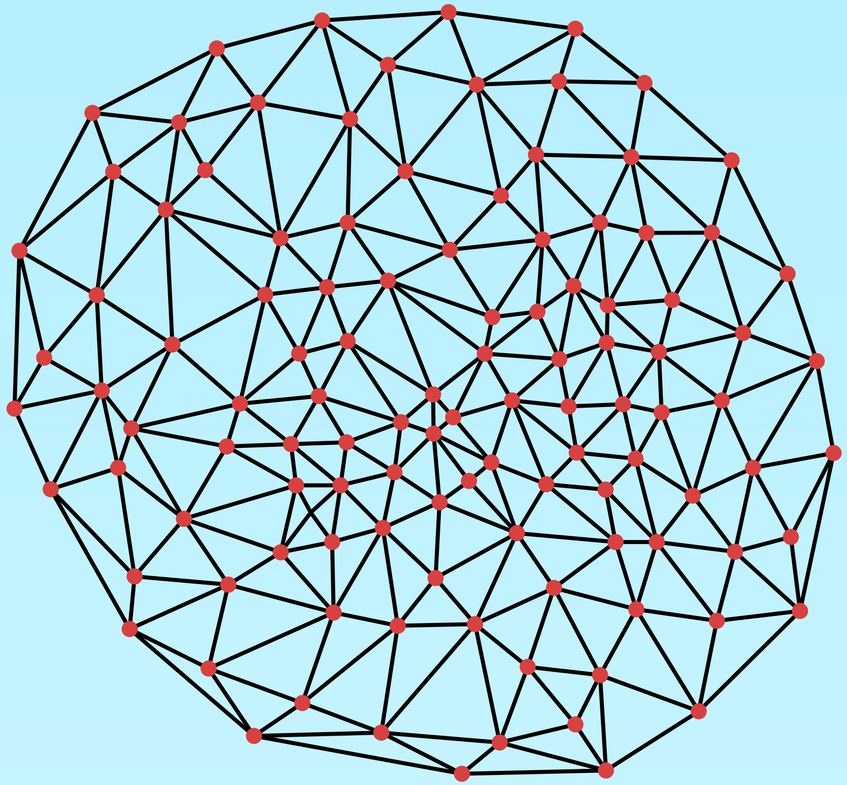

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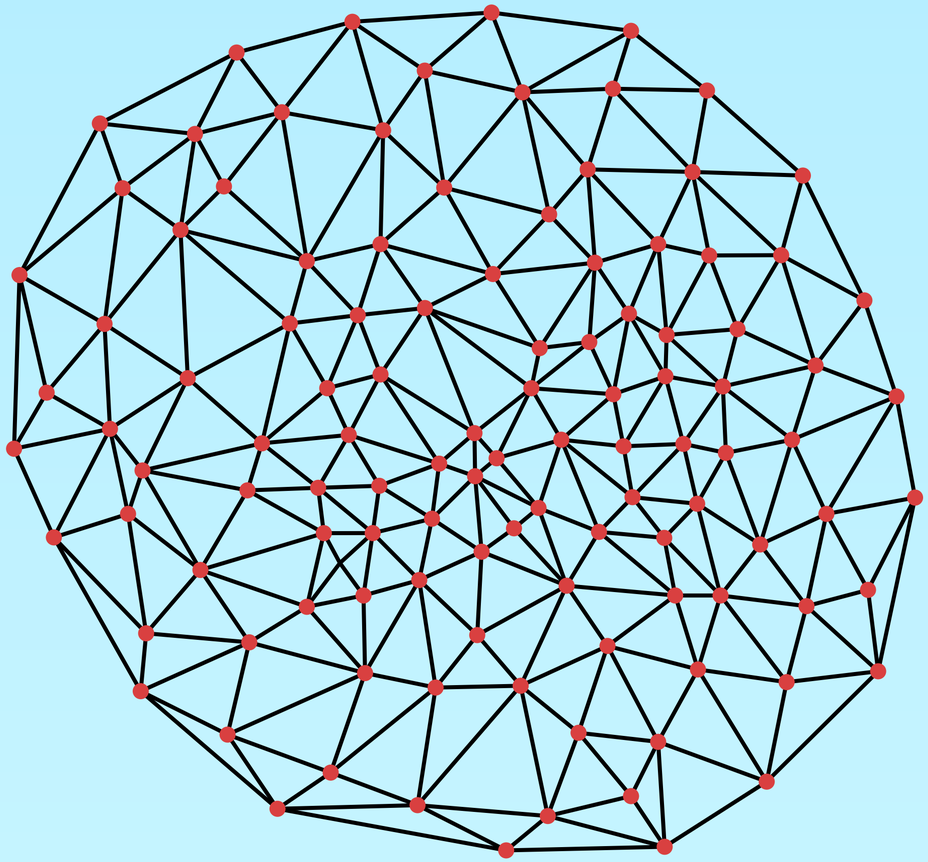

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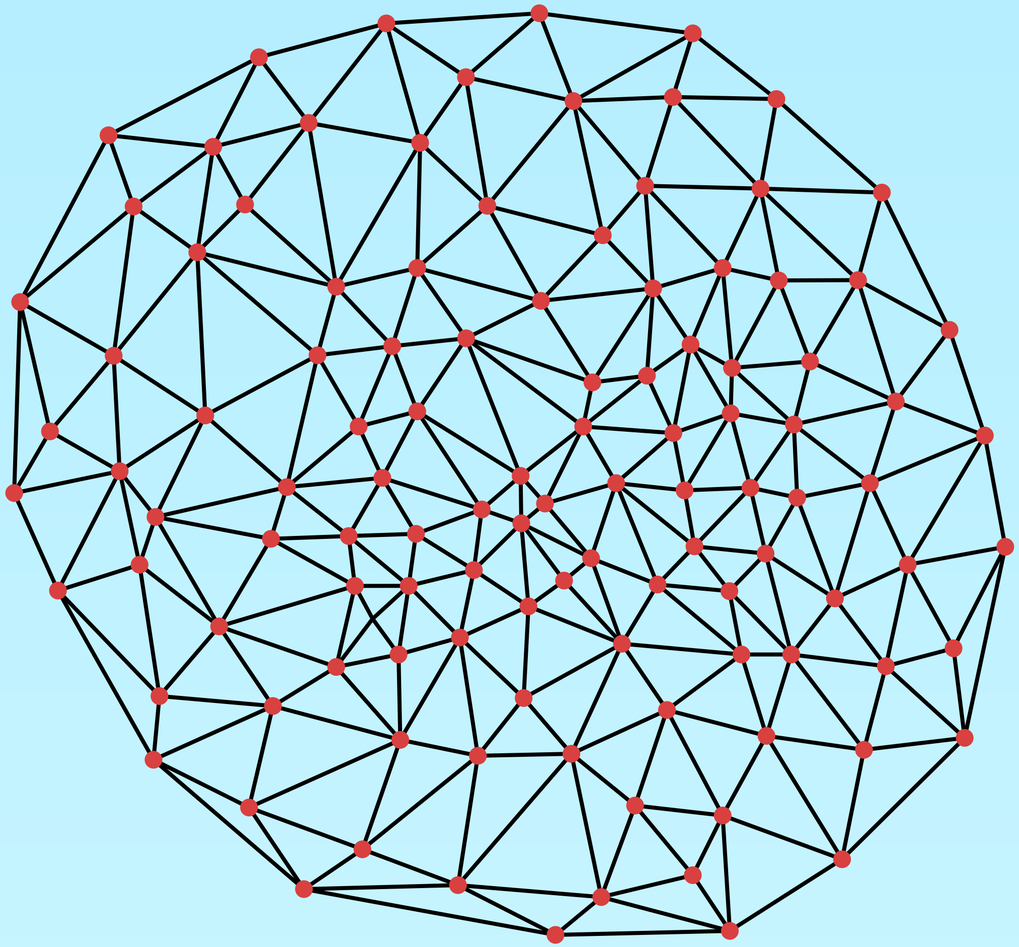

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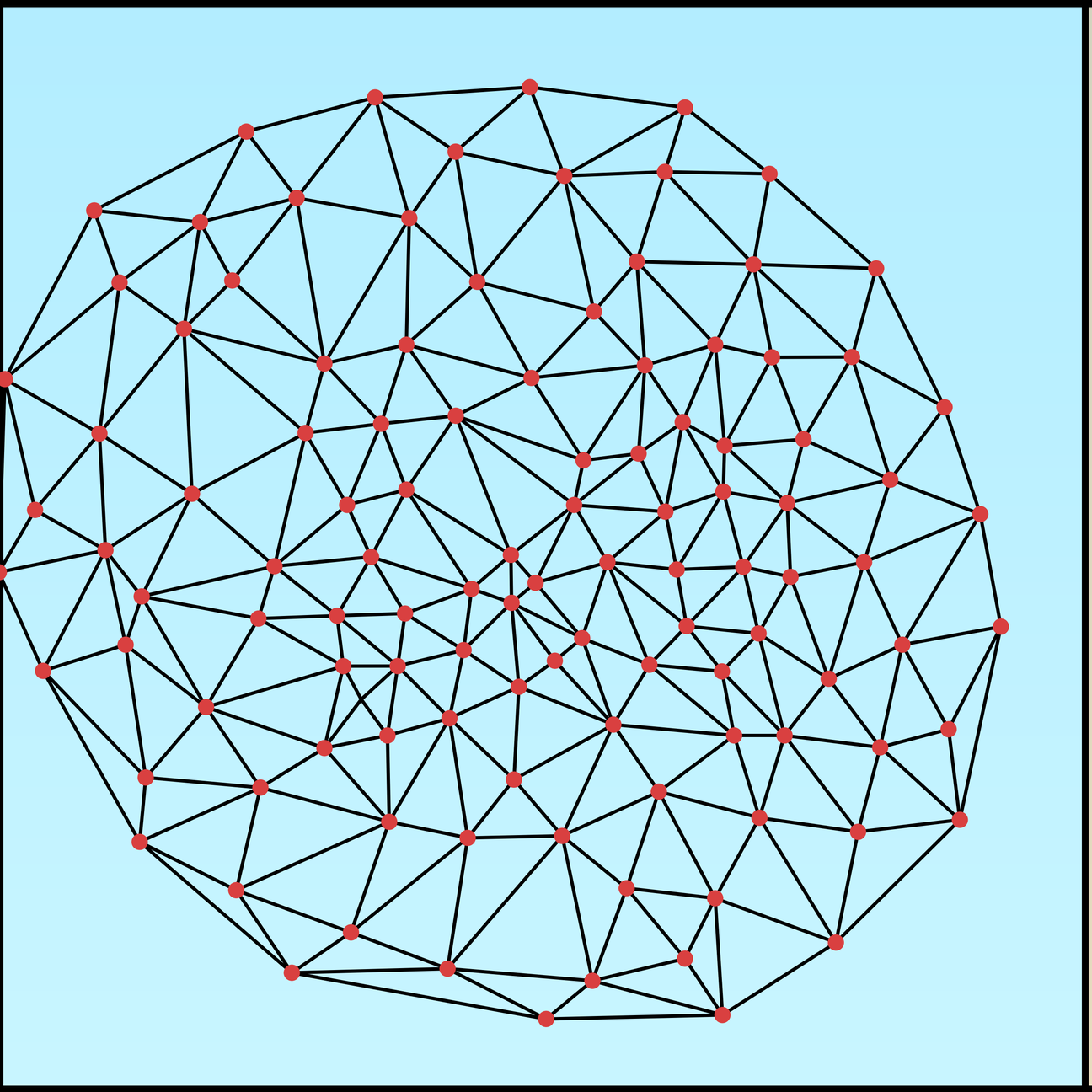

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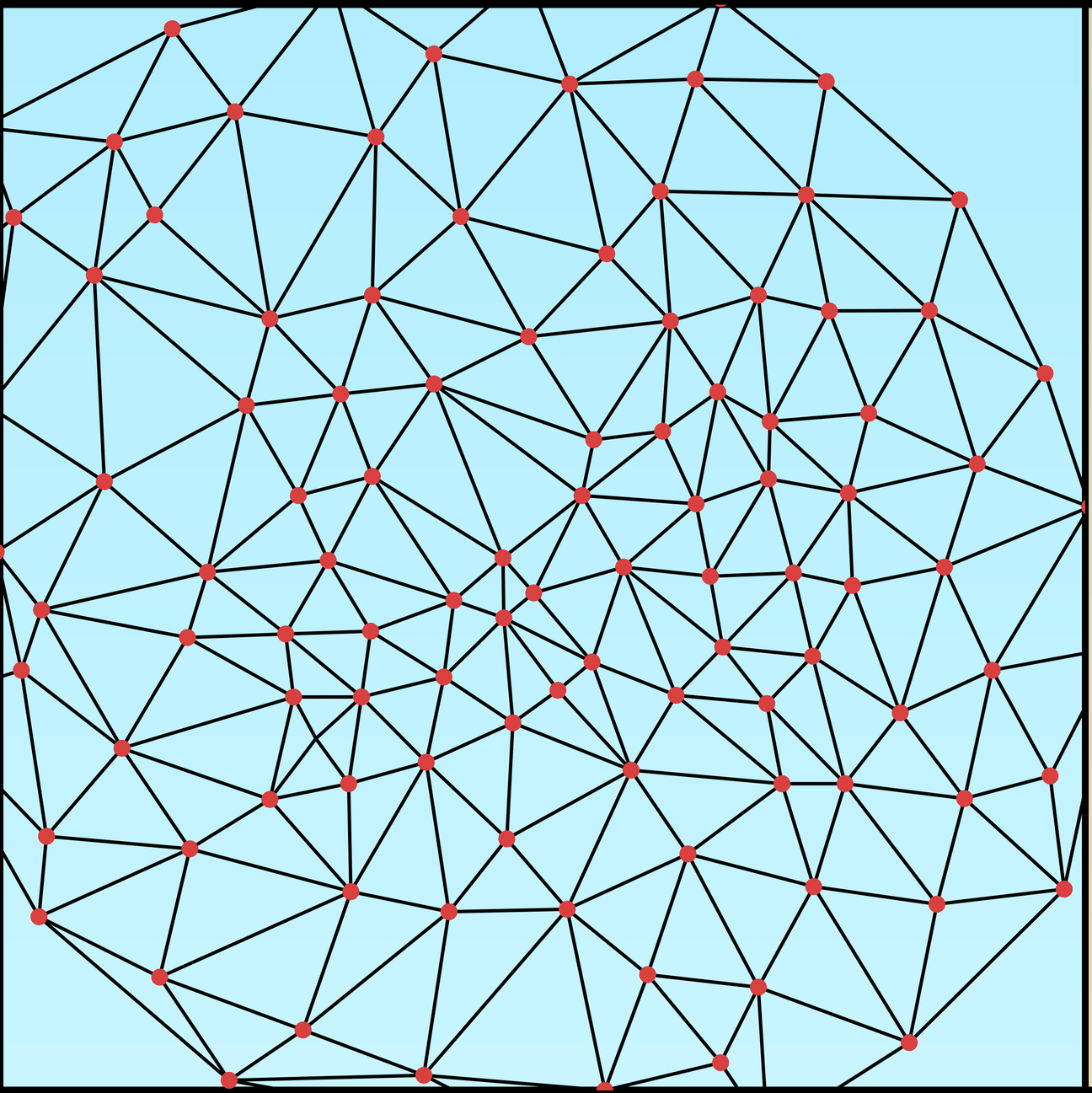

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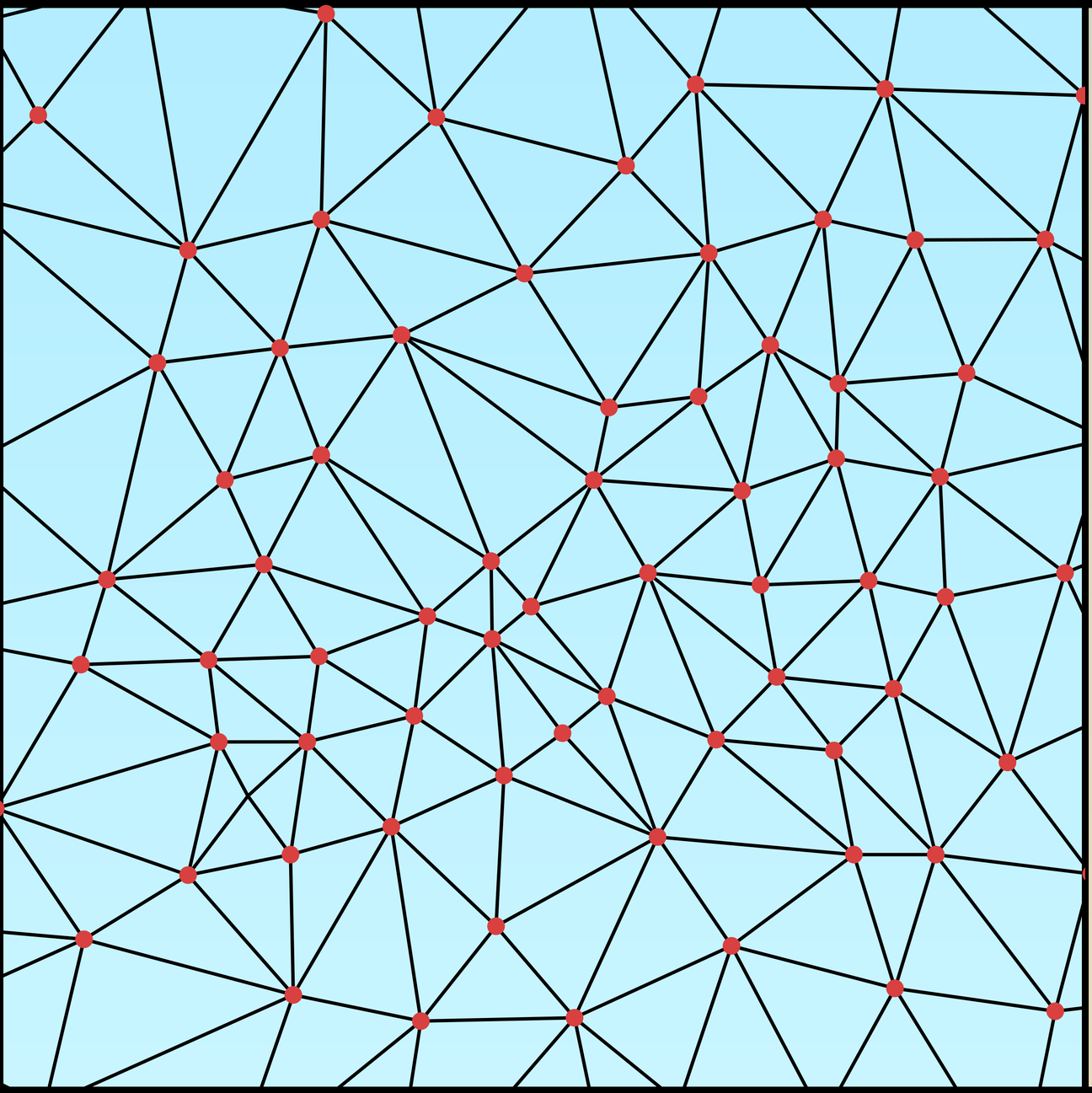

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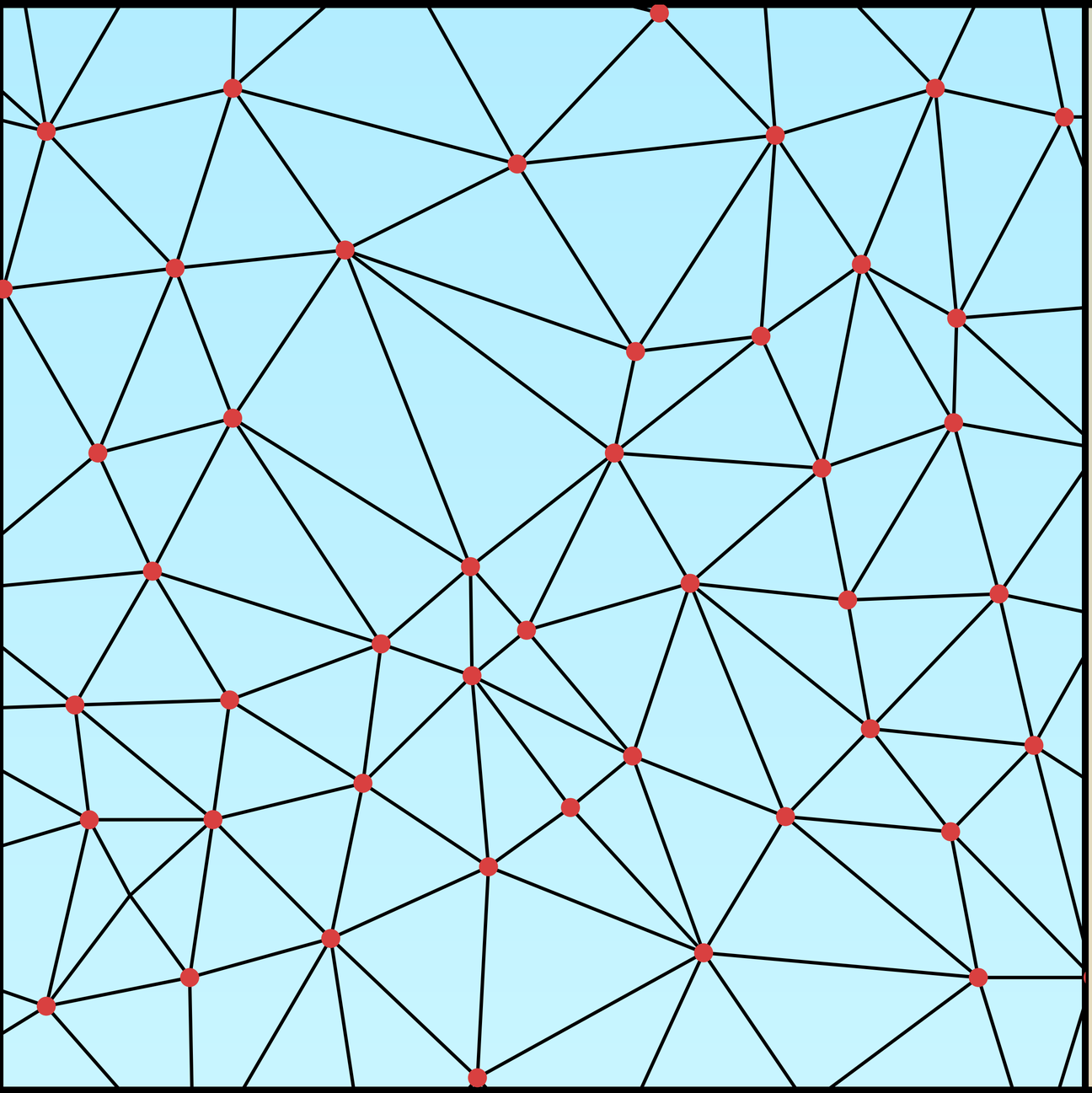

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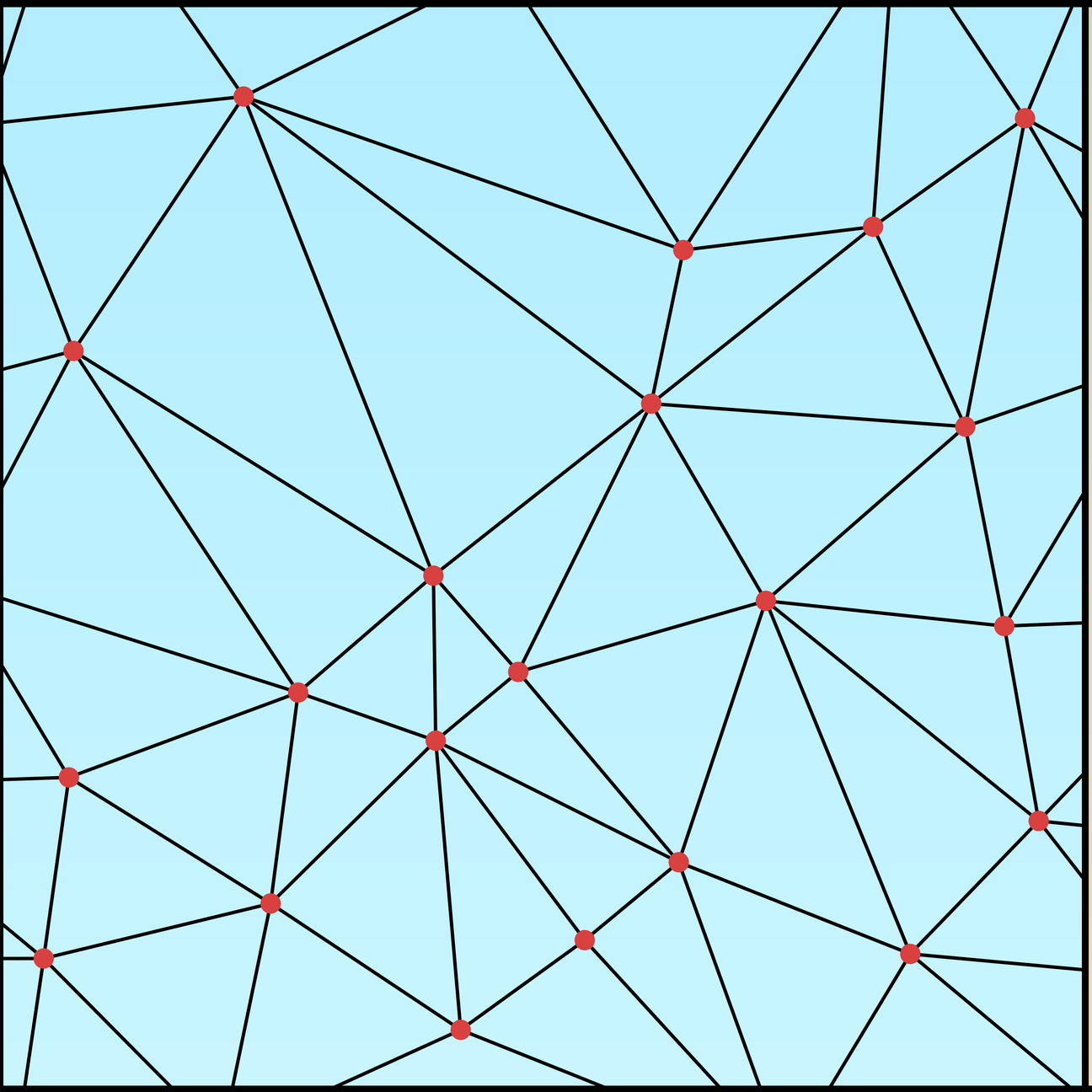

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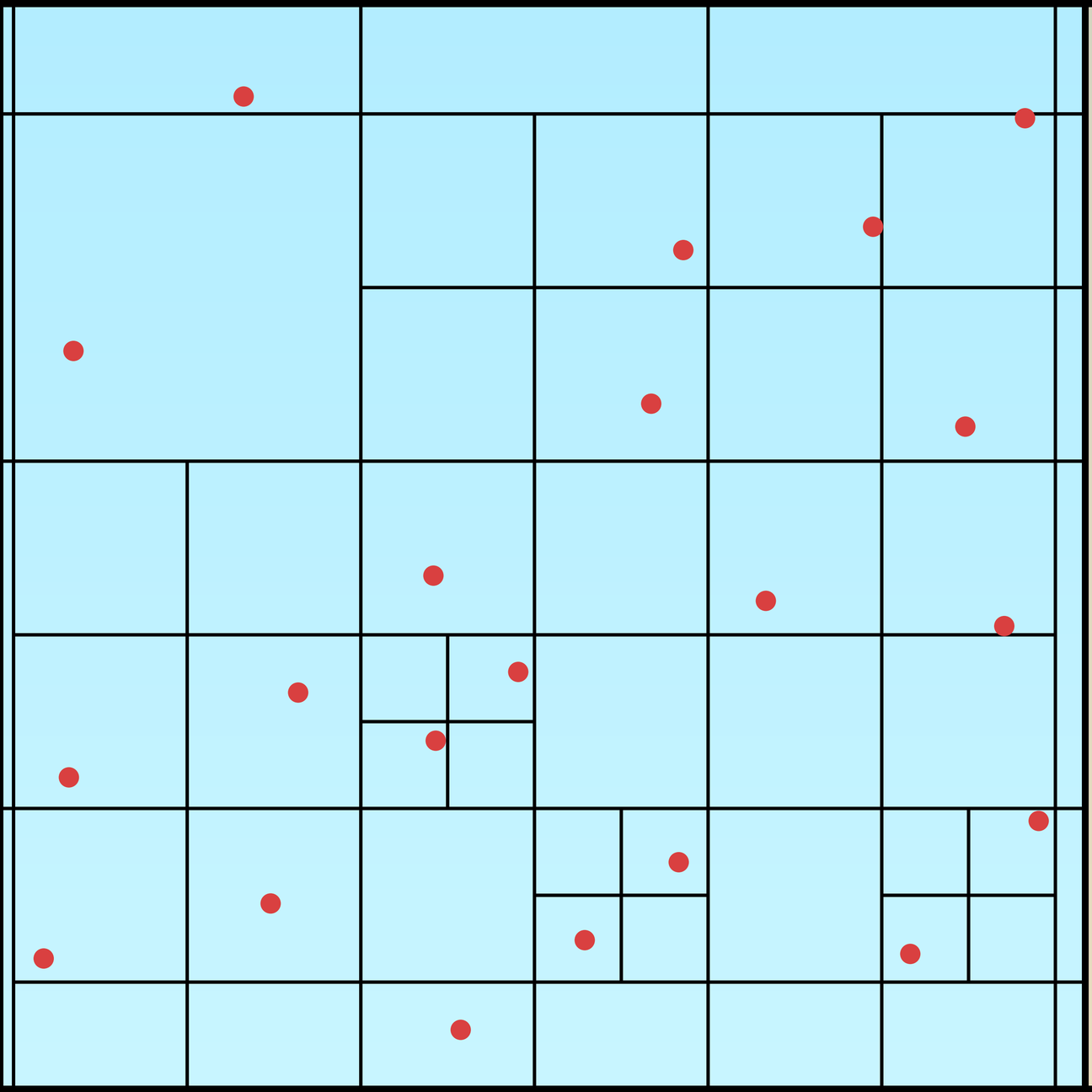

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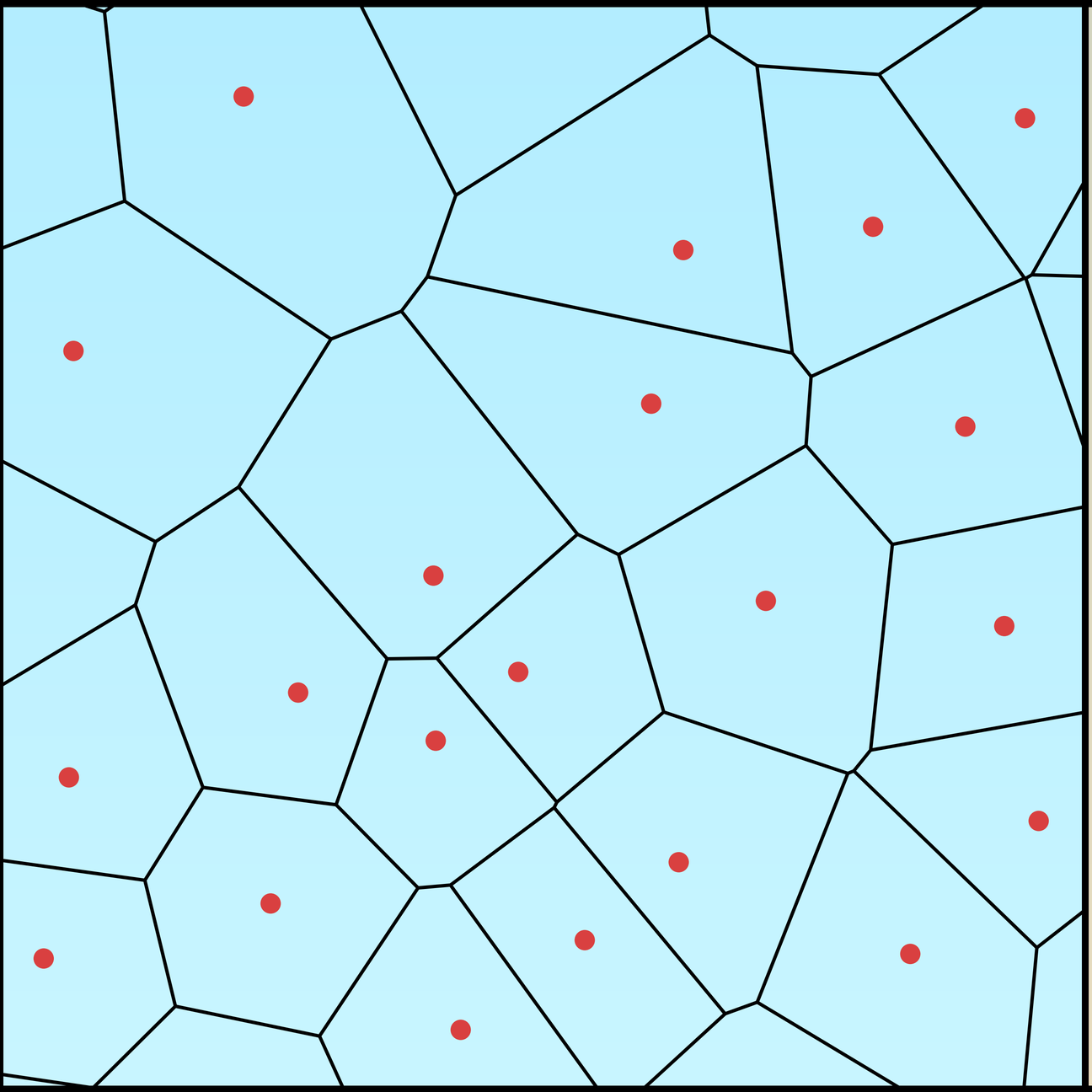

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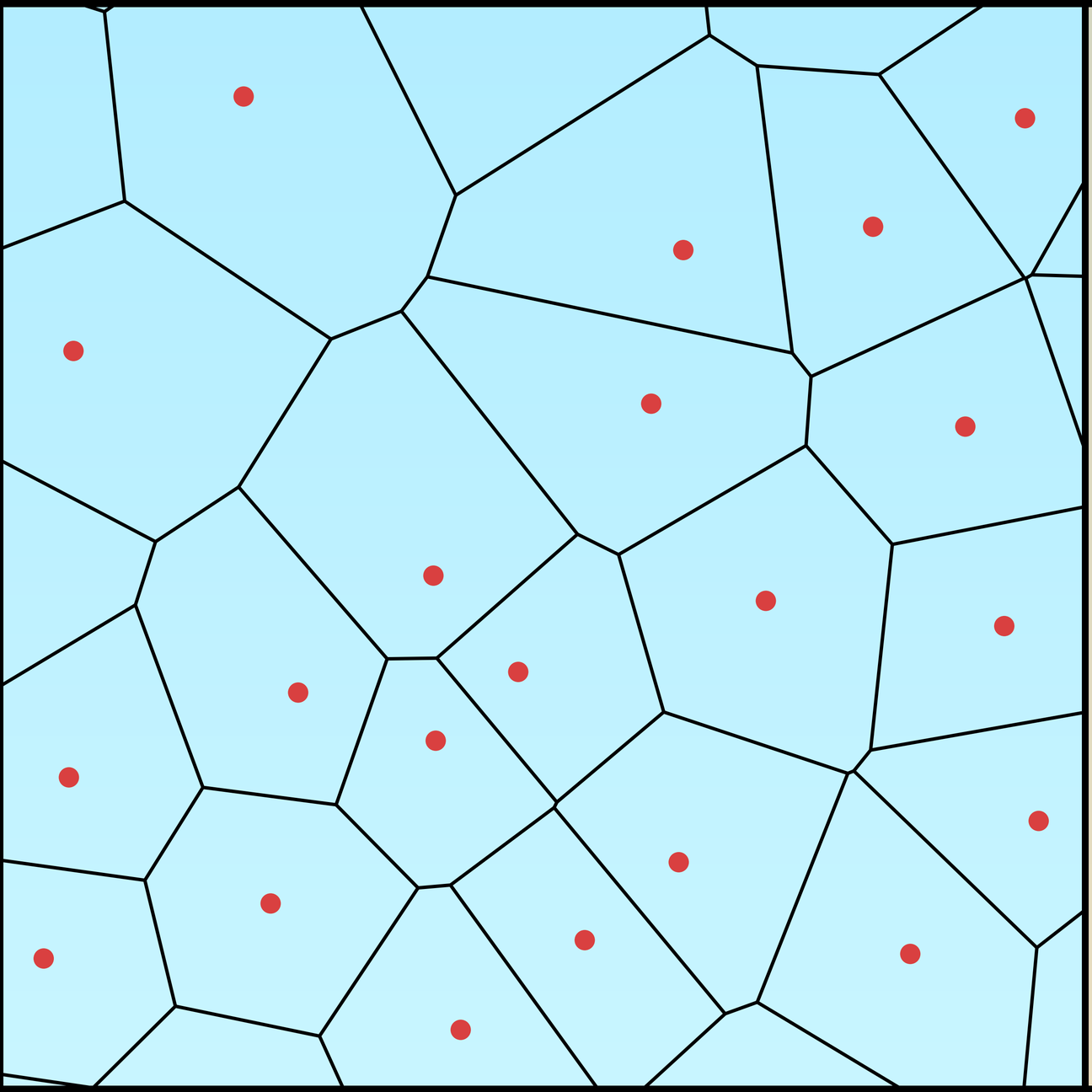

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Because defined locally, complexity should be $O(n)$.

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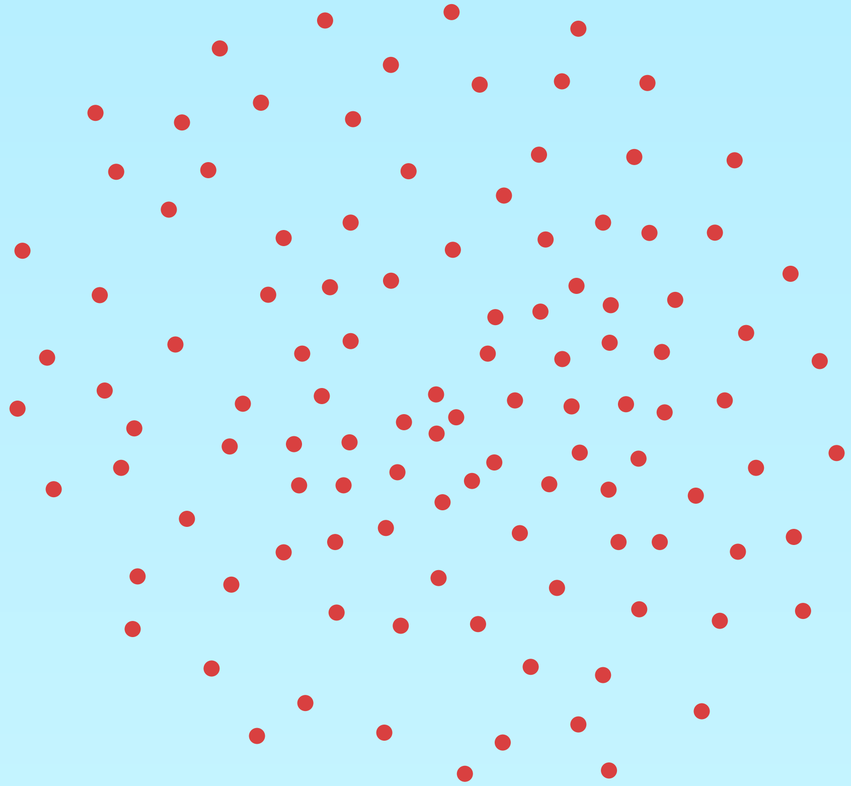

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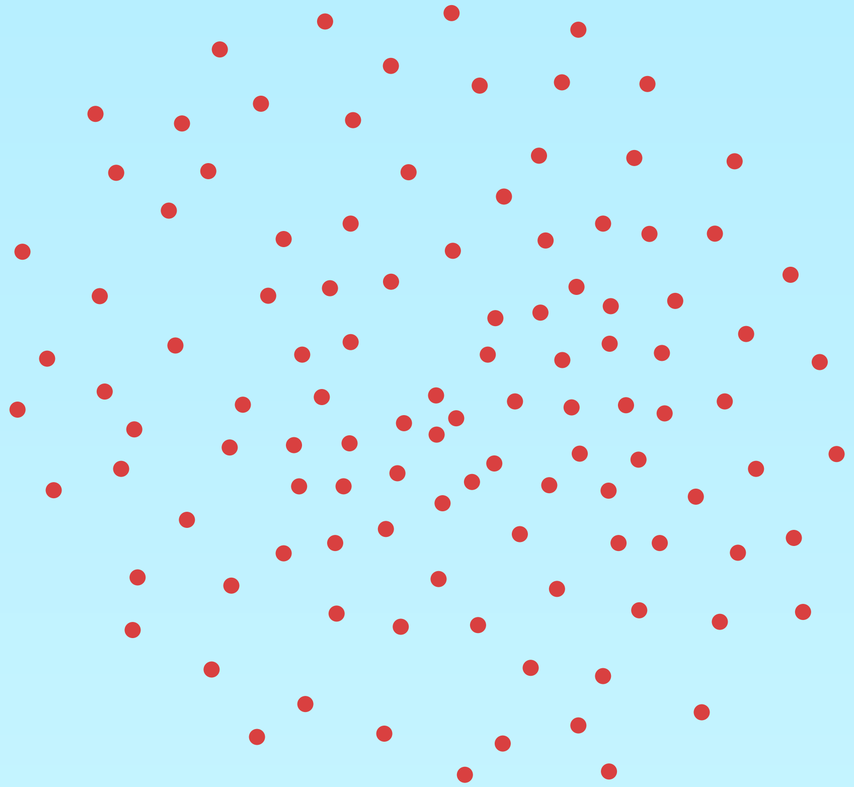

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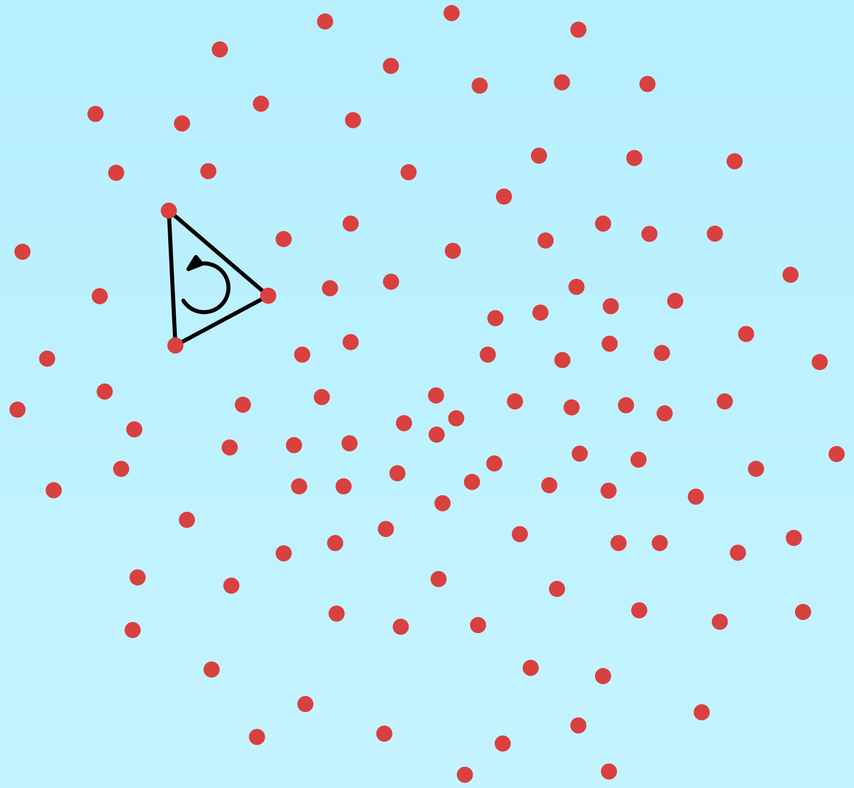

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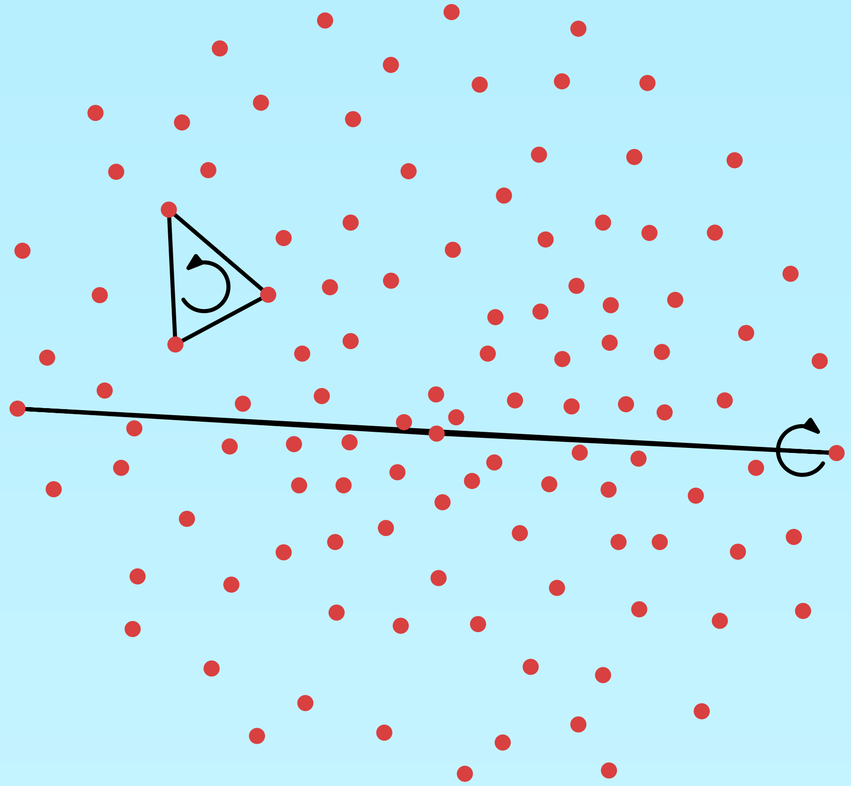

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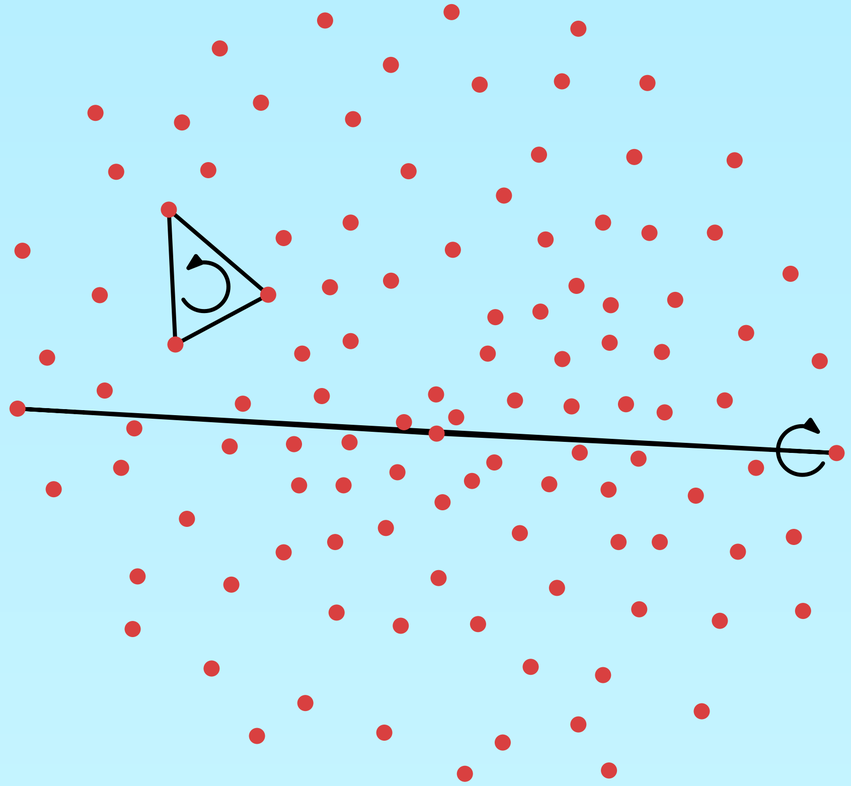

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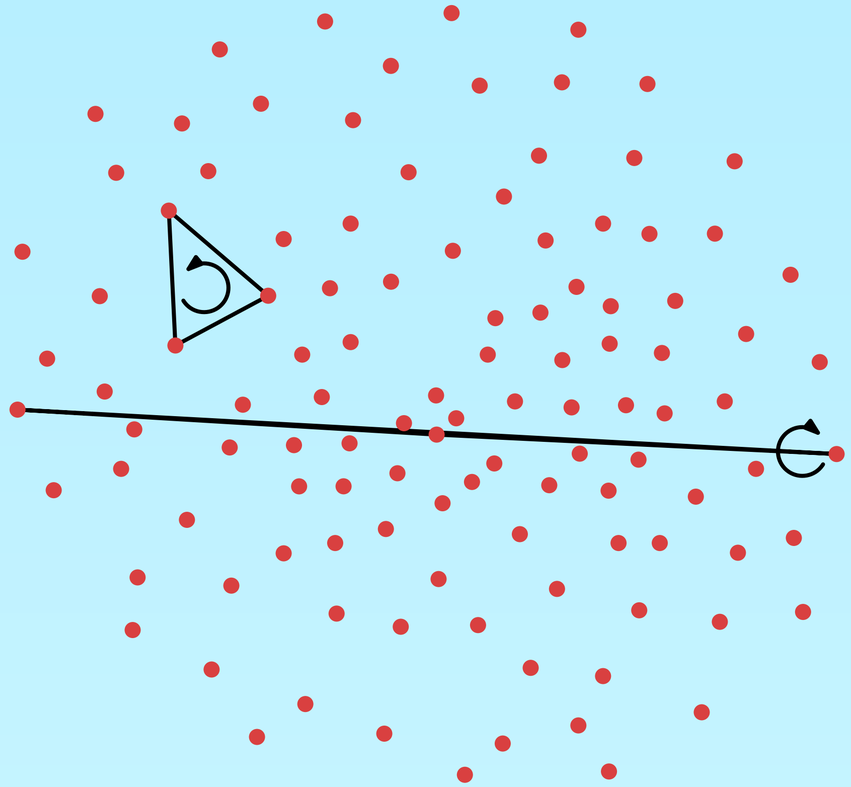

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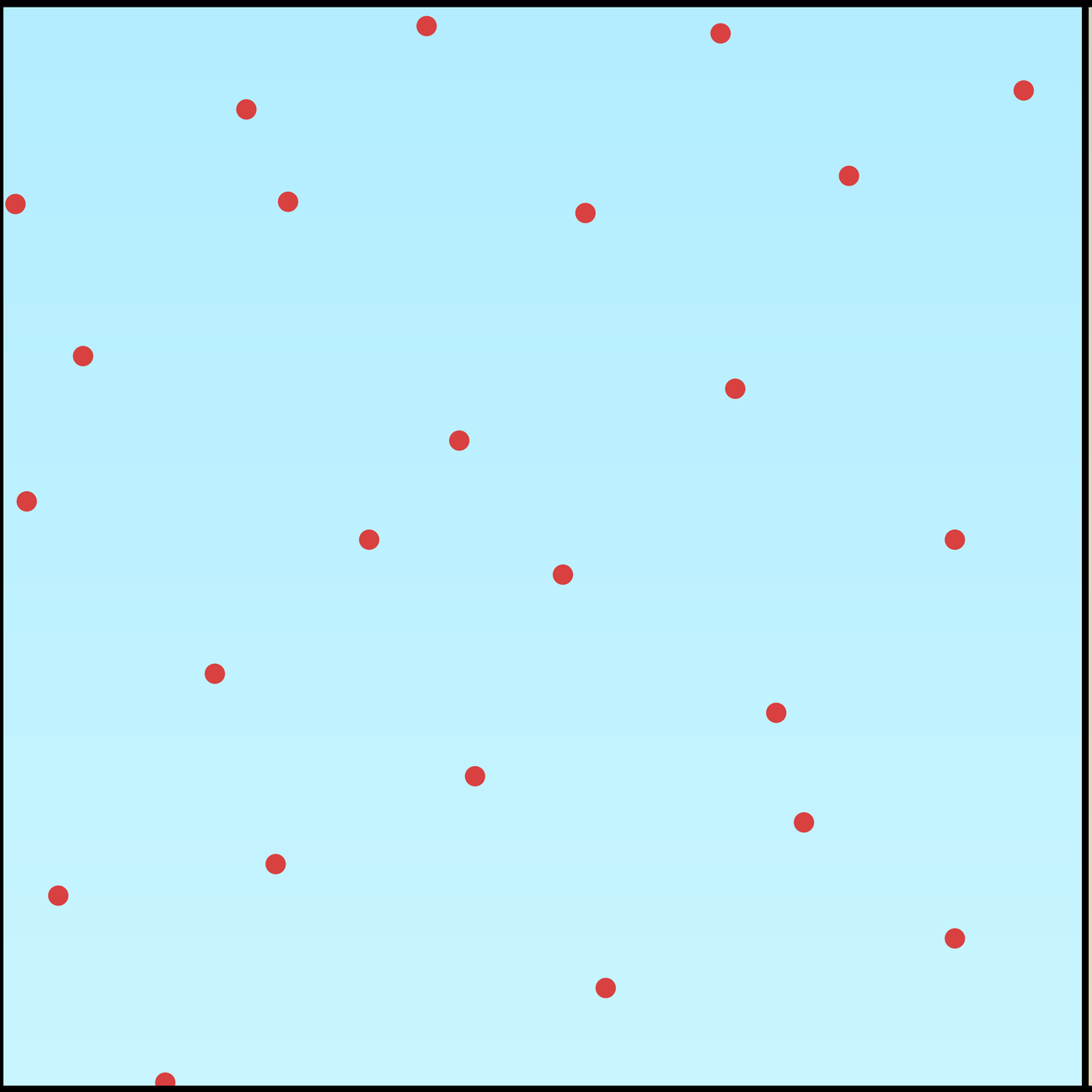

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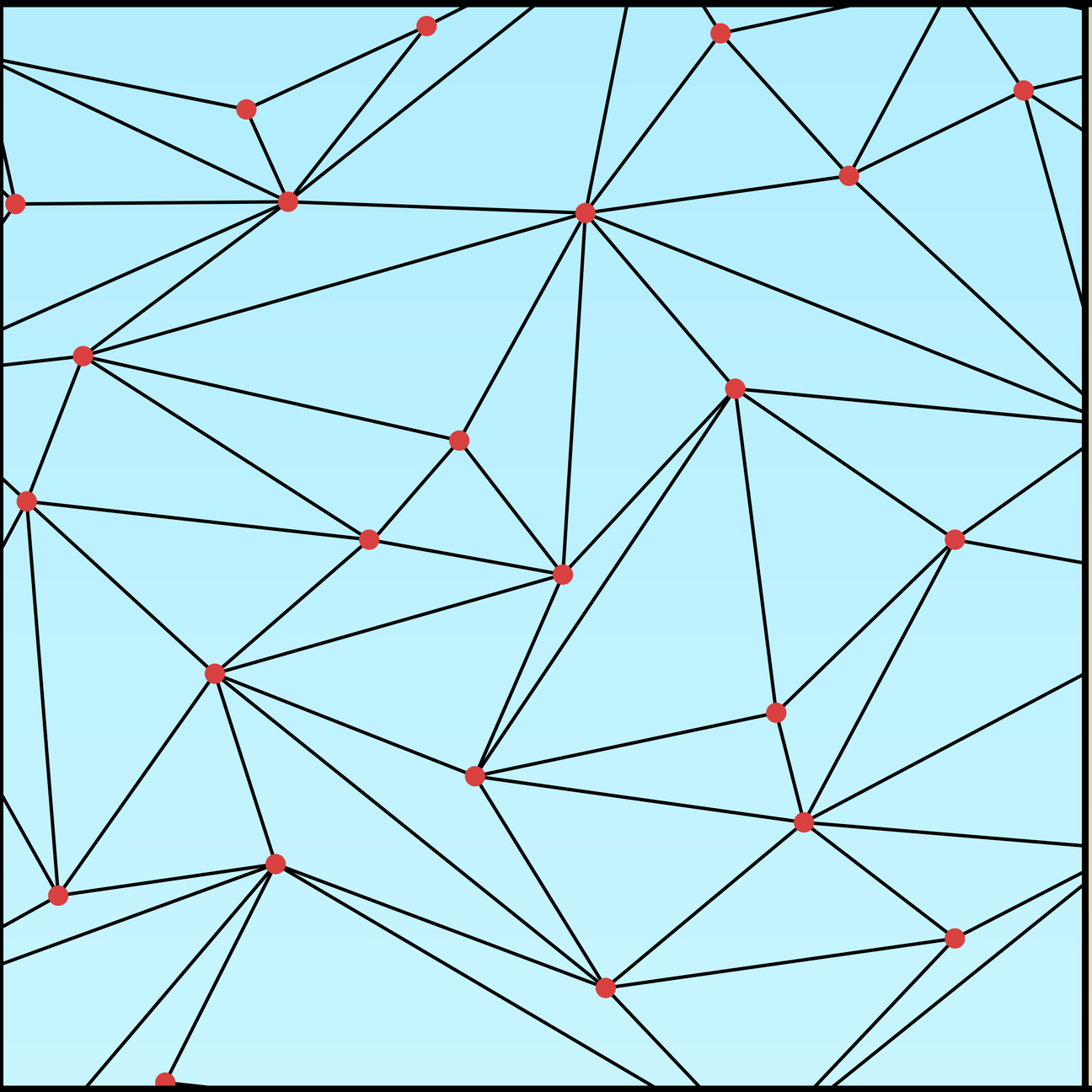

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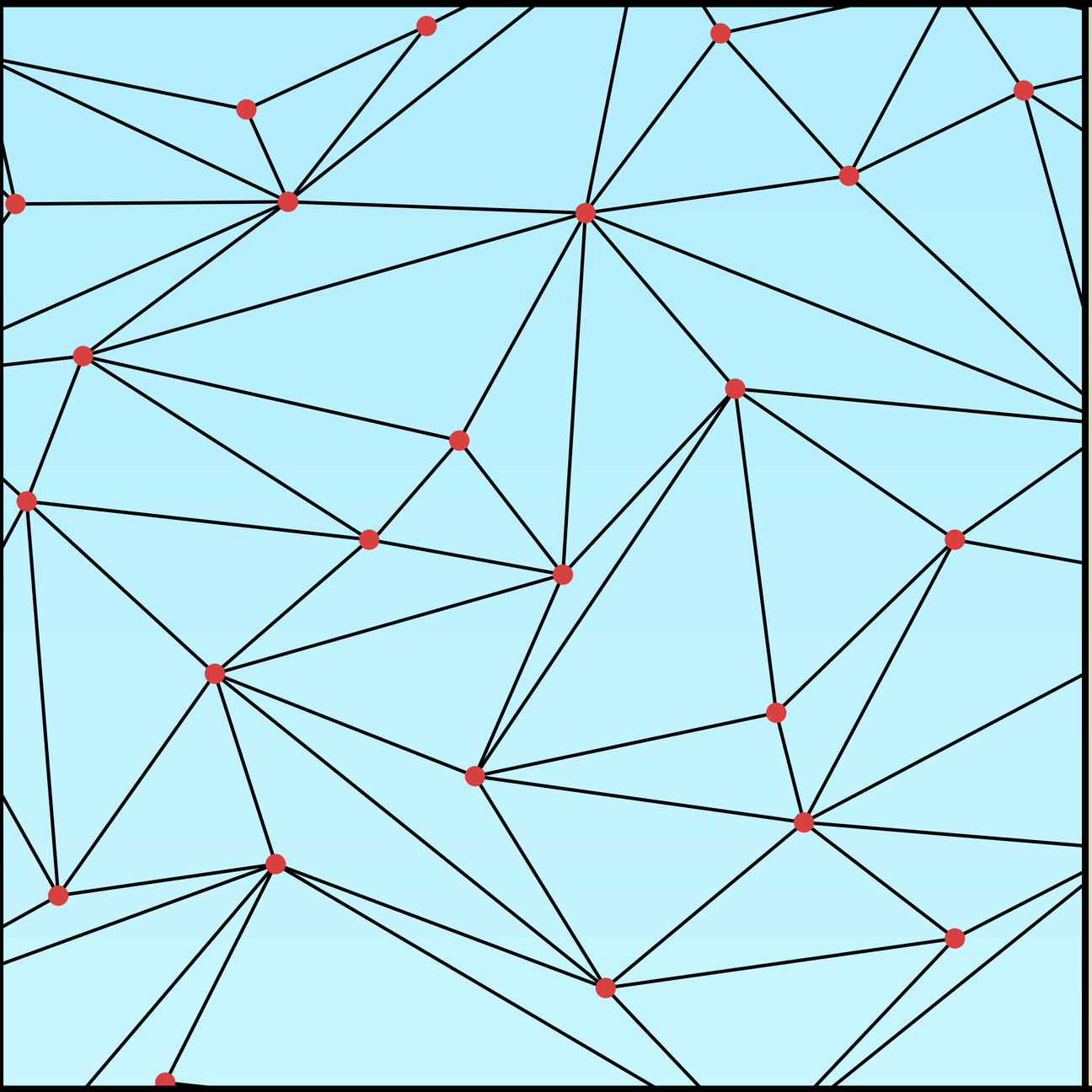

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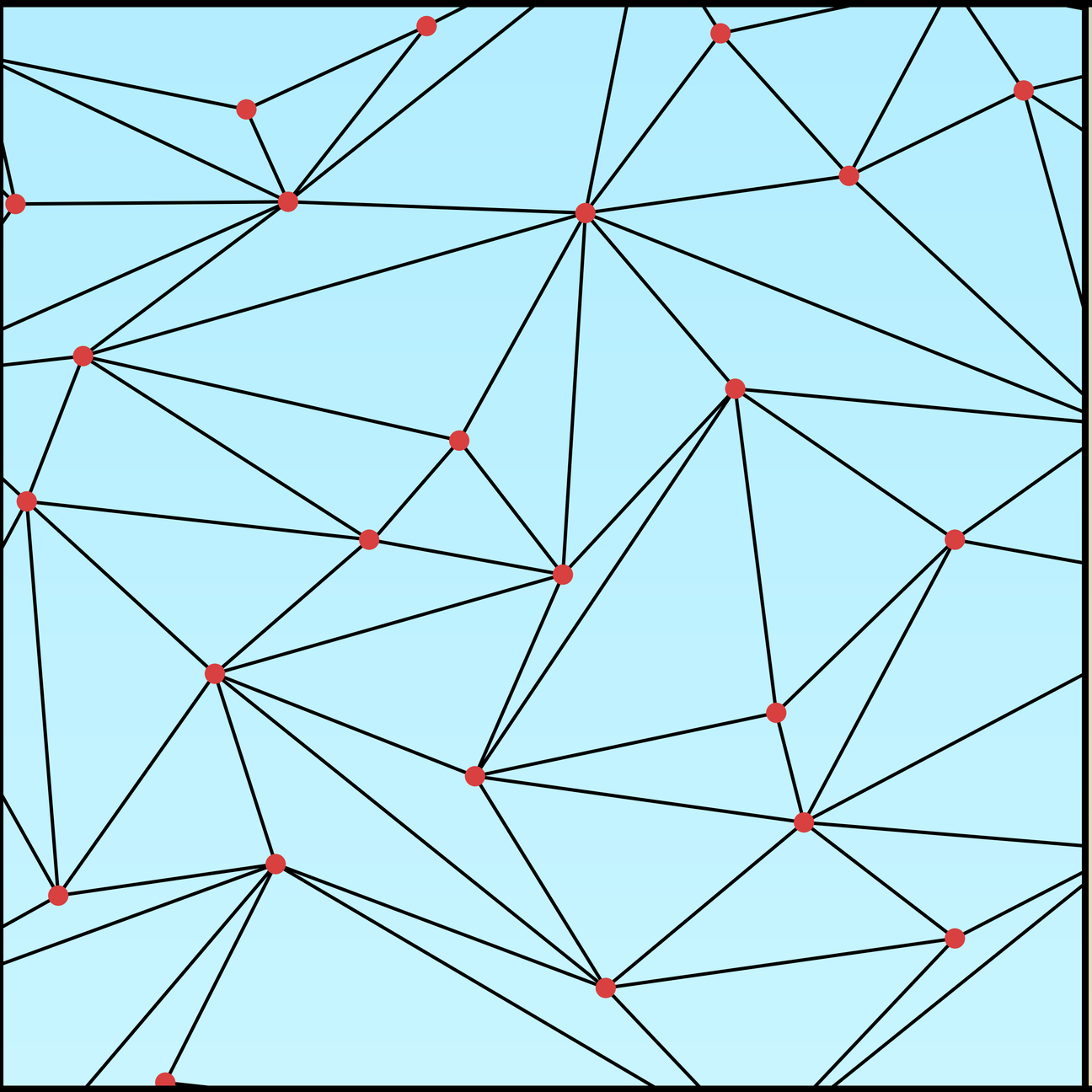

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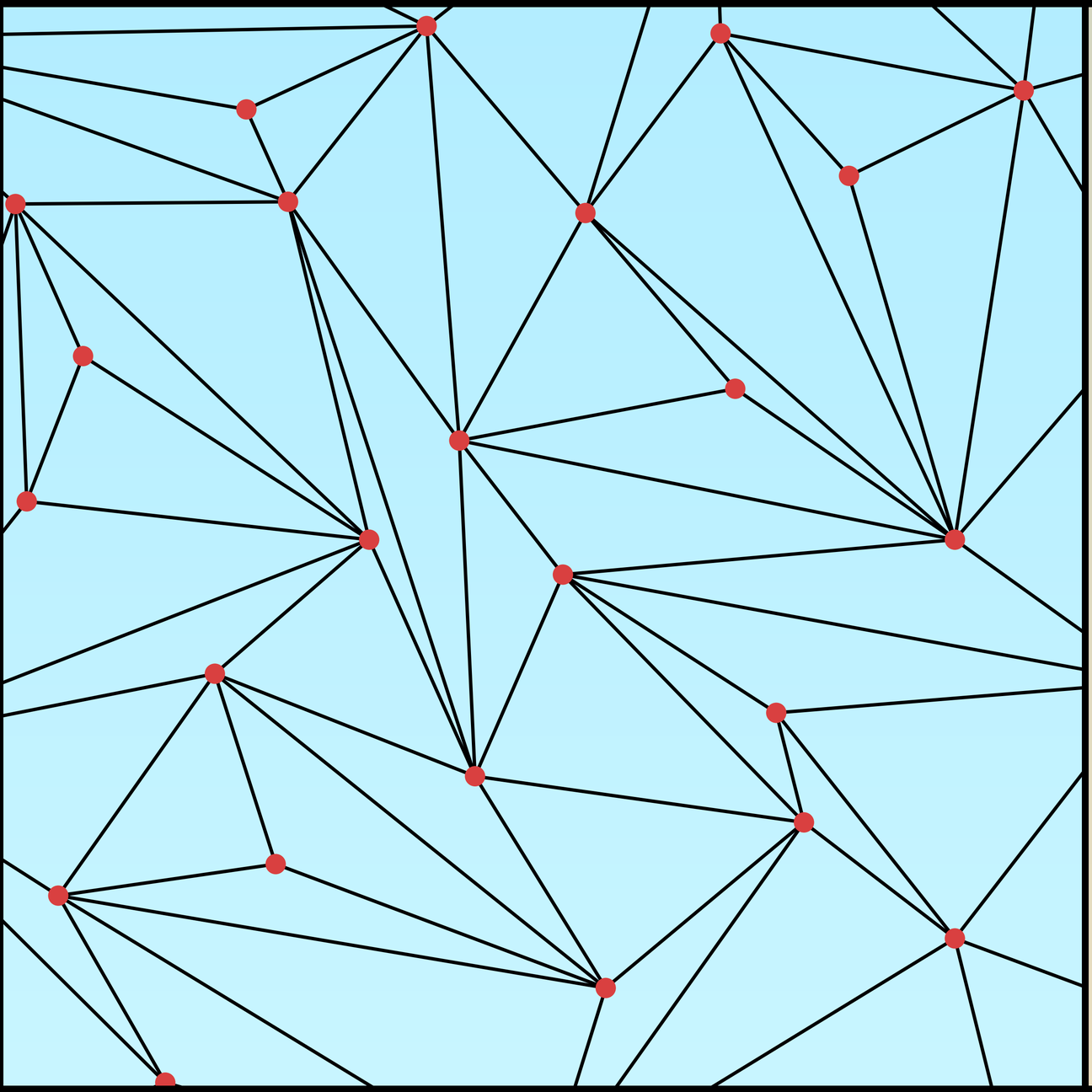

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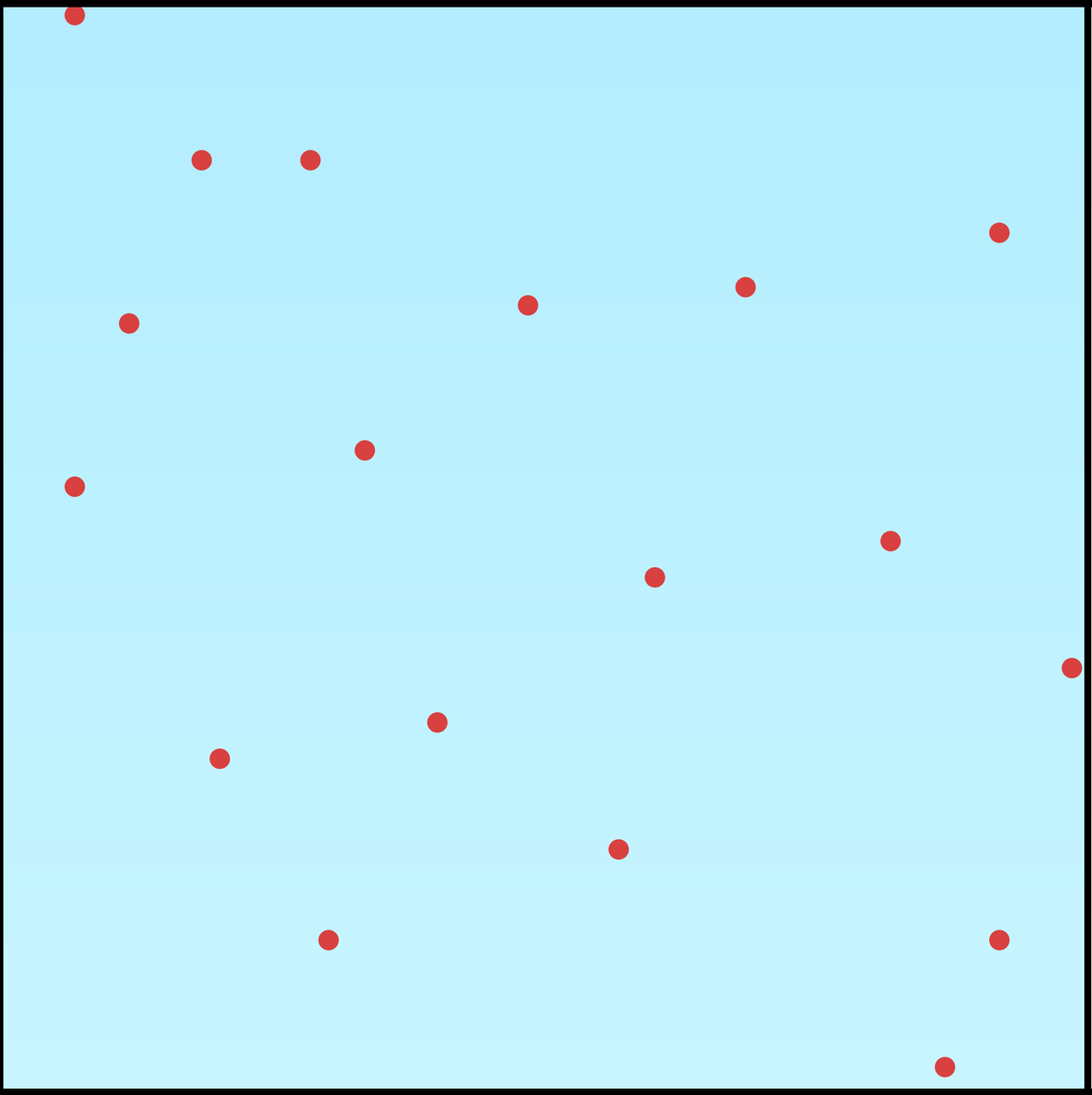

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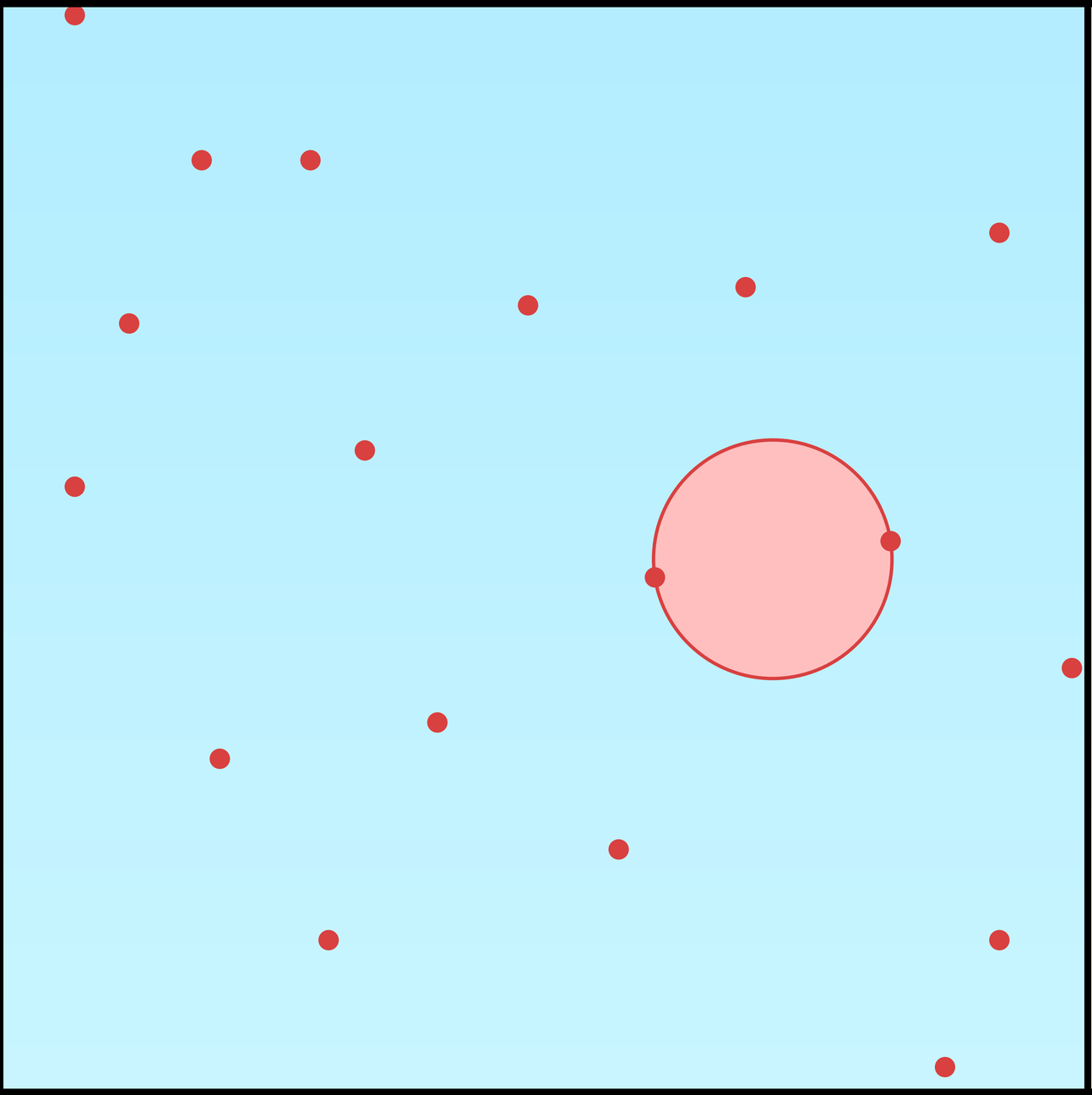
[Delaunay, 1934]



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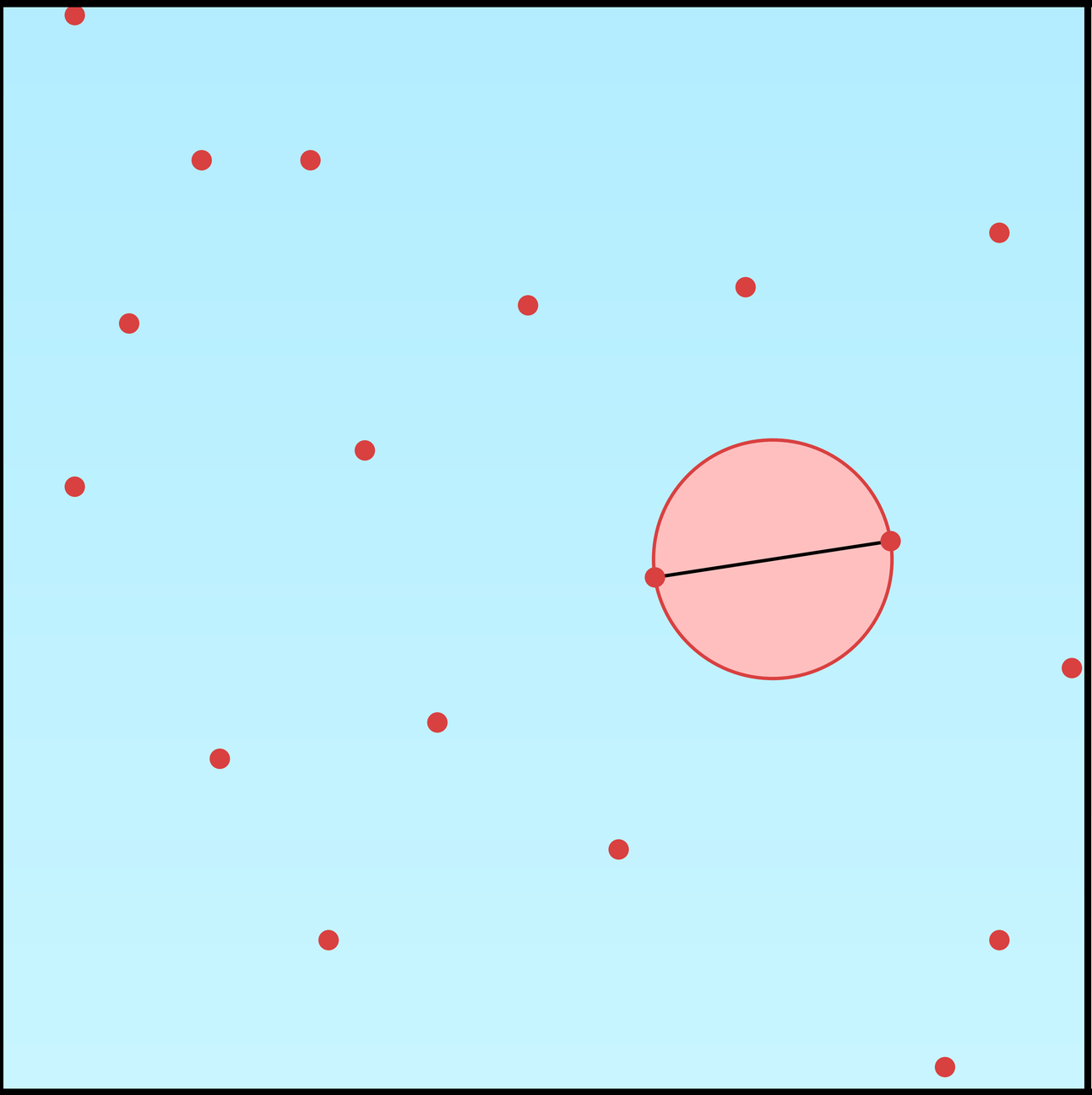

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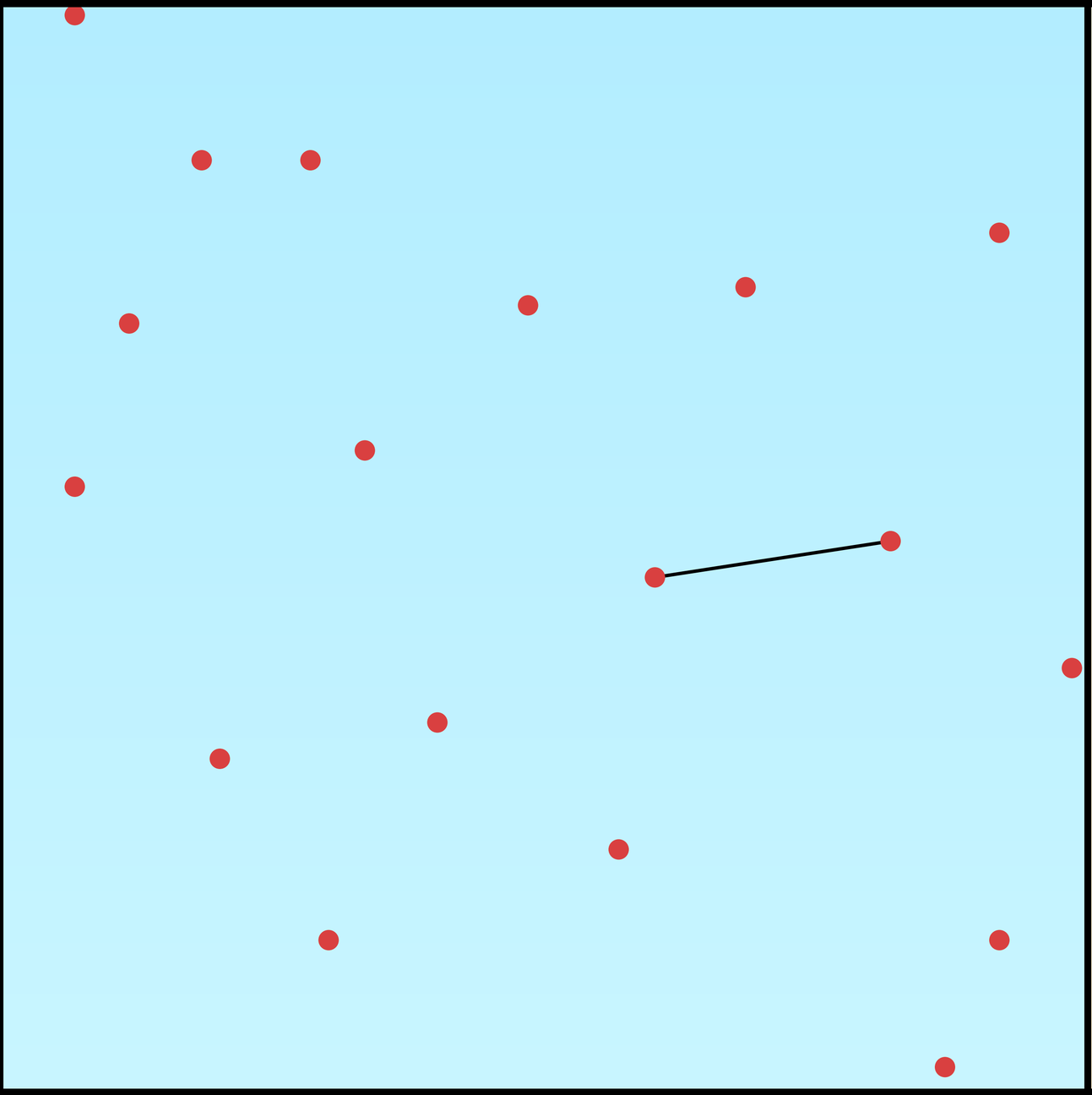

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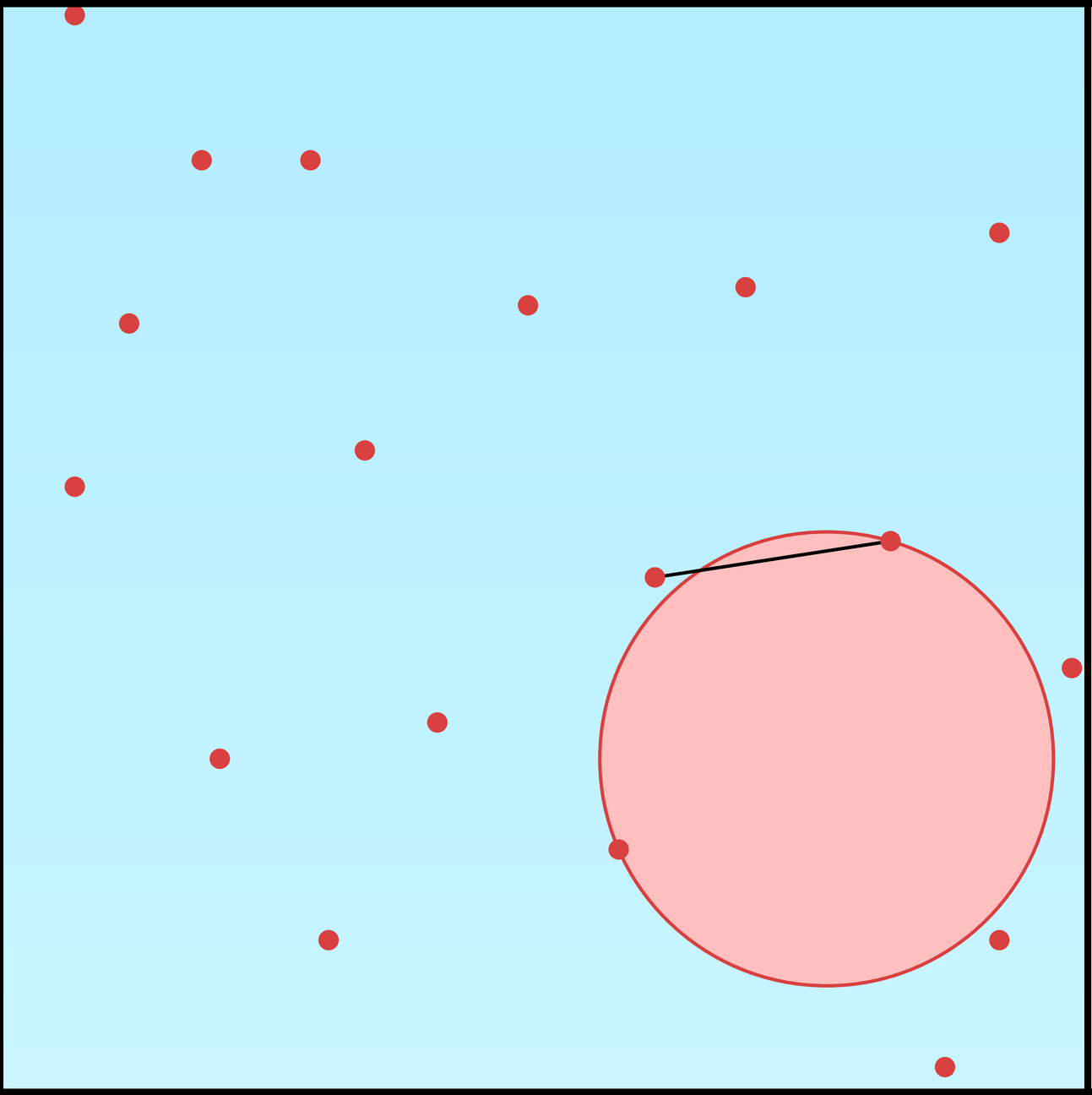

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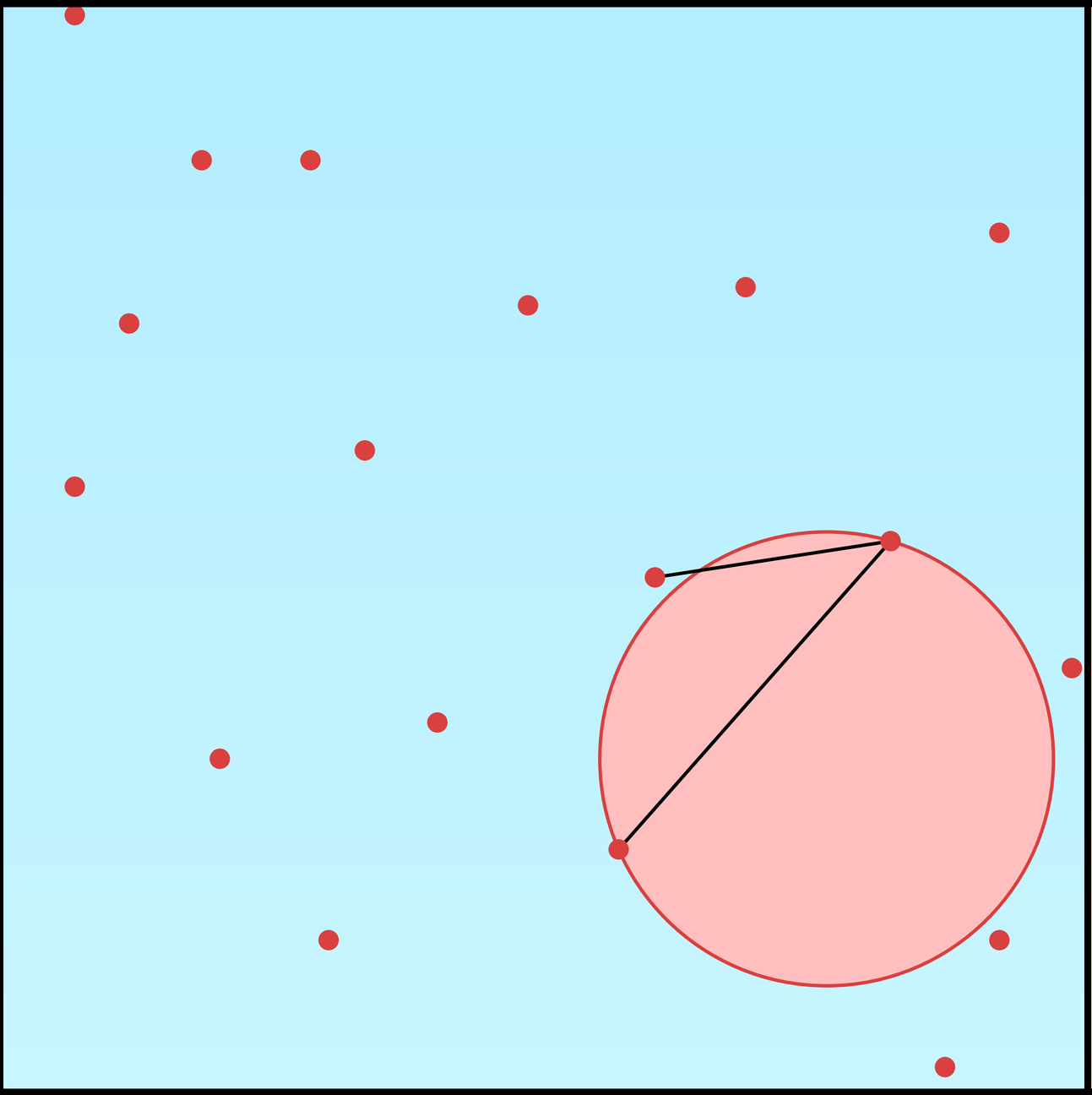

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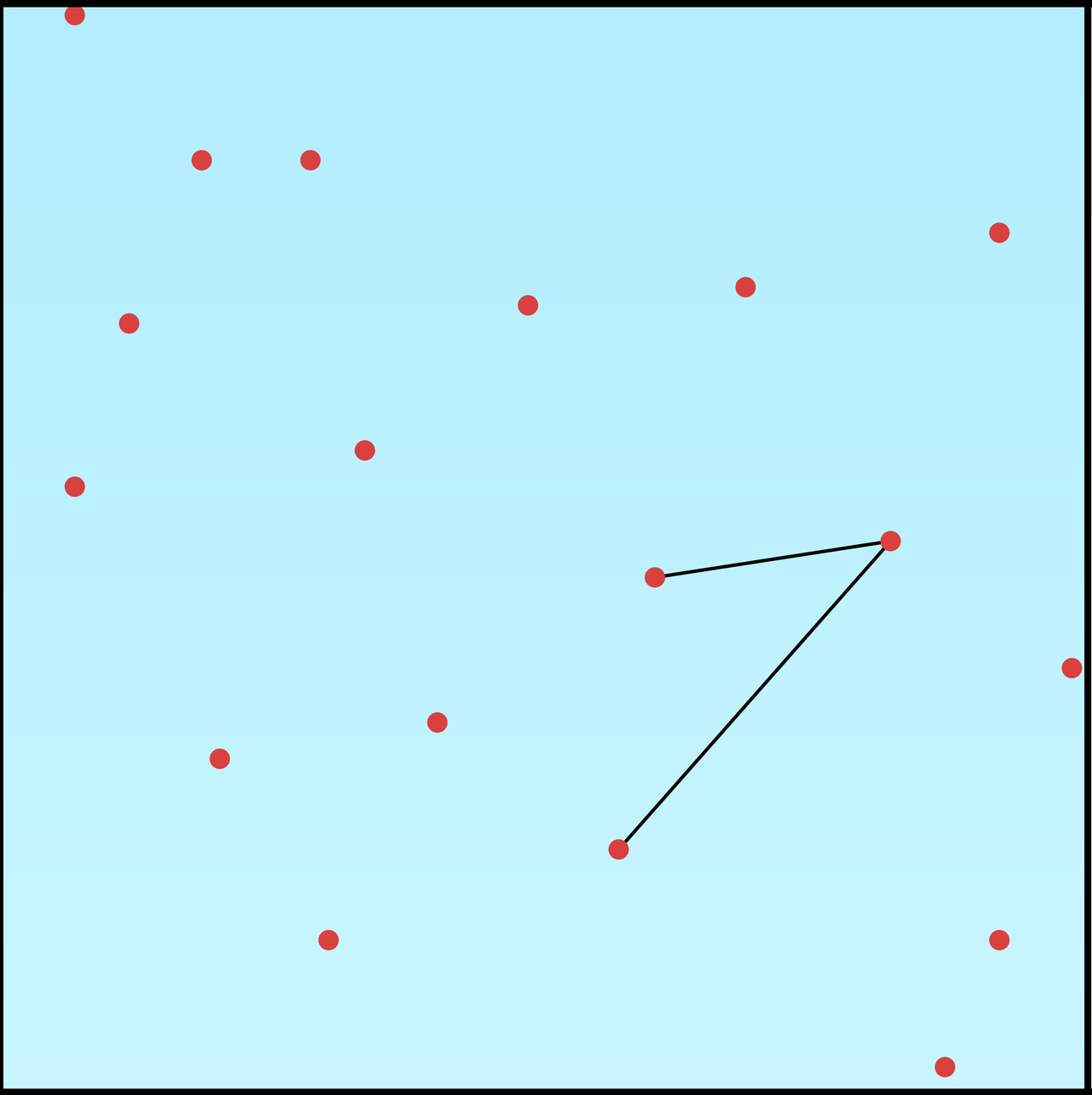

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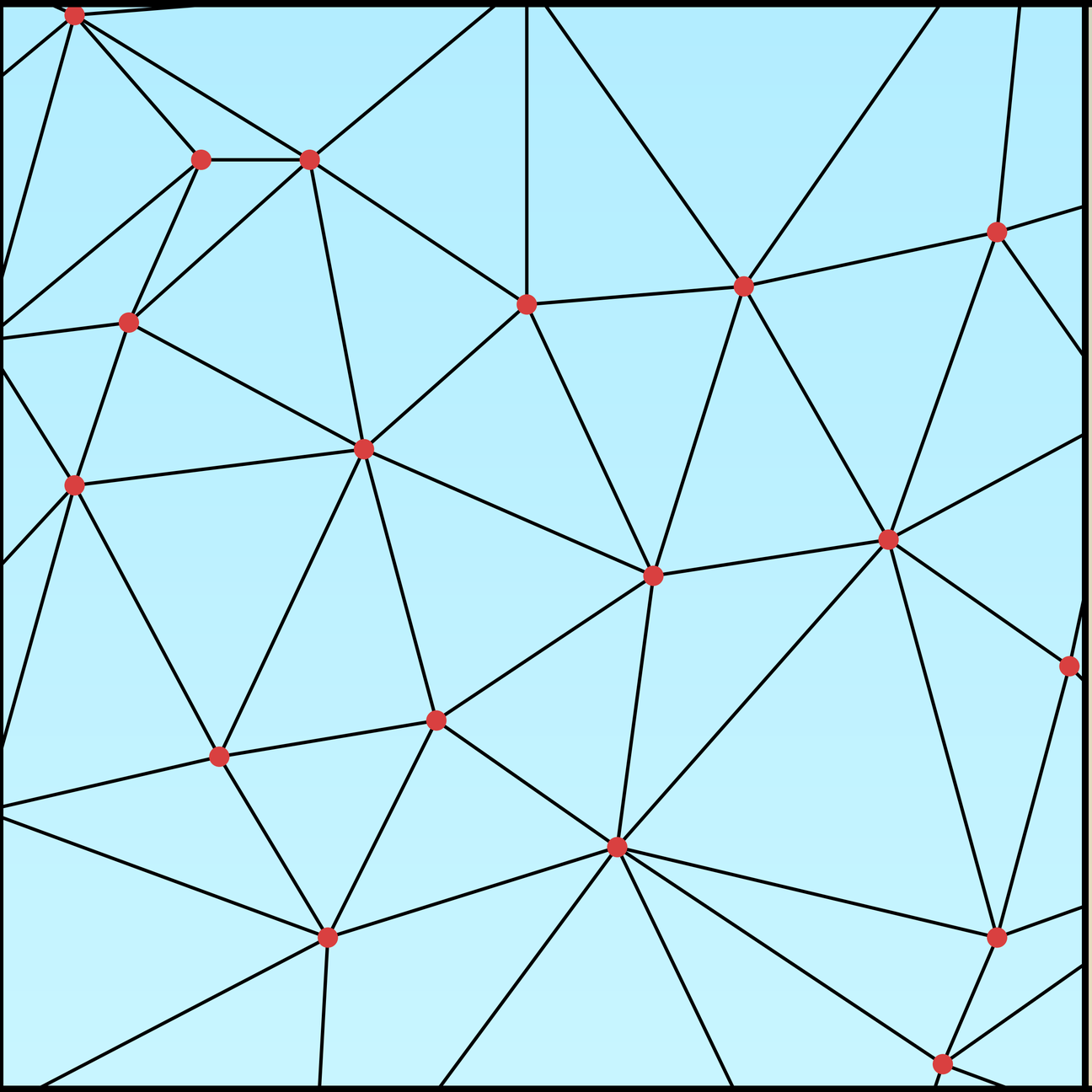

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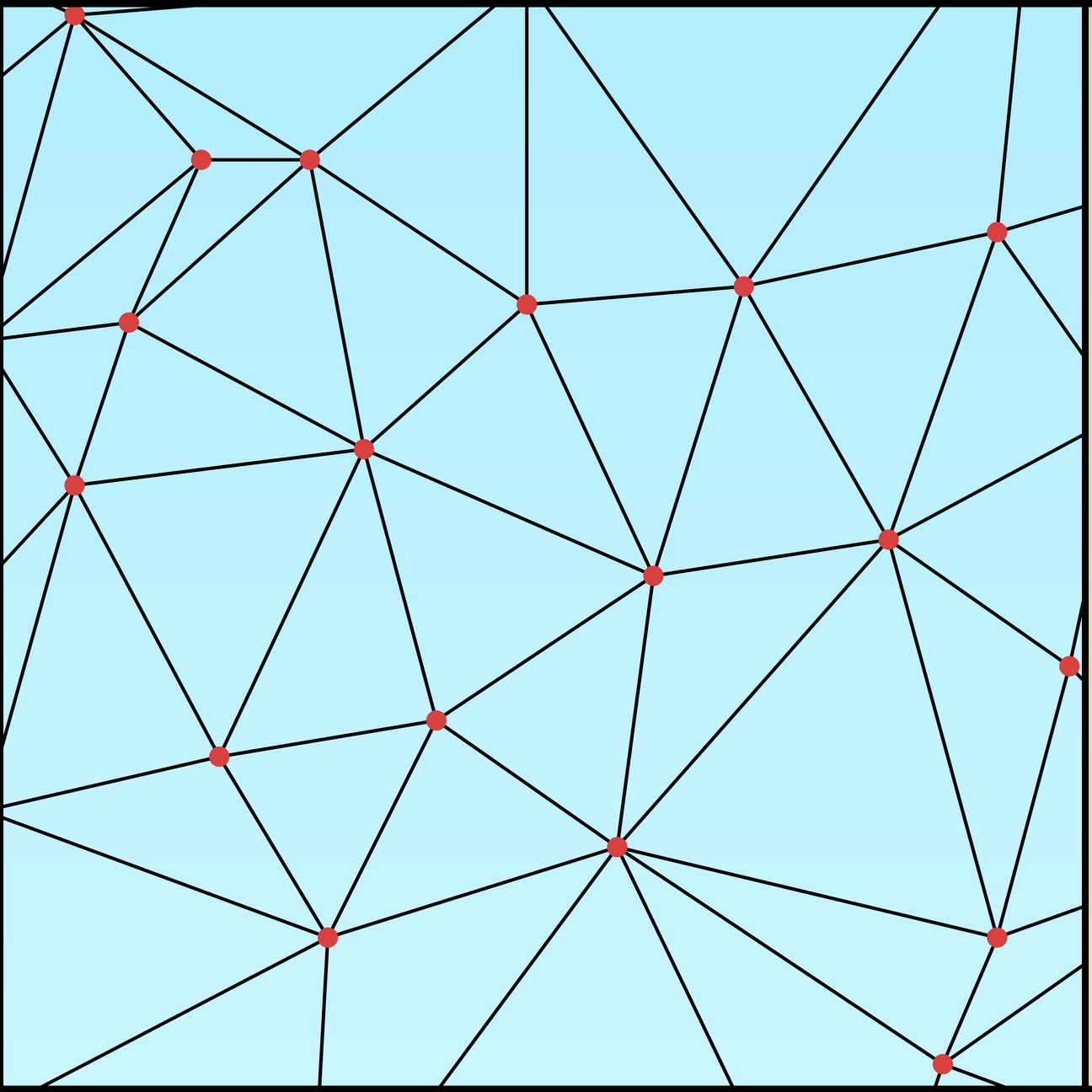

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If $P$ is in *general position*, the Delaunay triangulation is uniquely defined.

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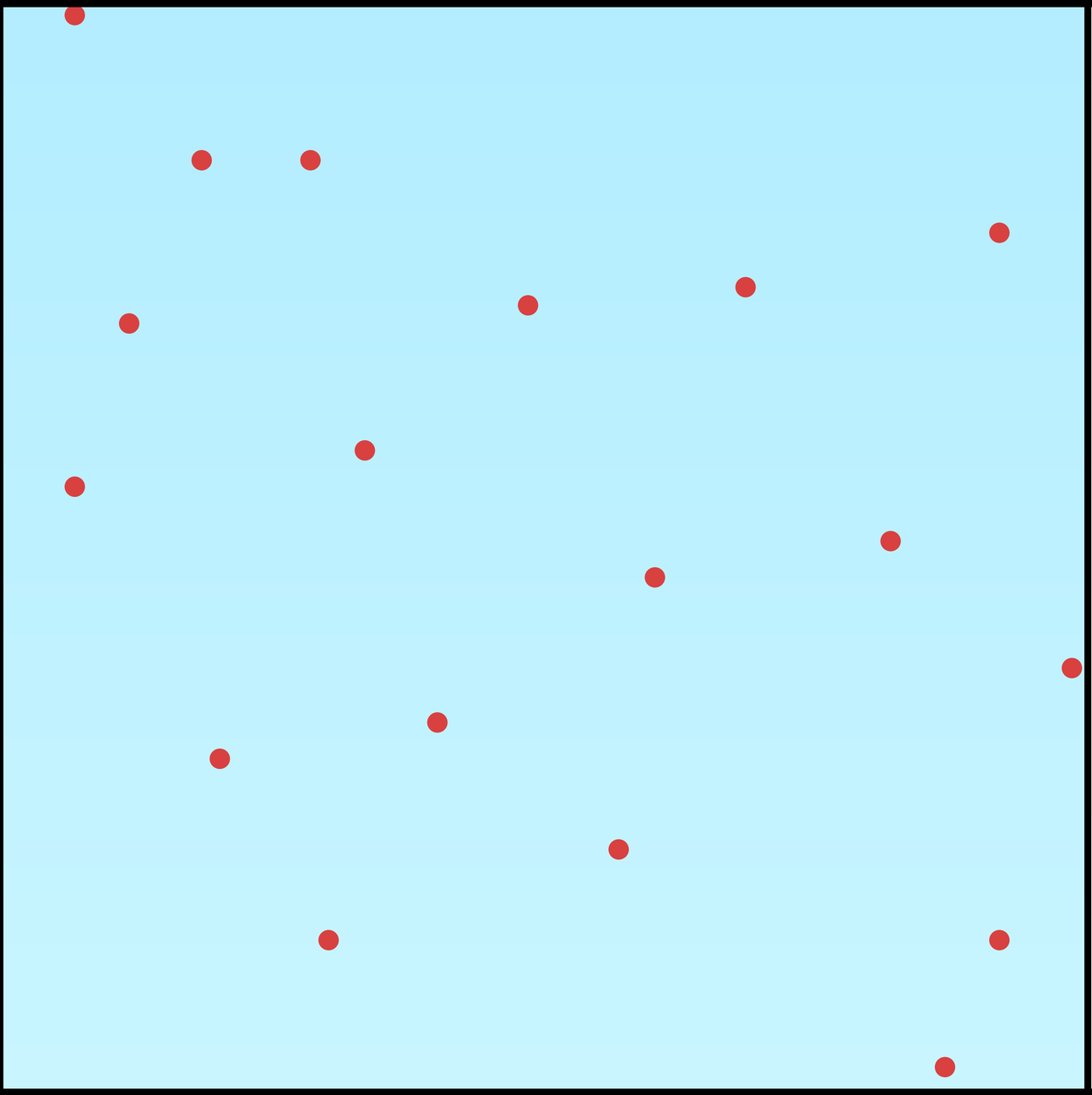

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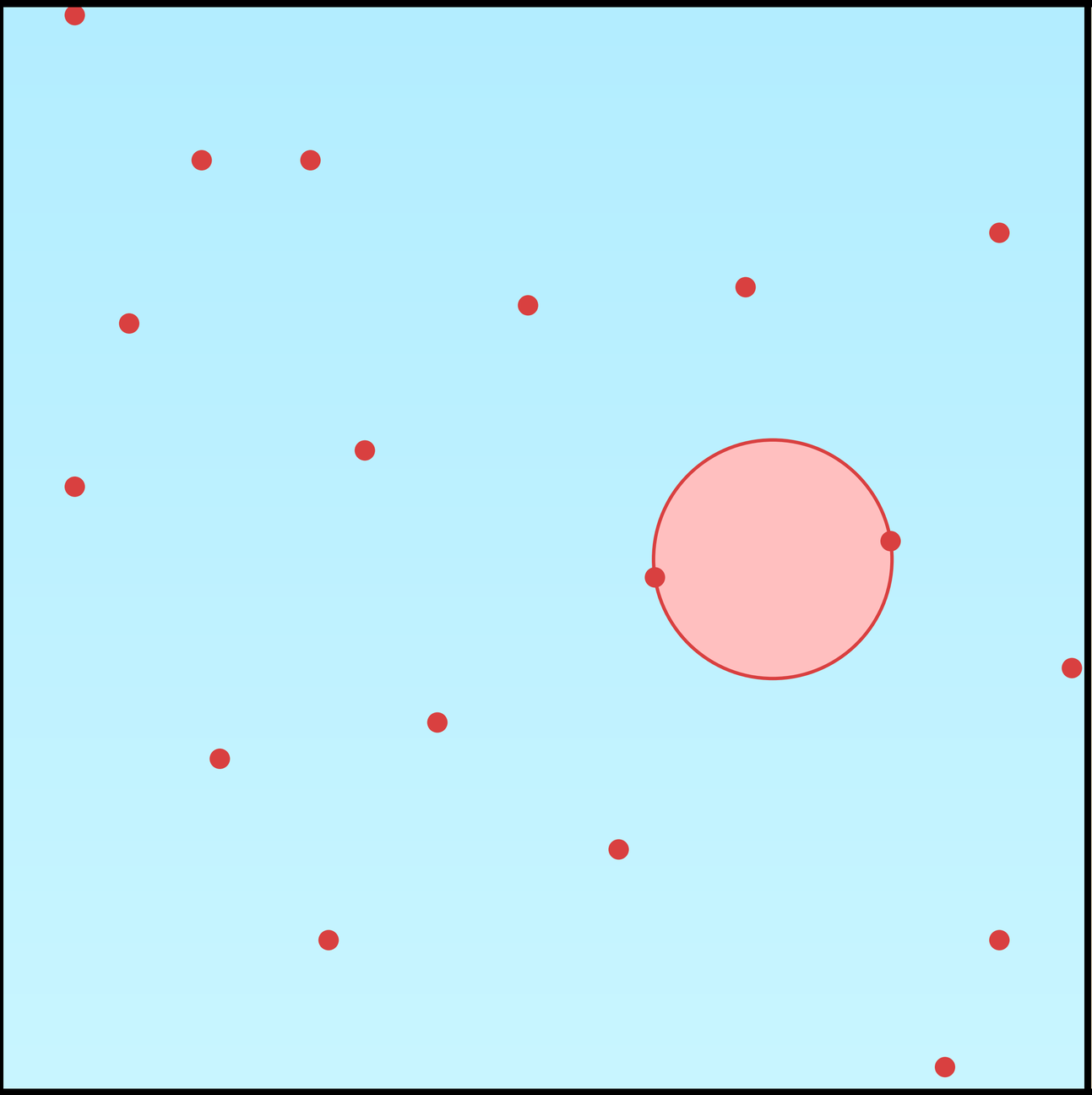

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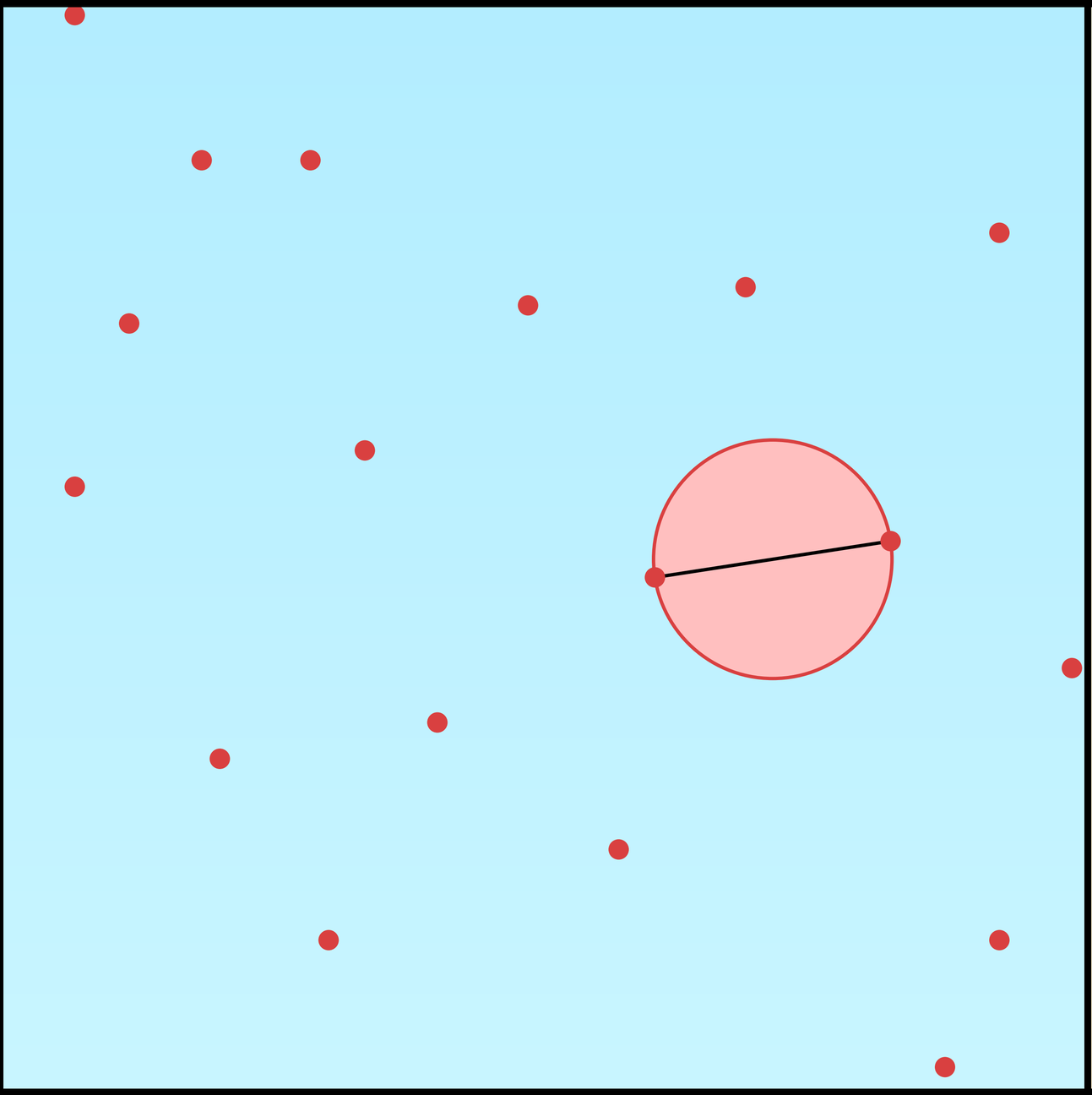

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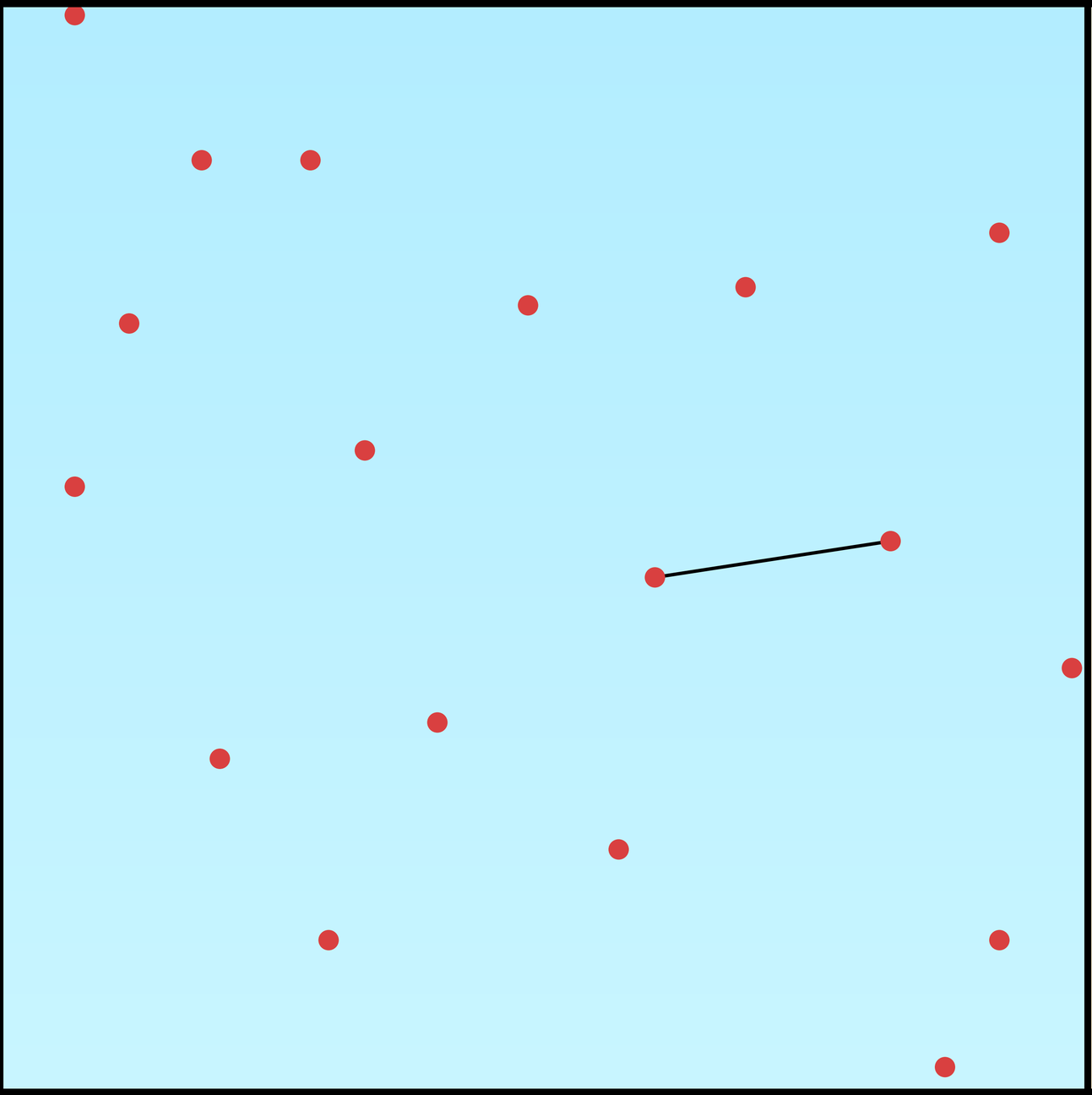

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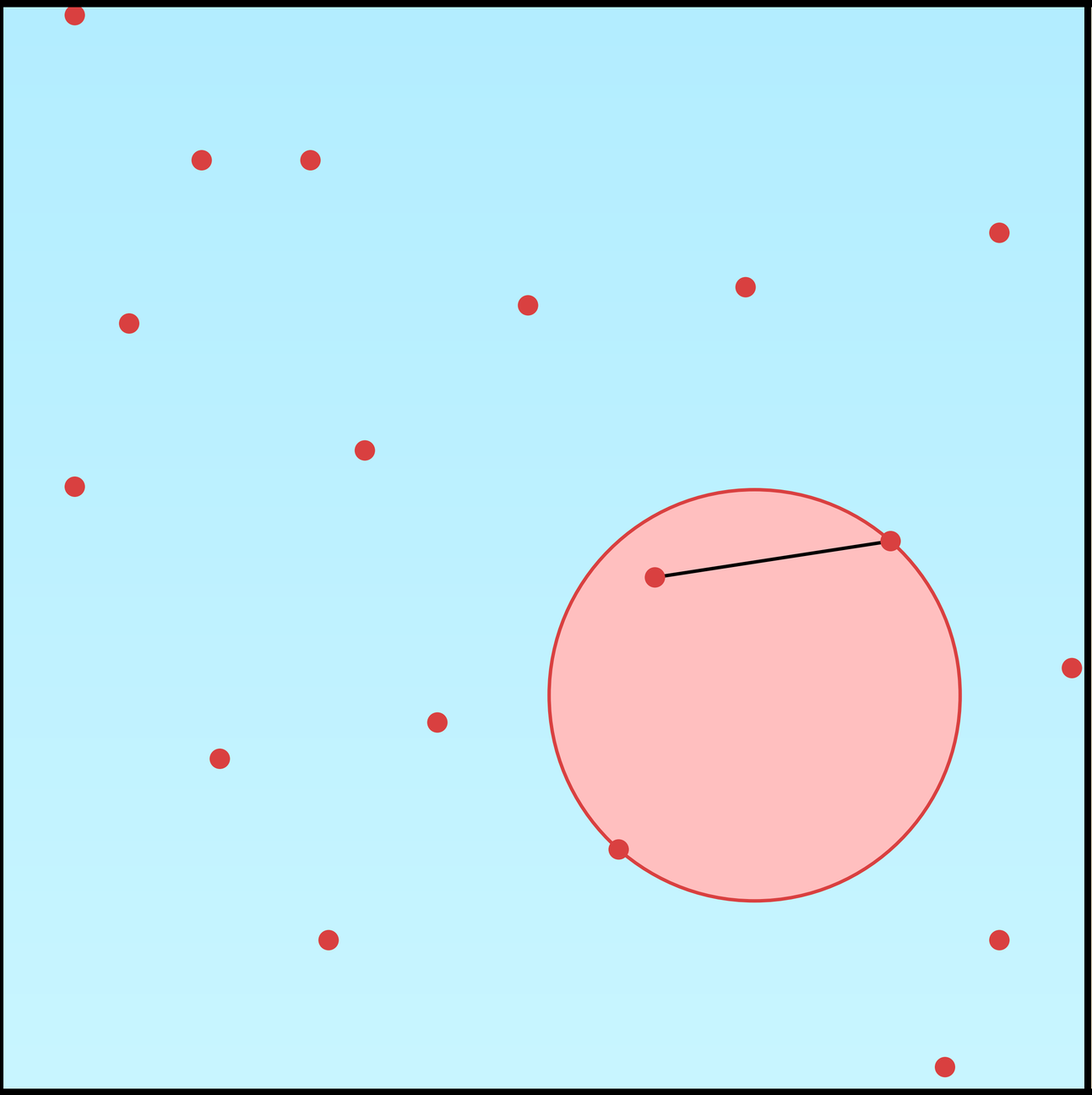

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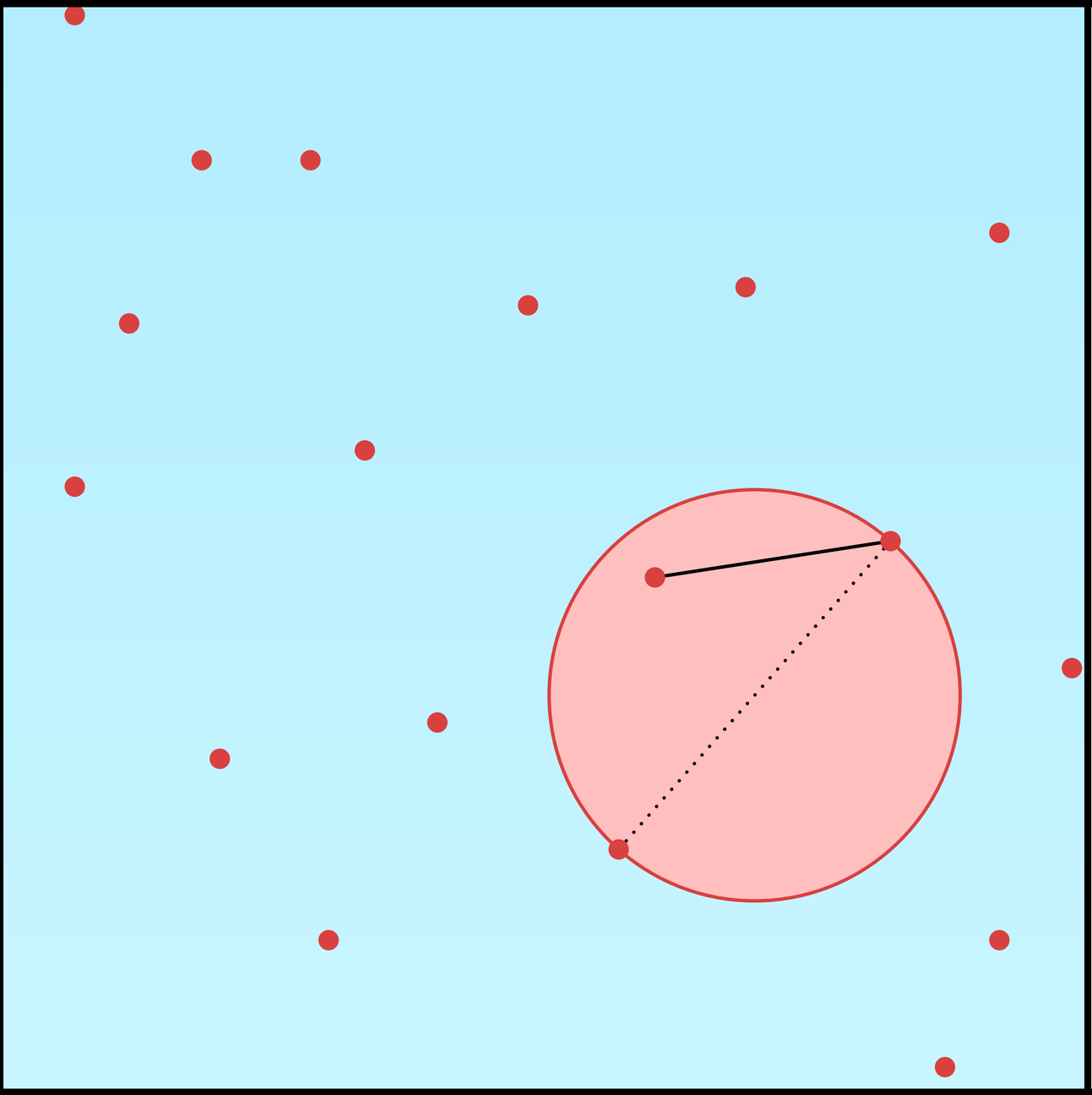

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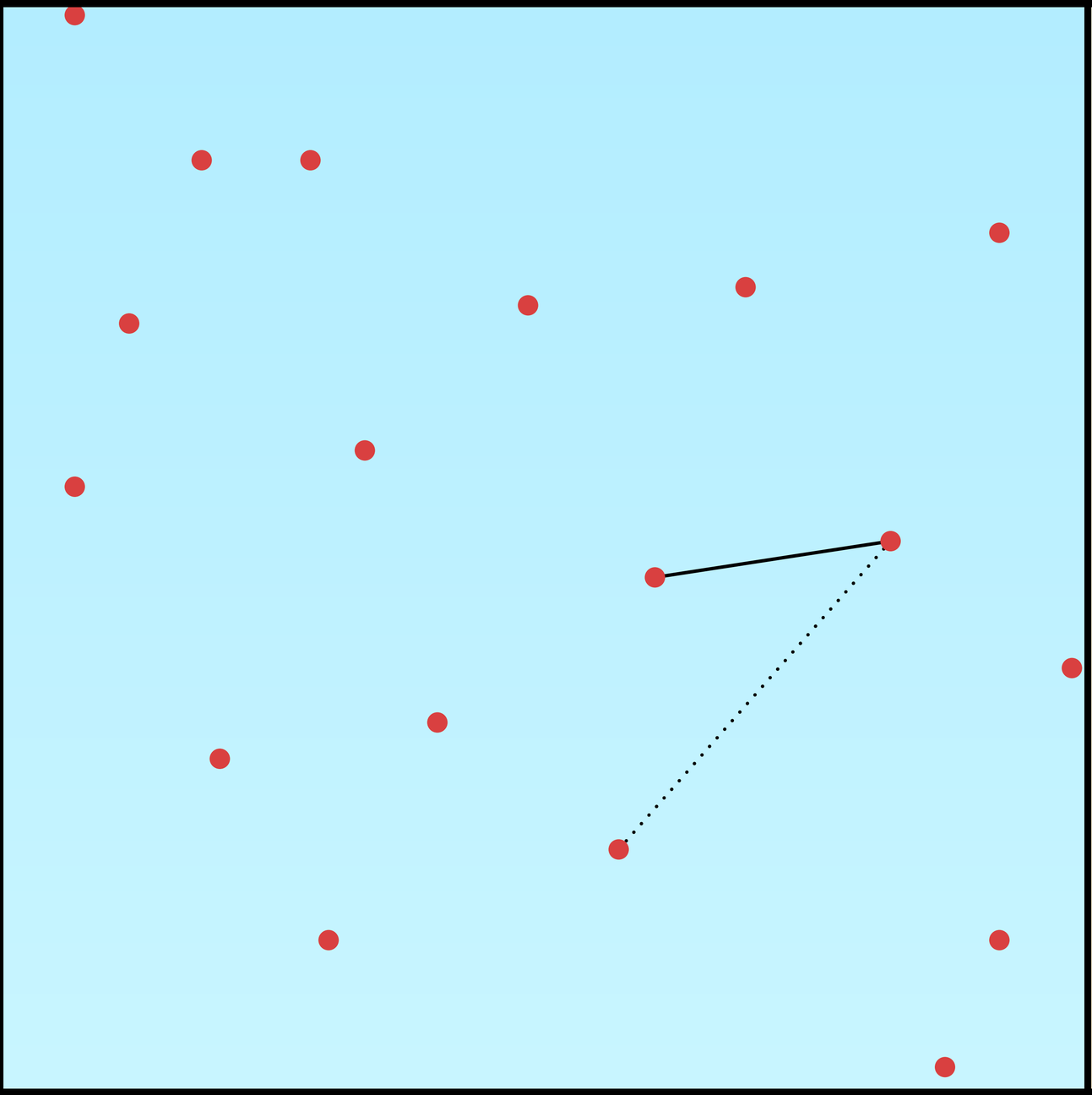

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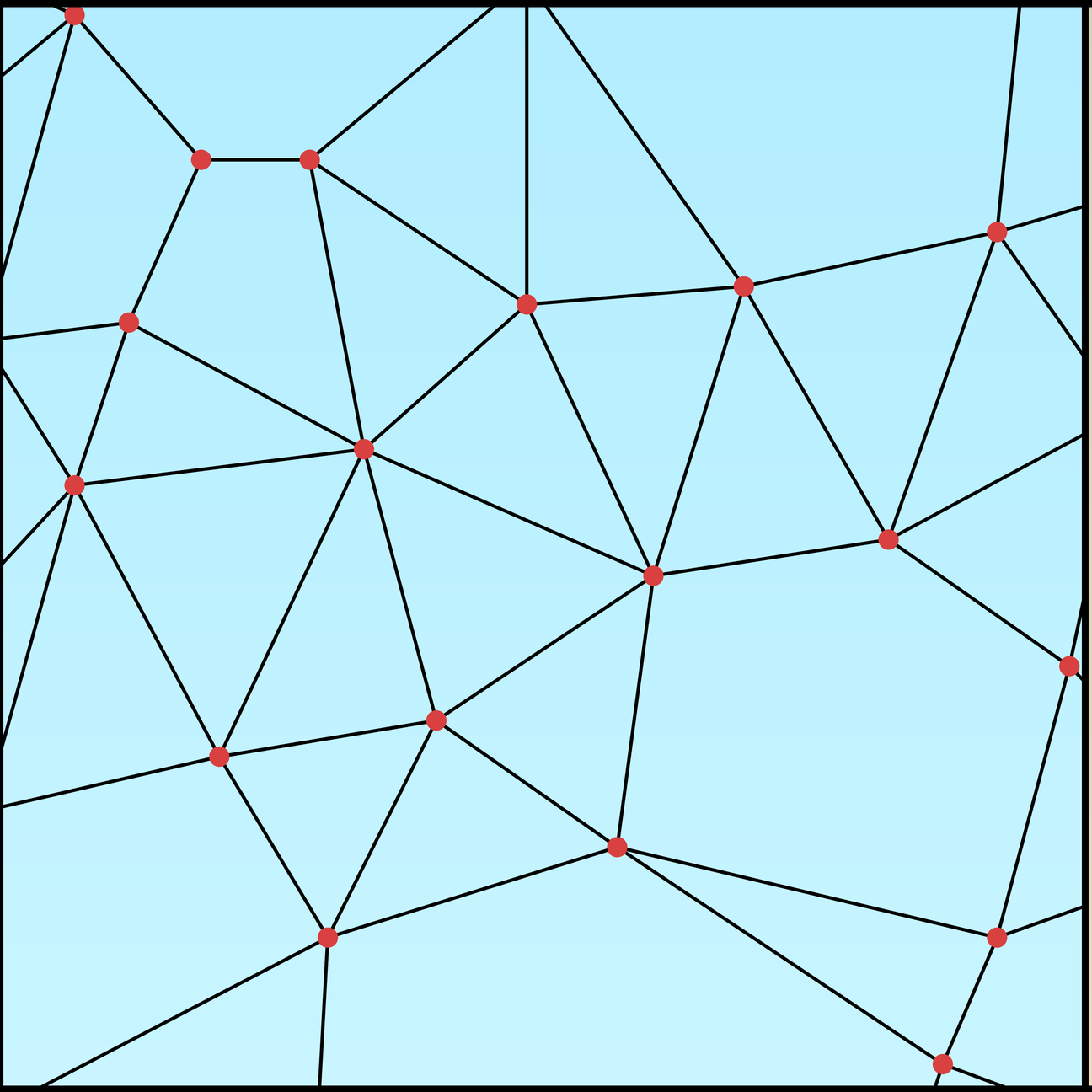

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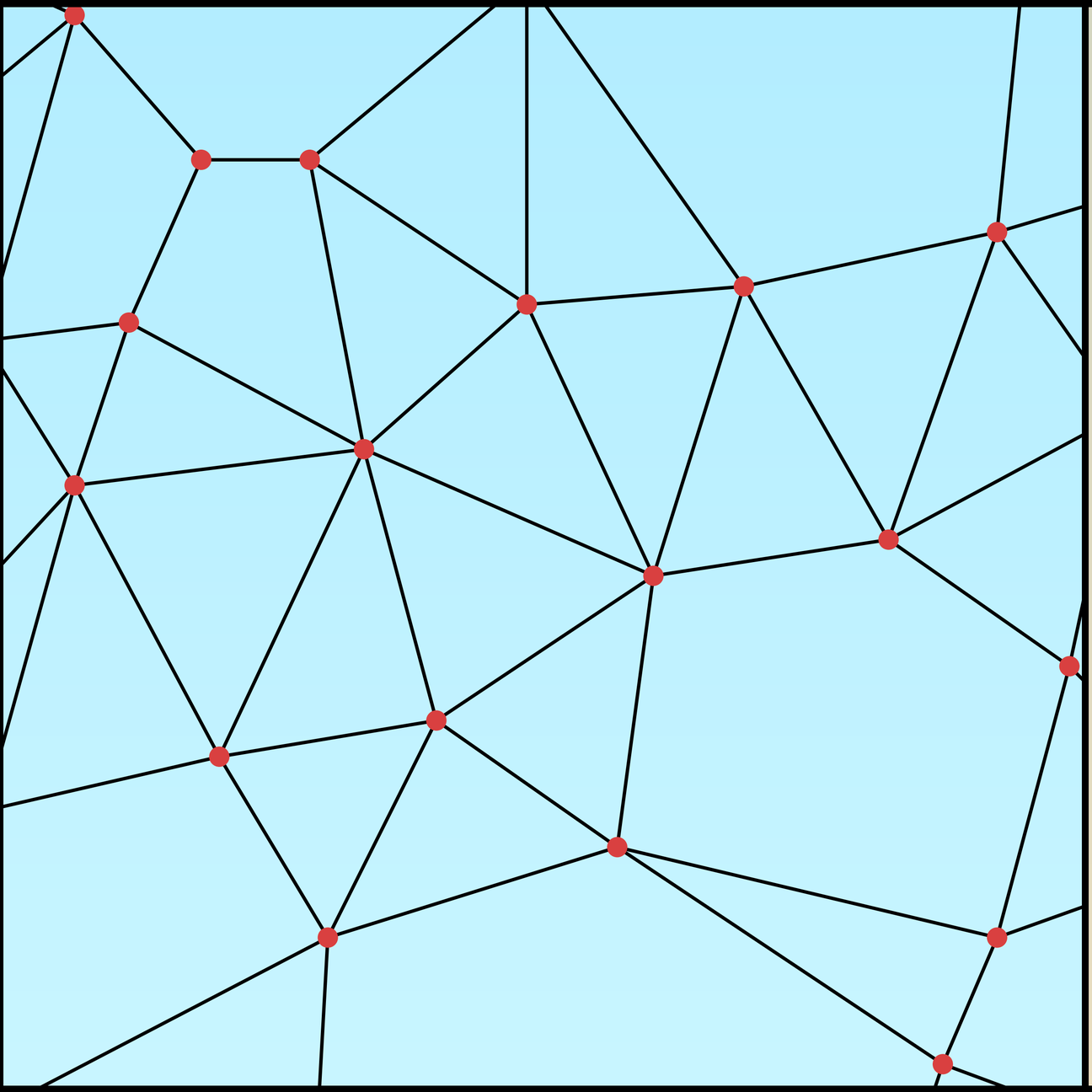

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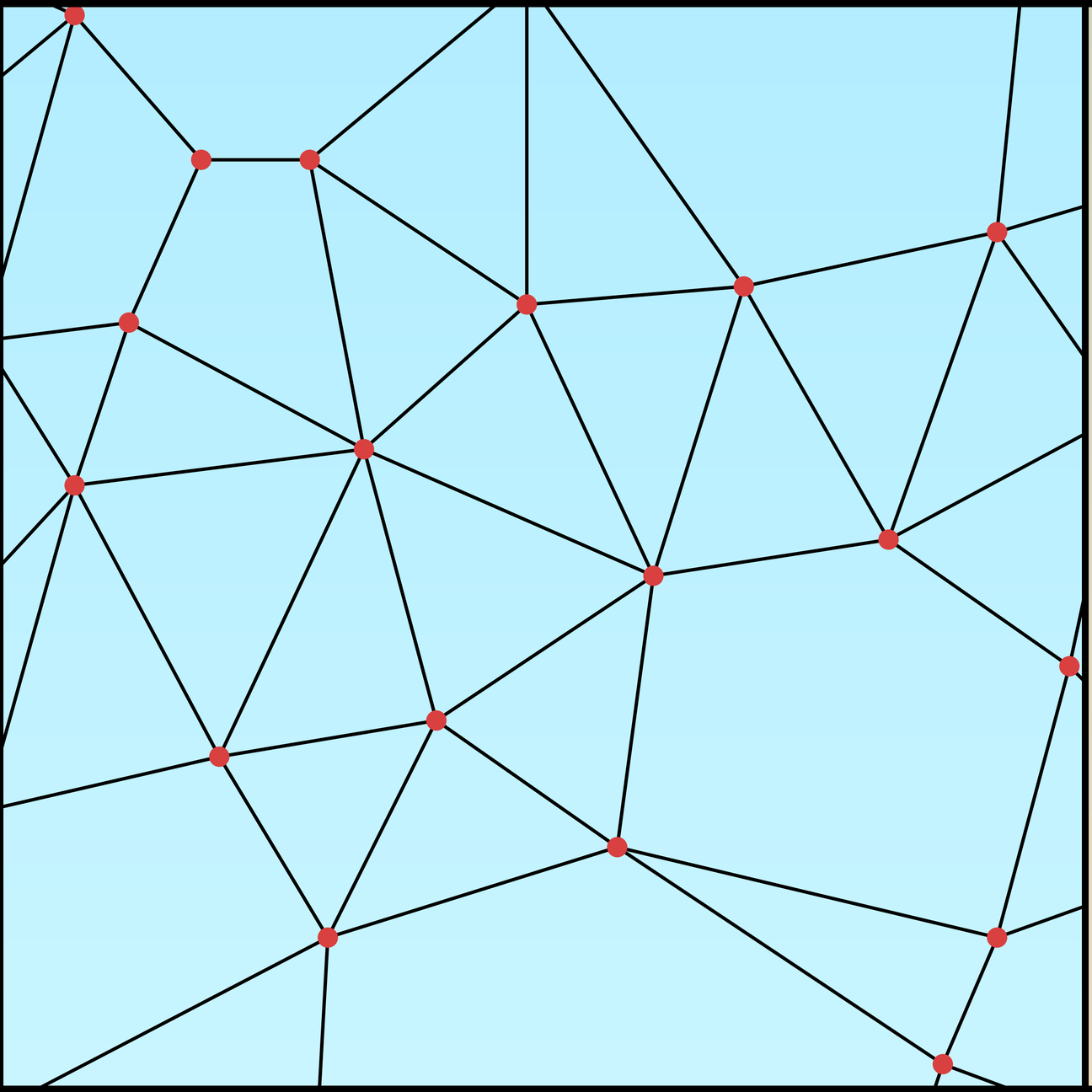

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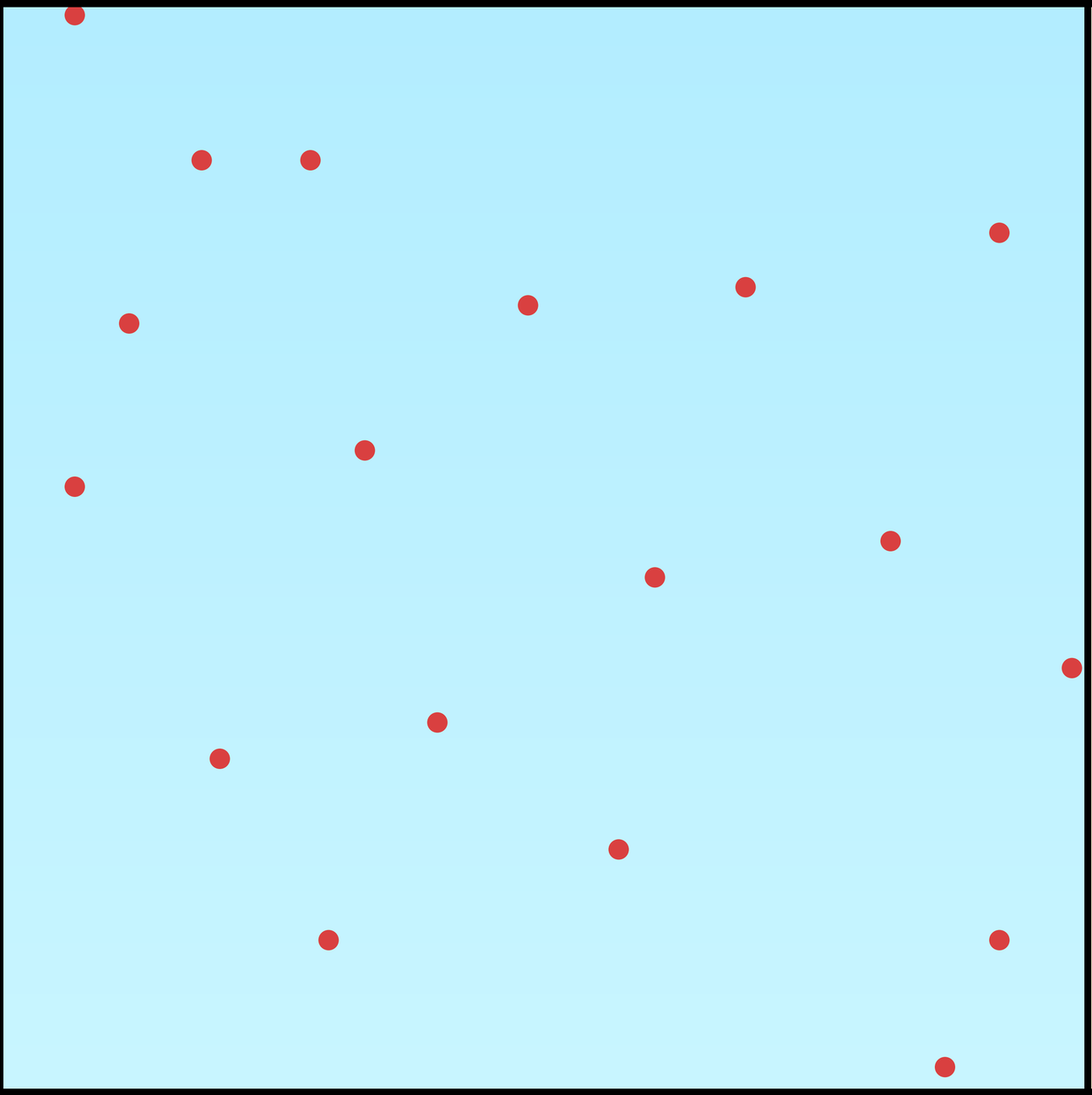
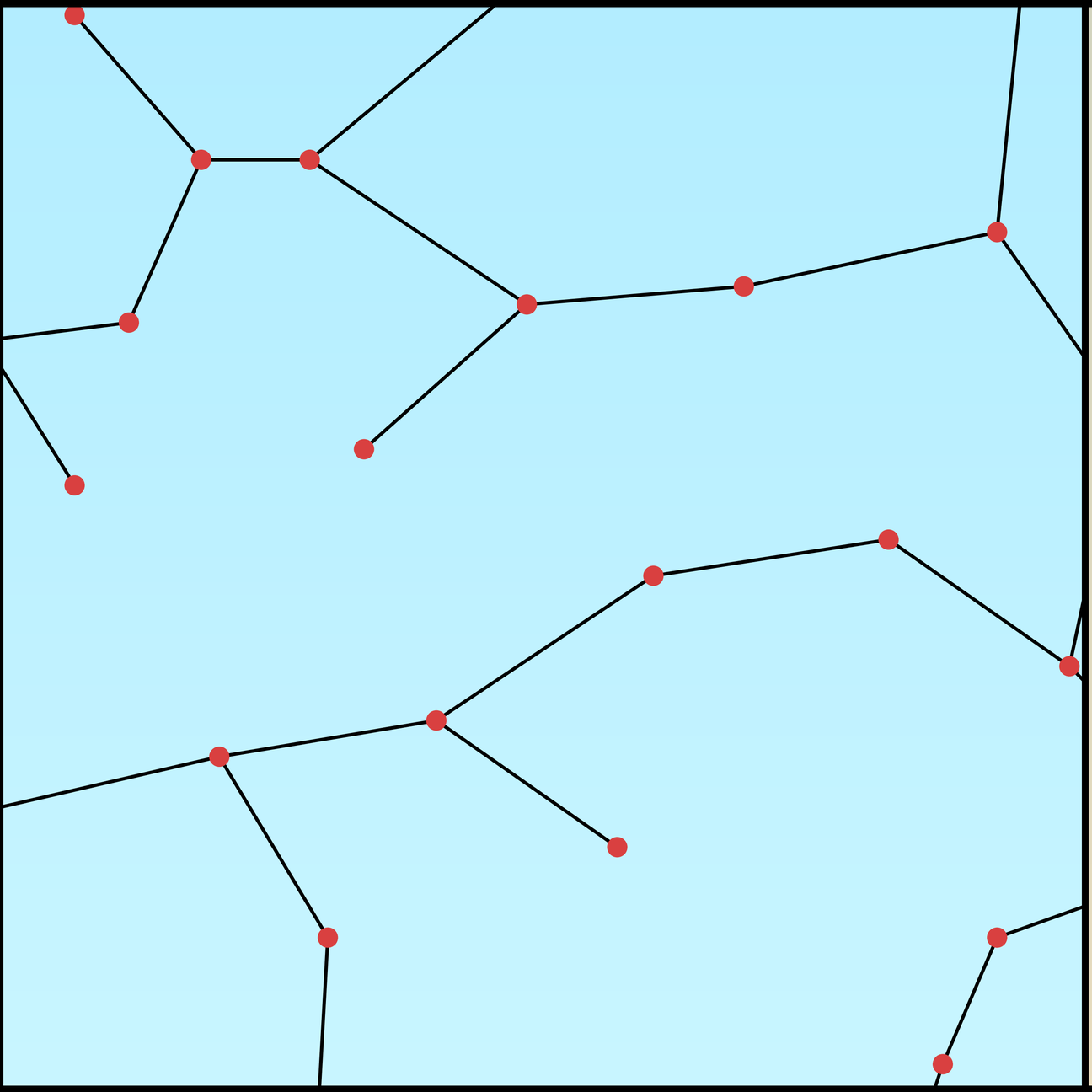
The *Gabriel graph* is always connected!

[Gabriel, 1969]



The *Euclidean minimum spanning tree (EMST)* is the minimum spanning tree of the complete graph on a point set $P$.



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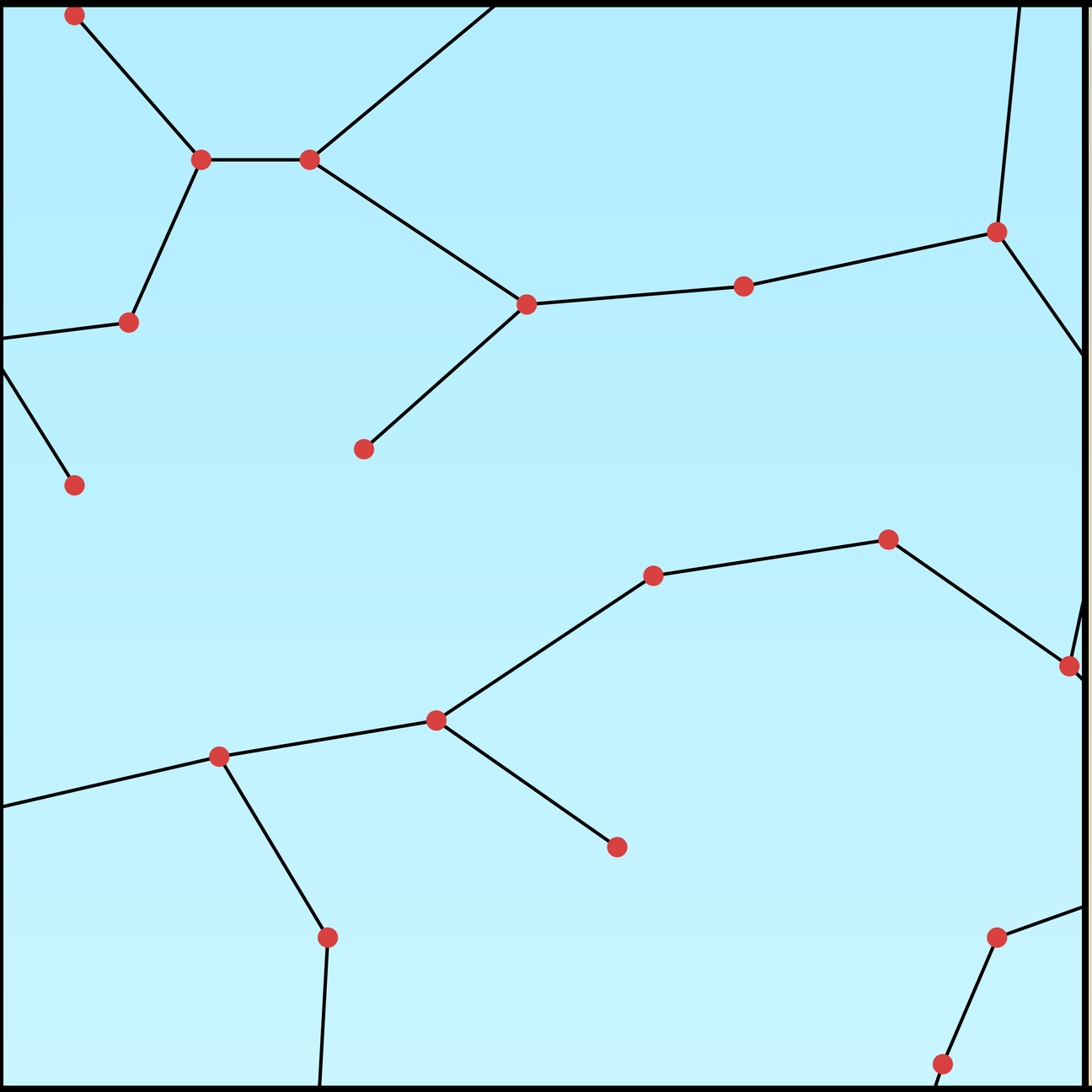

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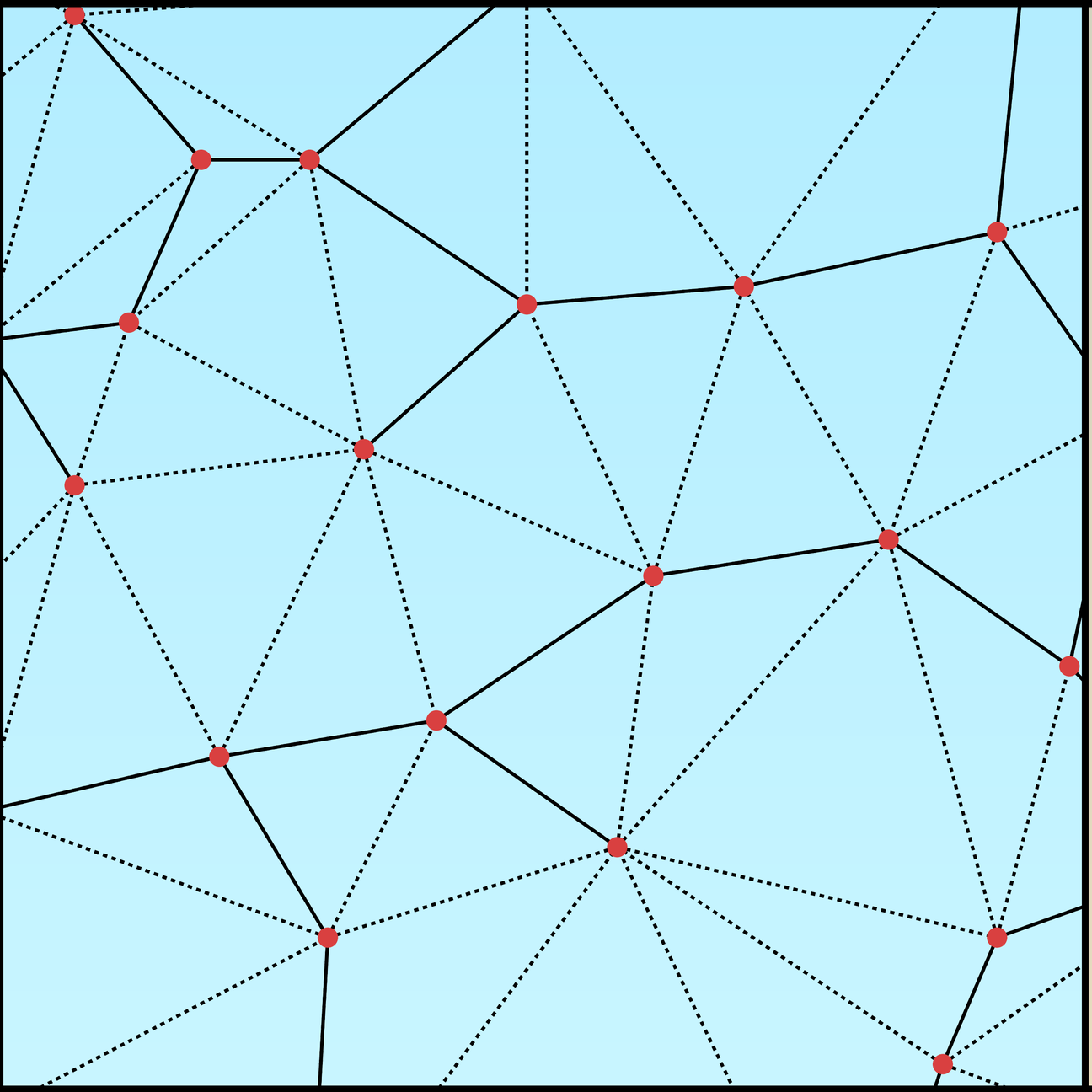

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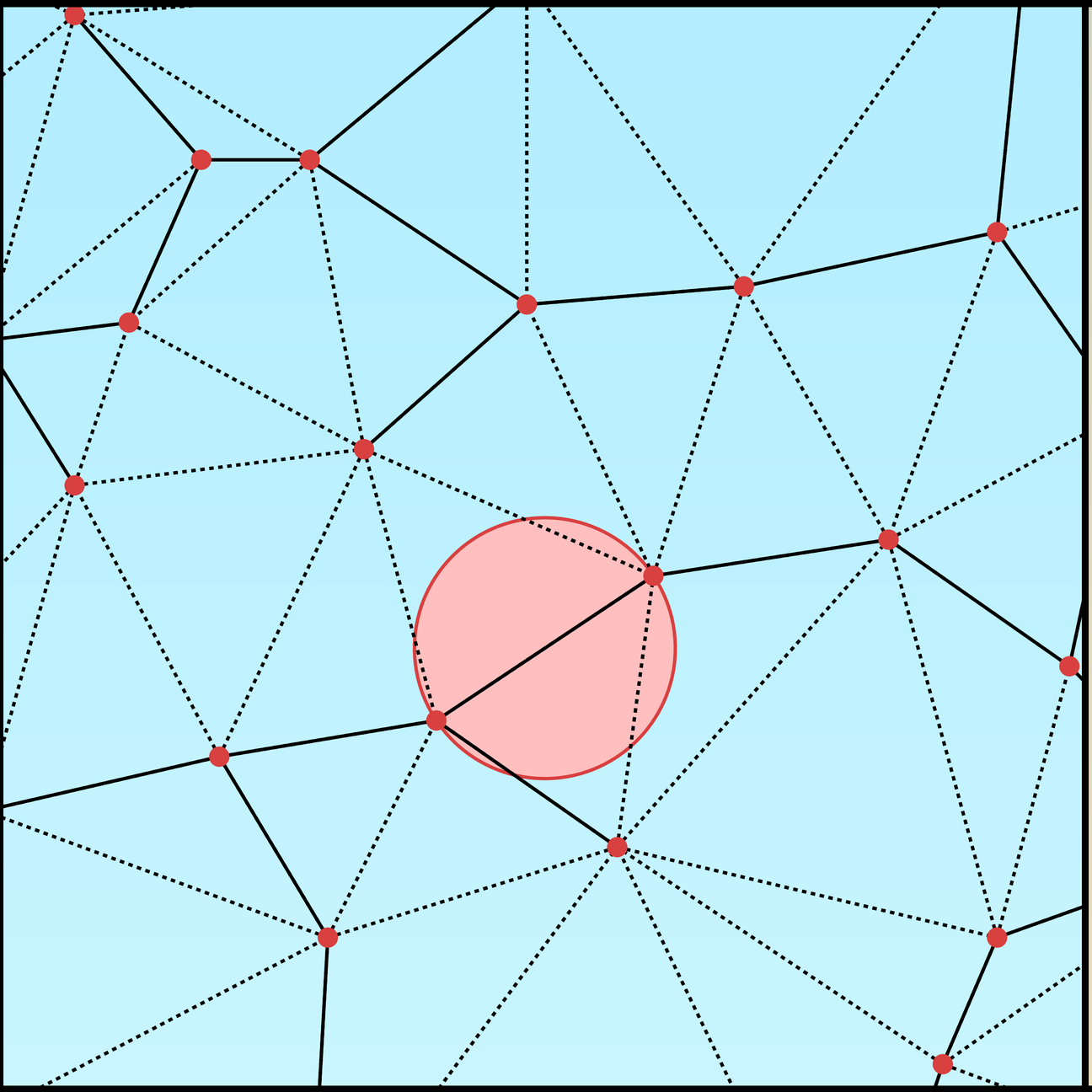

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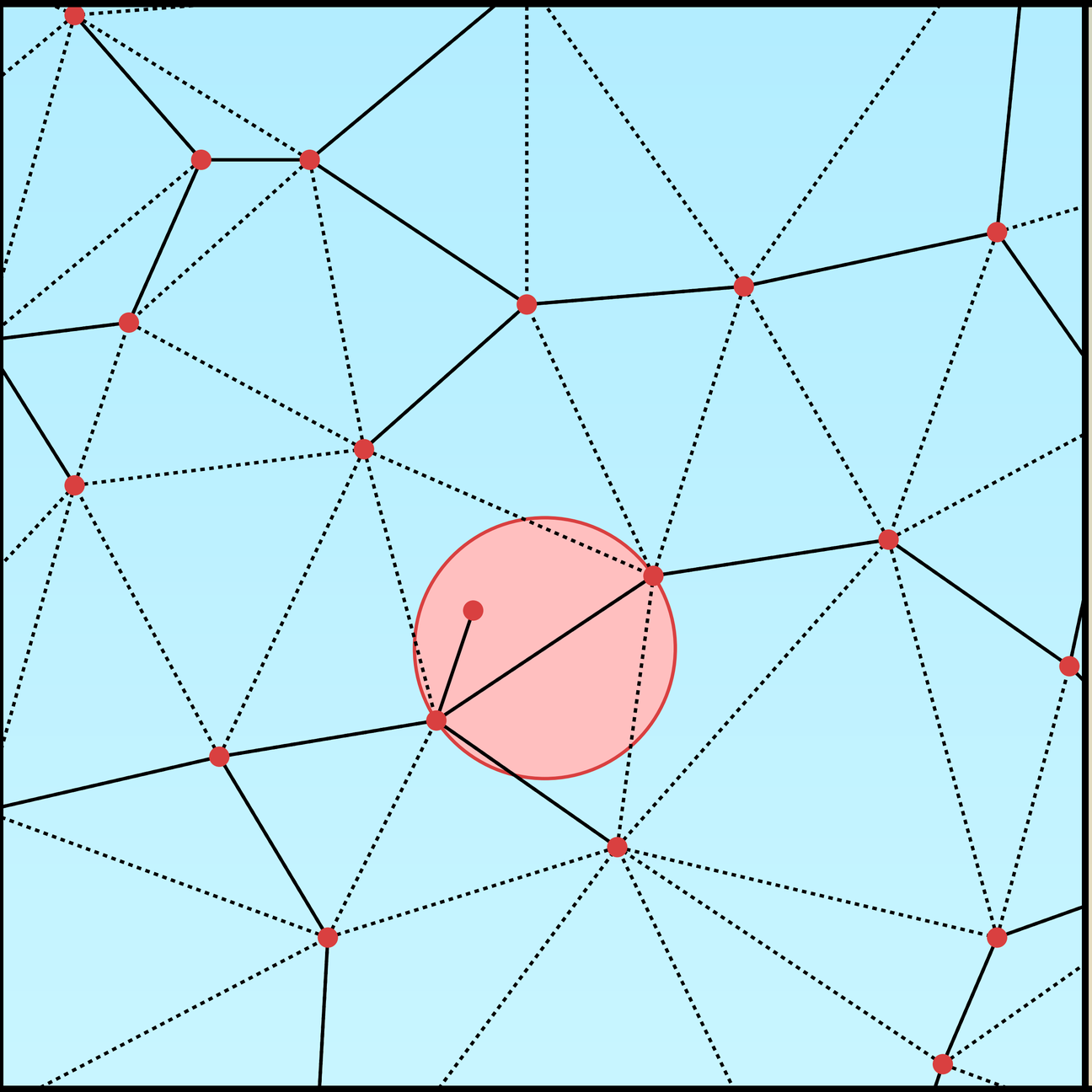

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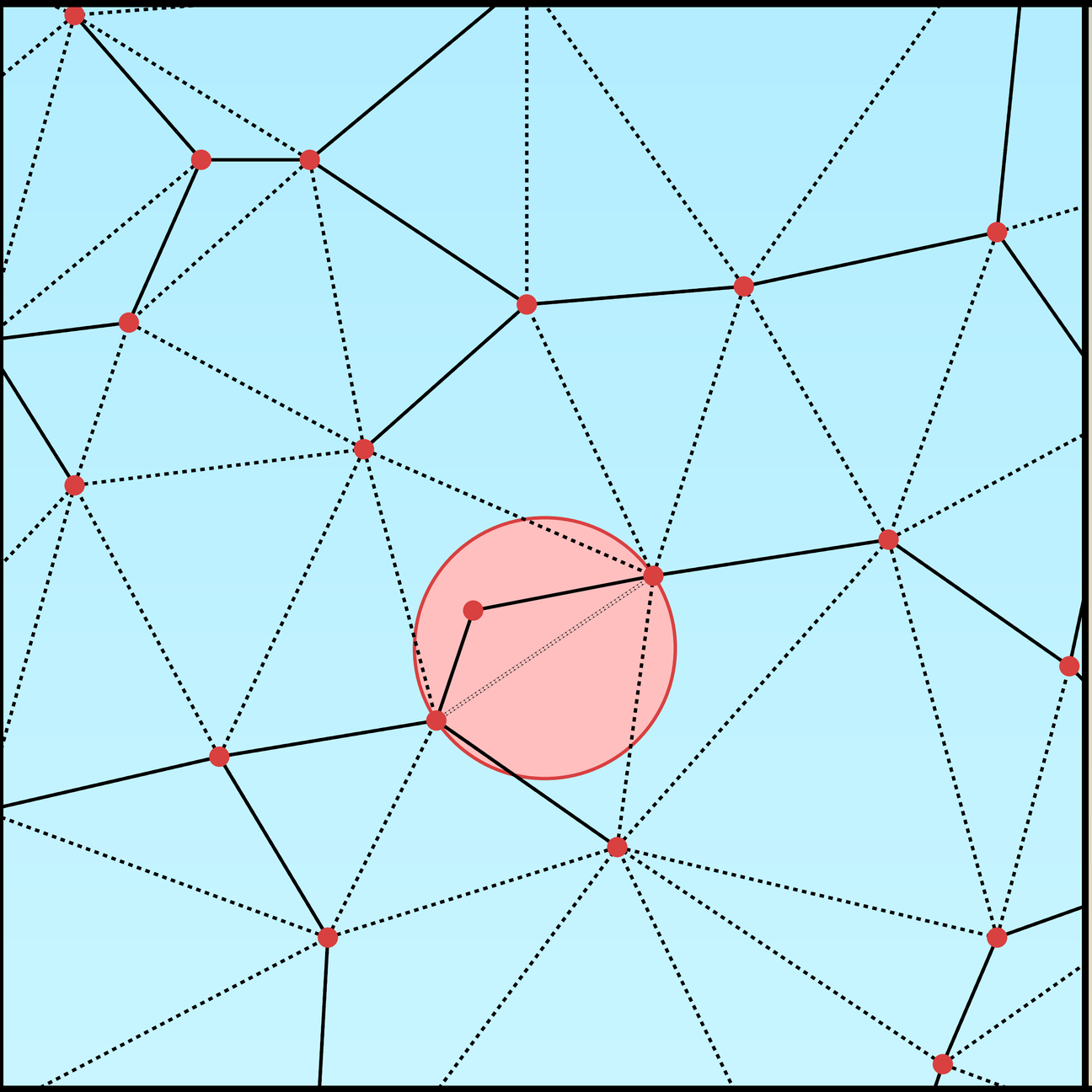

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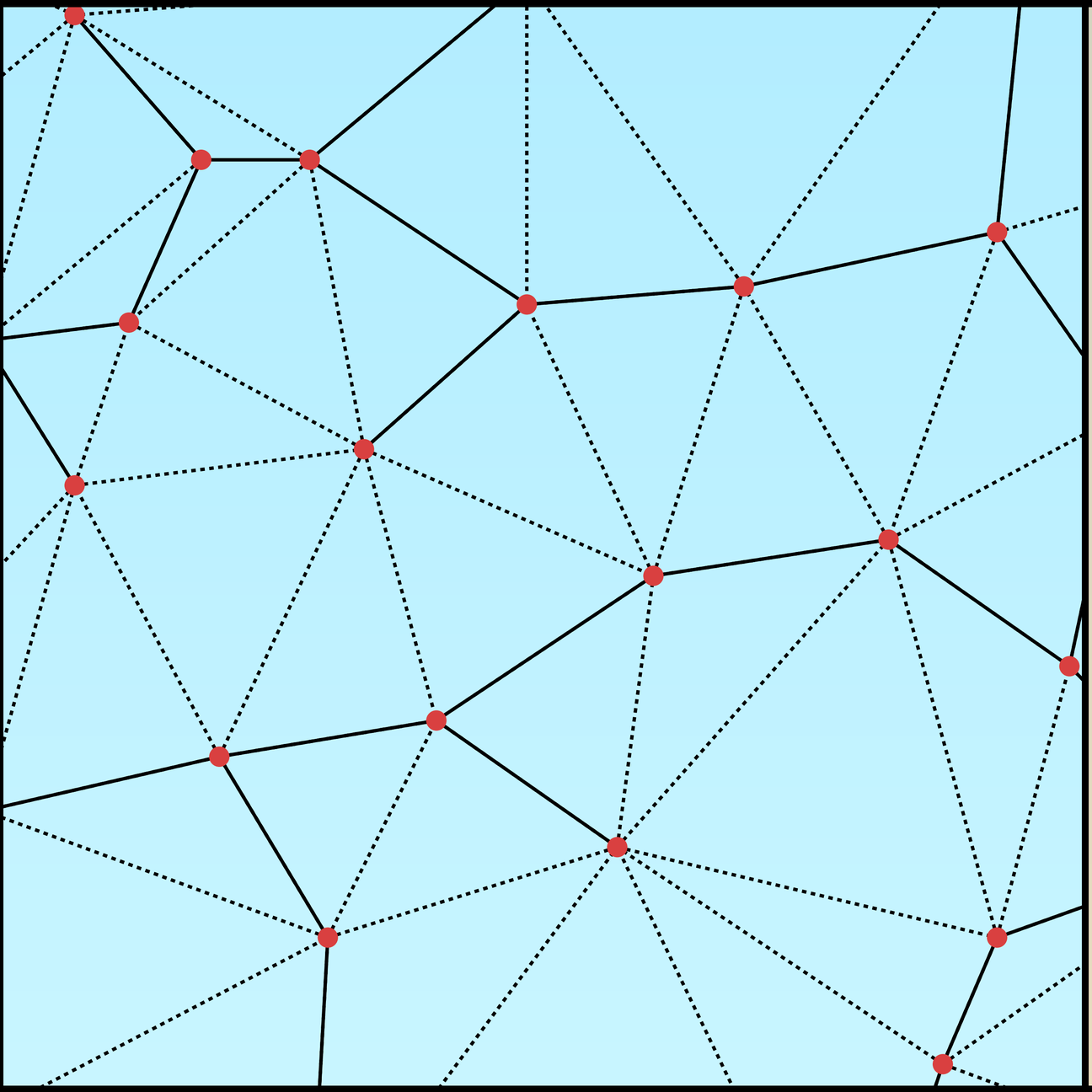

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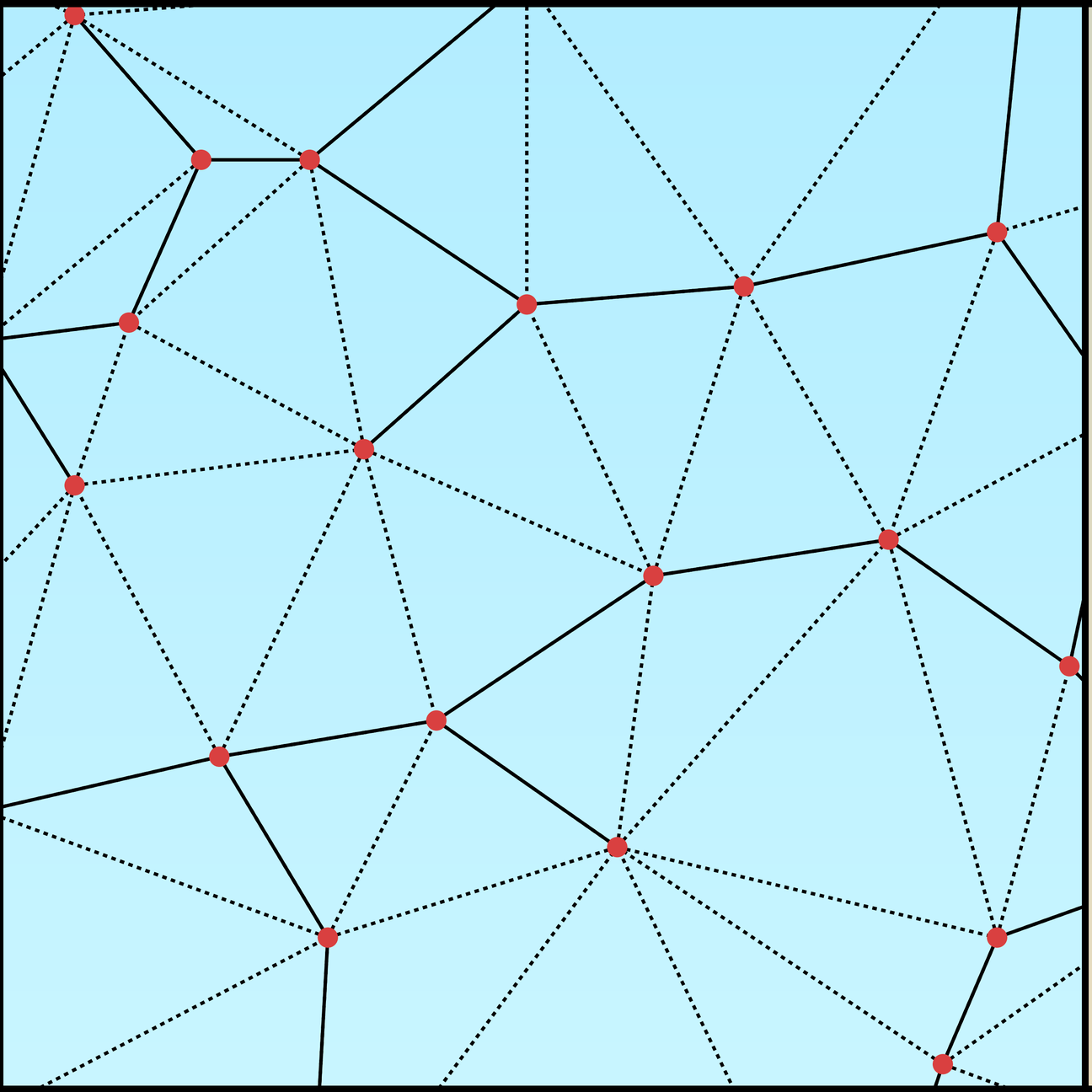

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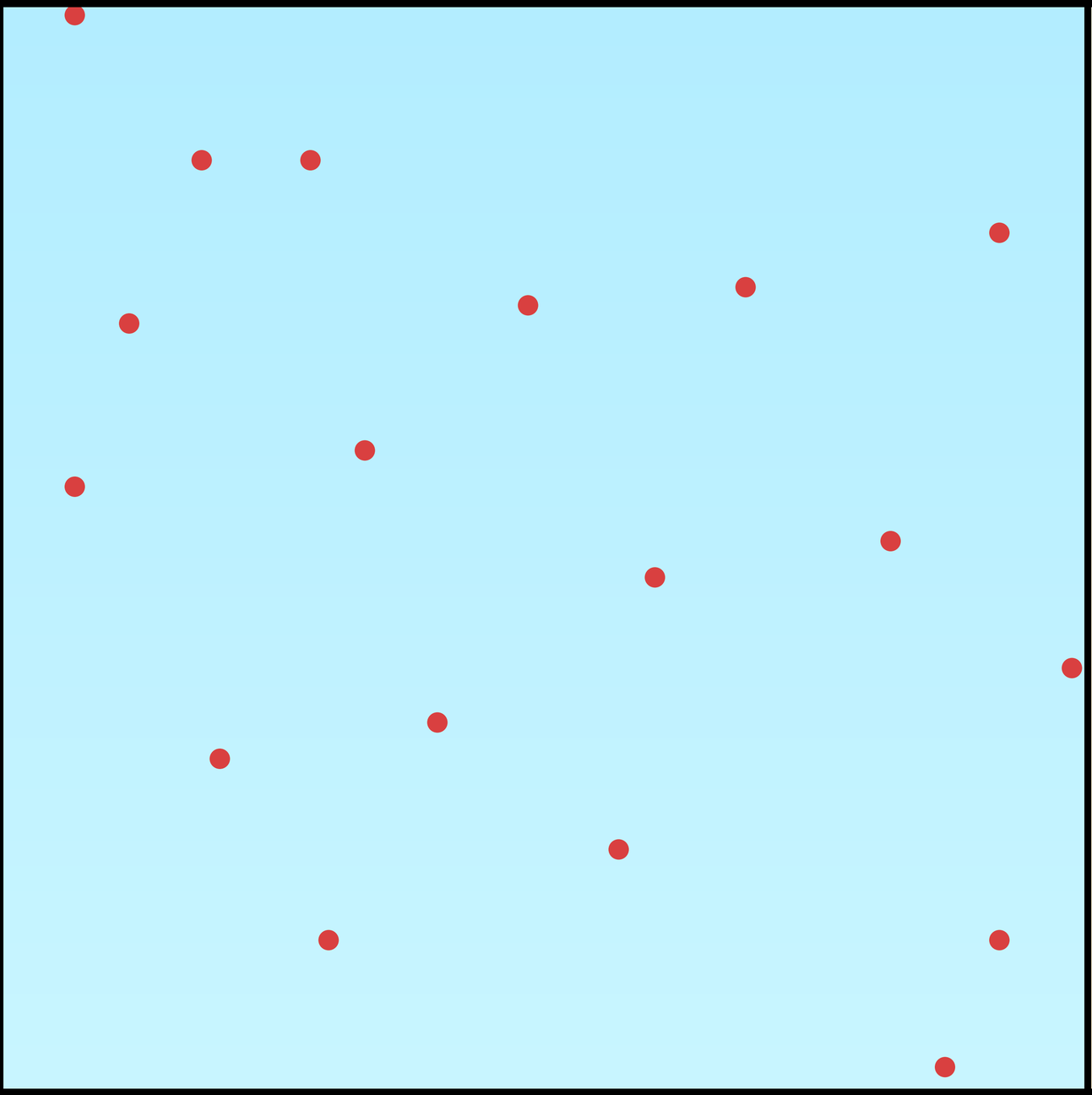
The EMST is not really a proximity structure, since it is not completely defined locally!

The *Nearest-Neighbour graph* is the graph that stores for every point in a point set $P$ its nearest neighbour.



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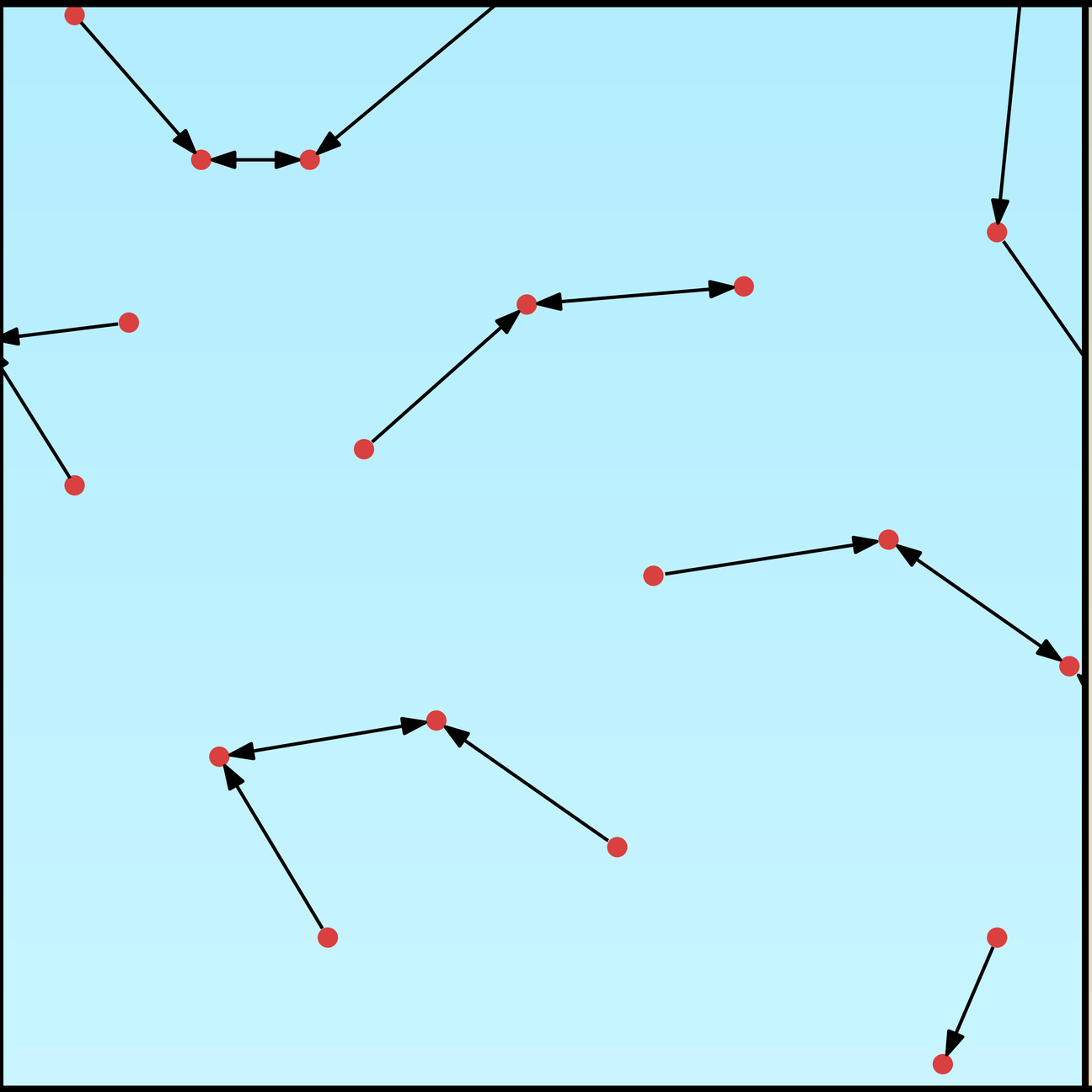

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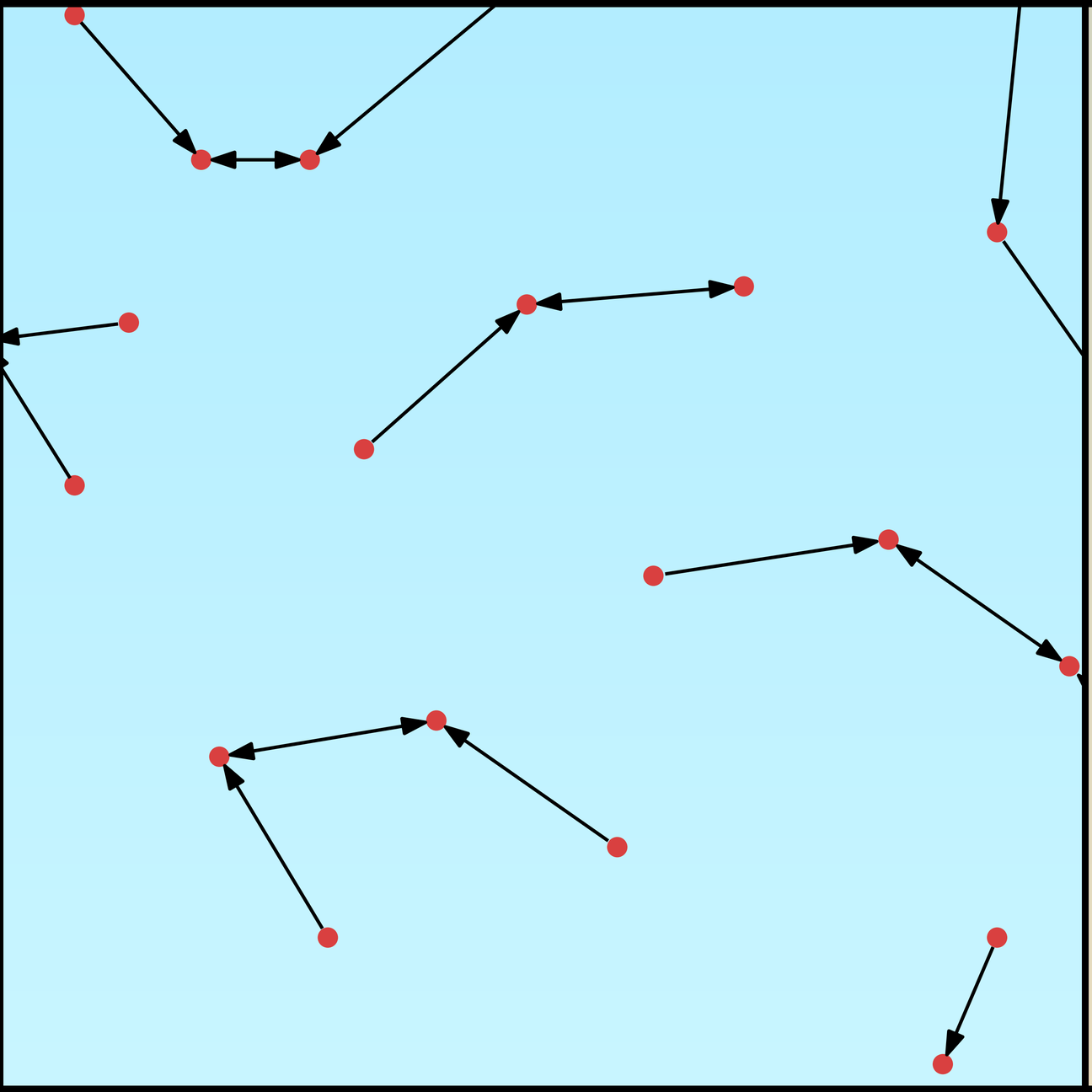

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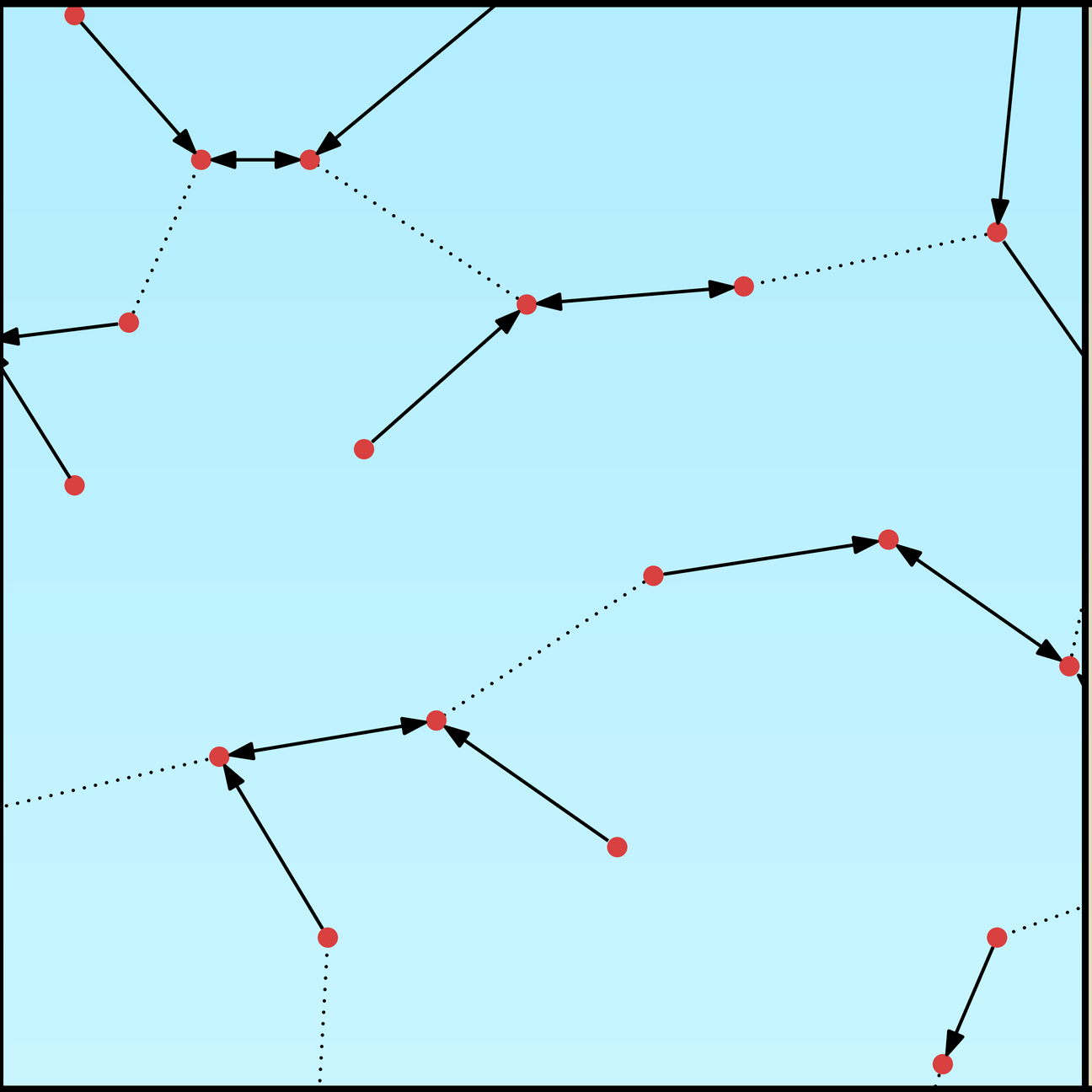

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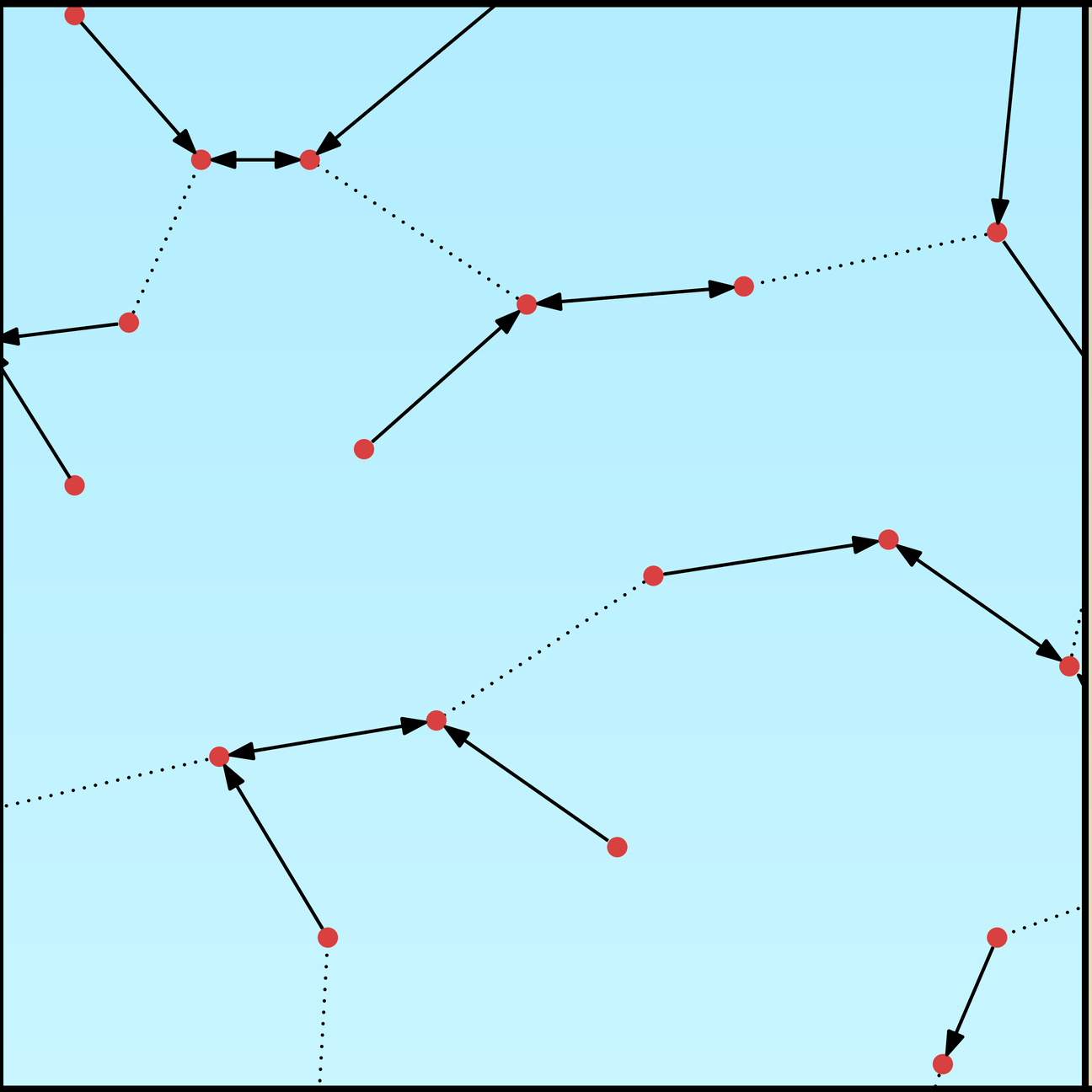

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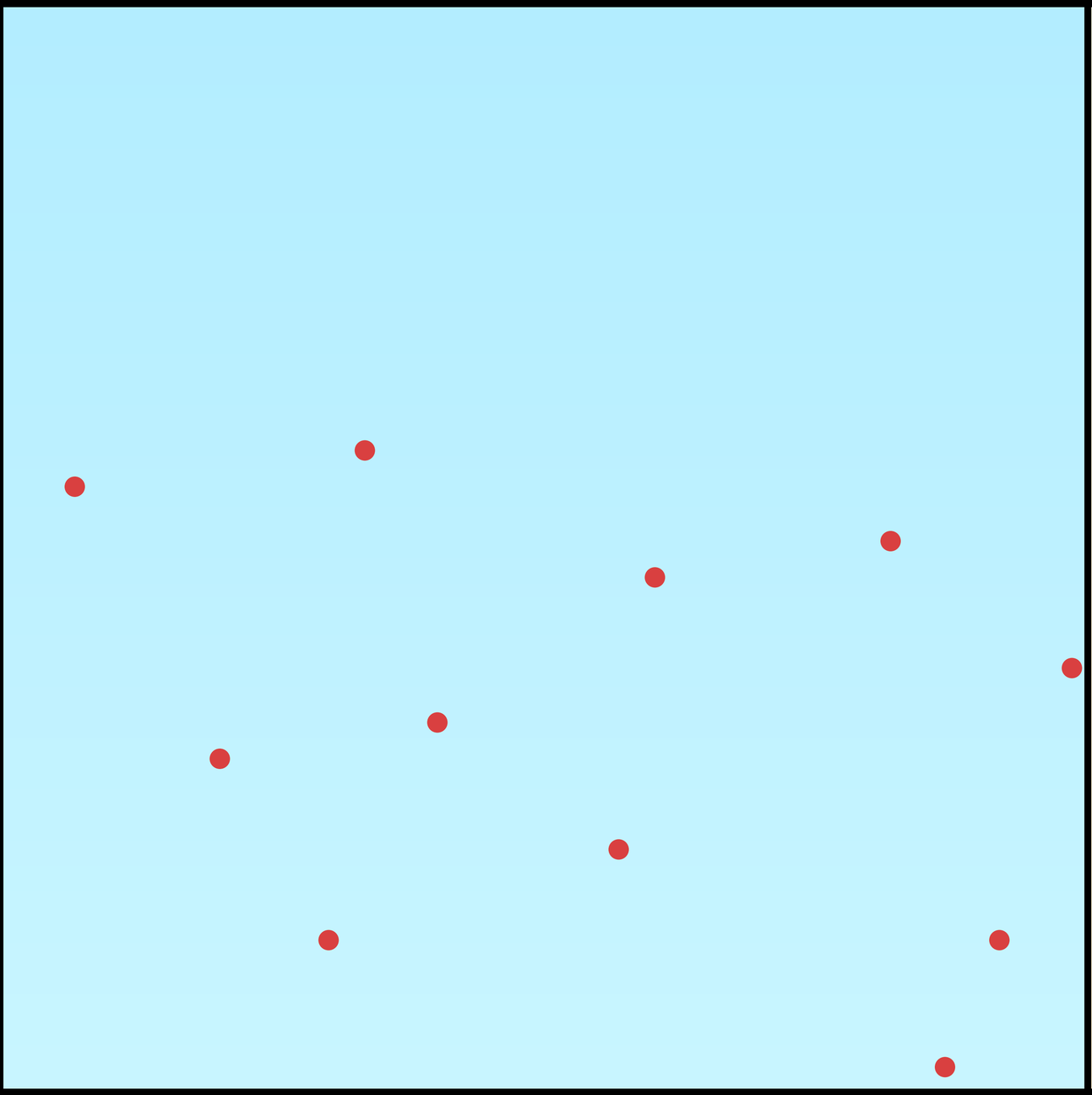
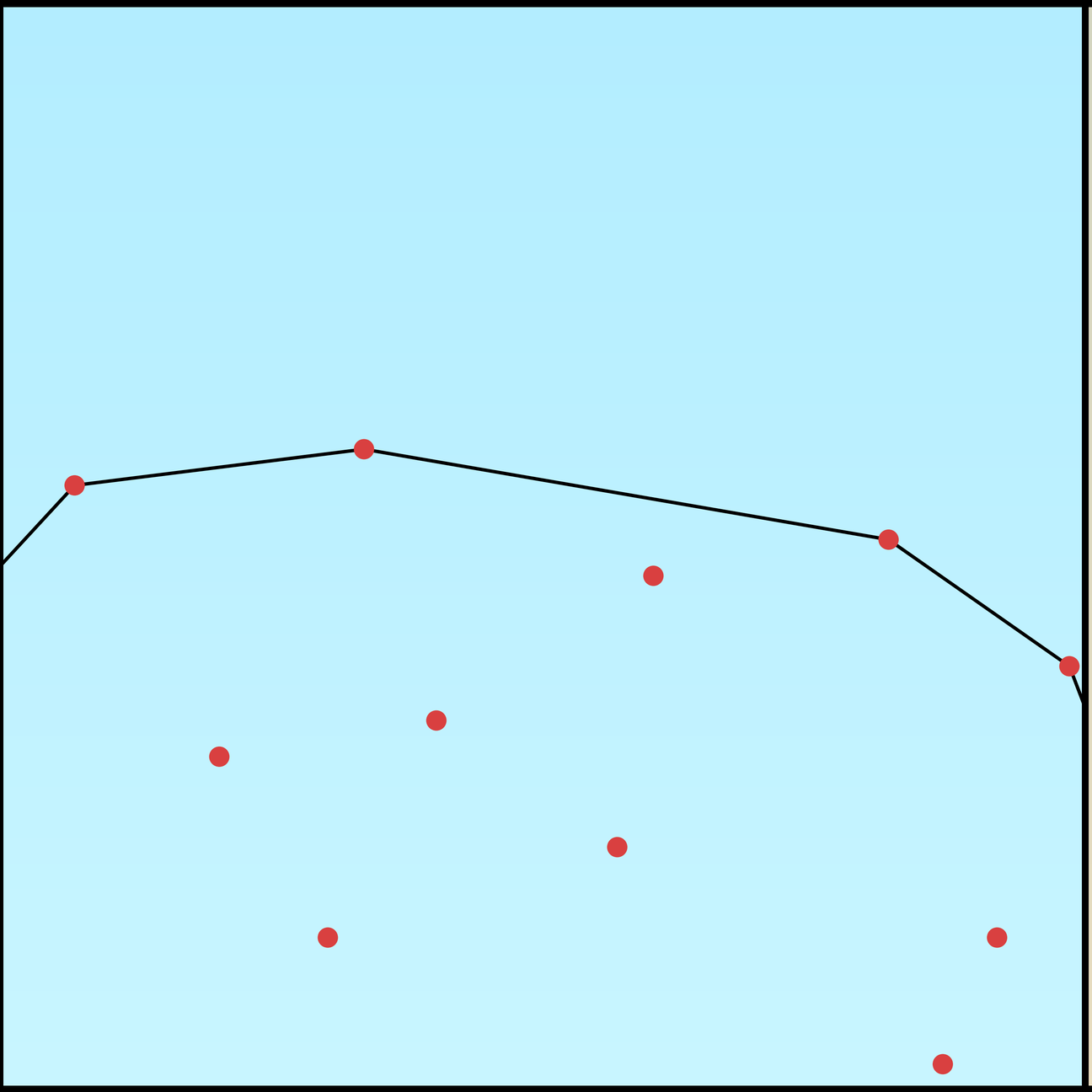
The NN graph is locally defined, but not necessarily connected.



Slight different: the *convex hull* of a point set $P$ is the smallest convex set that contains $P$.



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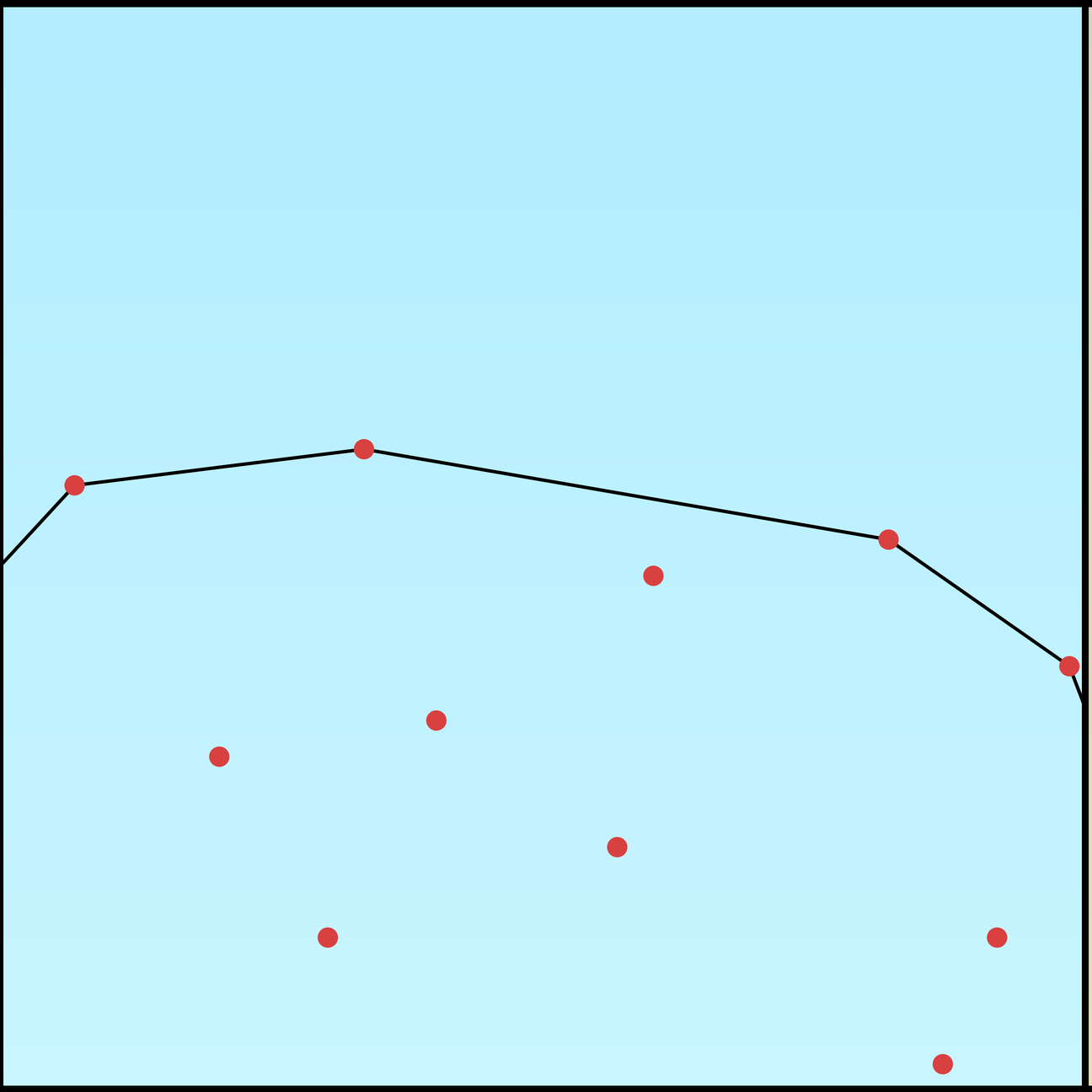

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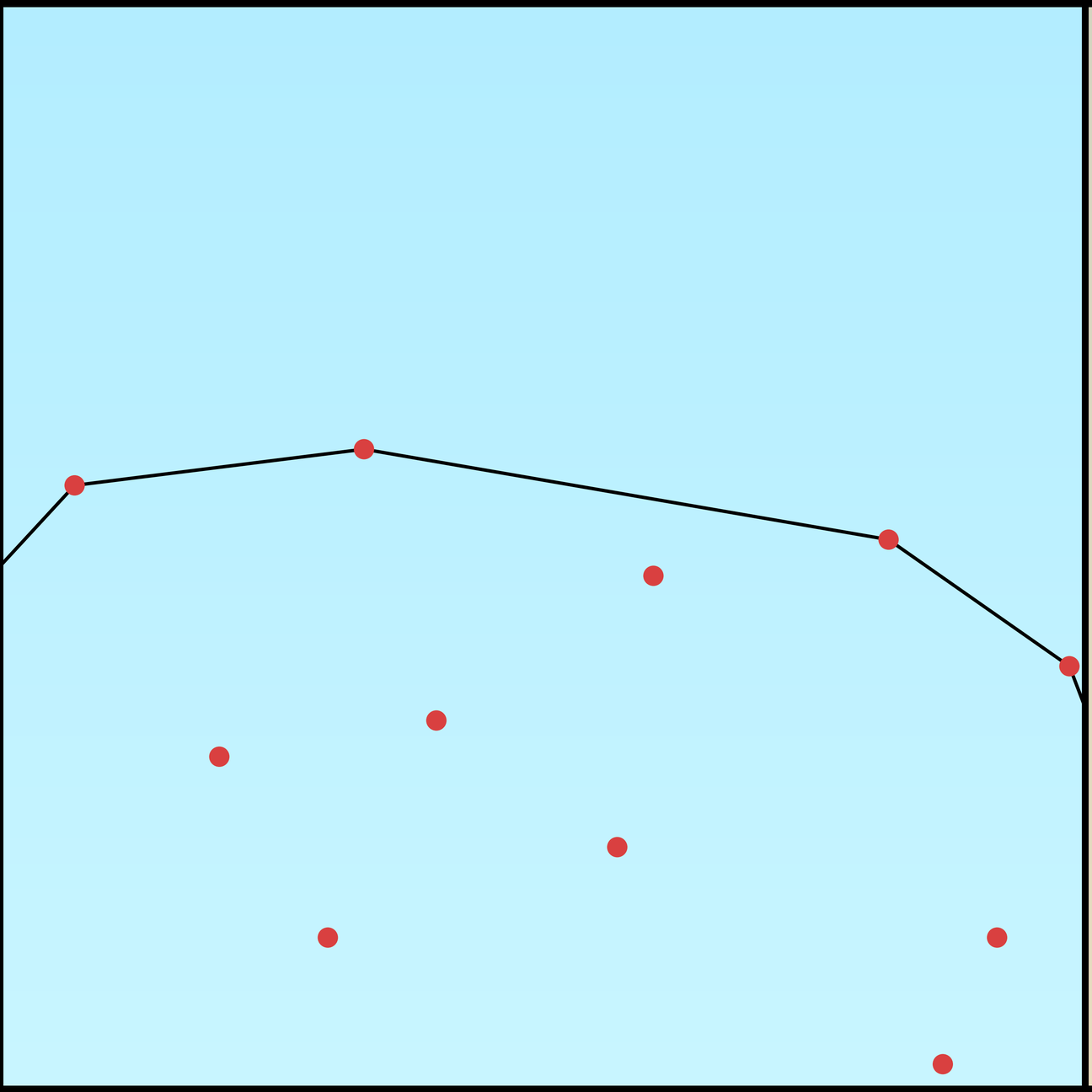

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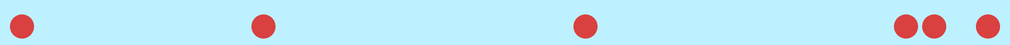

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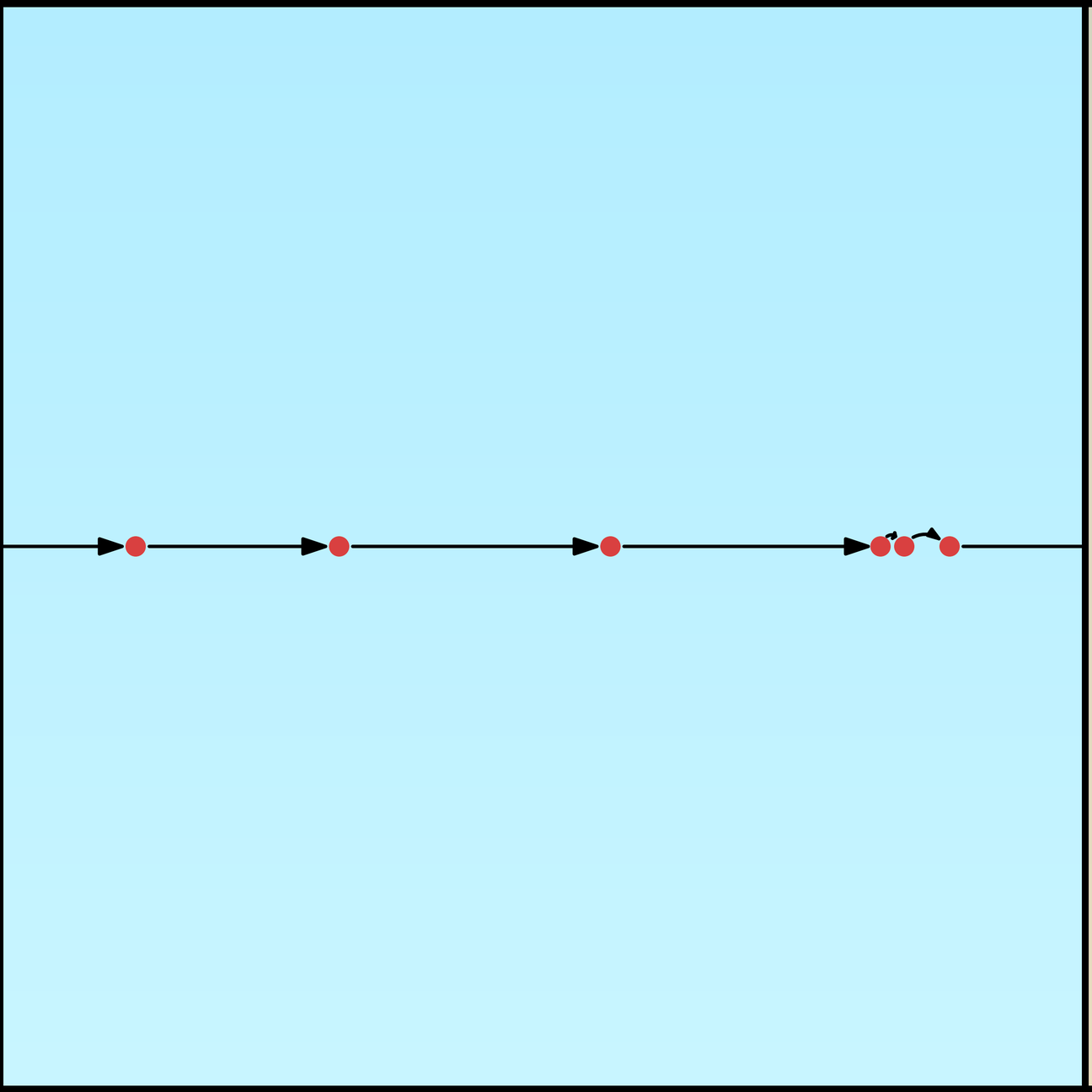

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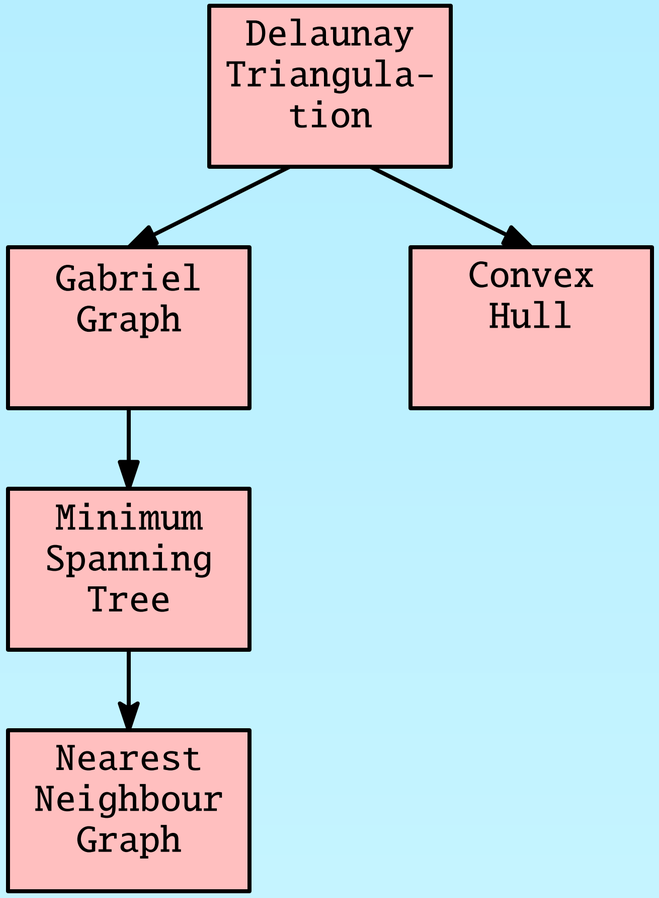


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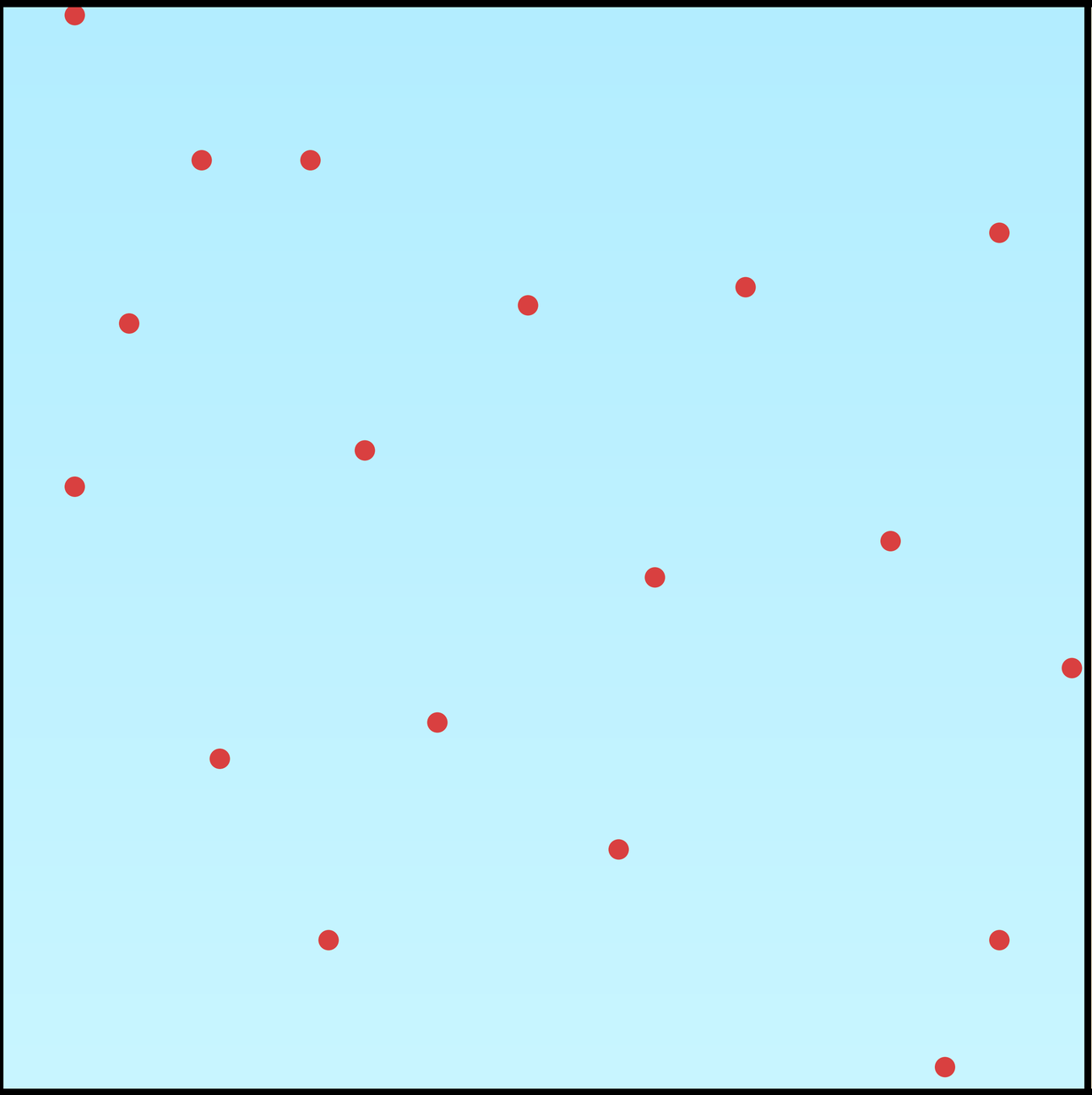

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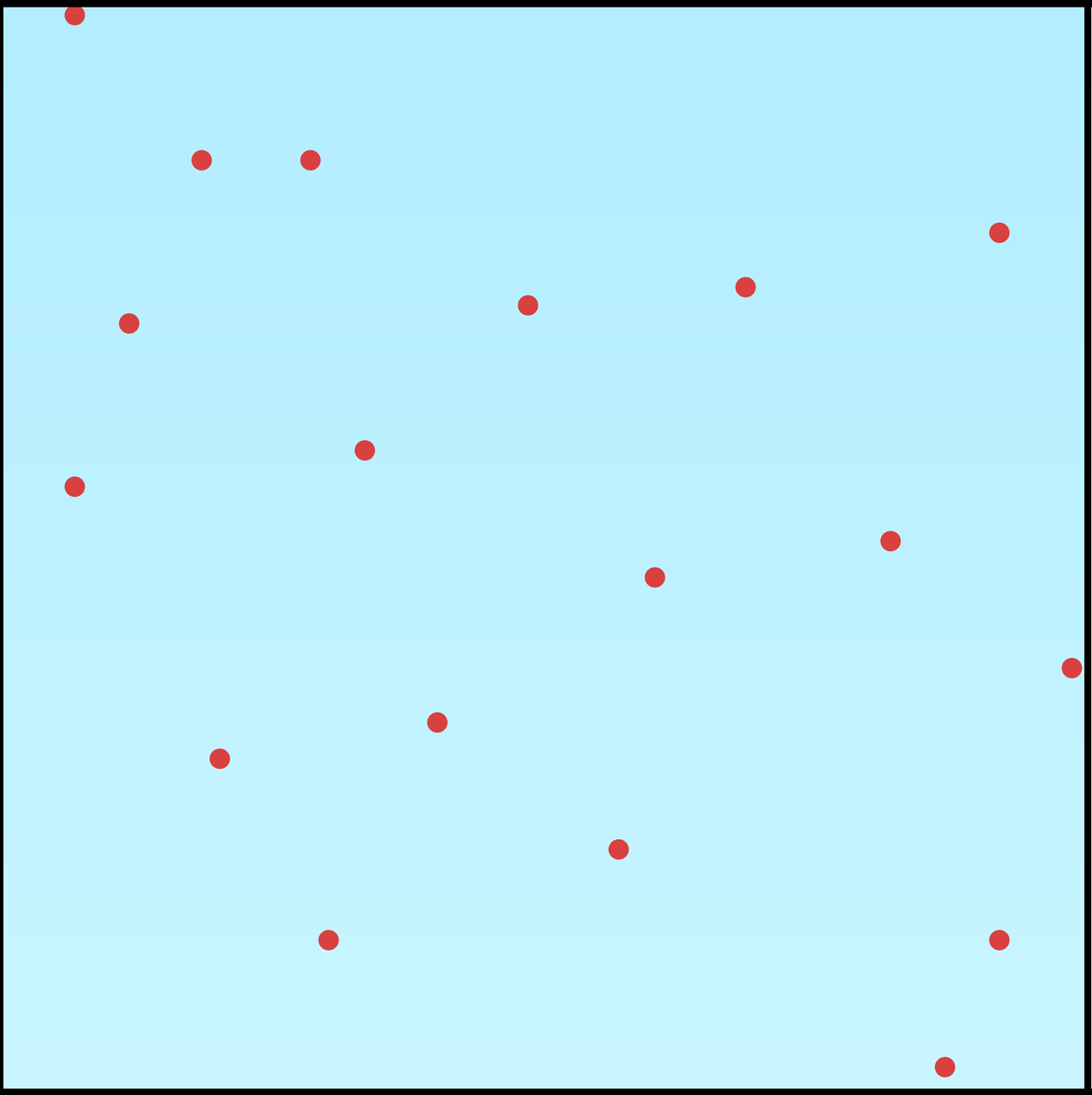

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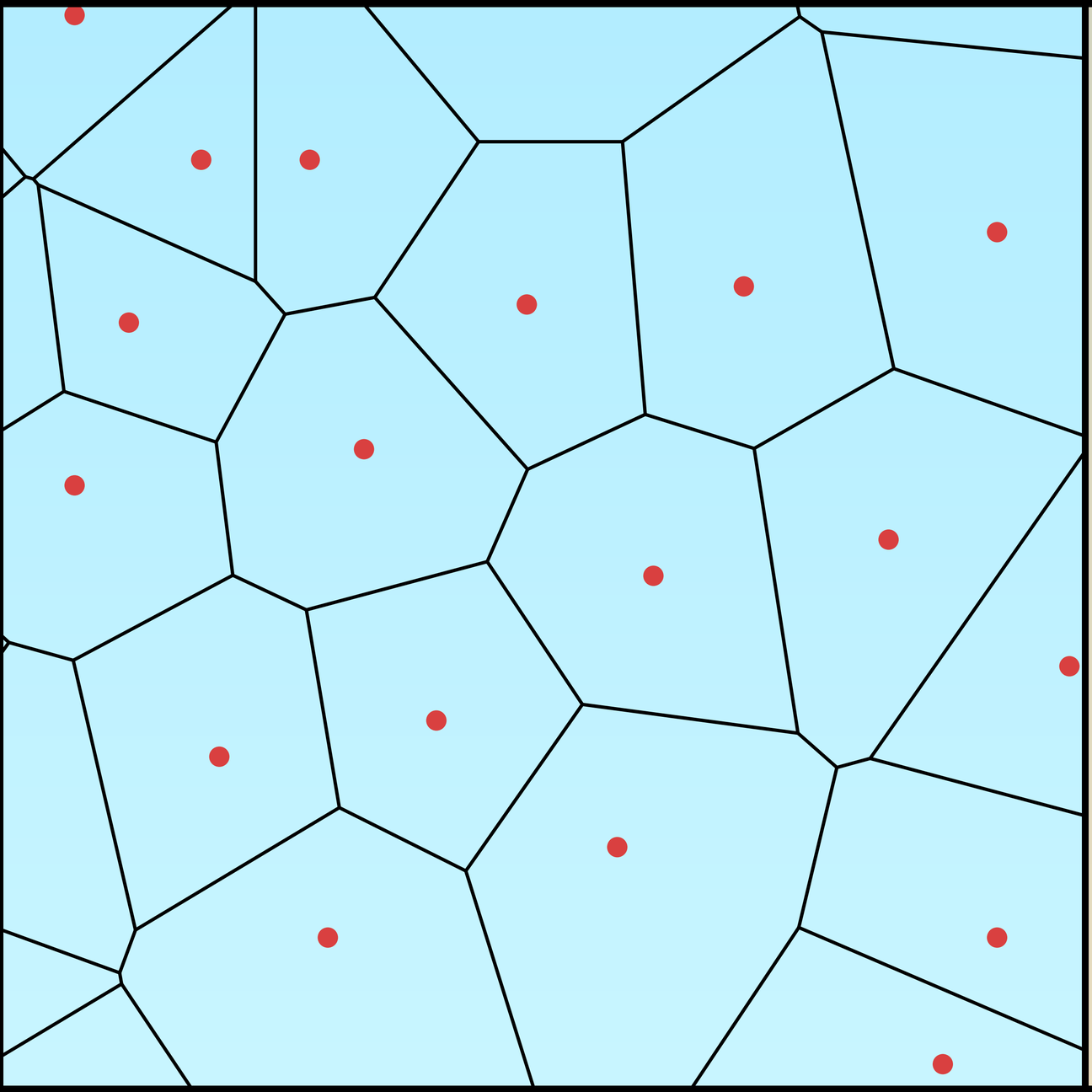
The *Voronoï diagram* of $P$ is the subdivision of $\mathbb{R}^2$ into cells for which the same point of $P$ is the closest.

[Dirichlet, 1850]



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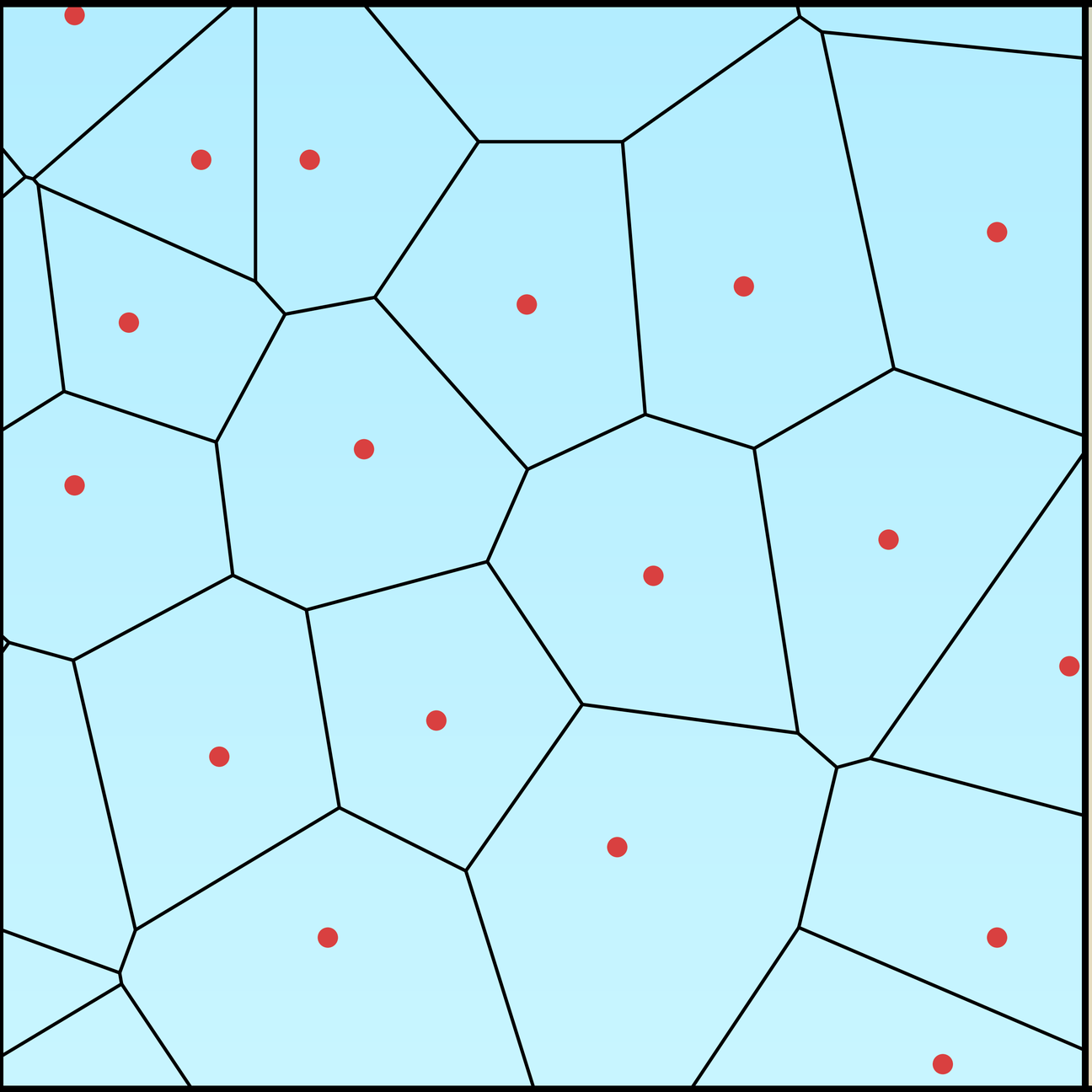

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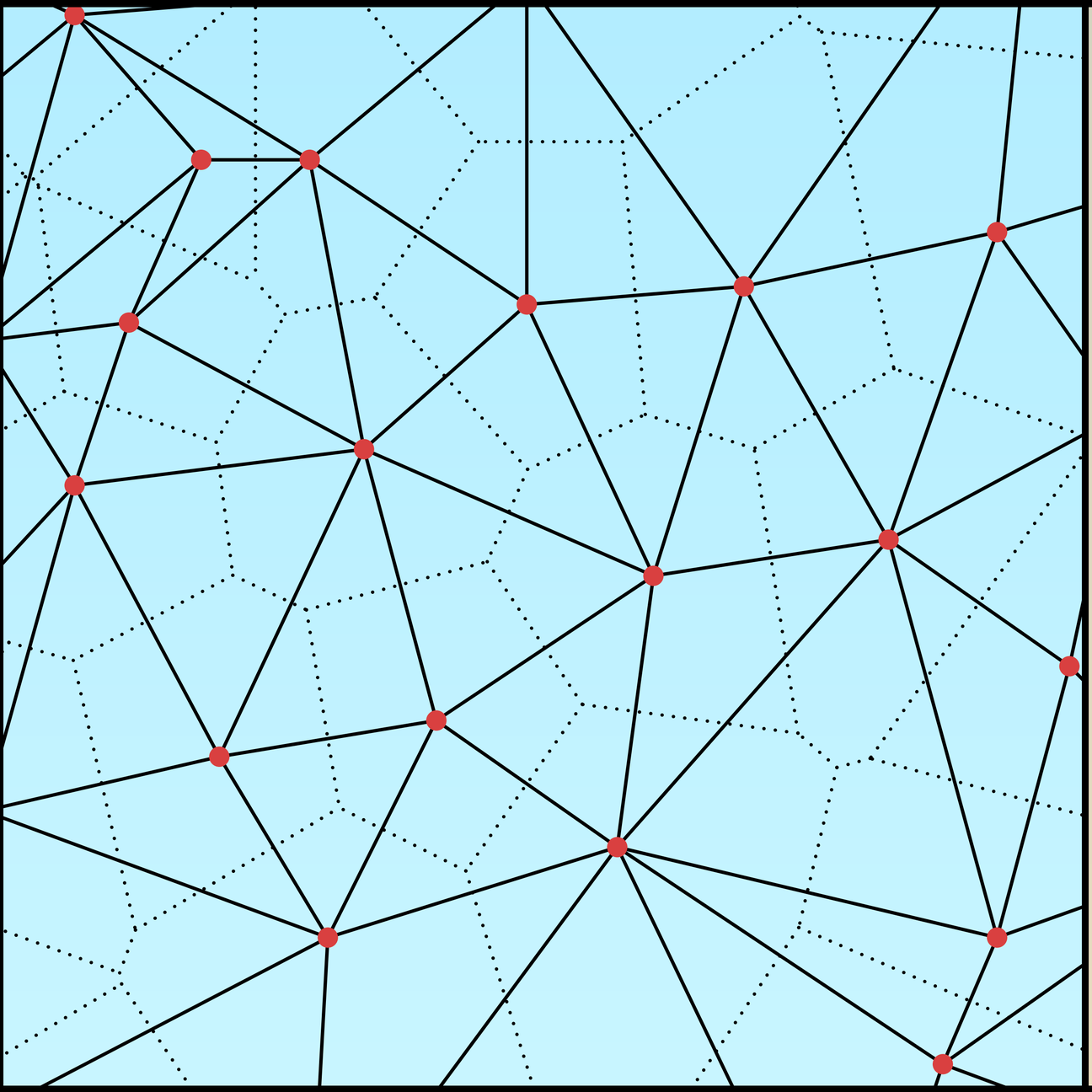

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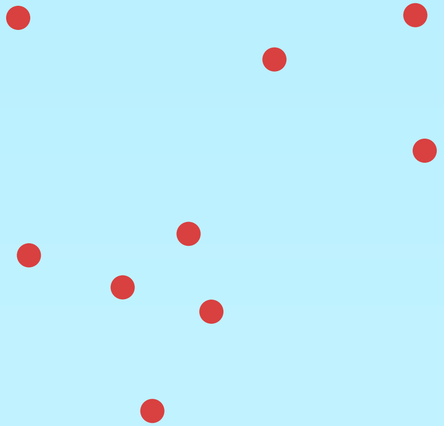

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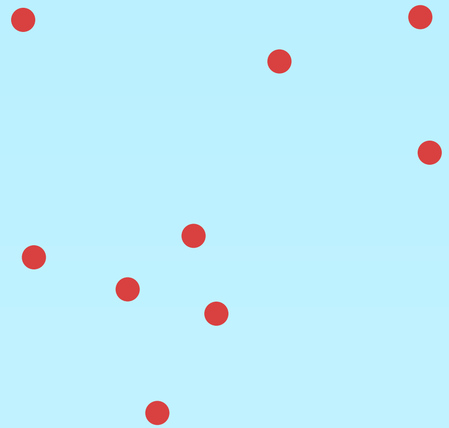

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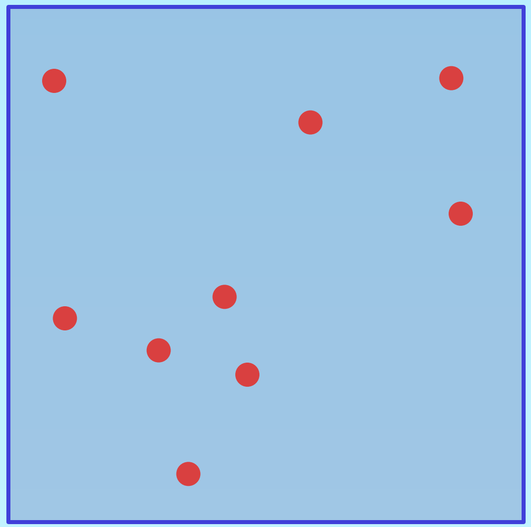

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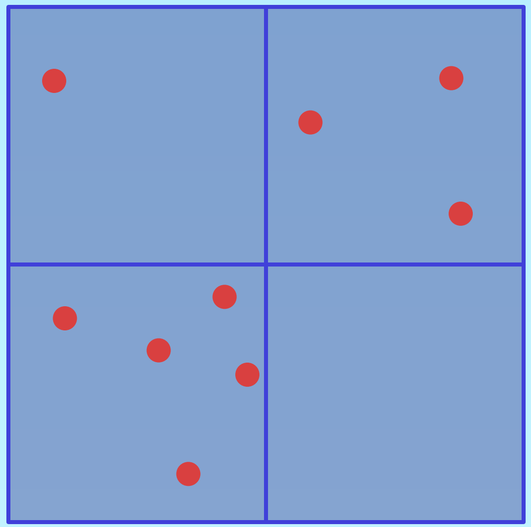

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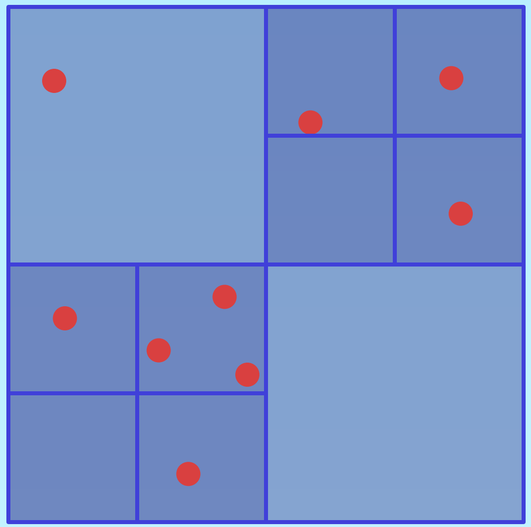

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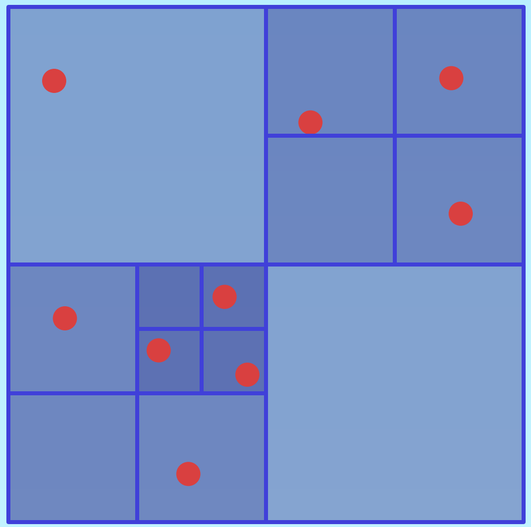

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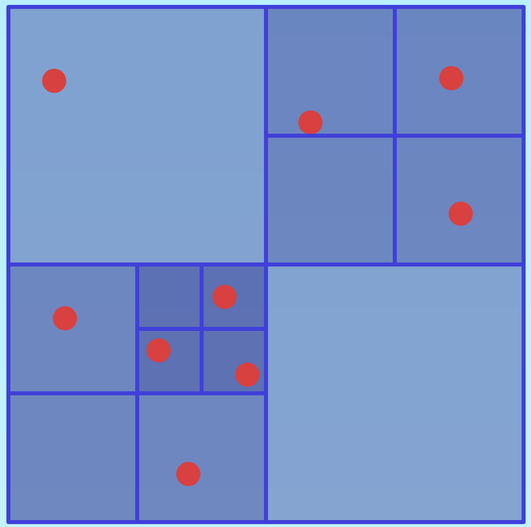

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For point sets with large spread, a quadtree needs to be *compressed*.

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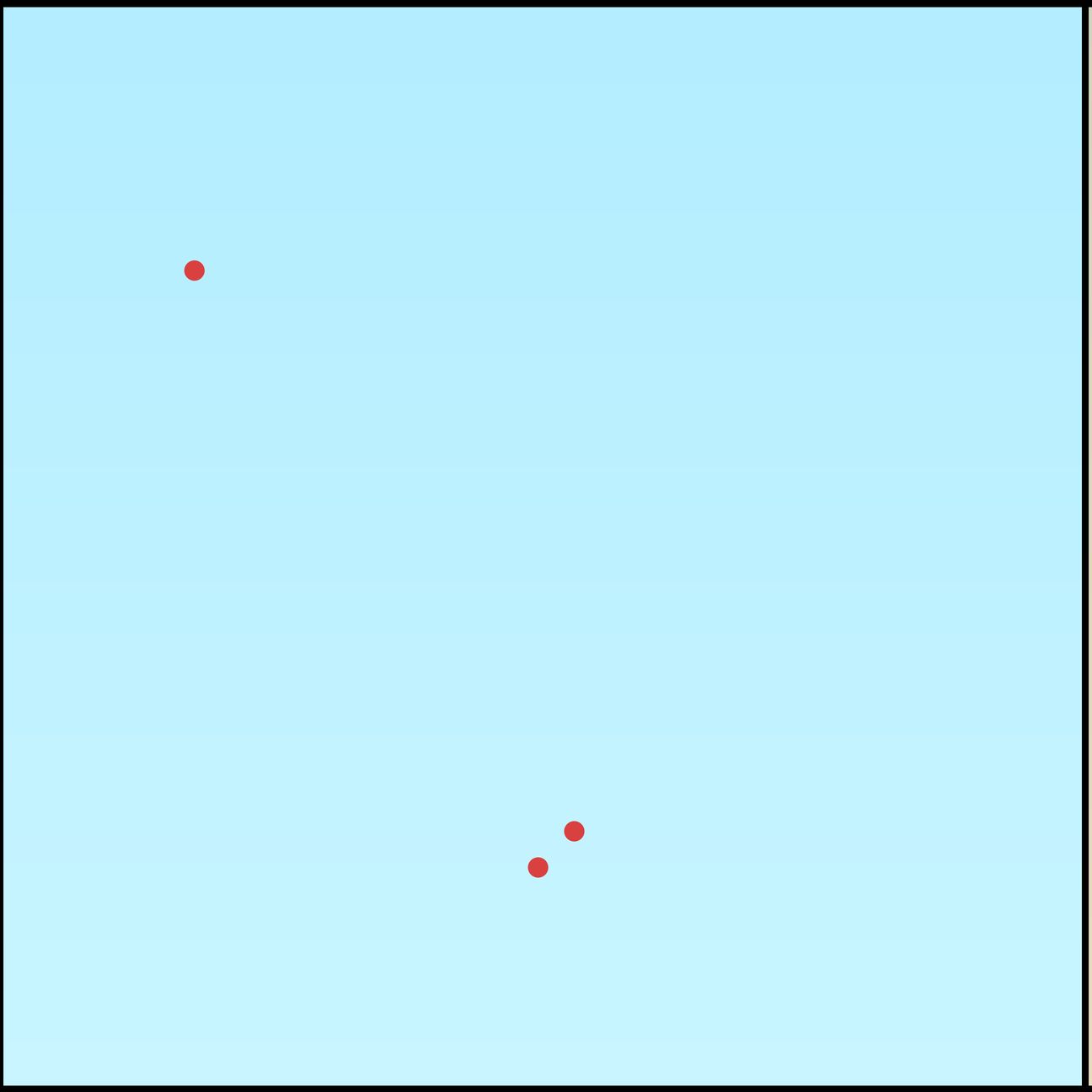

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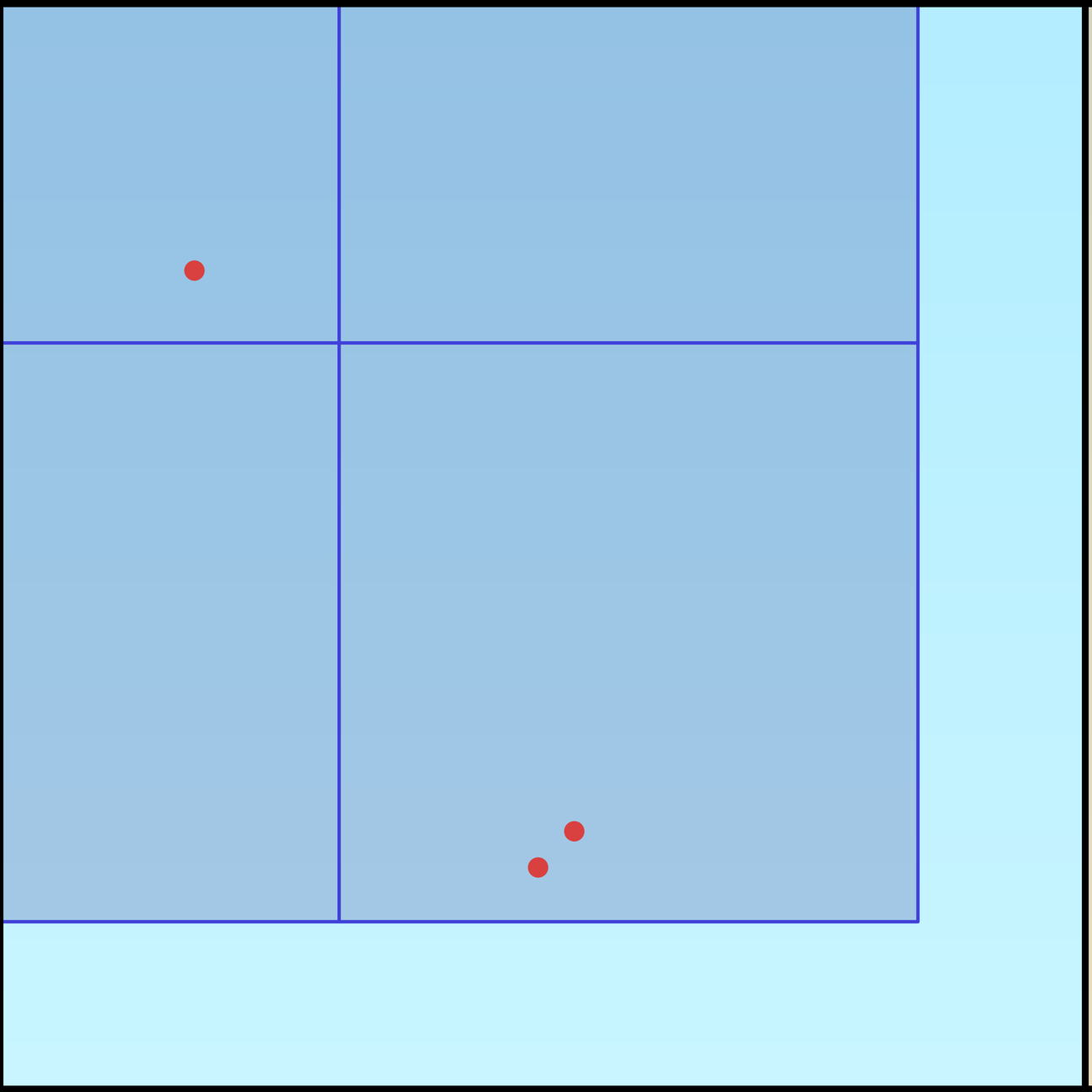

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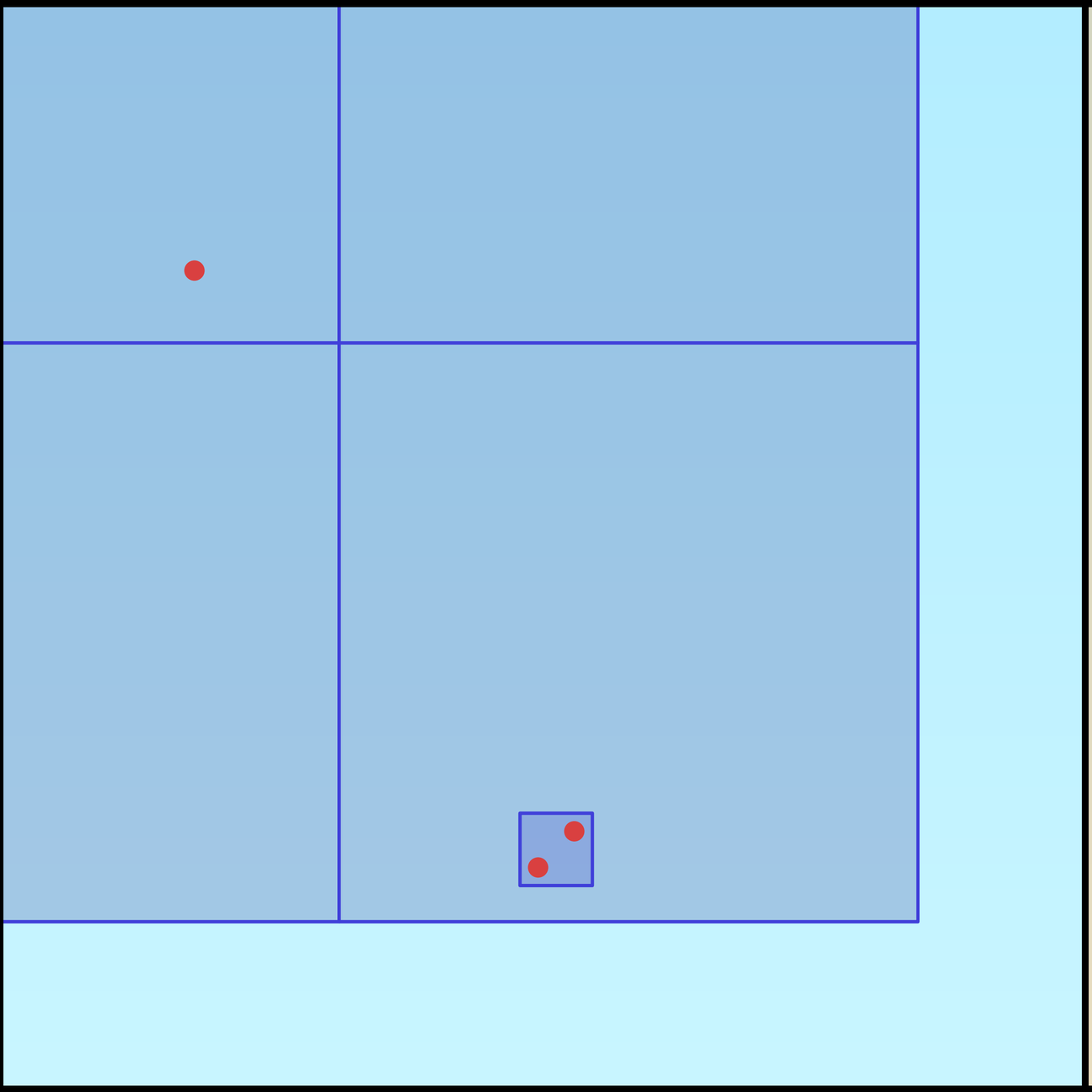

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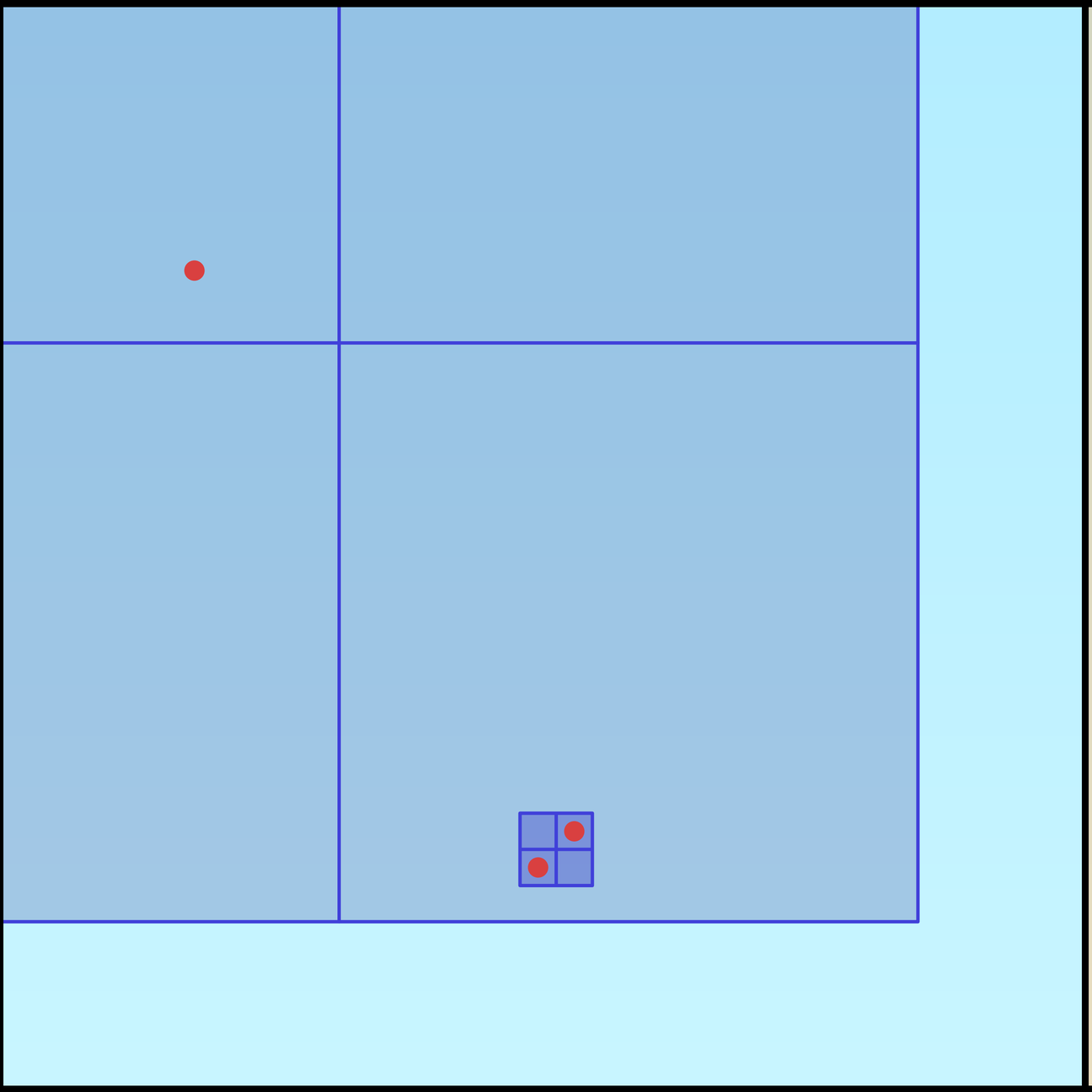

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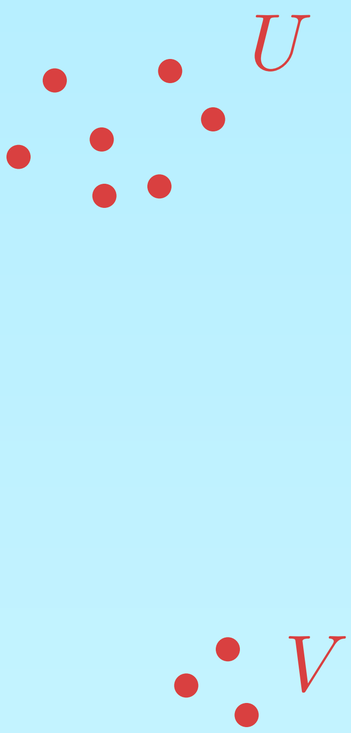
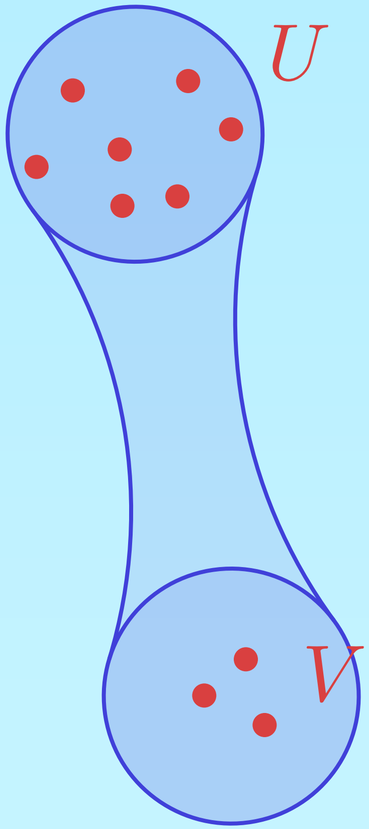



A *$c$-well-separated pair* is a pair of point sets $U, V$ whose distance to each other is at least $c$ times their diameters.

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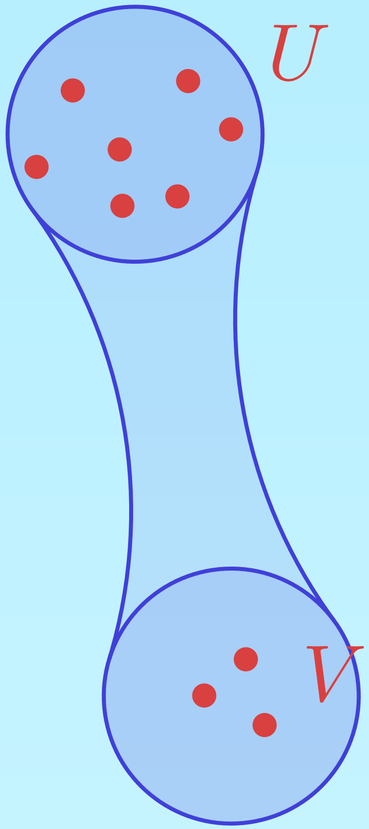


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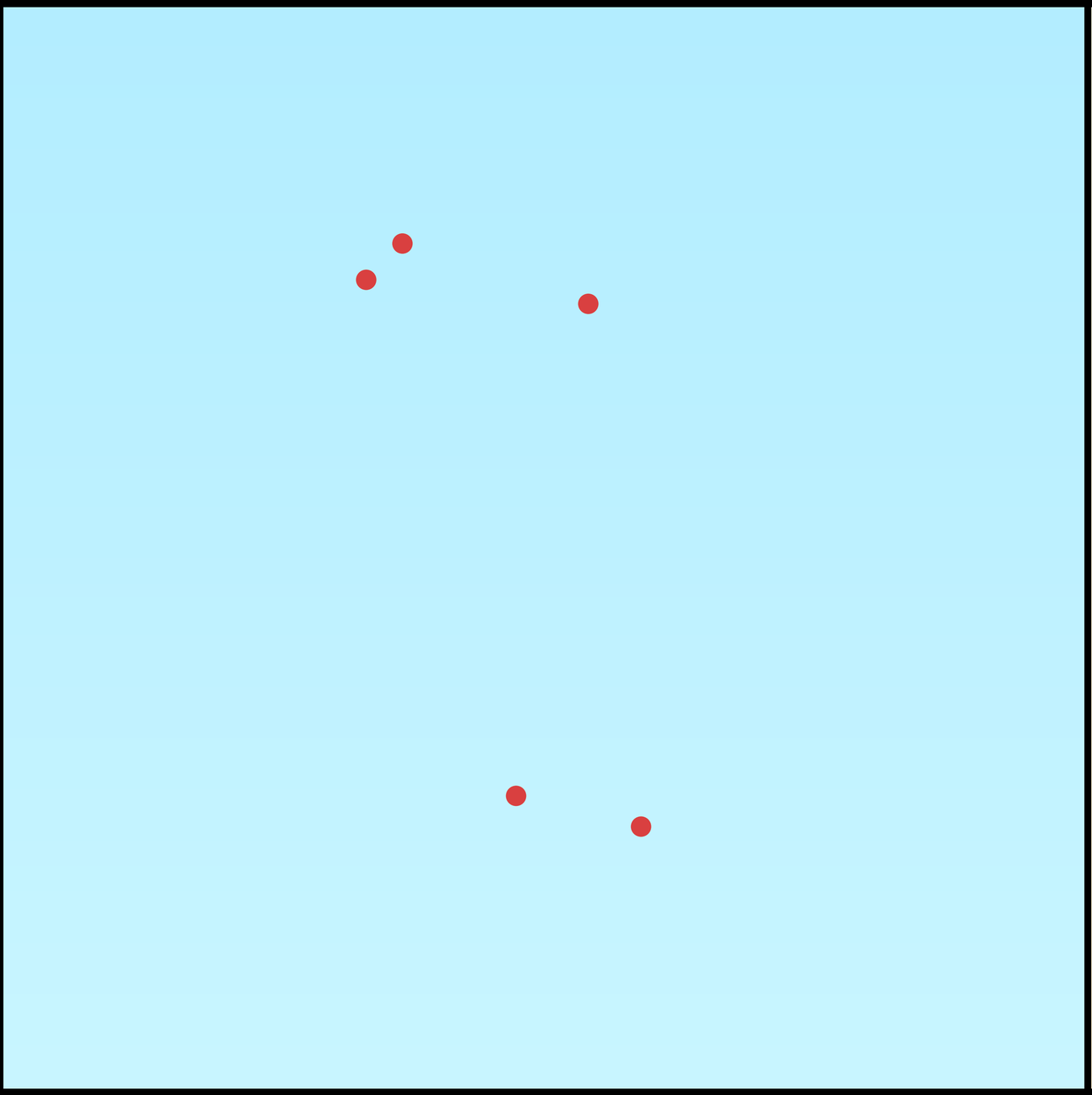

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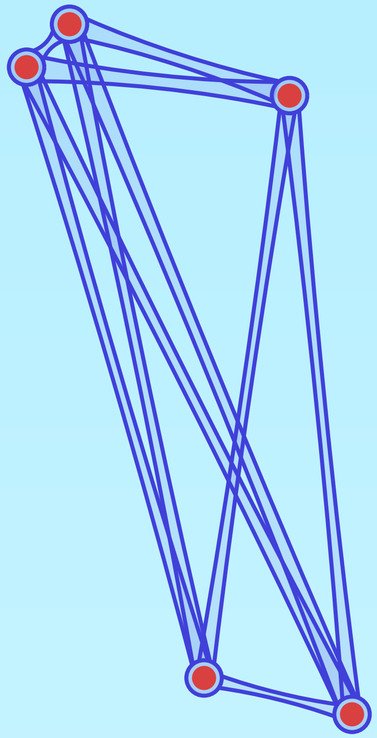

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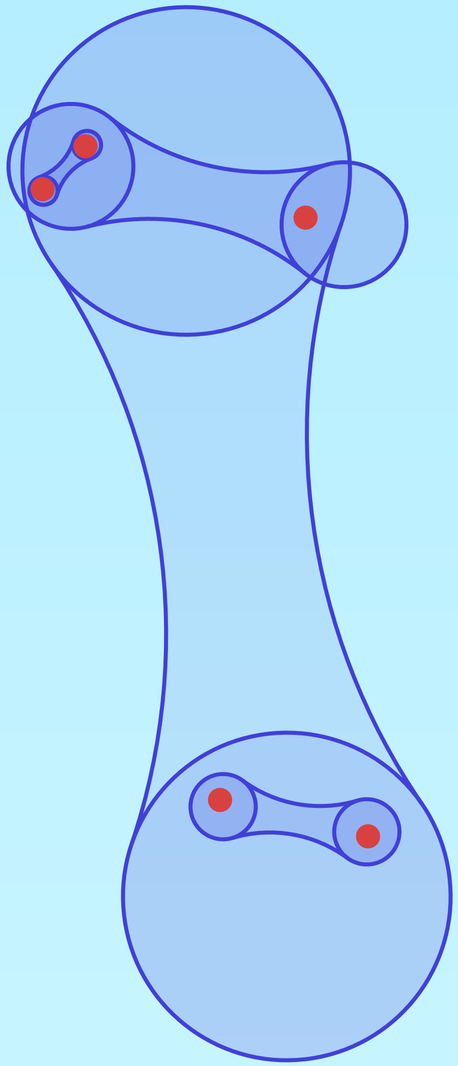

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# Chapter 1
## *PROXIMITY STRUCTURES*

Point sets and proximity

Computational questions

State of the art in 1D

State of the art in 2D

A *pointer machine* models how a computer can access and manipulate *data*.

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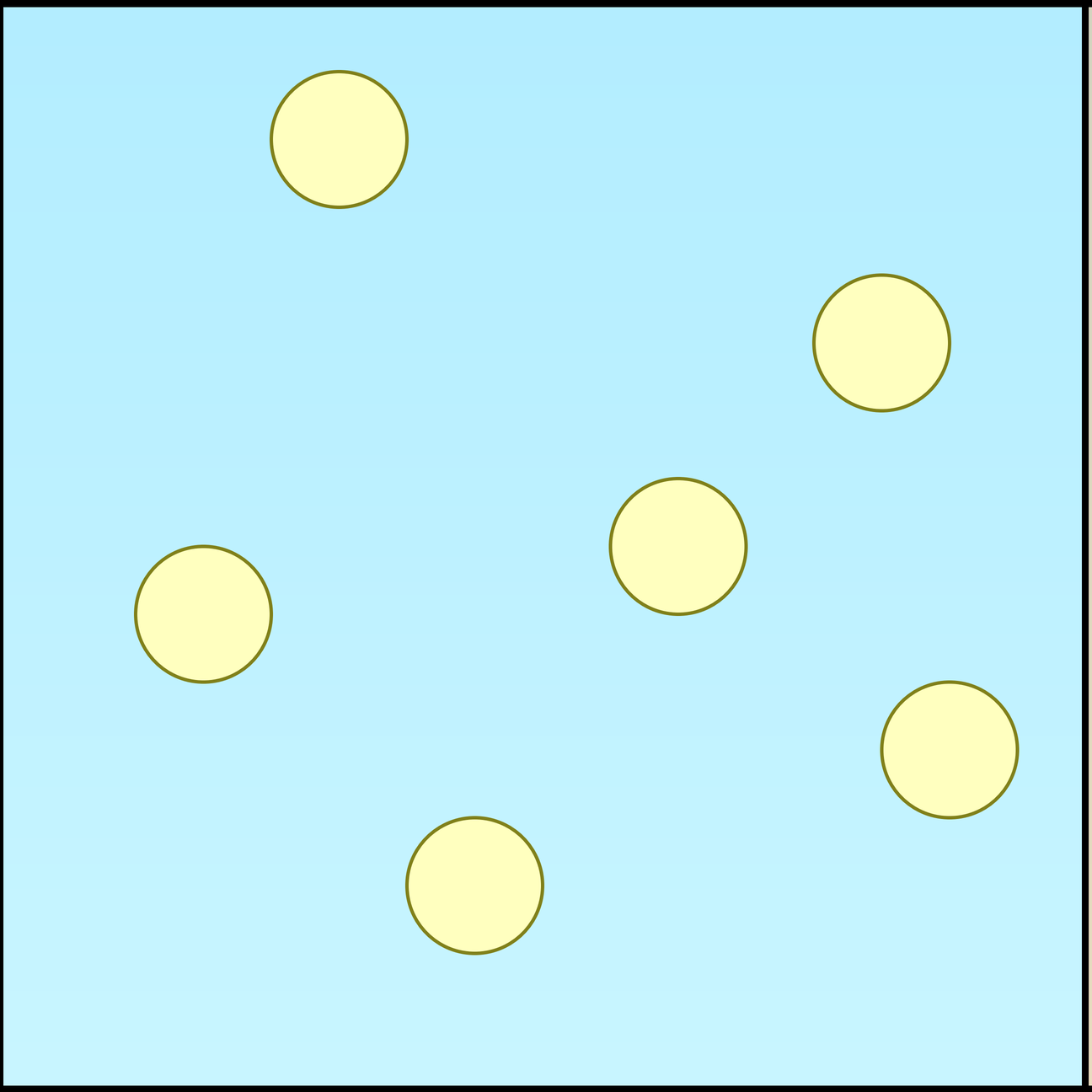
Data is stored in *cells*.



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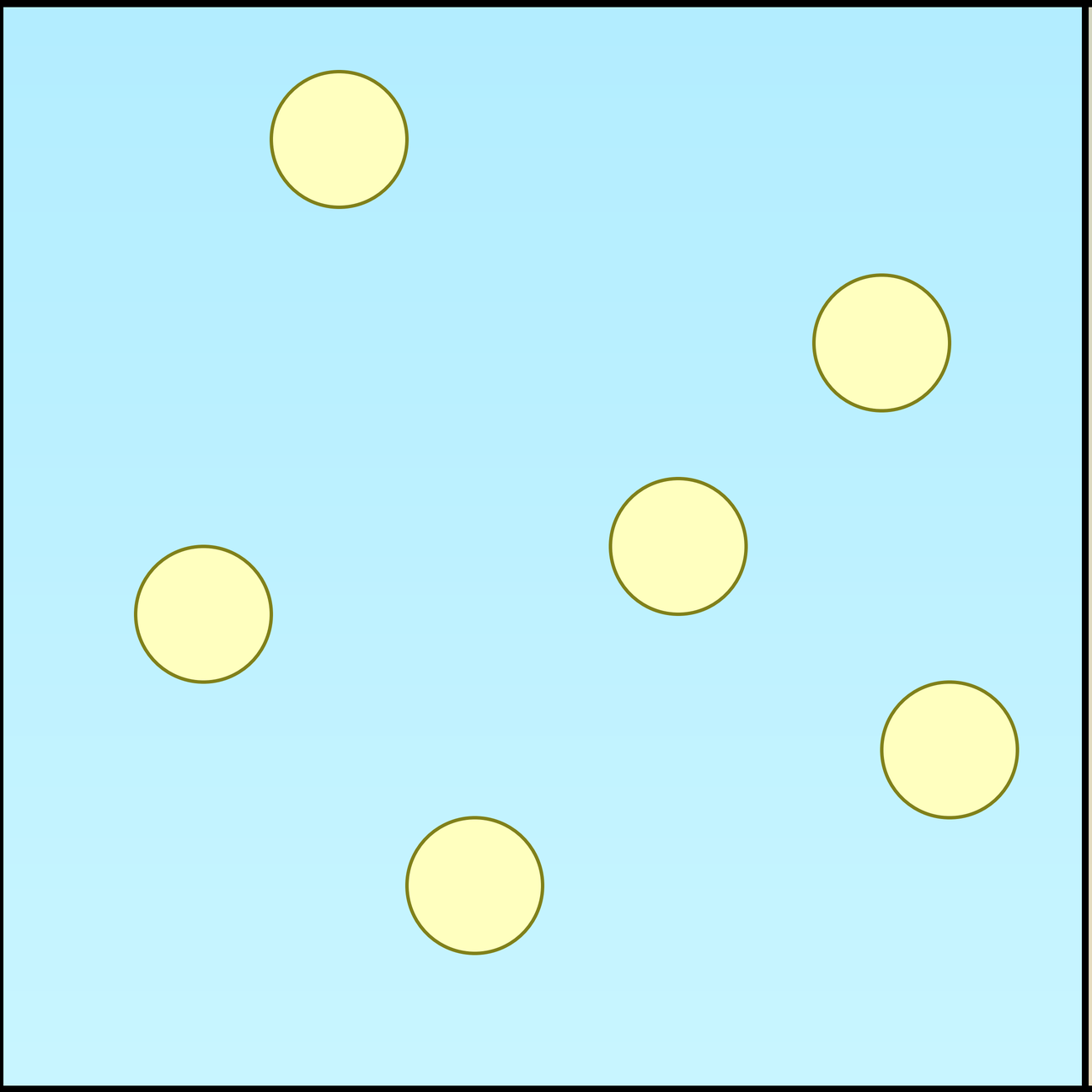

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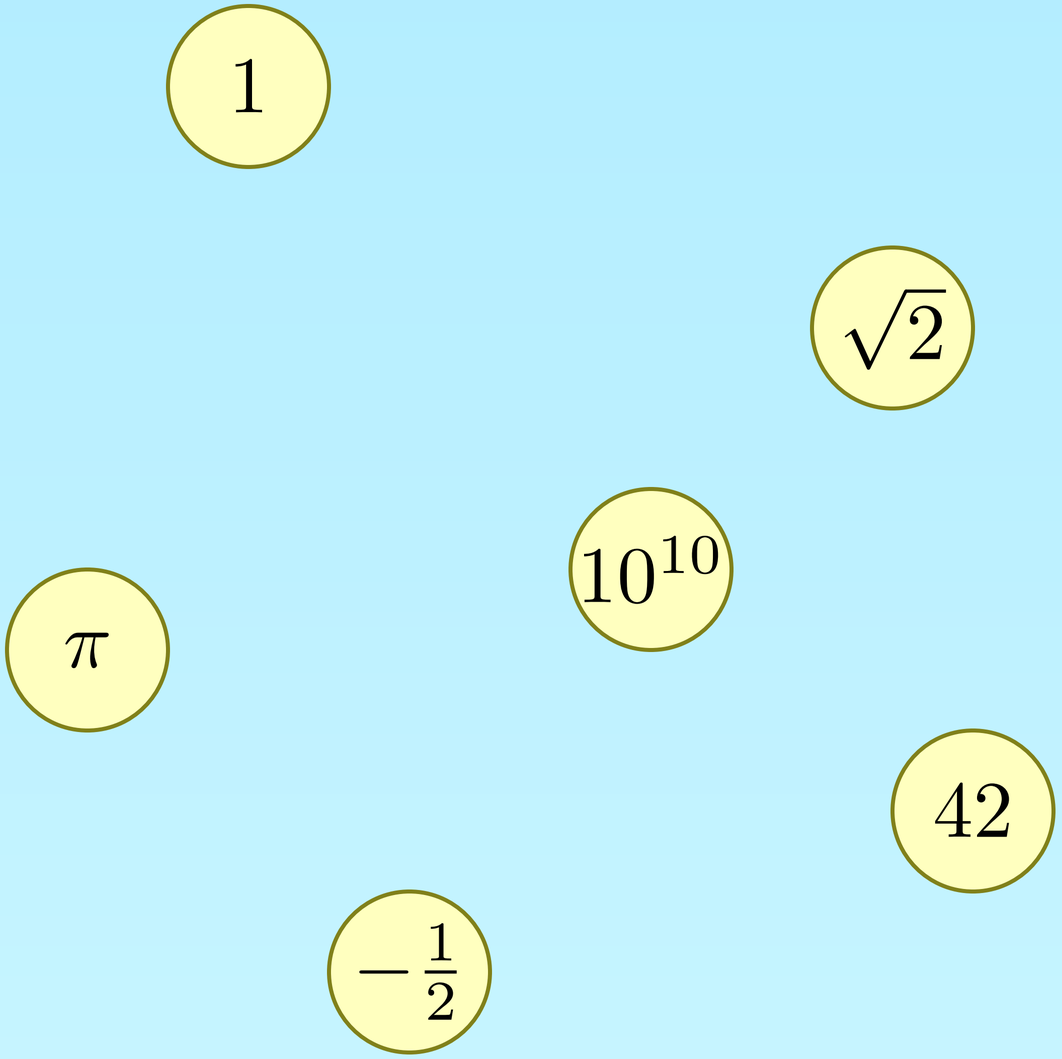


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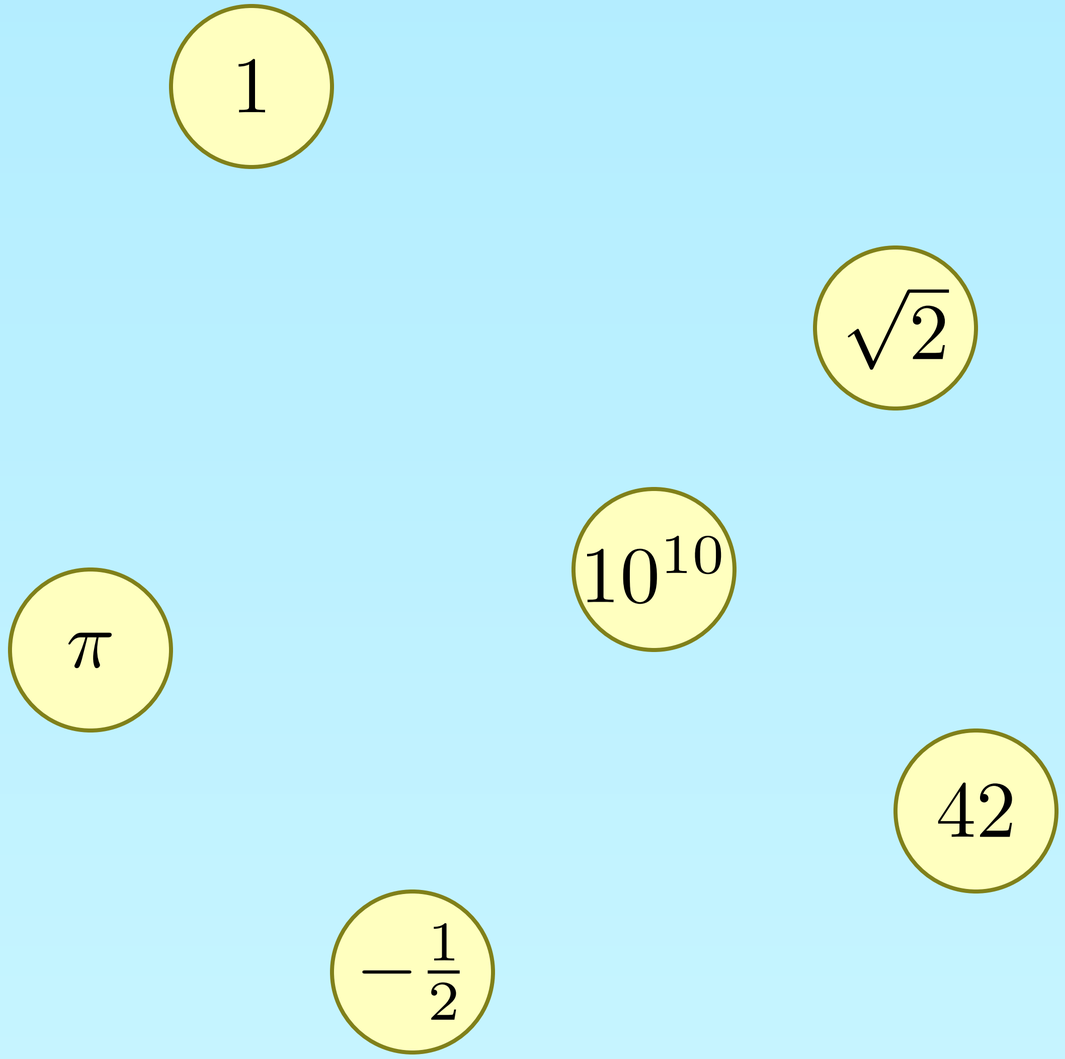


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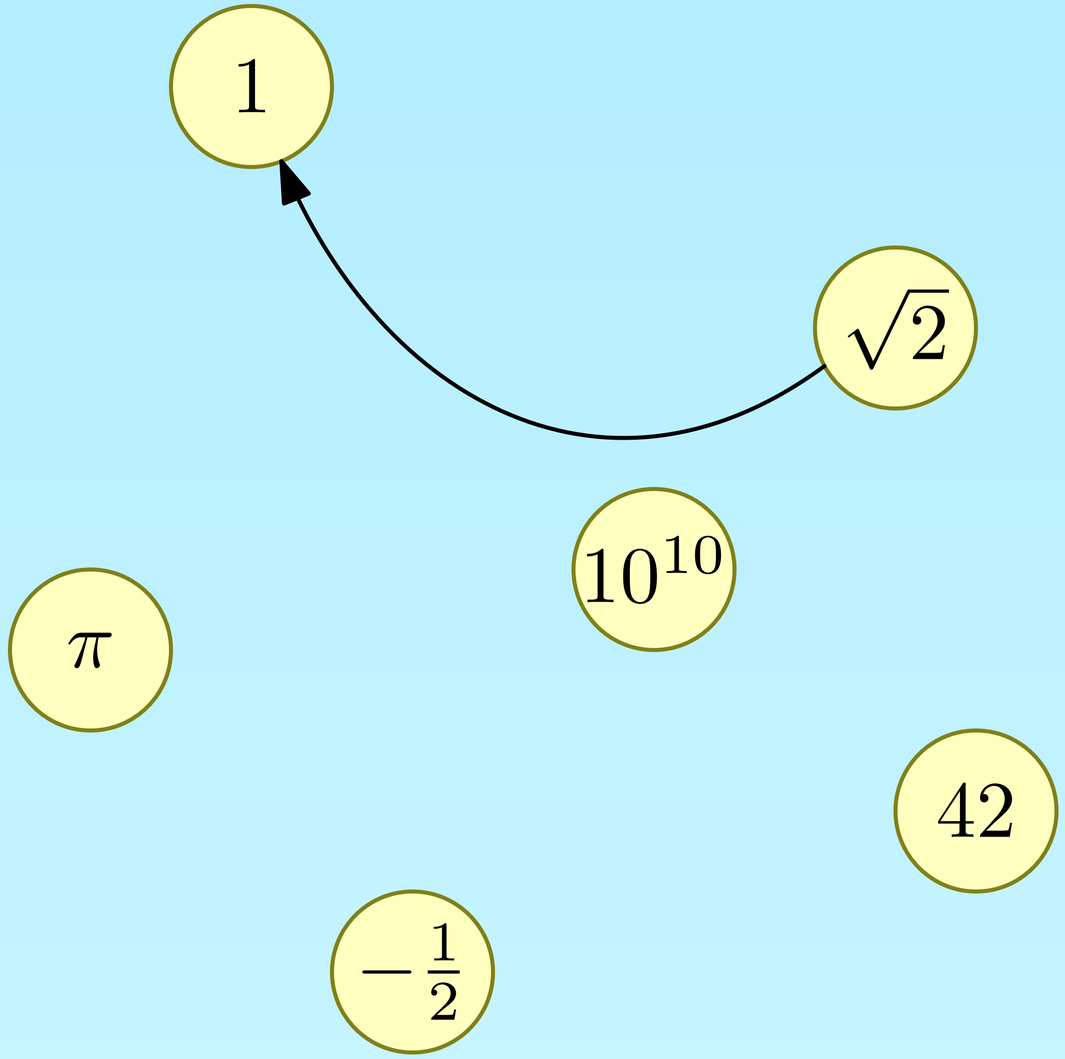


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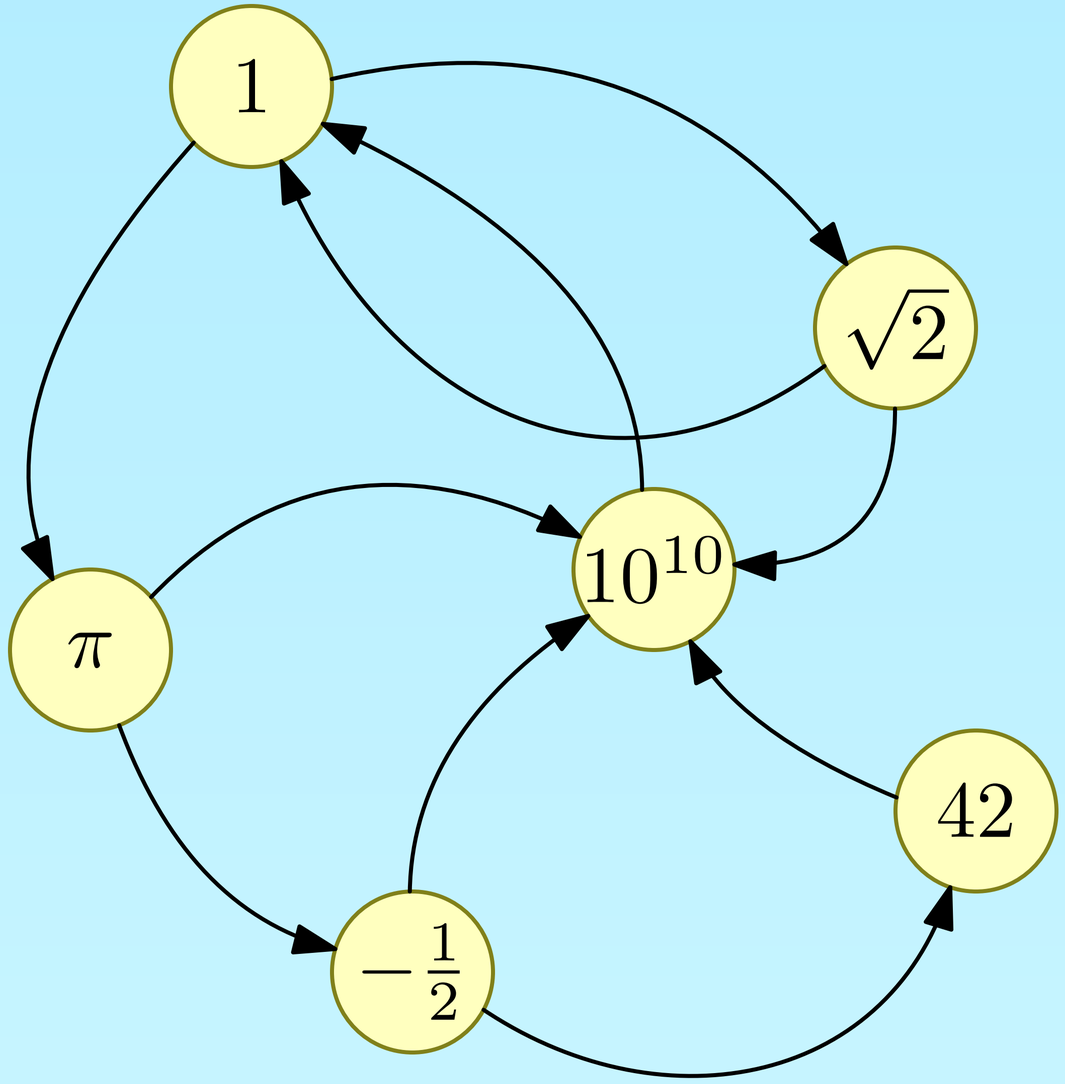


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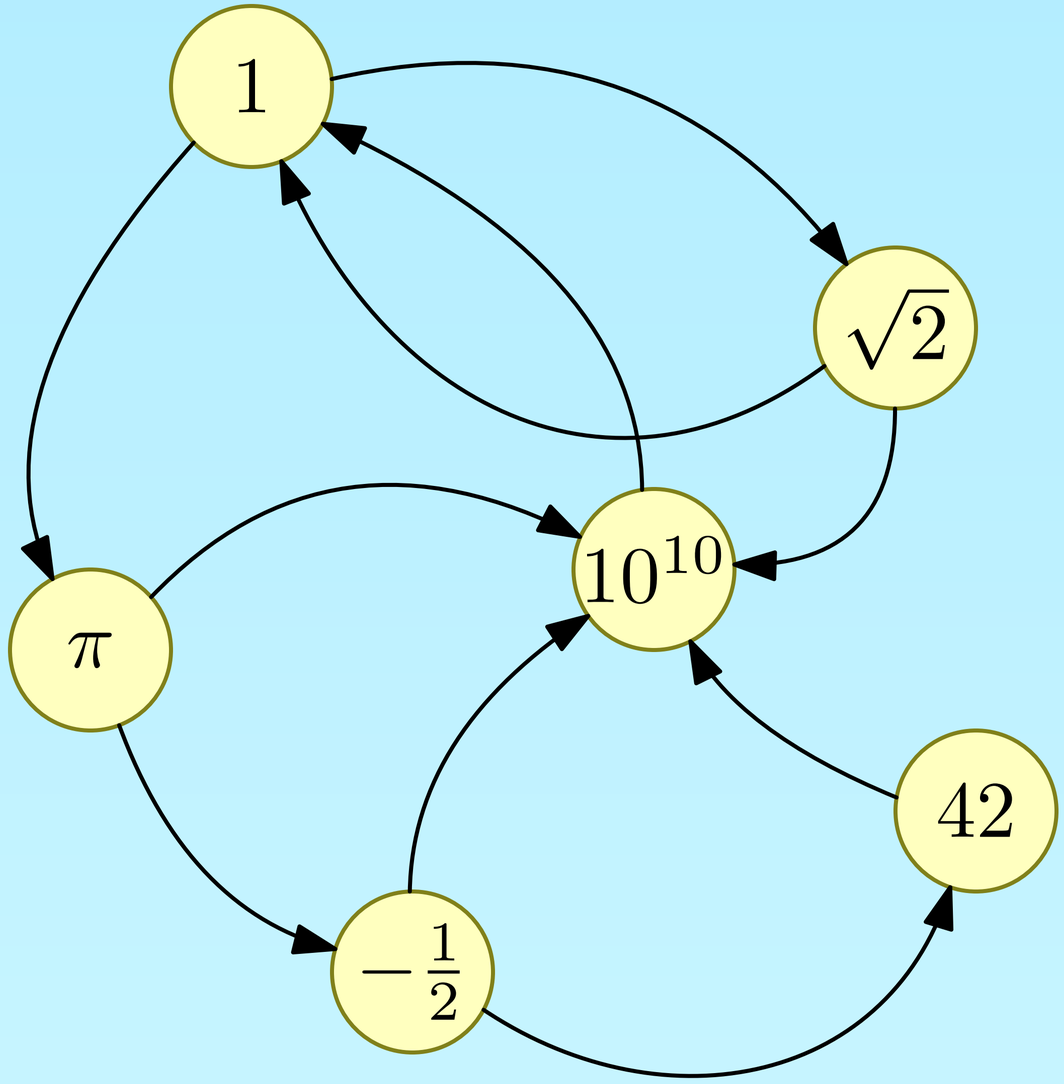


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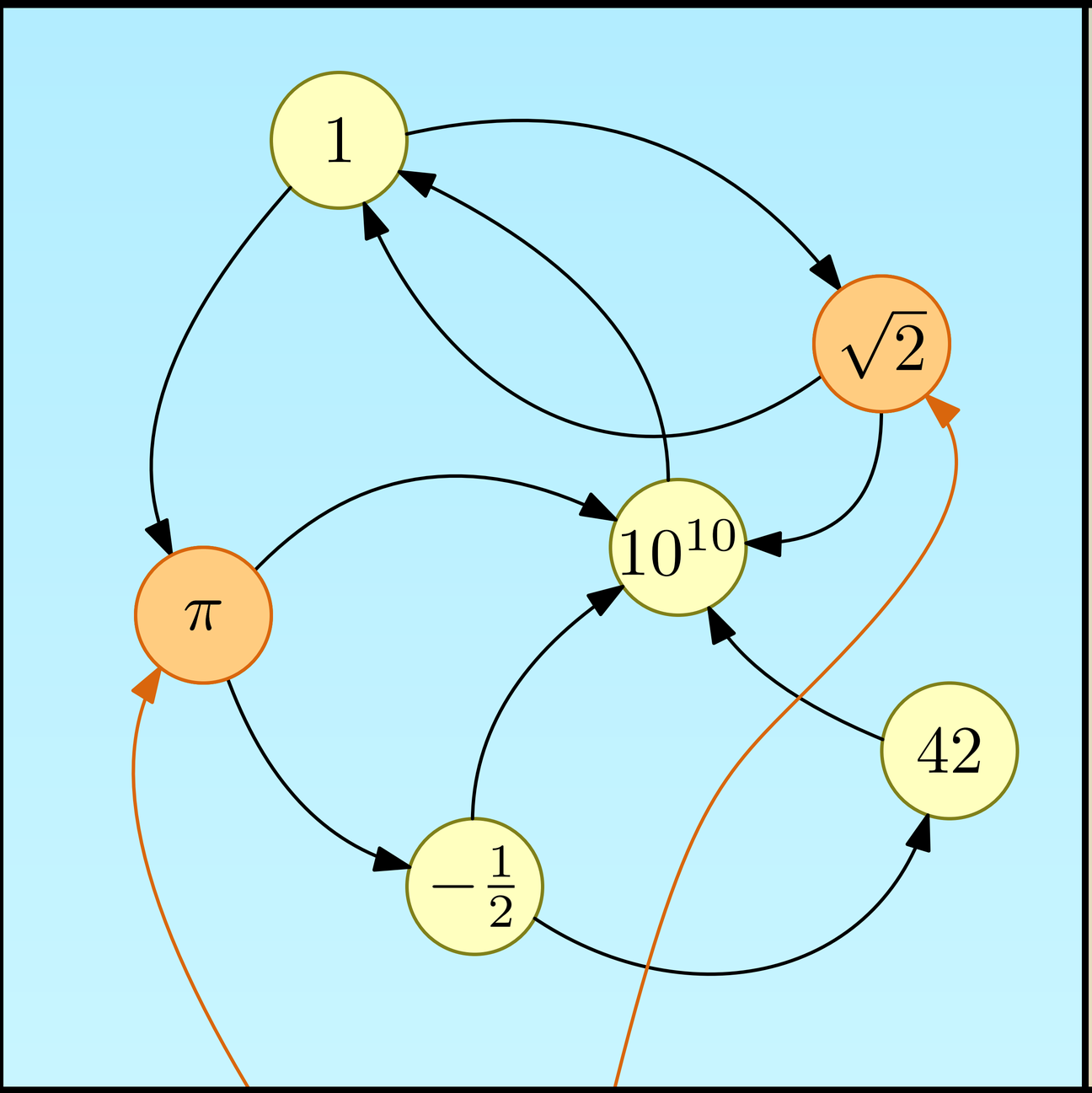


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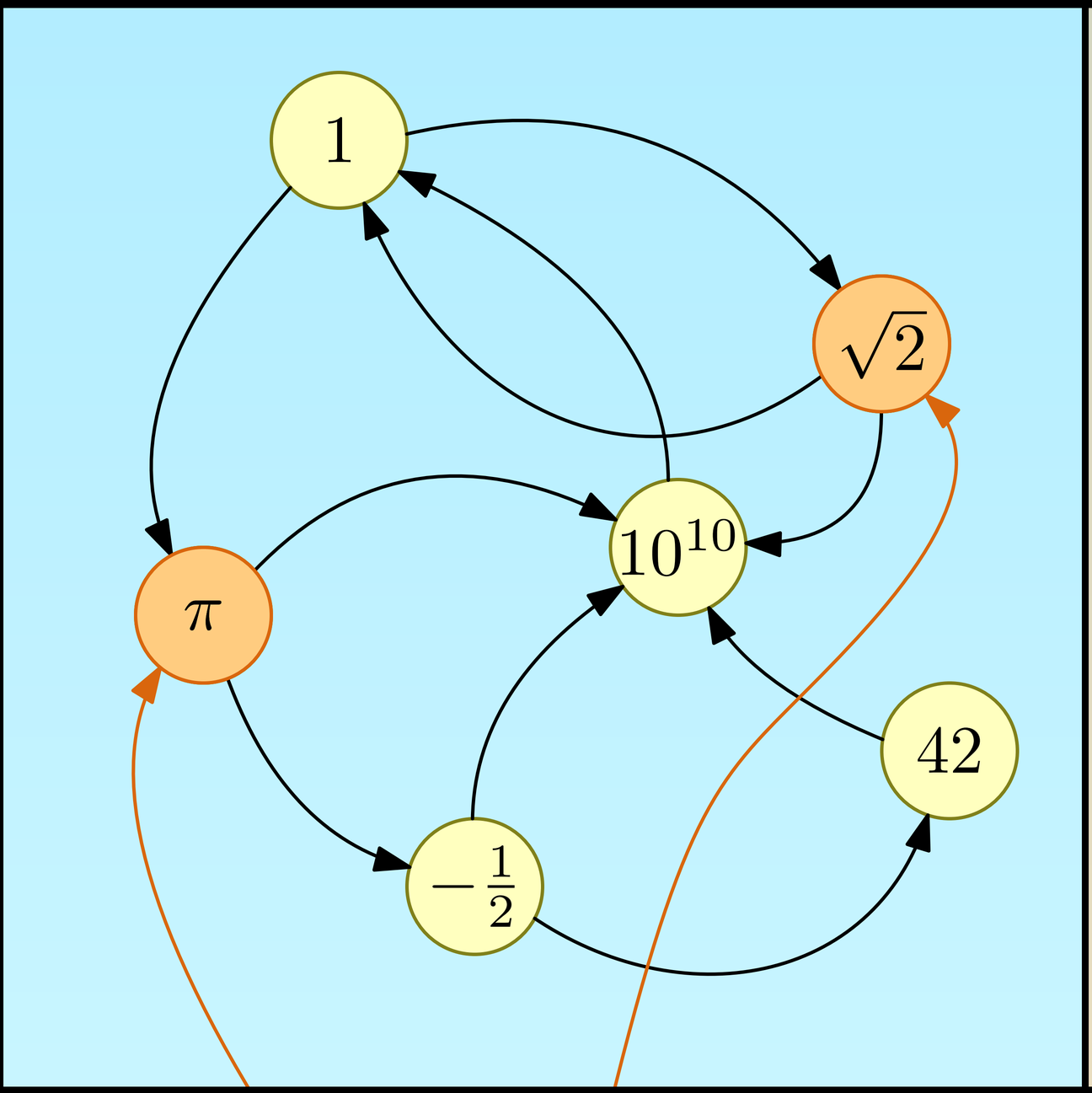


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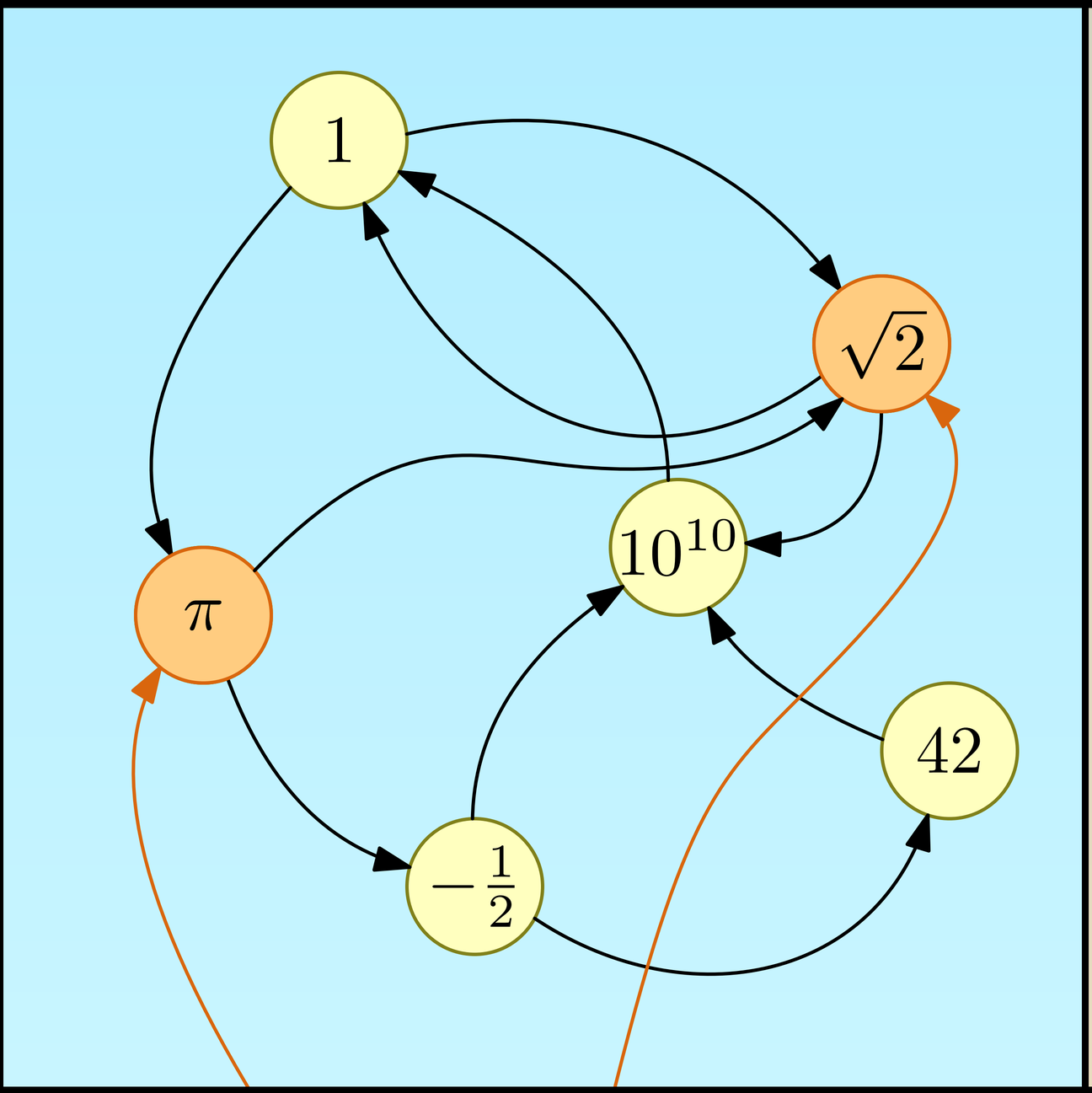


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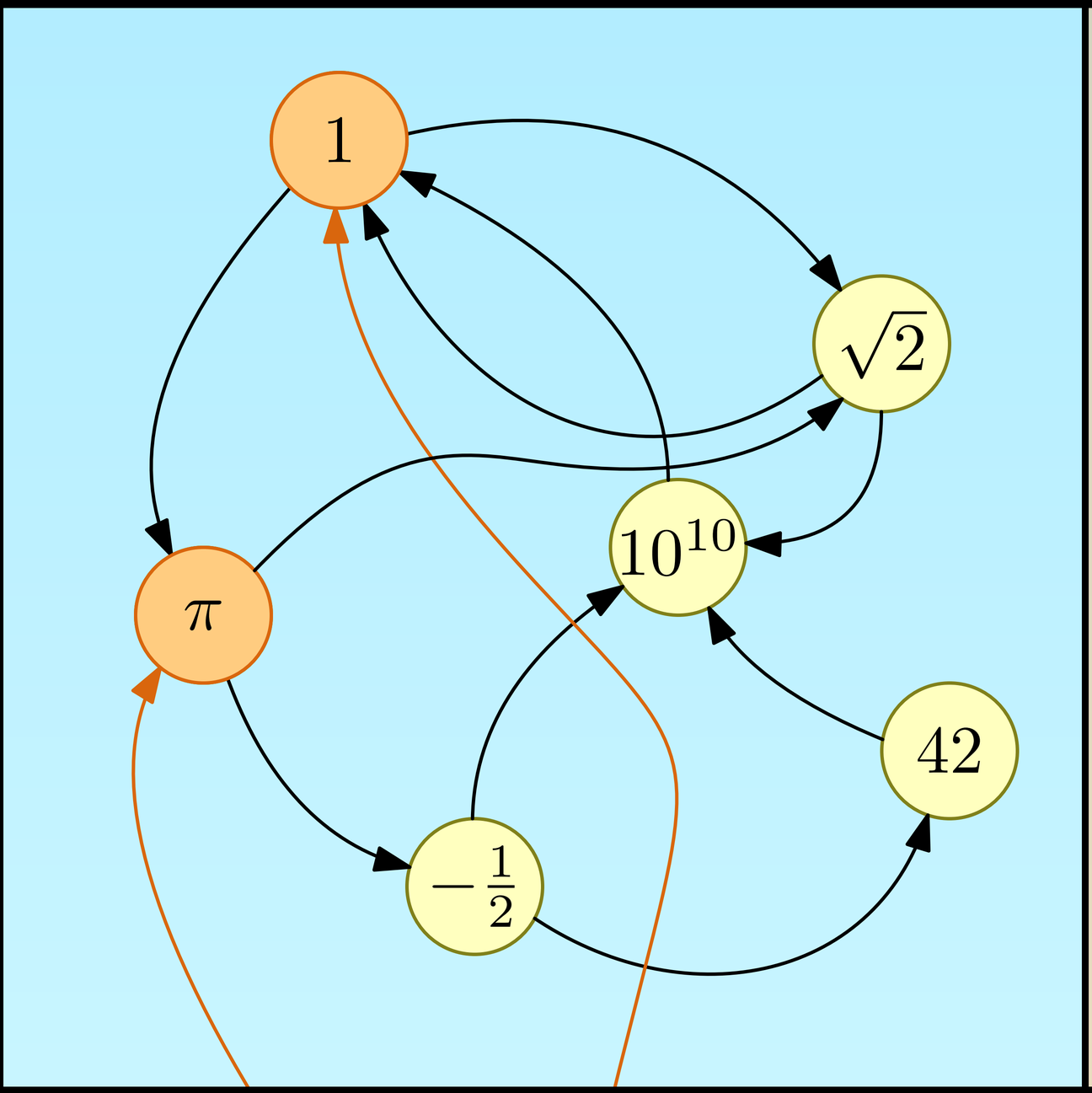


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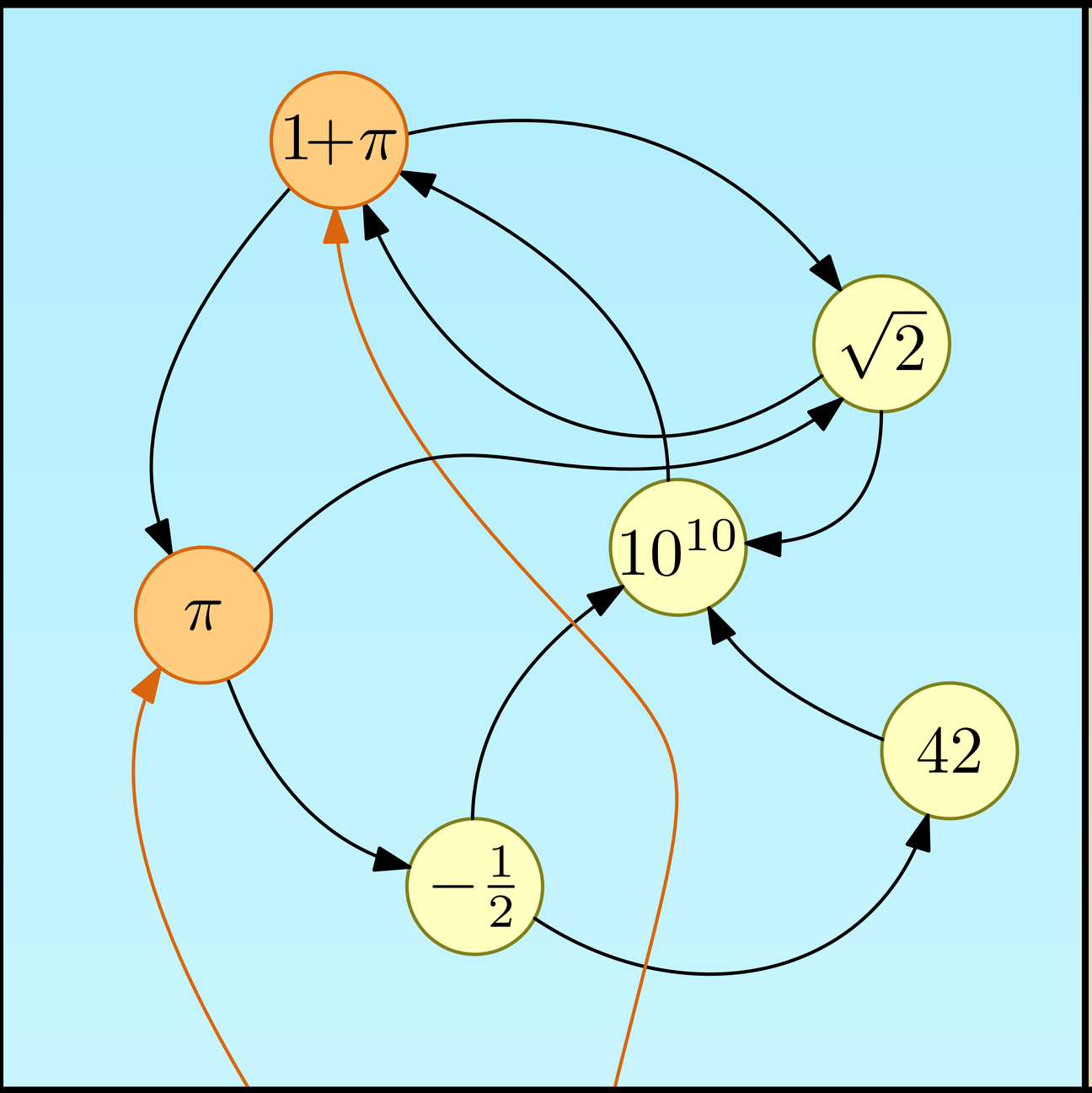


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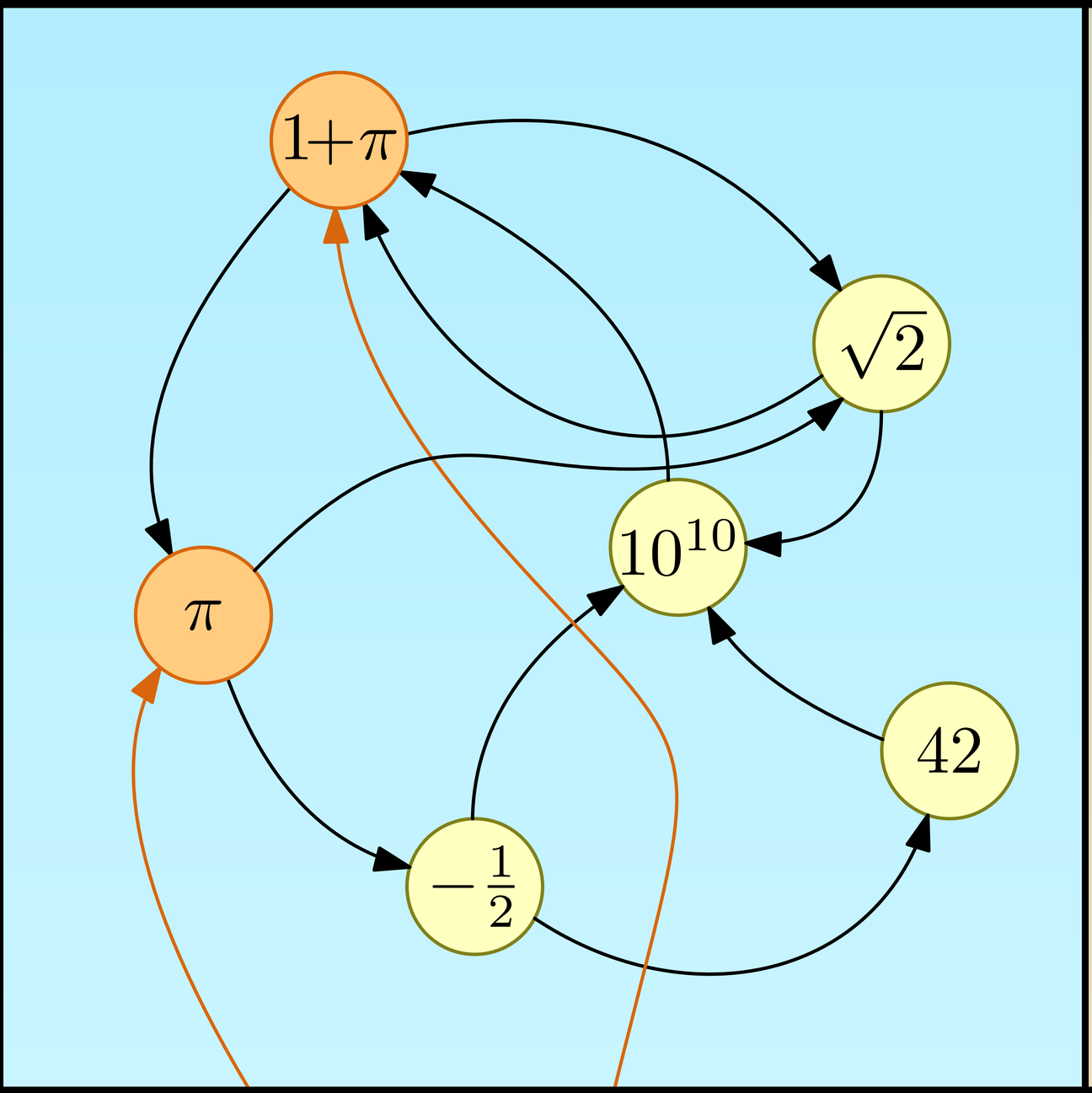


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In this model of computation, pointers are *not* numbers!

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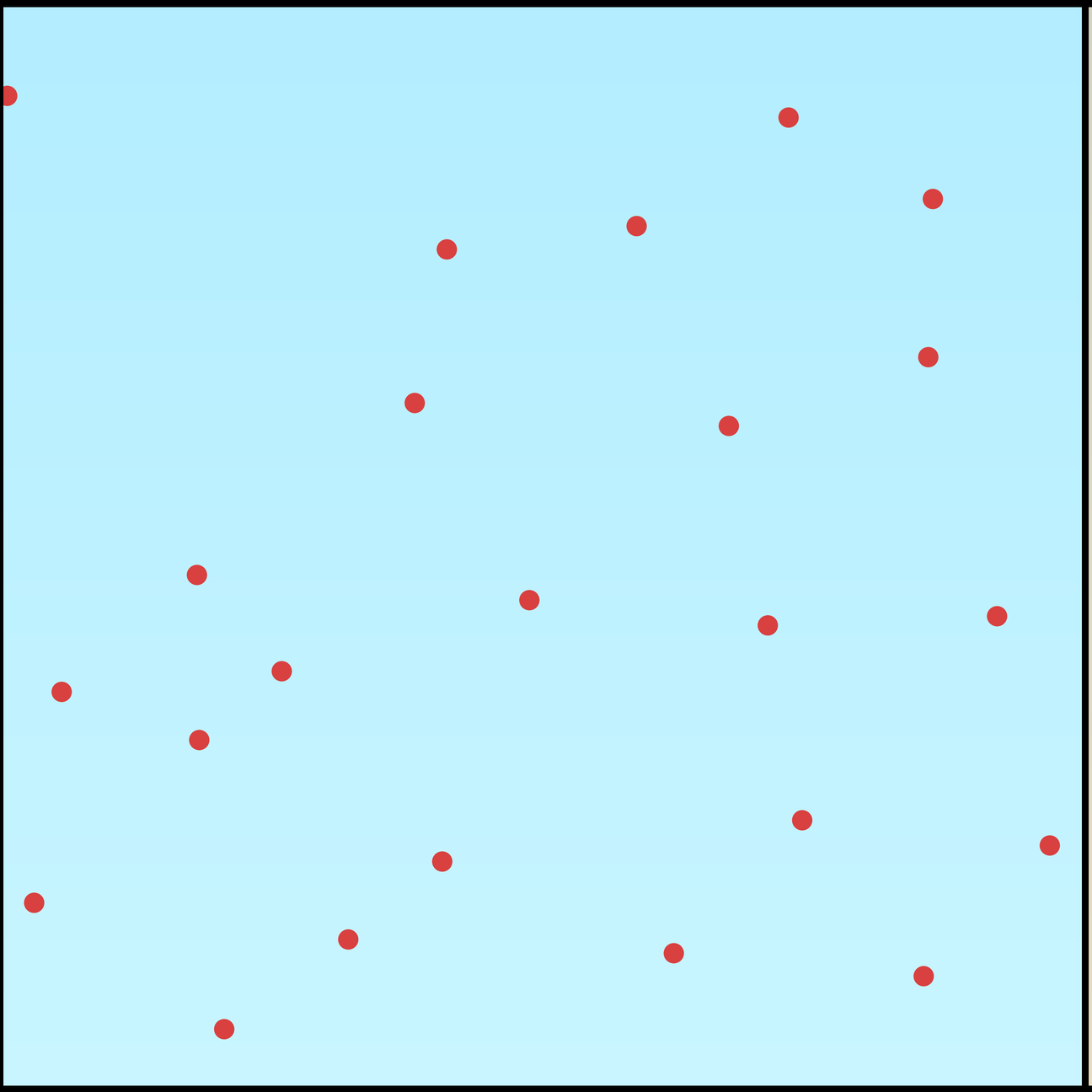

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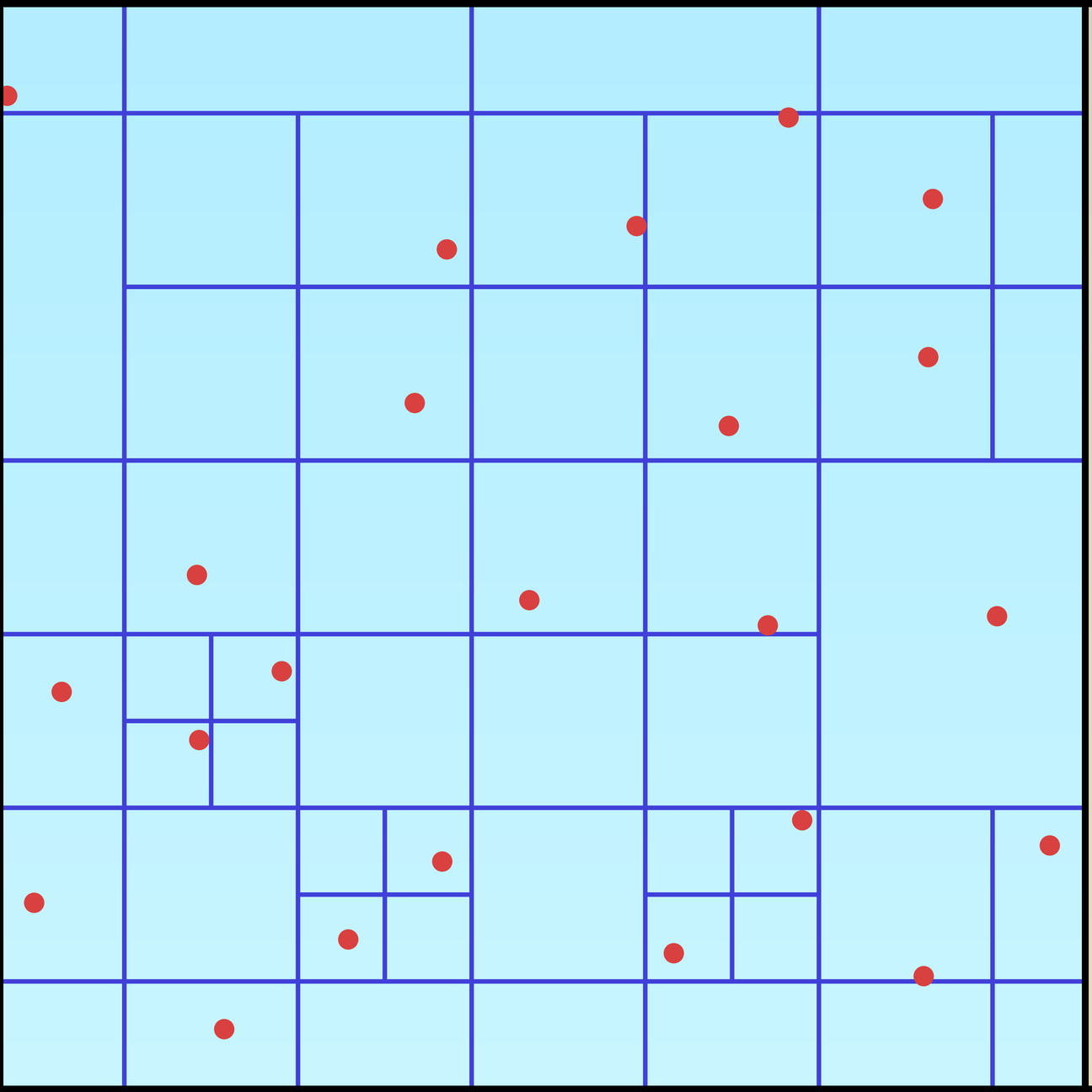

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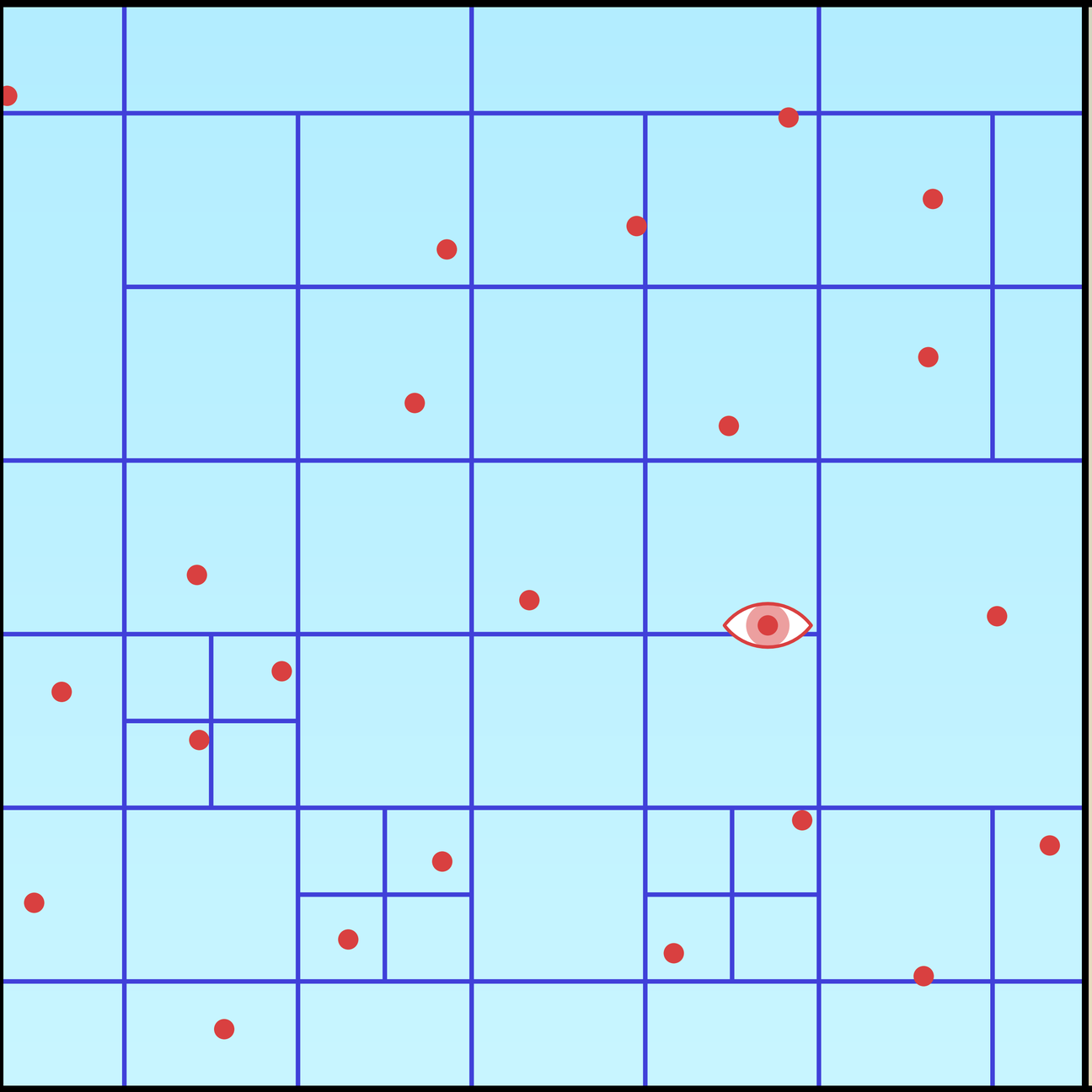

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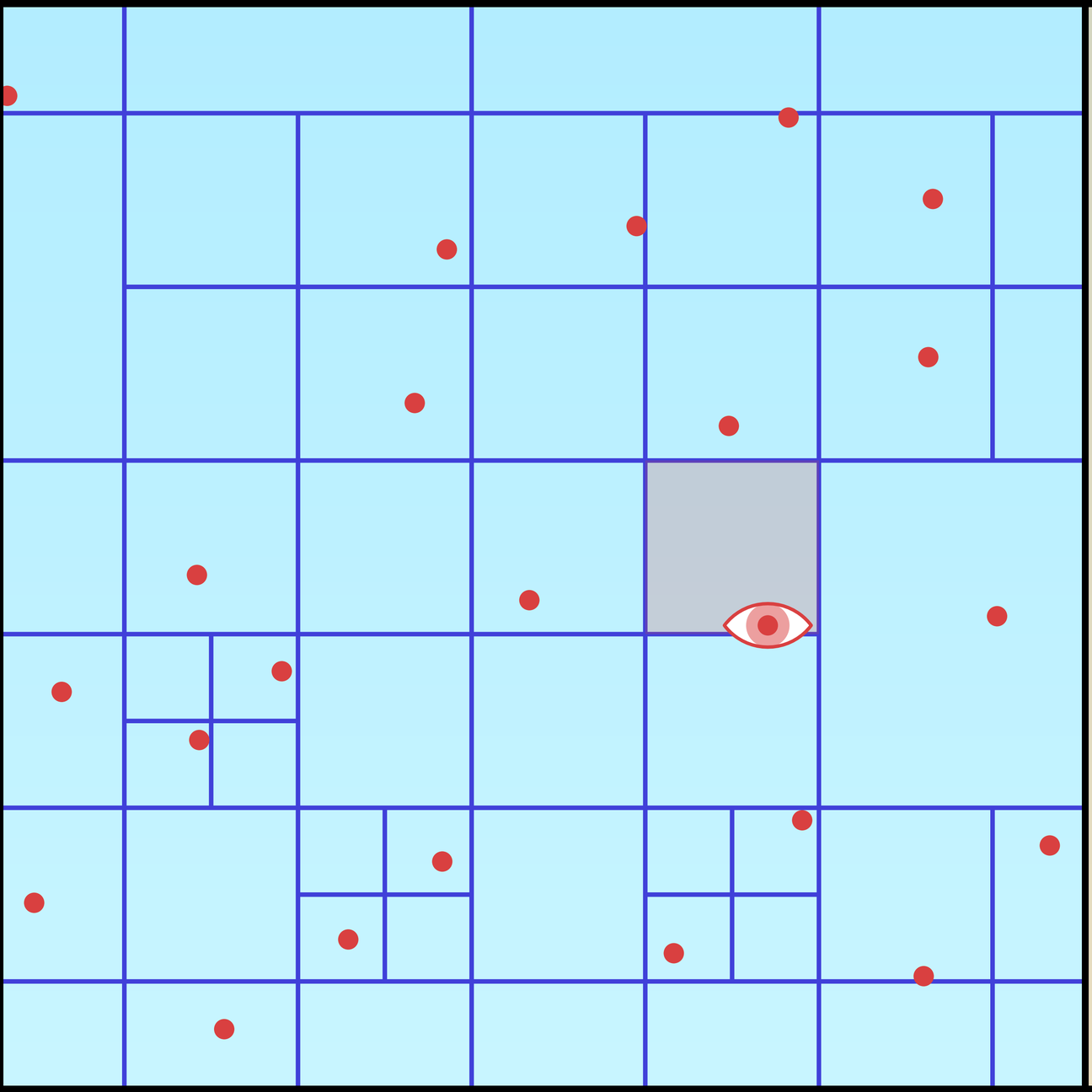

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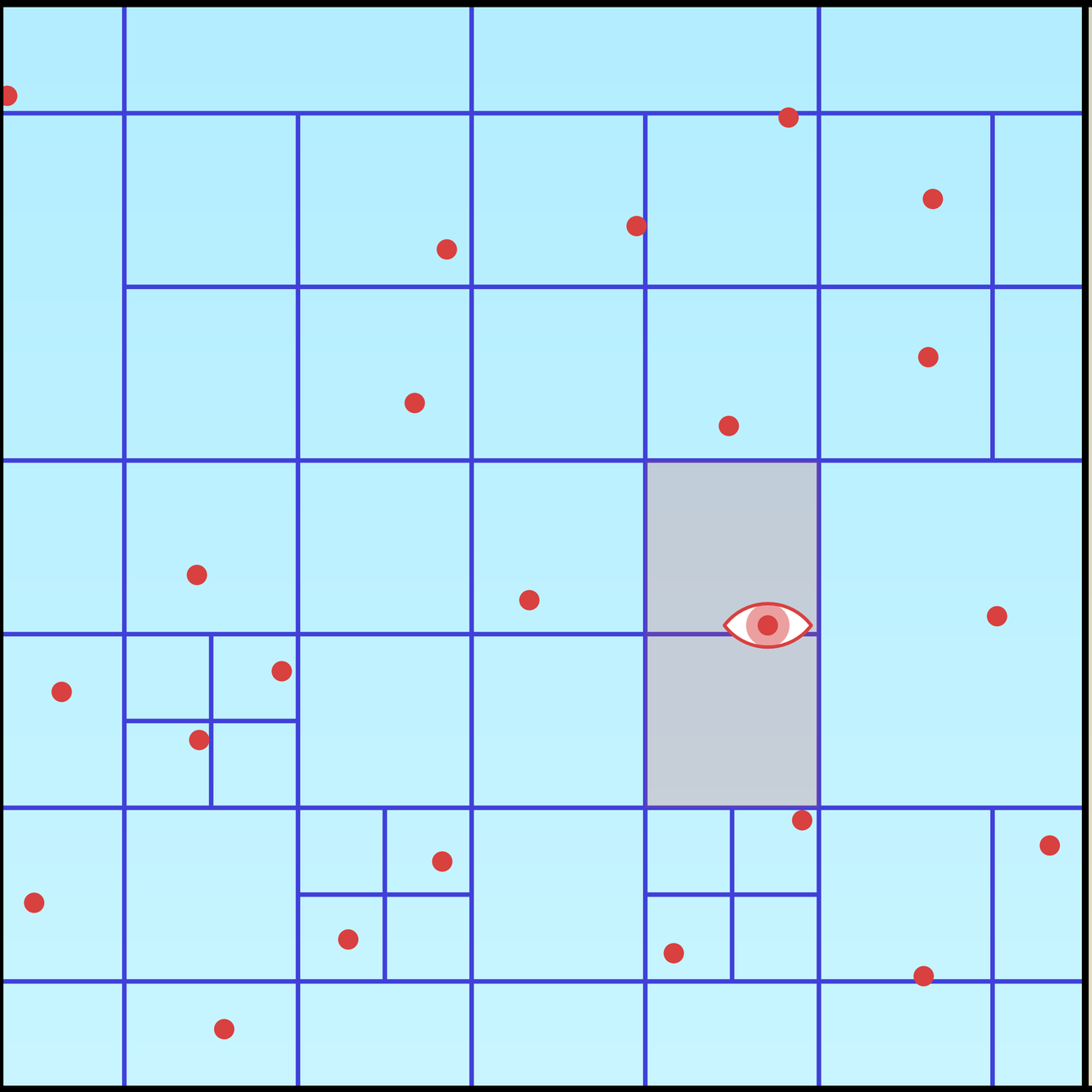

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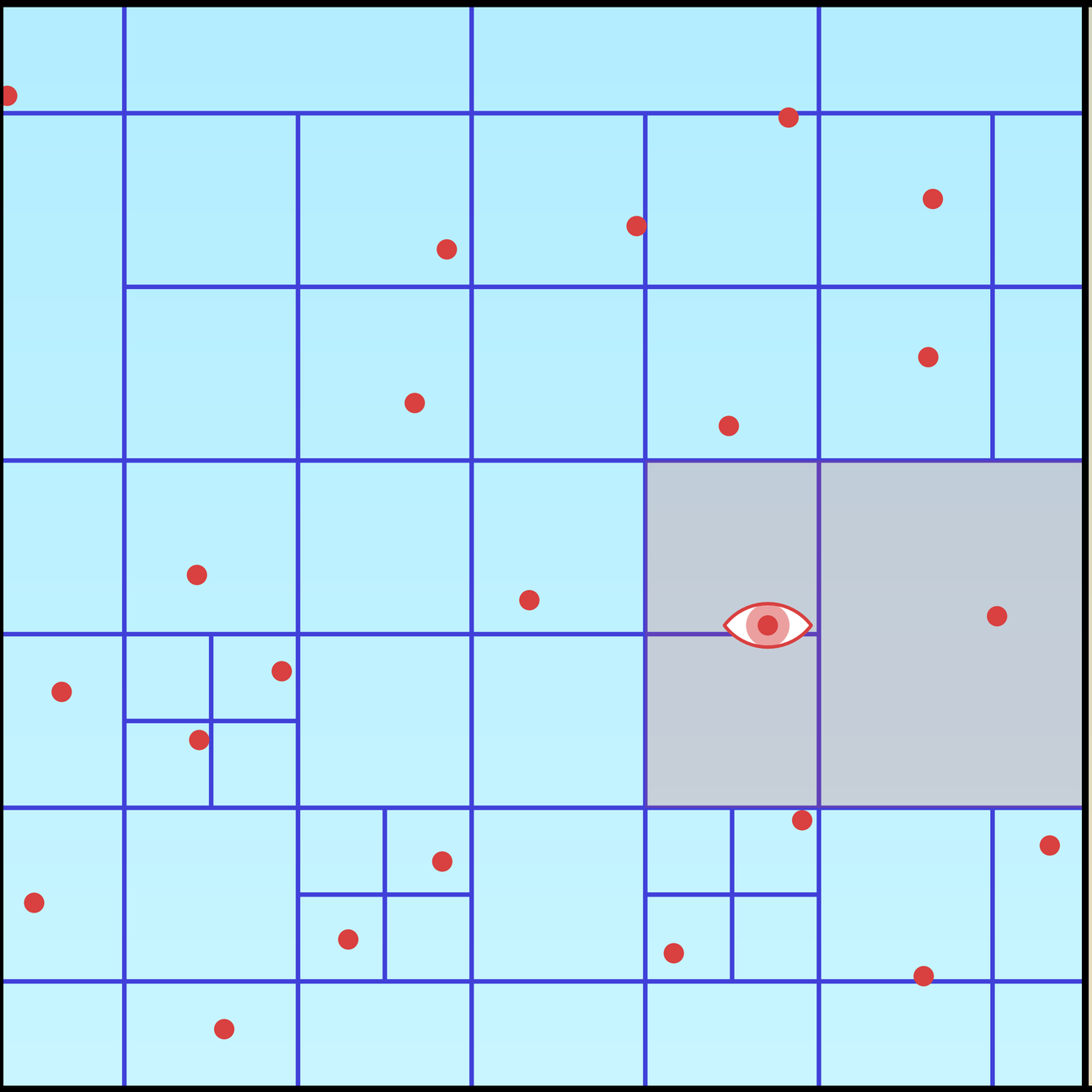

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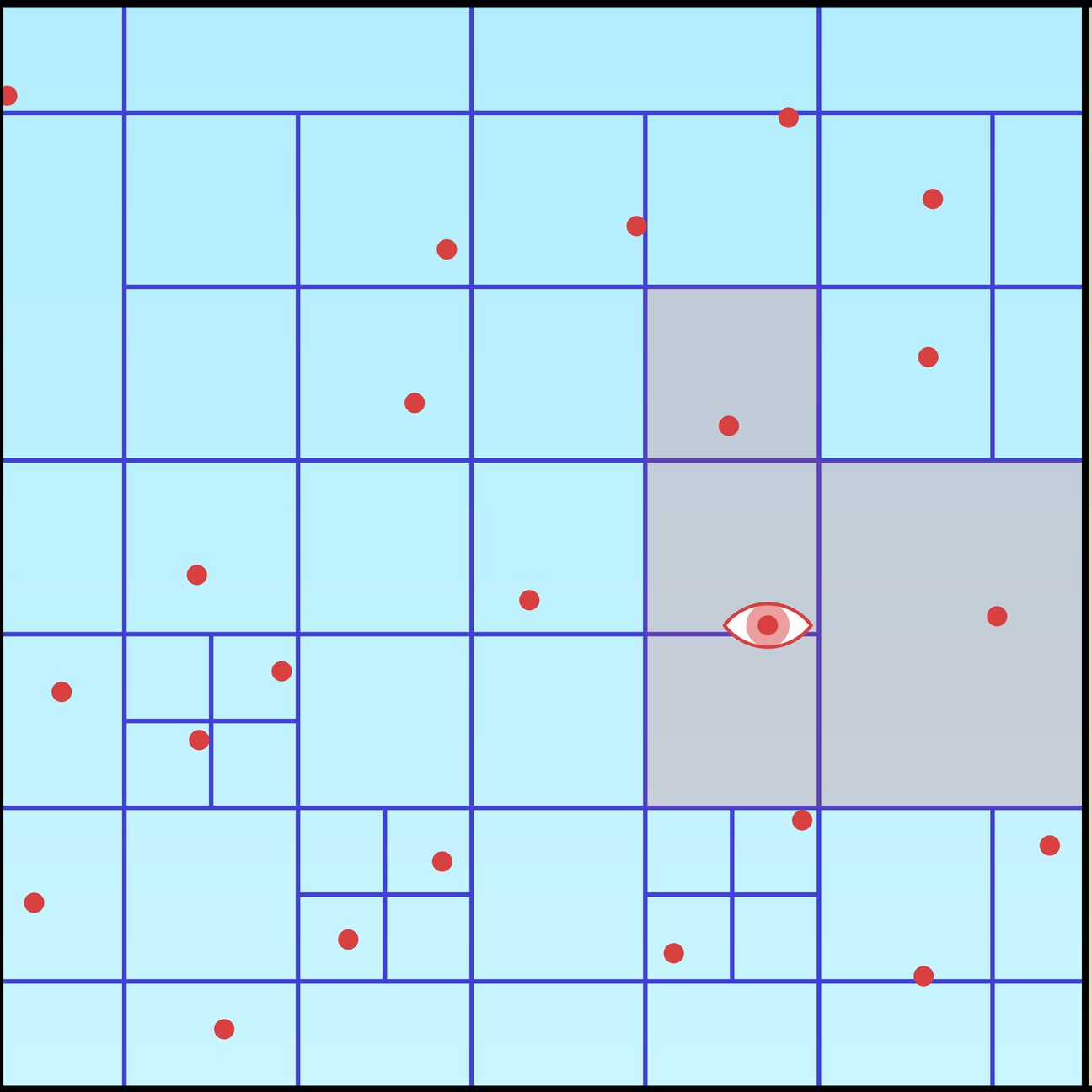

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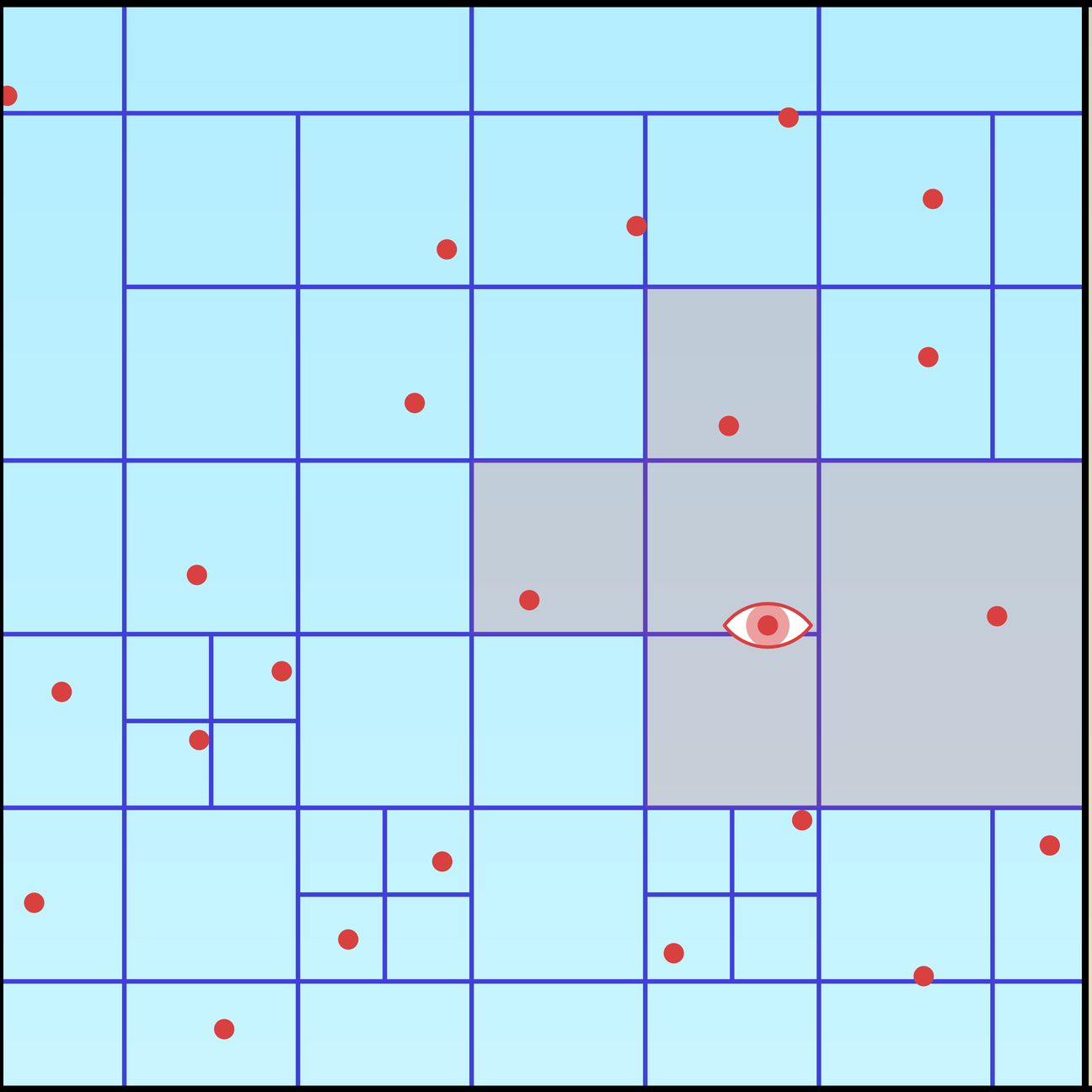

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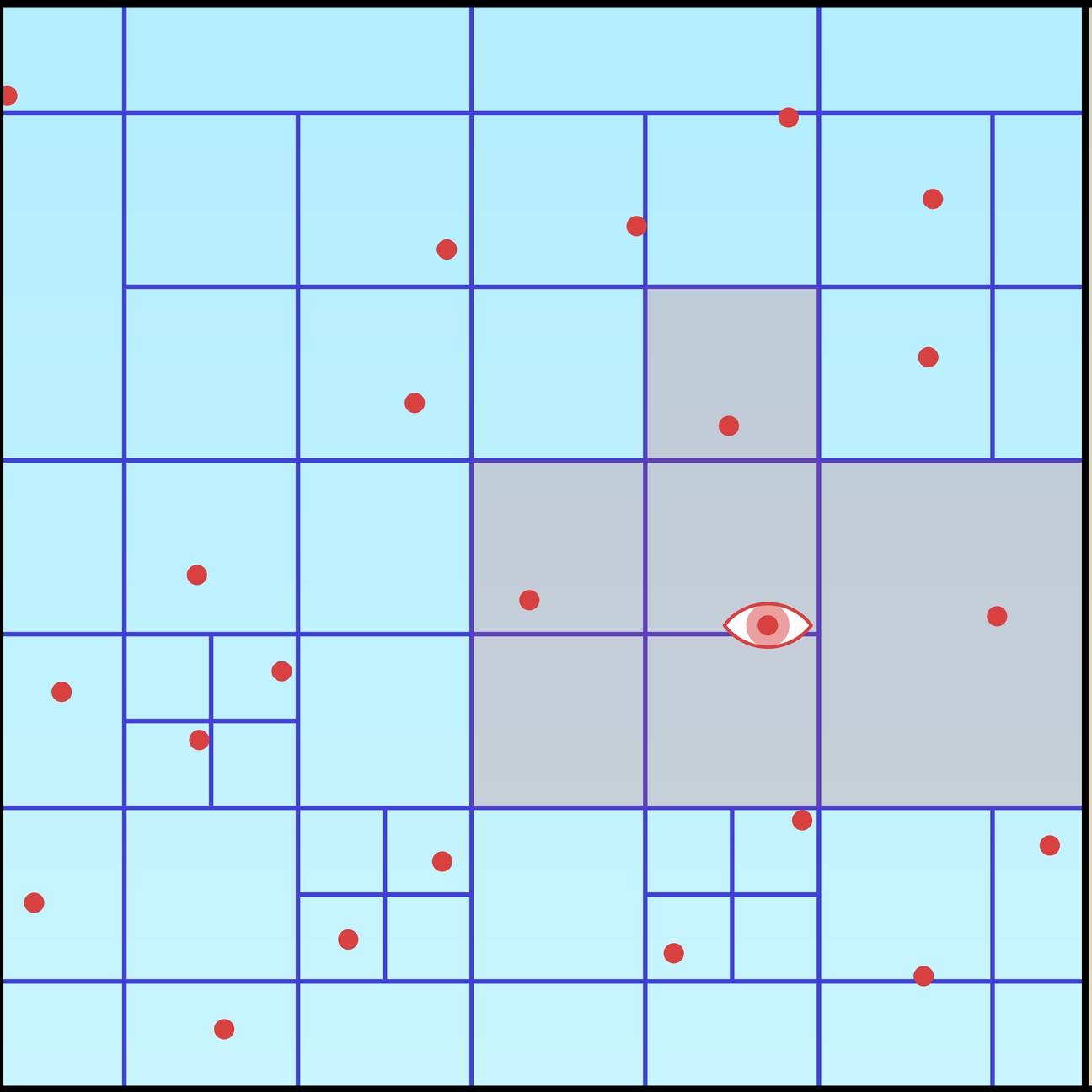

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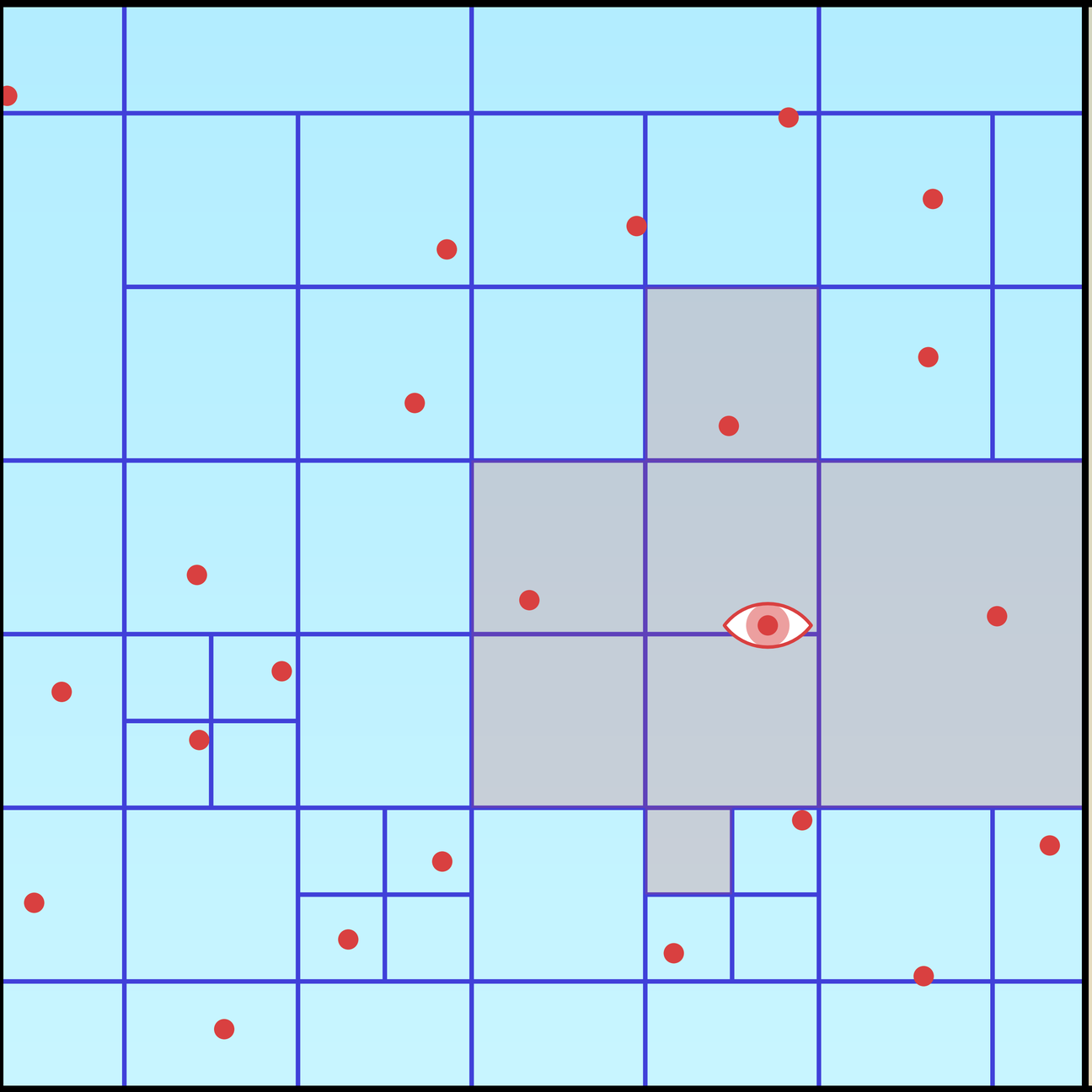

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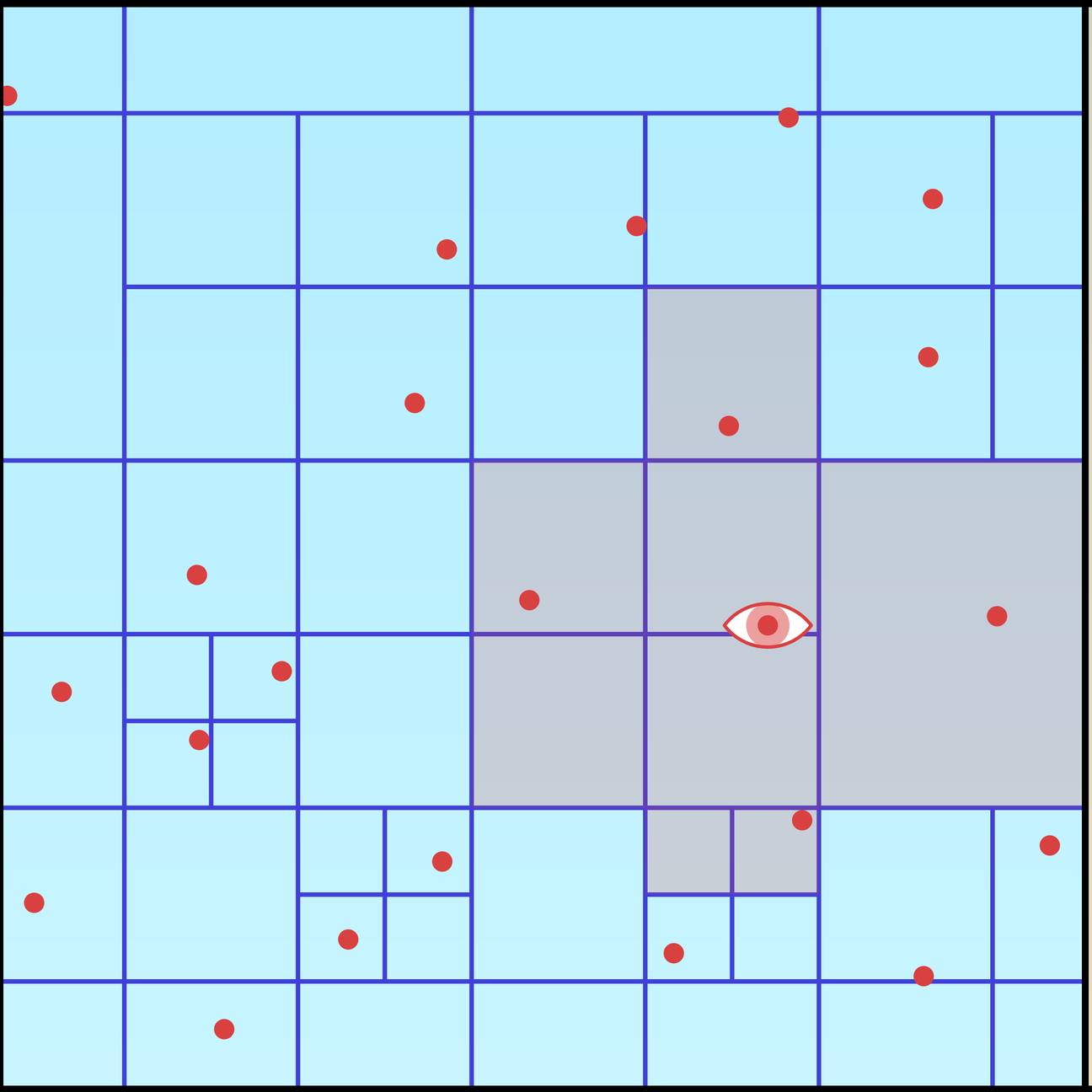

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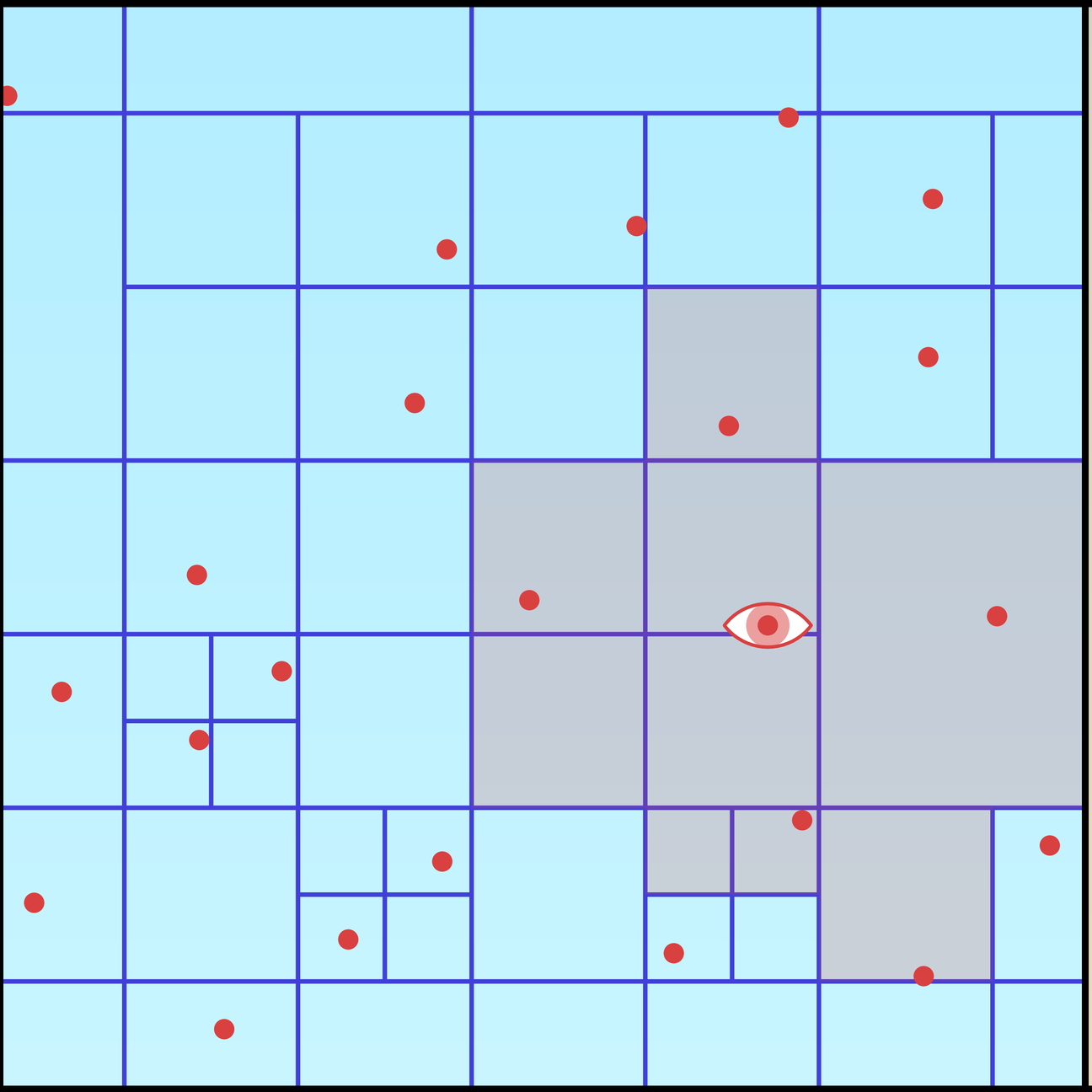

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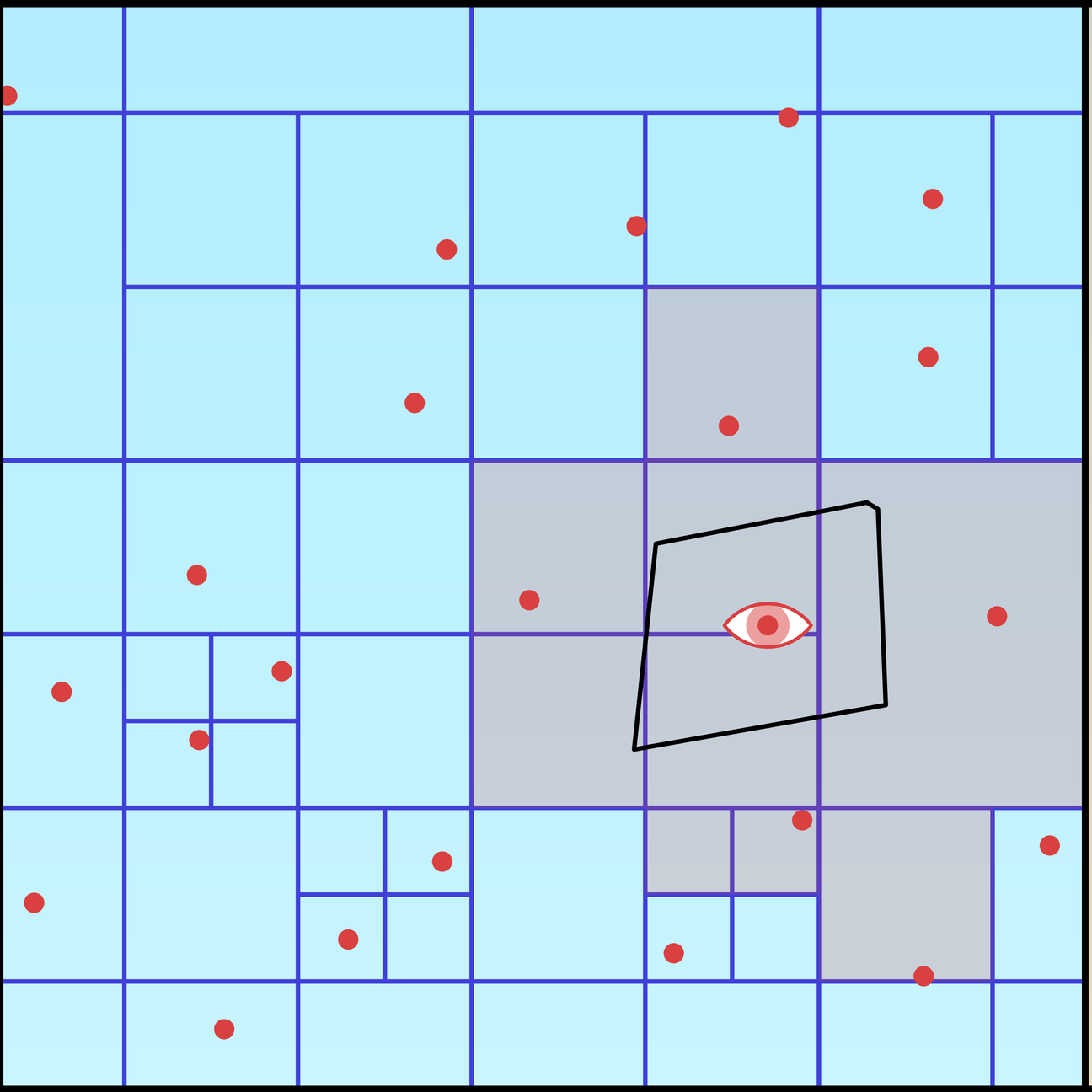

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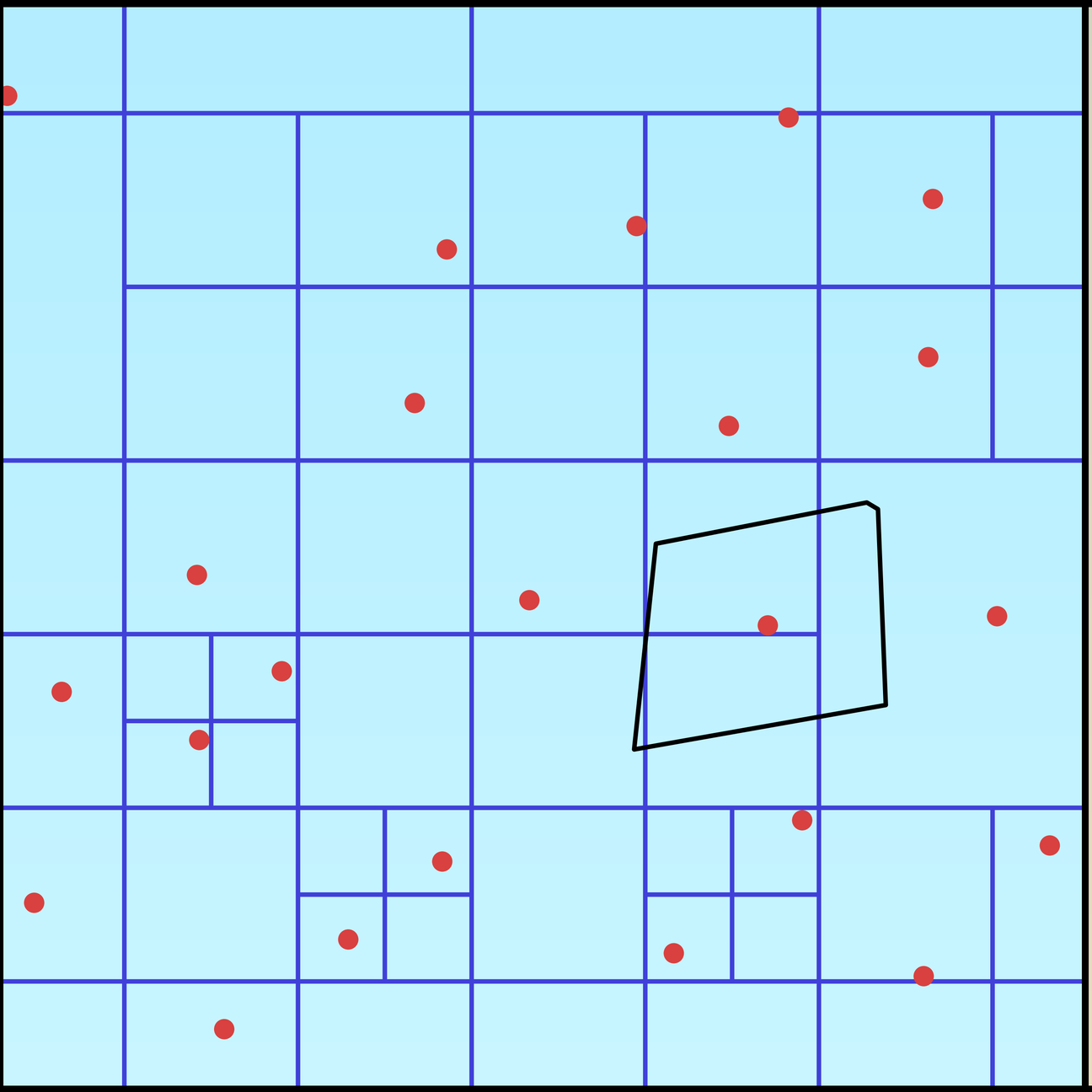

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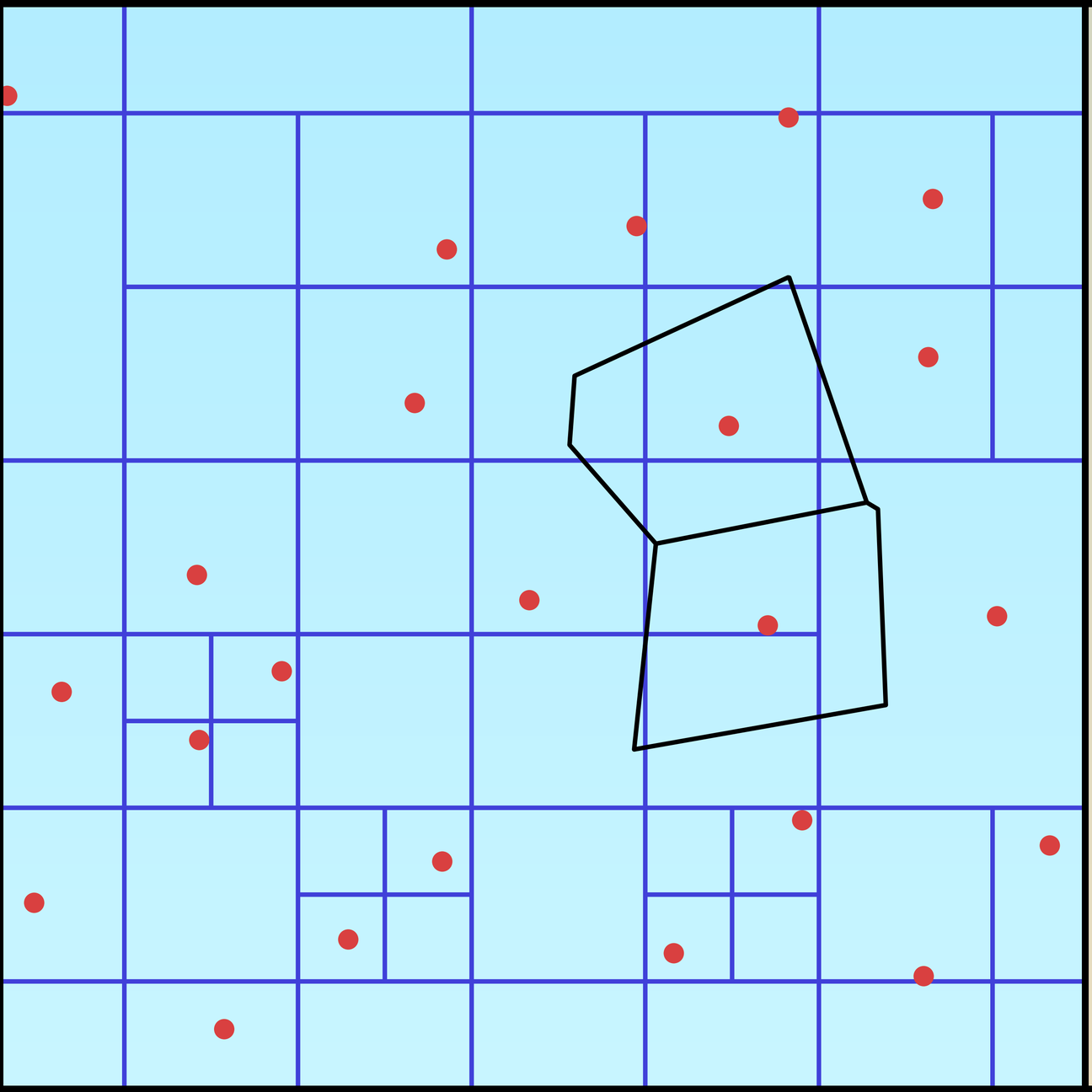

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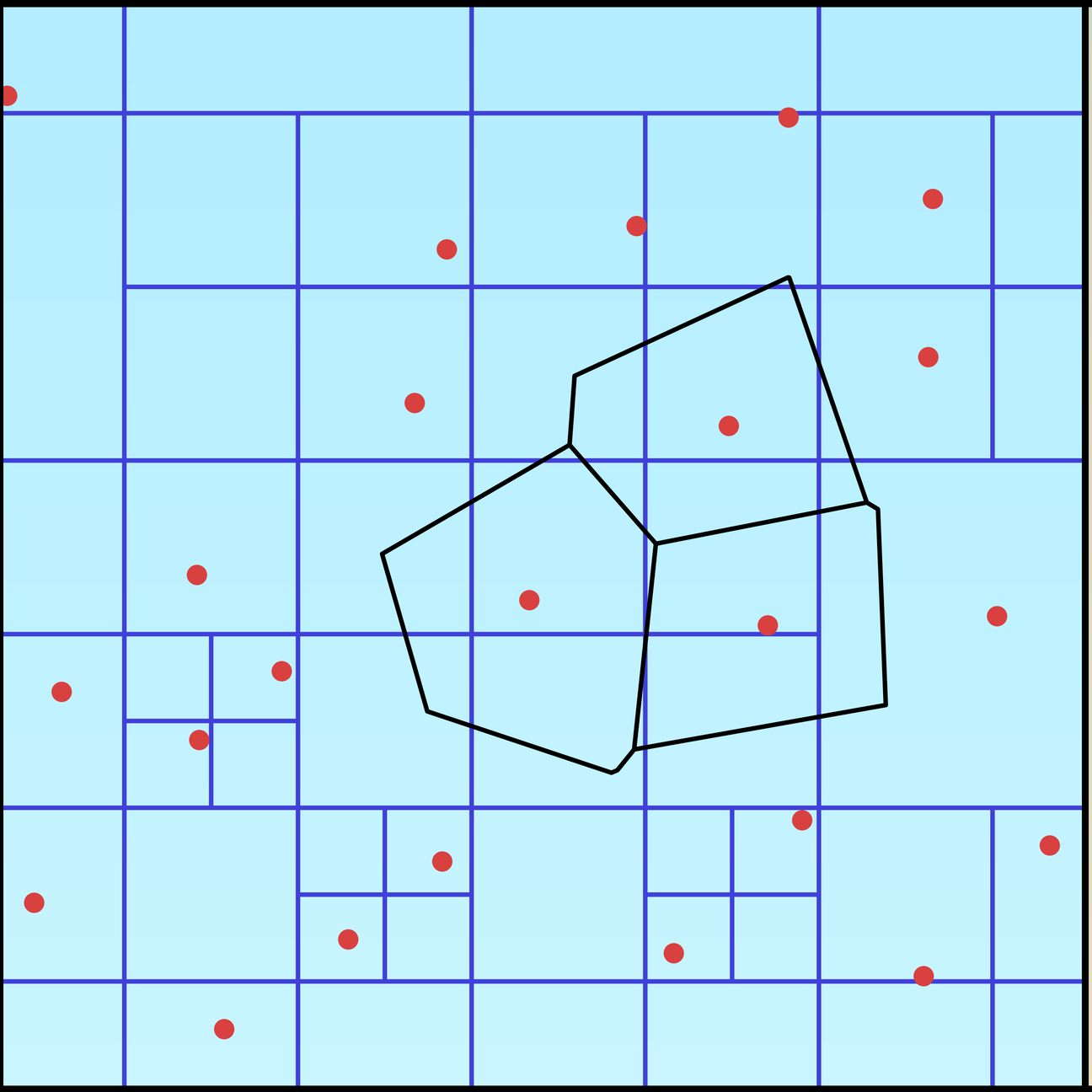

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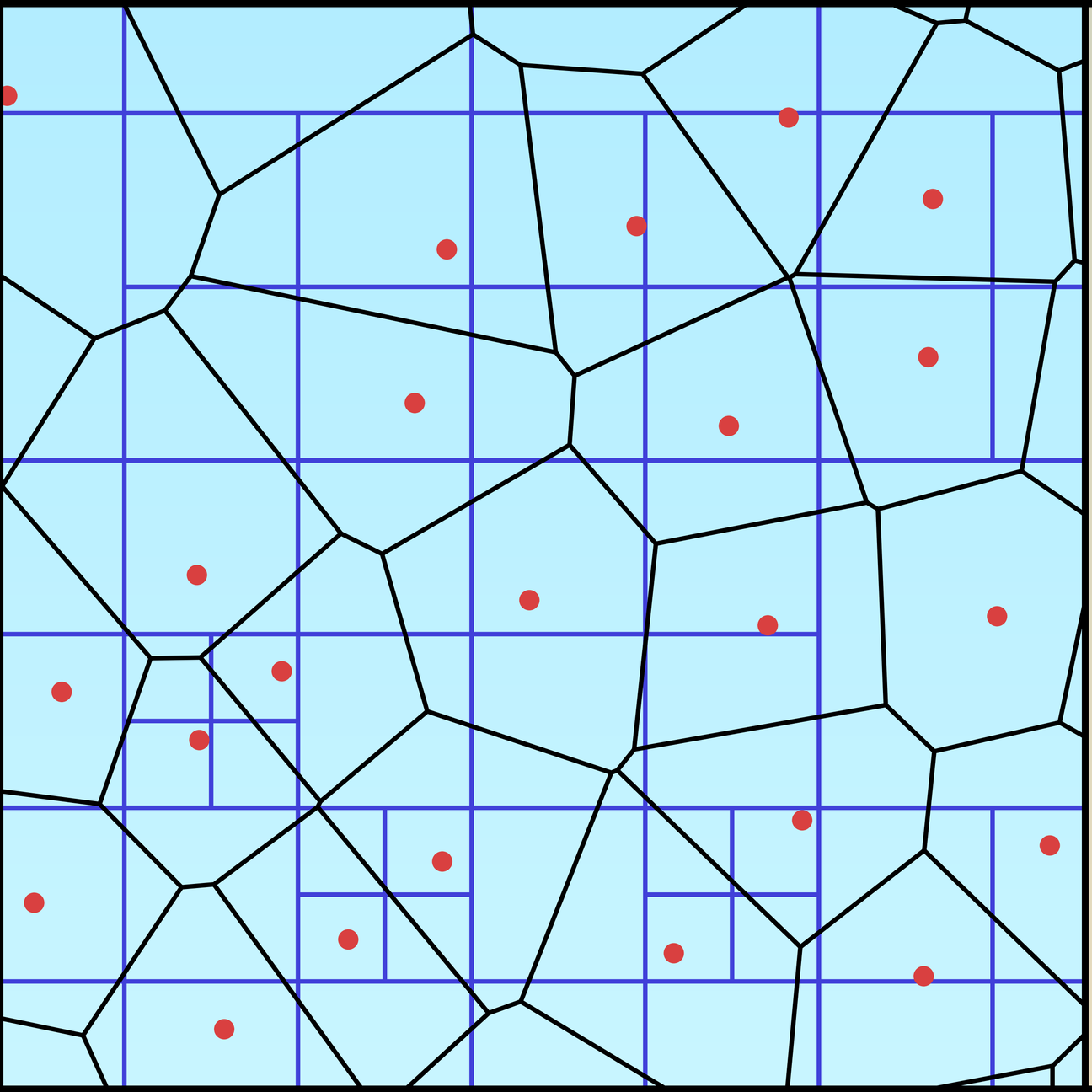

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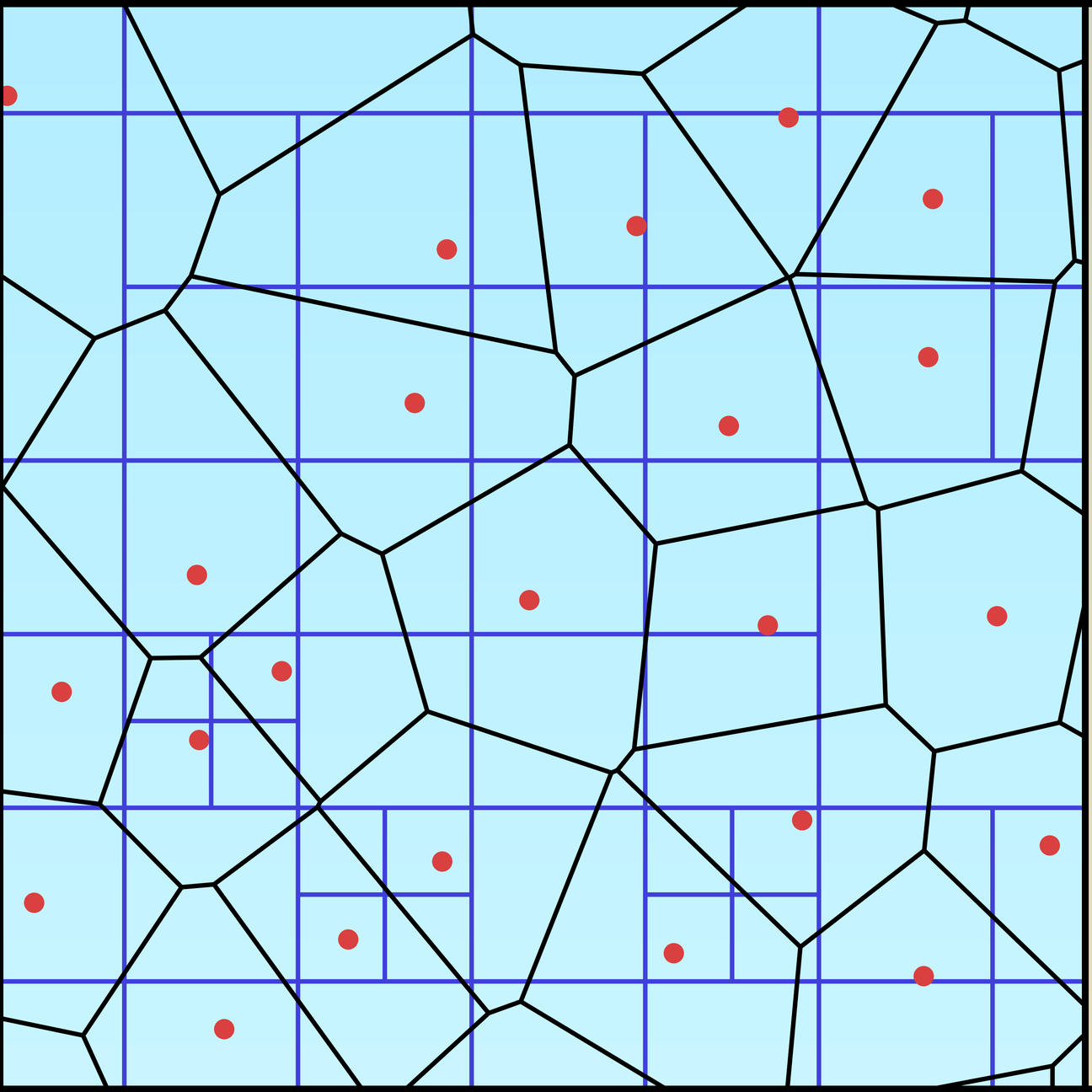

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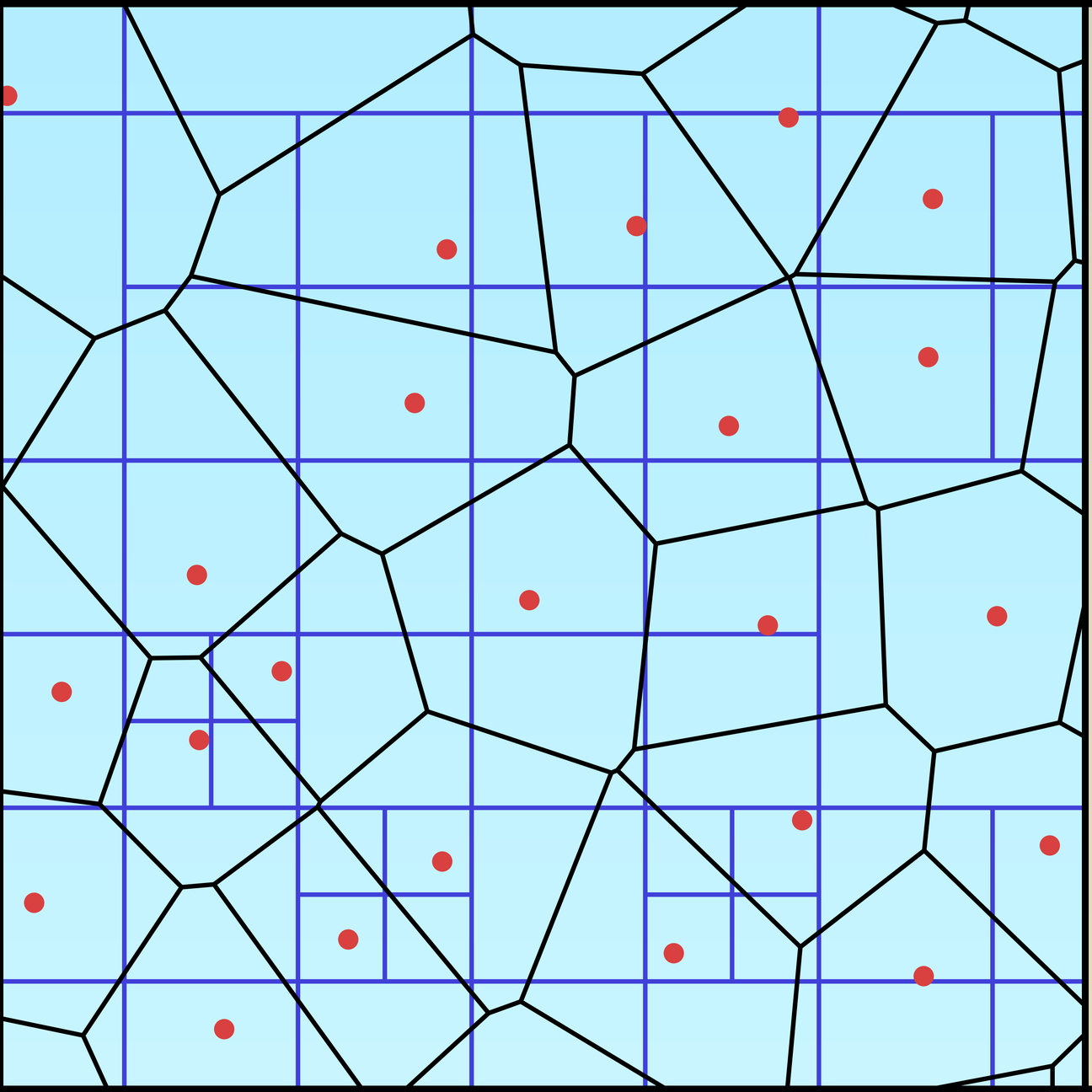
Since information is *local*, such a conversion should take $O(n)$ time.



We turn things around,
and define what we mean
by a proximity structure
“capturing the local
information” of $P$ in terms
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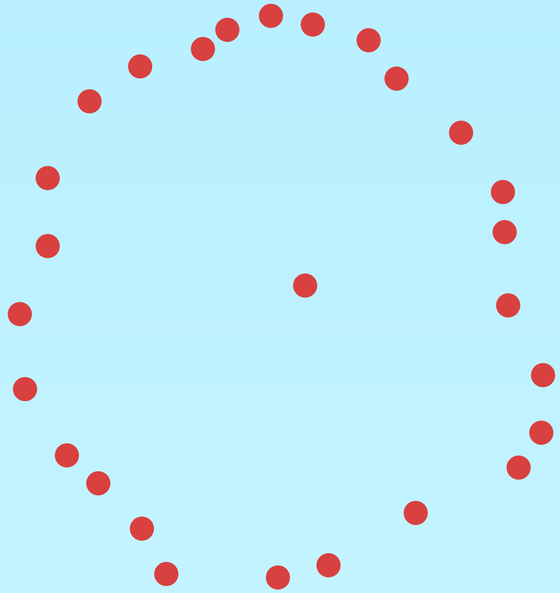

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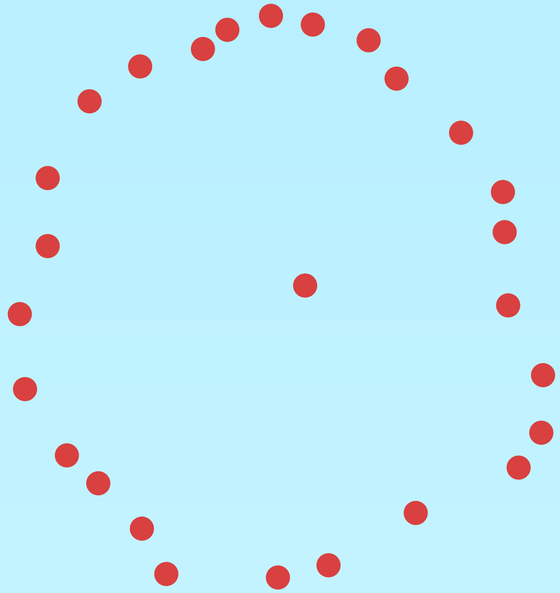

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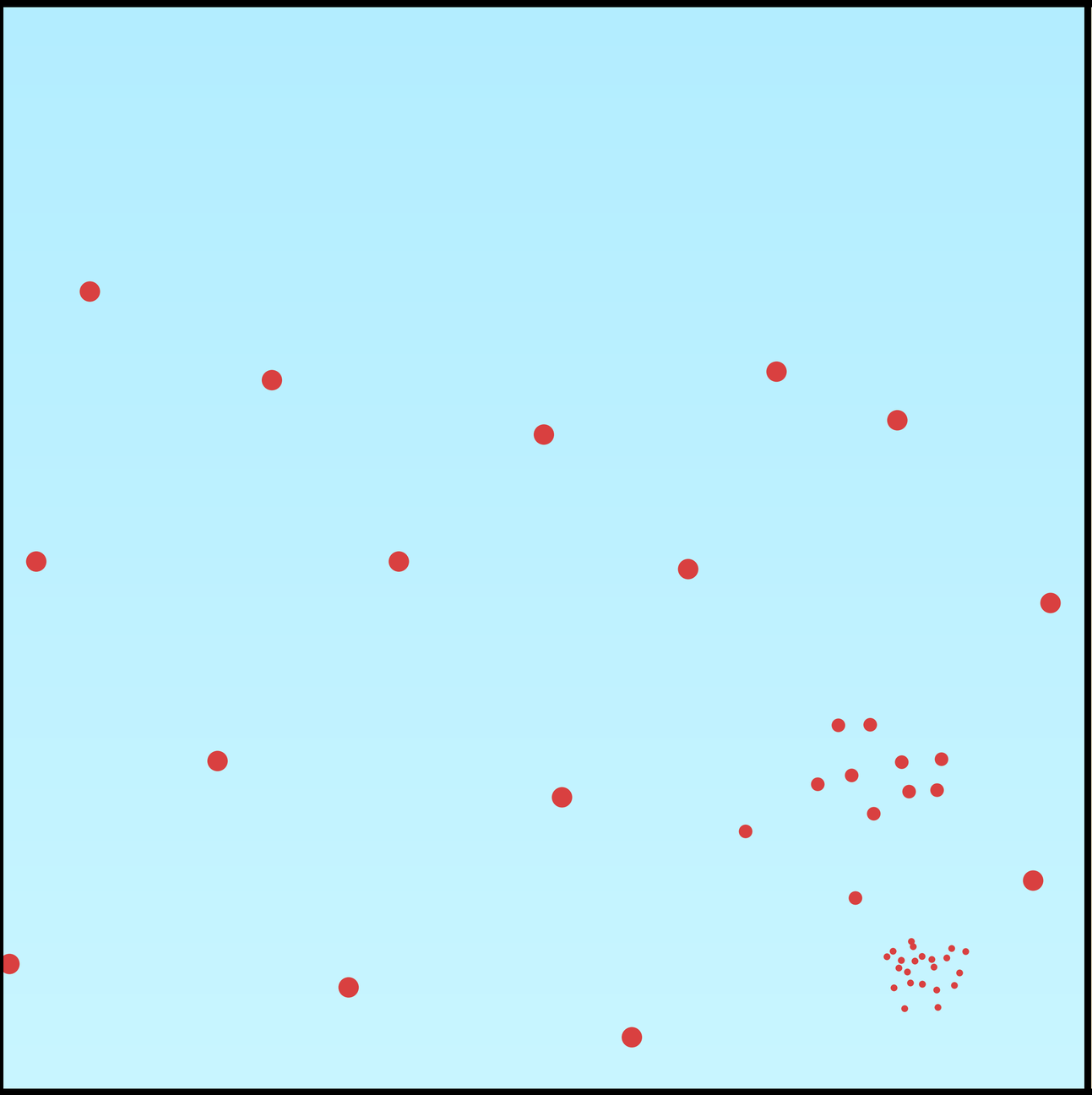
The scale of a point set need not be uniform.



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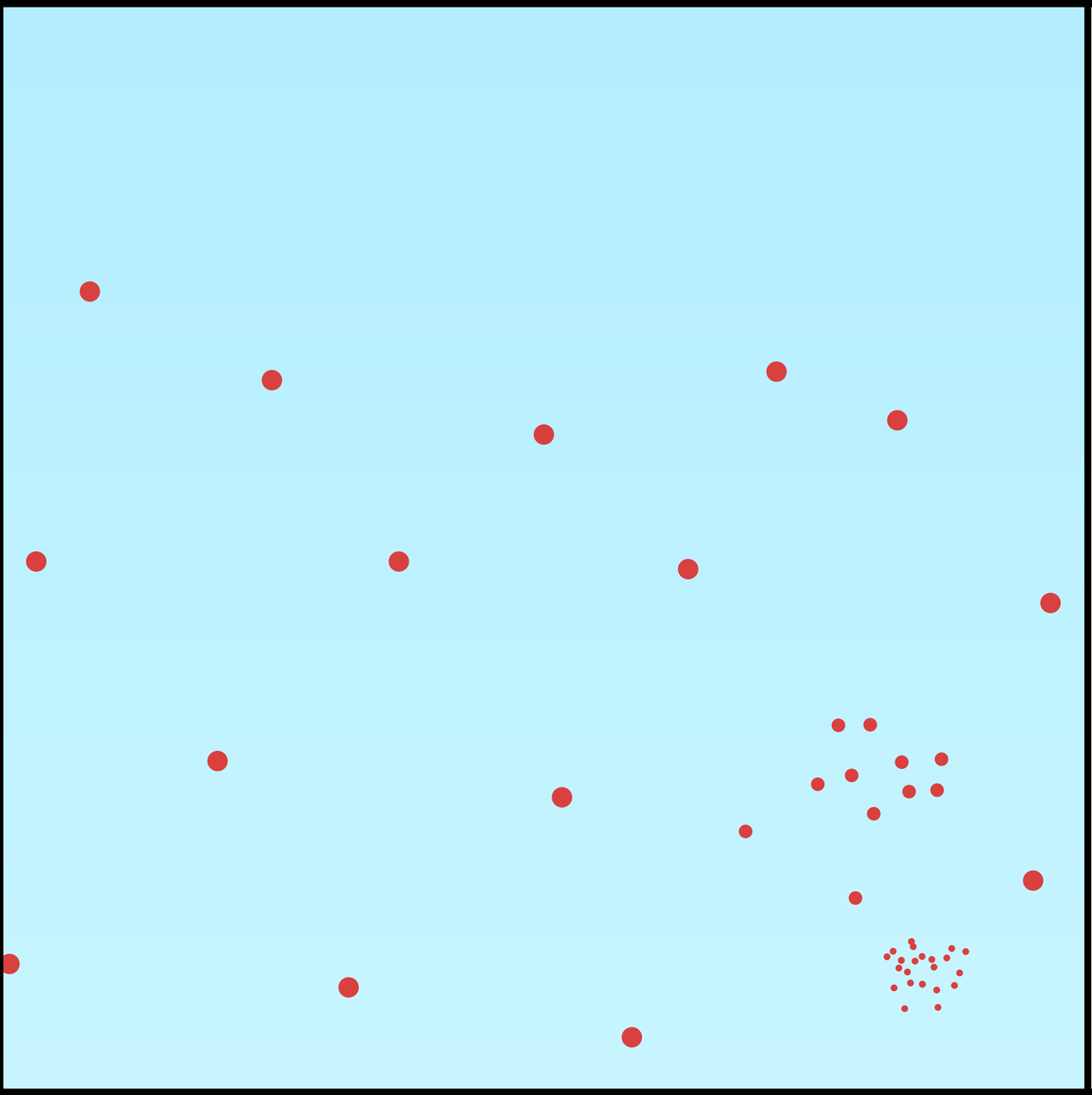

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*Or do we insist on being general, and design devilishly complicated algorithms that nobody will ever use, but that are able to handle arbitrary point sets?*

# Chapter 1
## *PROXIMITY STRUCTURES*

Point sets and proximity

Computational questions

State of the art in 1D

State of the art in 2D

Points in $\mathbb{R}^1$ are numbers.

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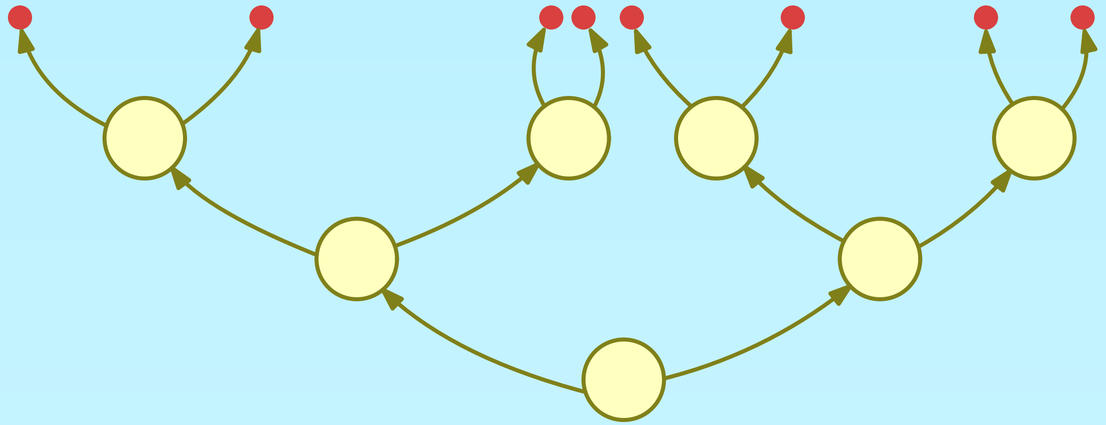

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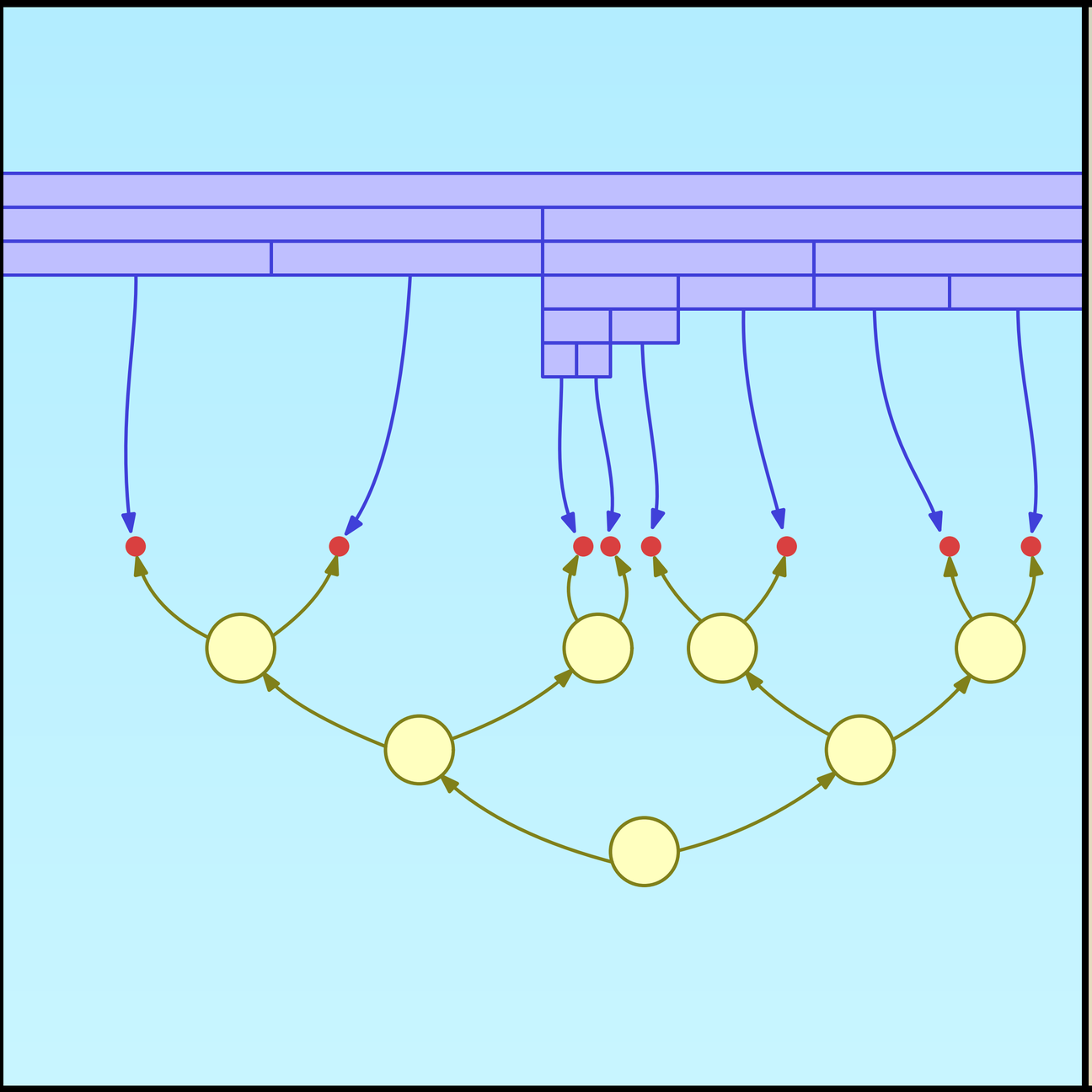

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Voronoi Diagram

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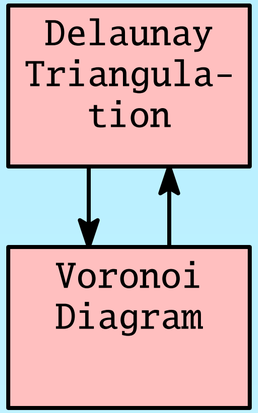


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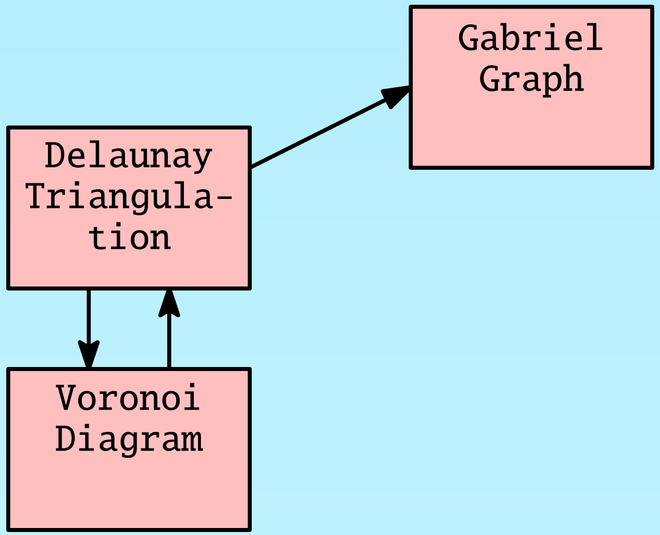


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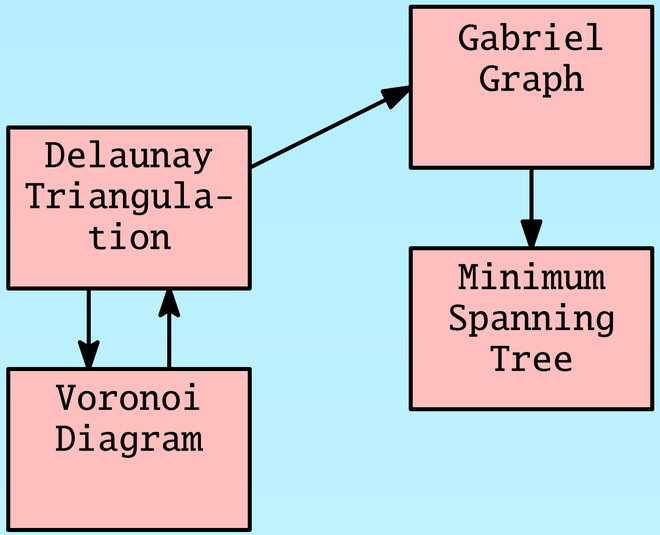


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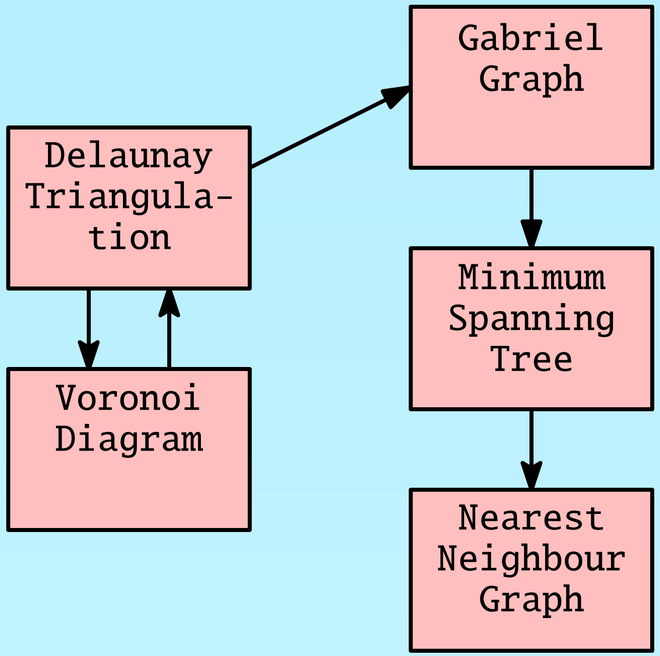


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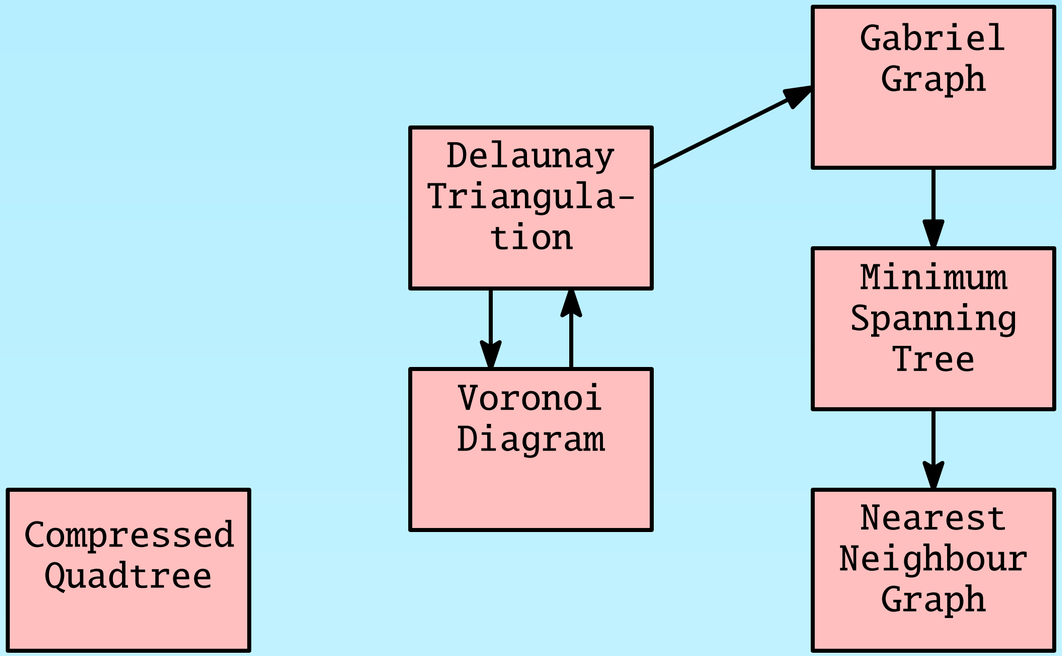


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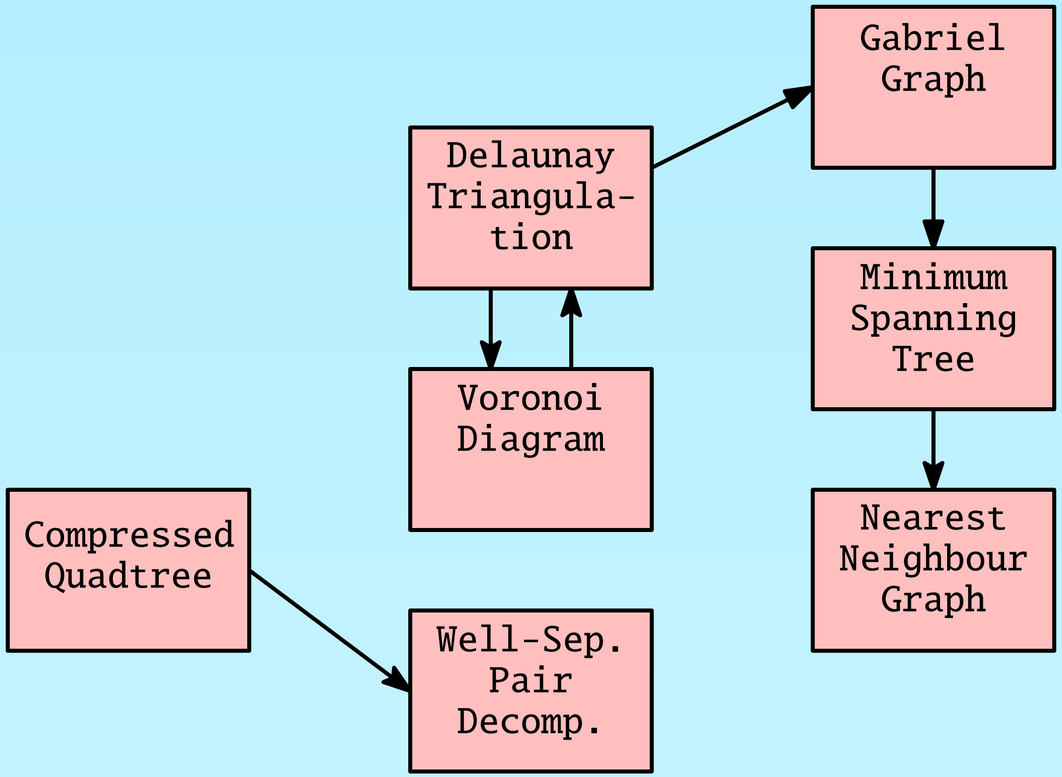


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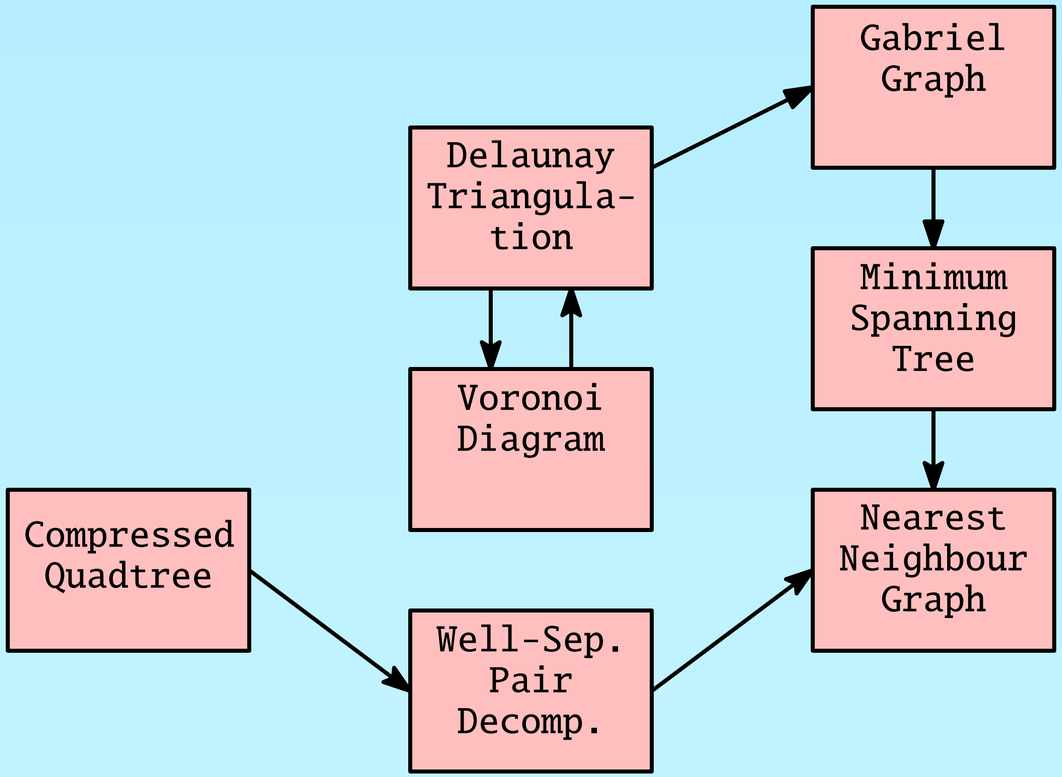


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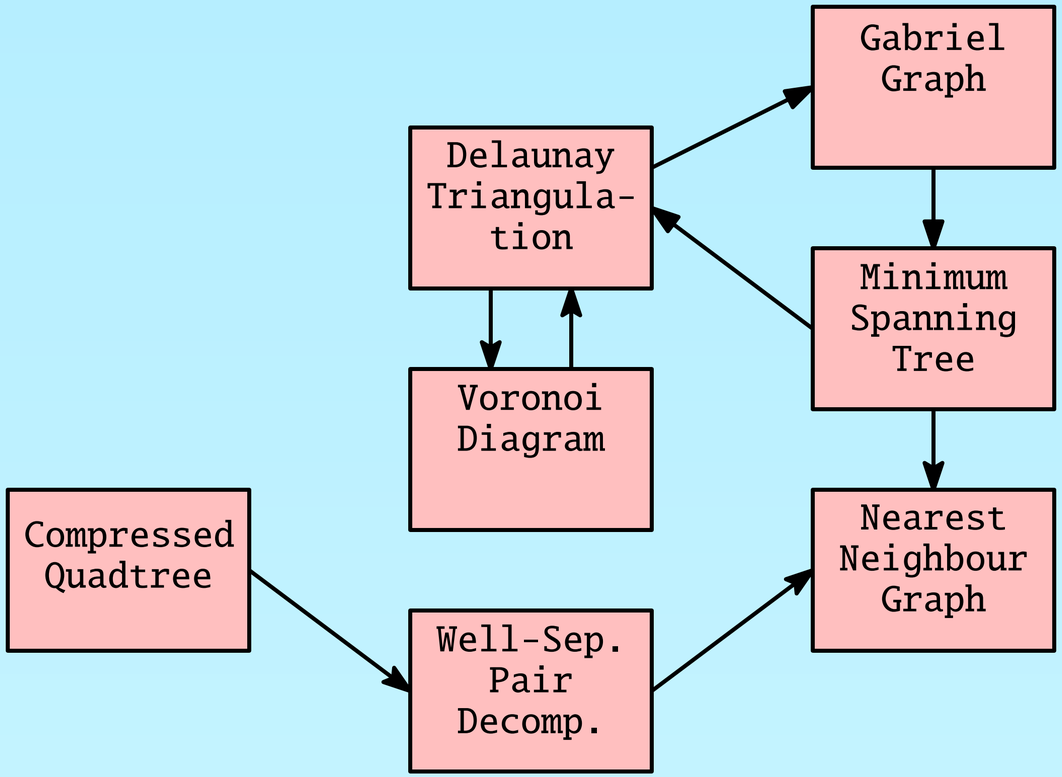


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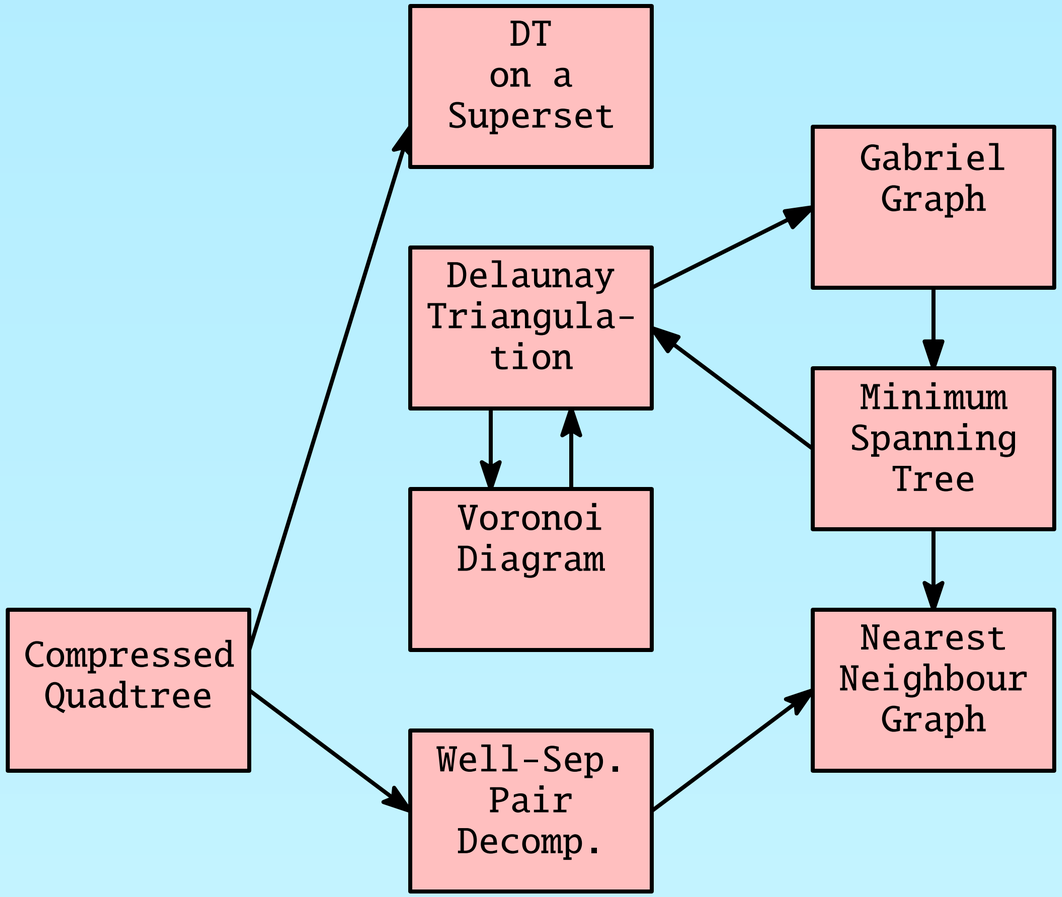


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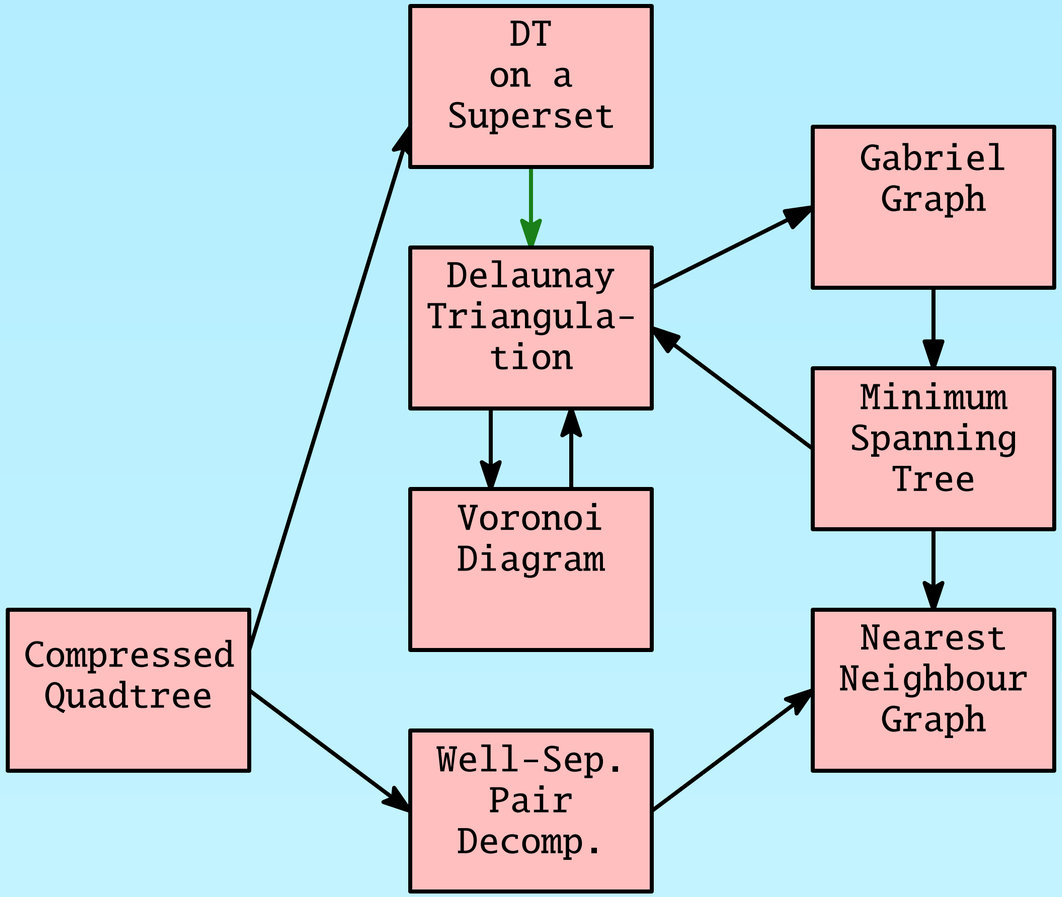


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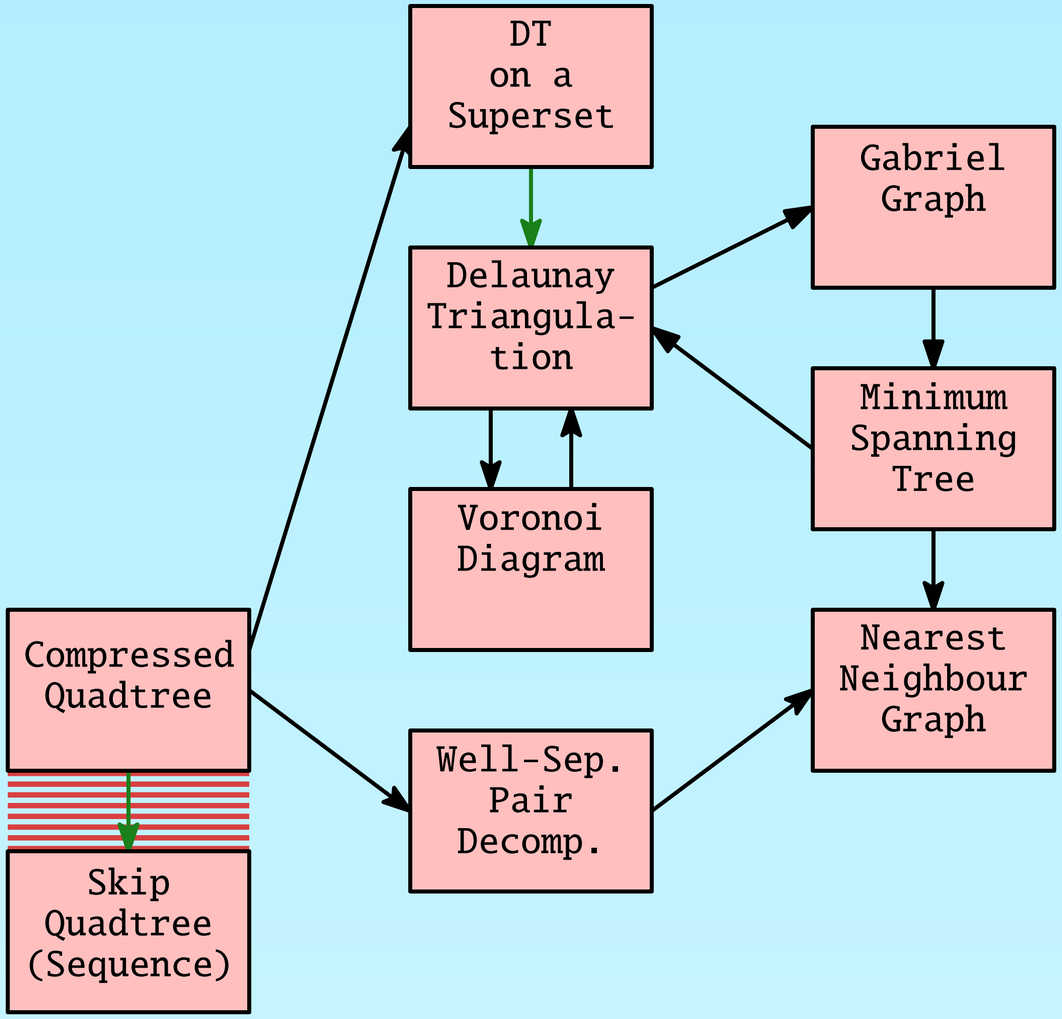


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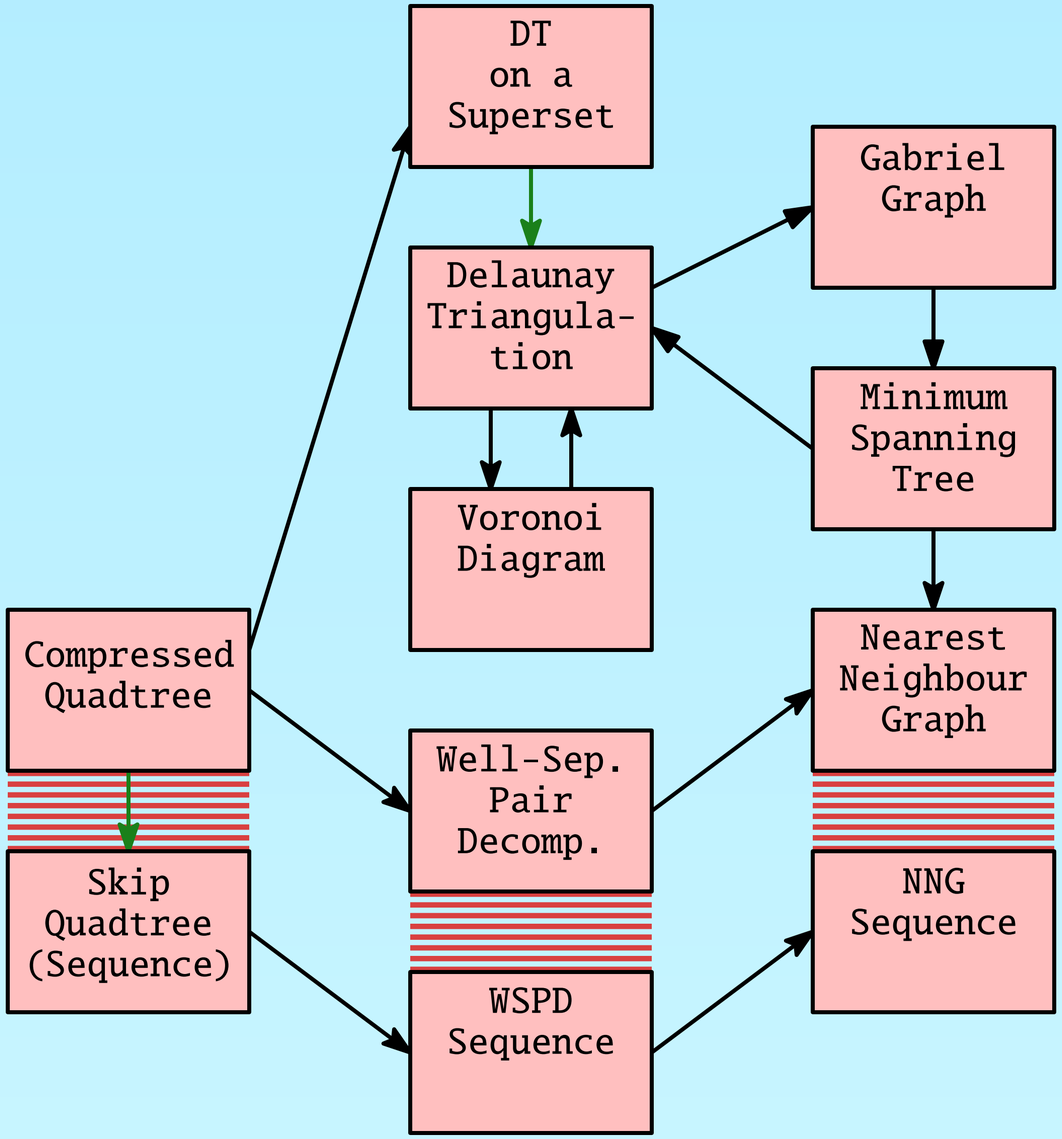


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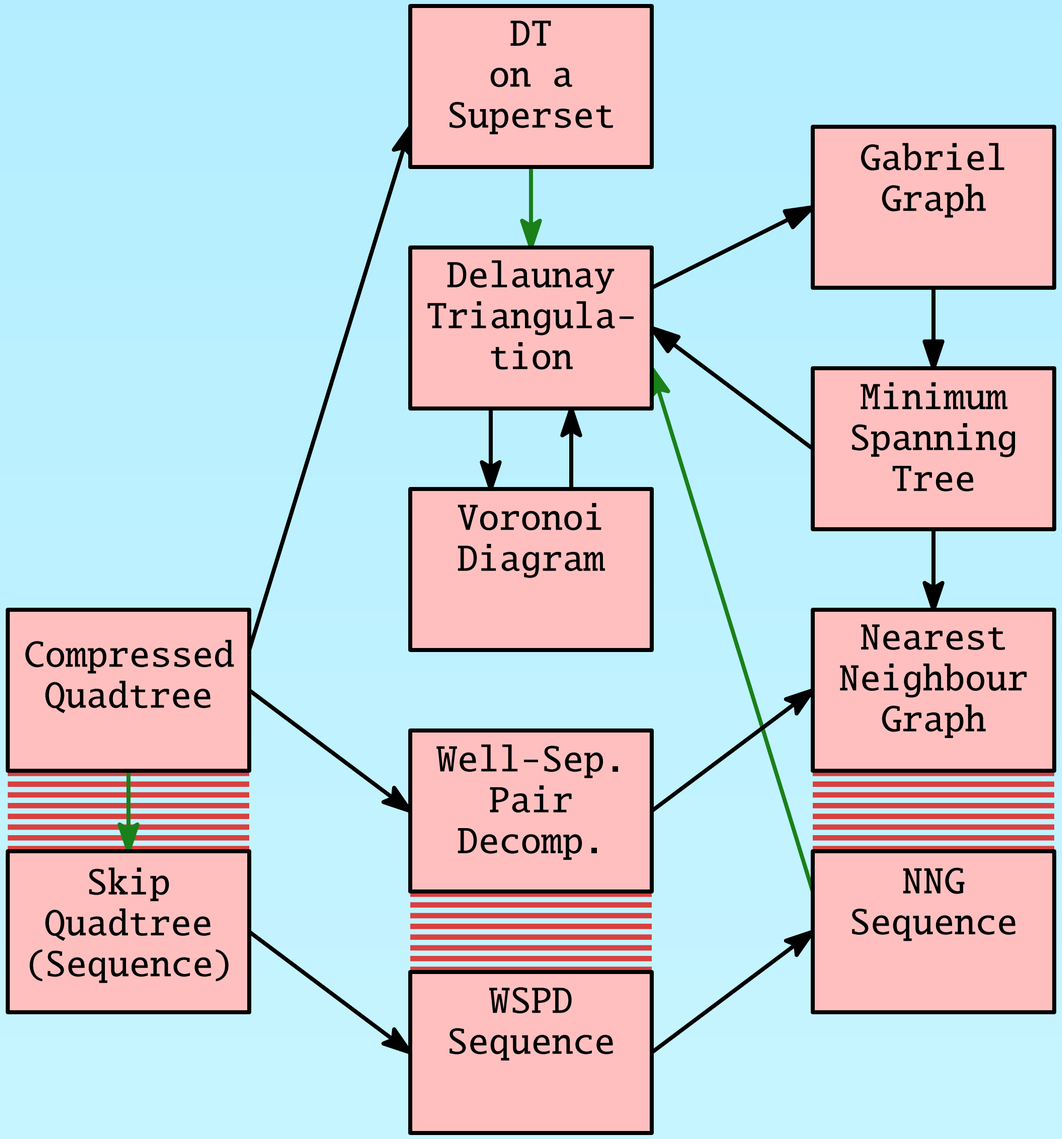


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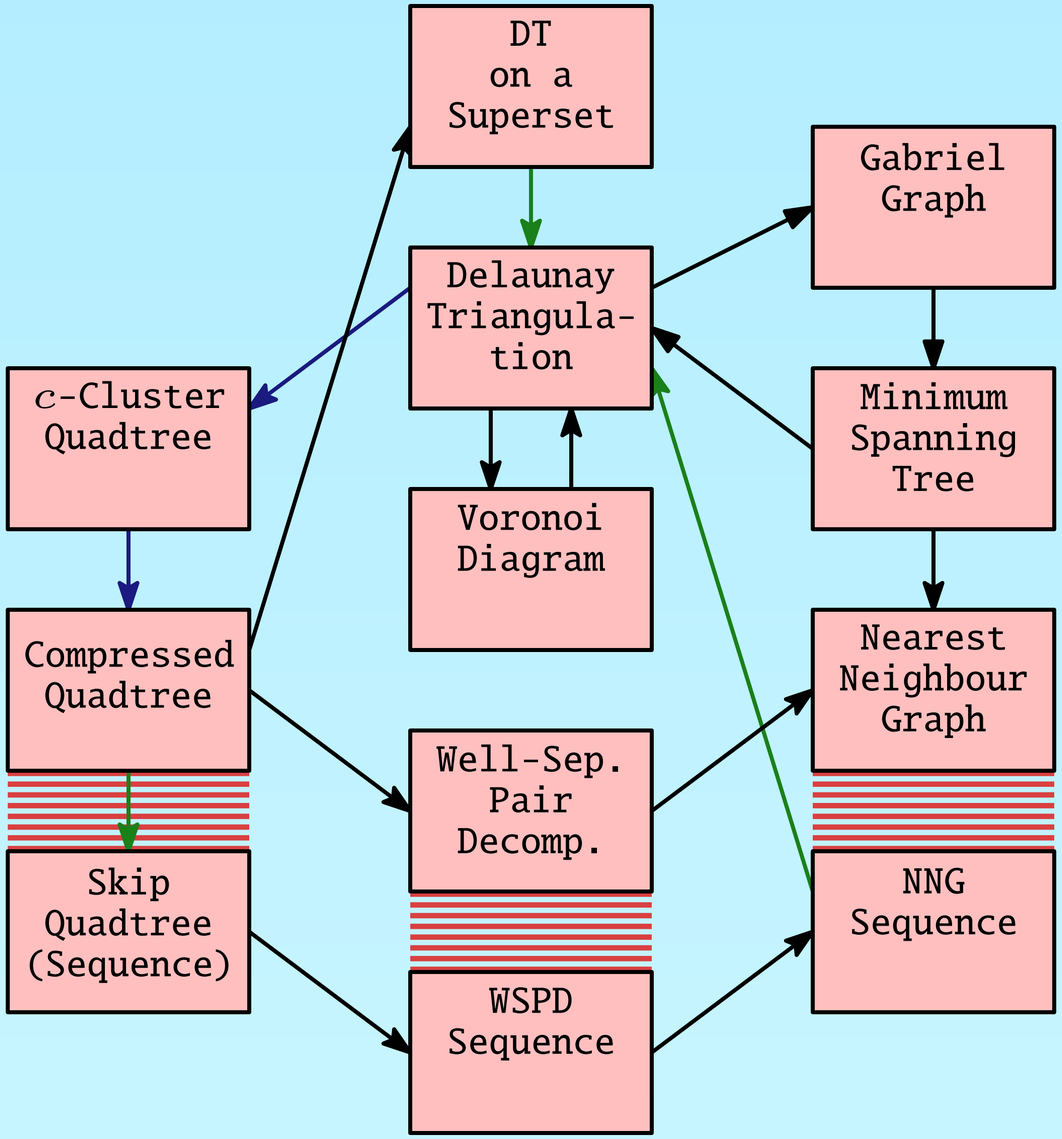


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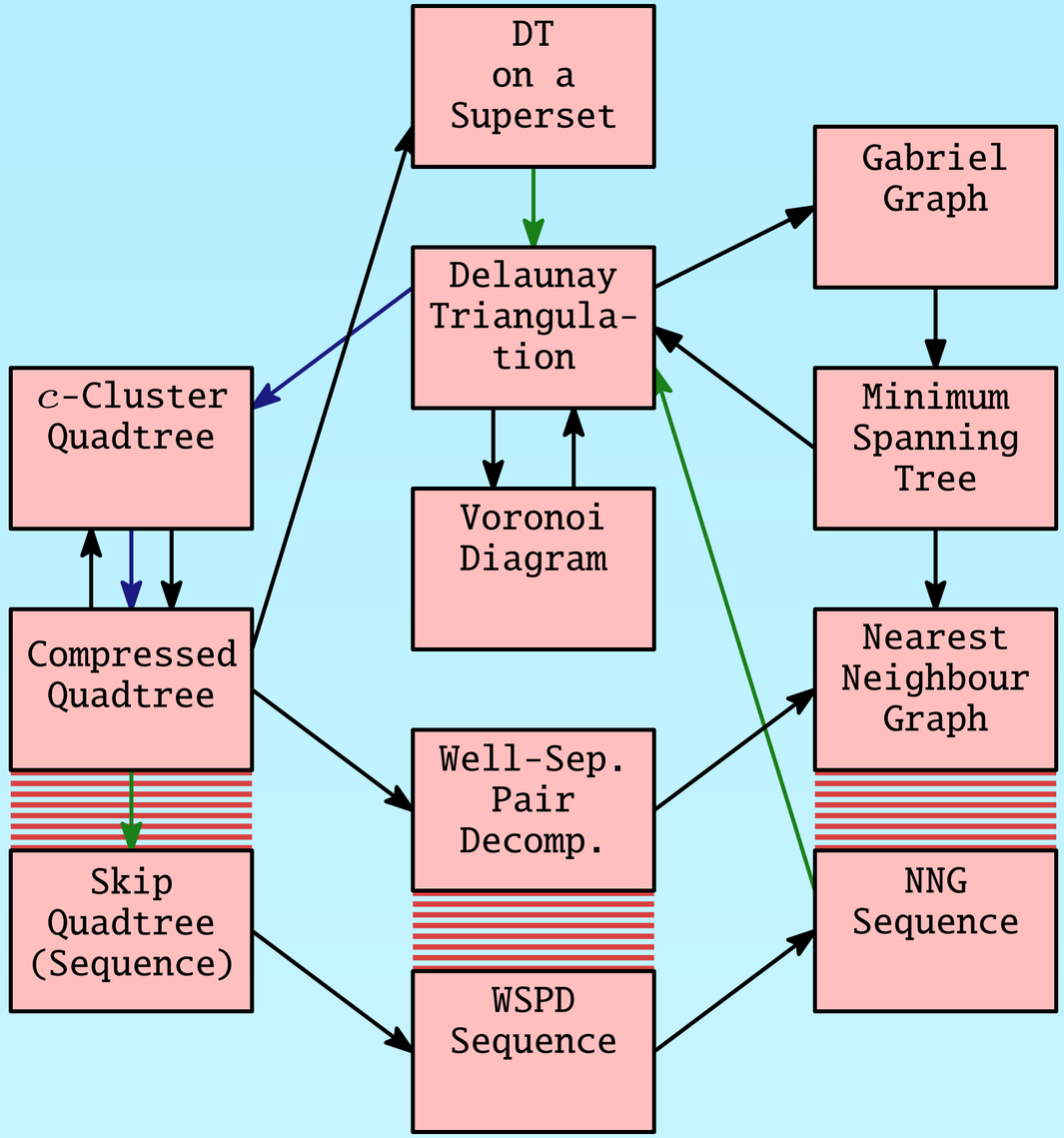


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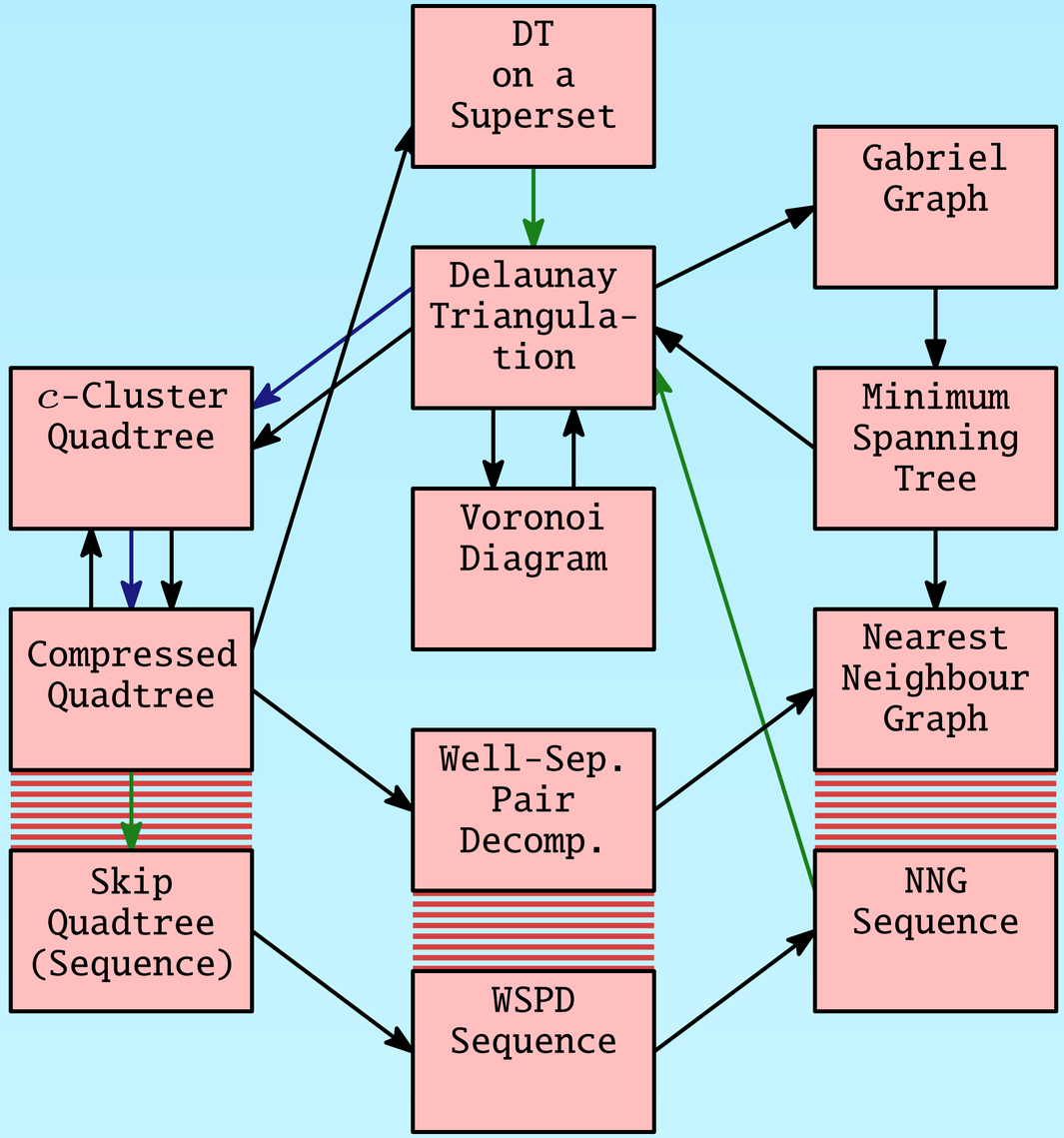


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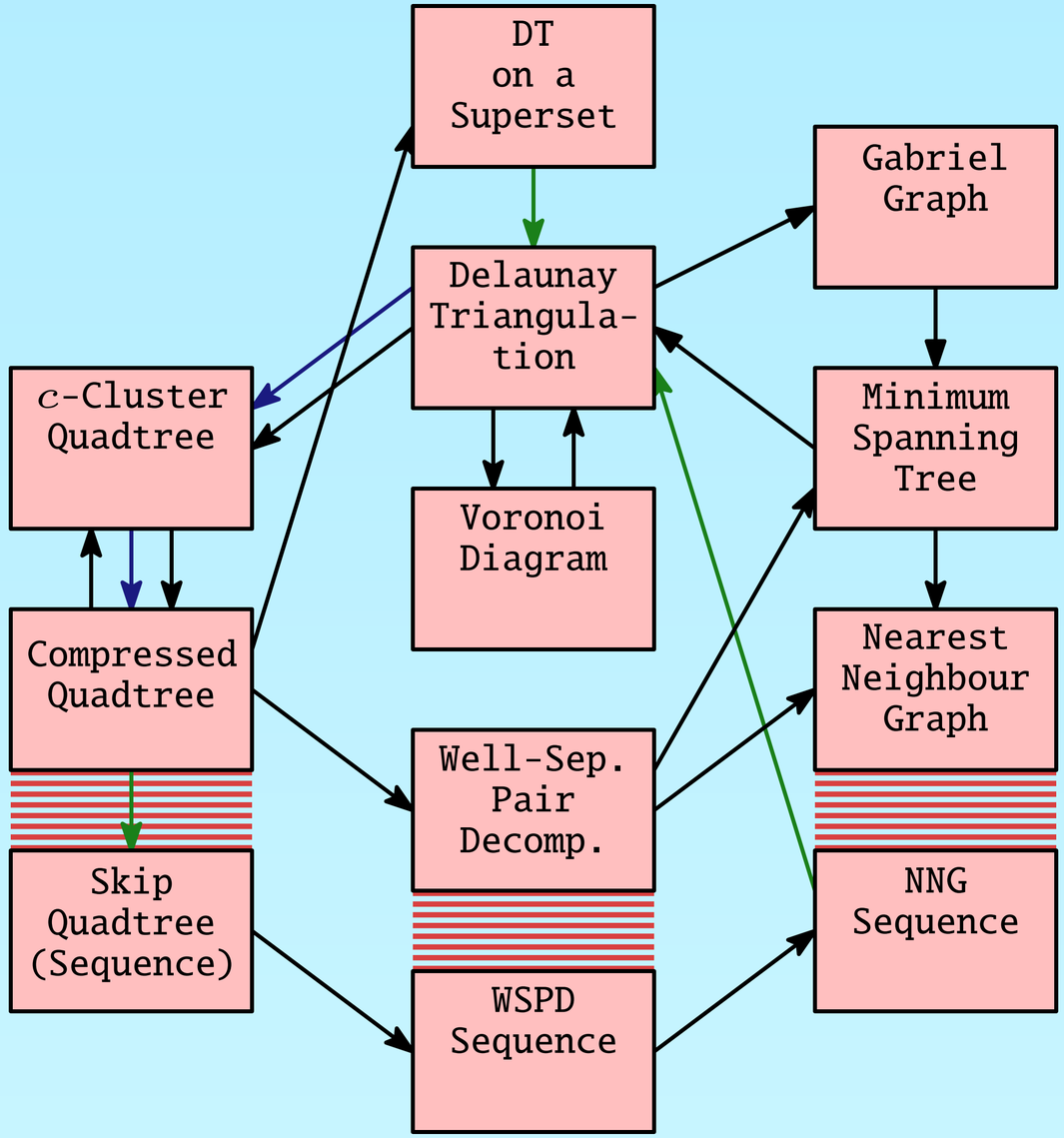


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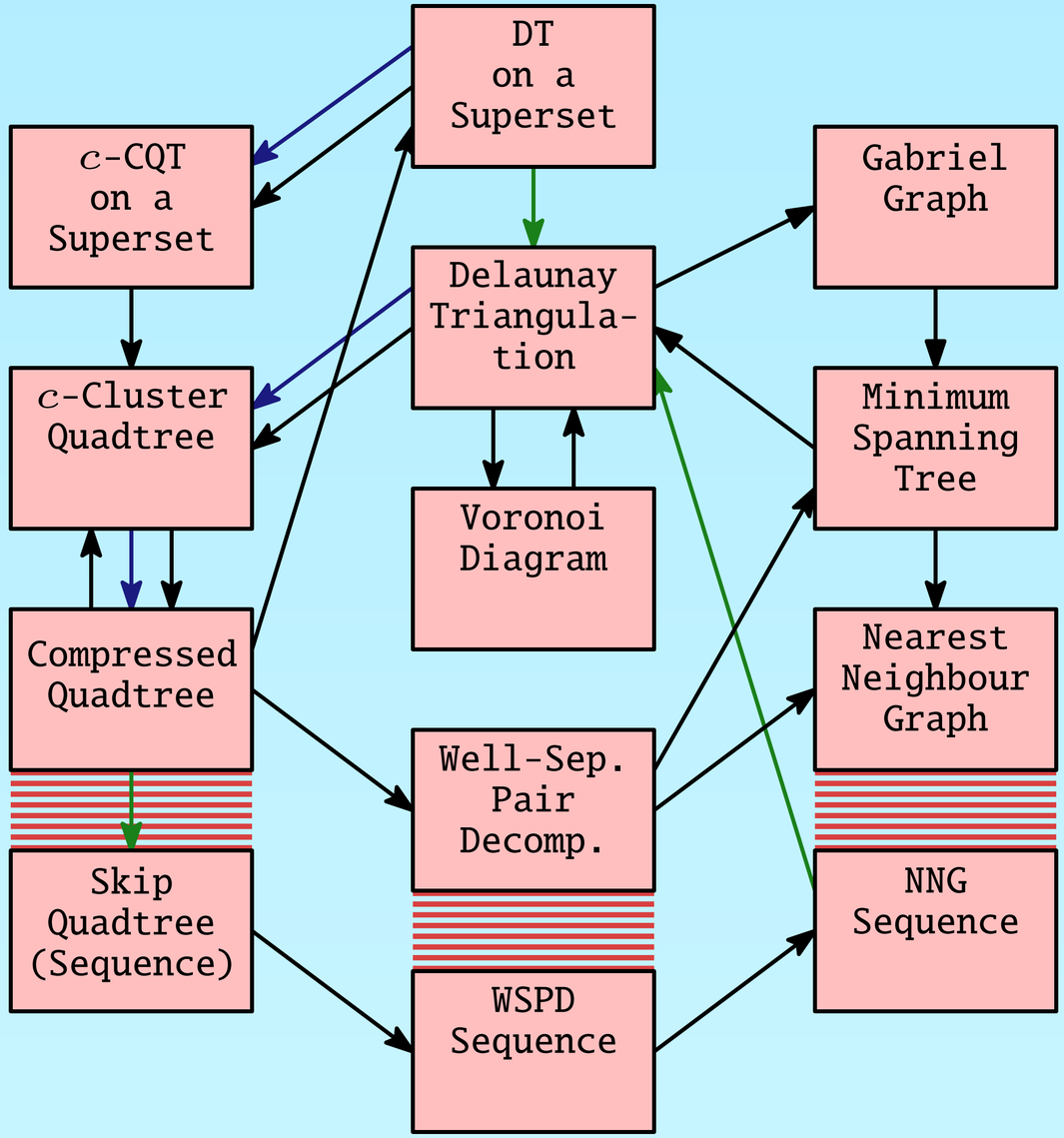


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# Chapter 2
*PARTIAL INFORMATION*

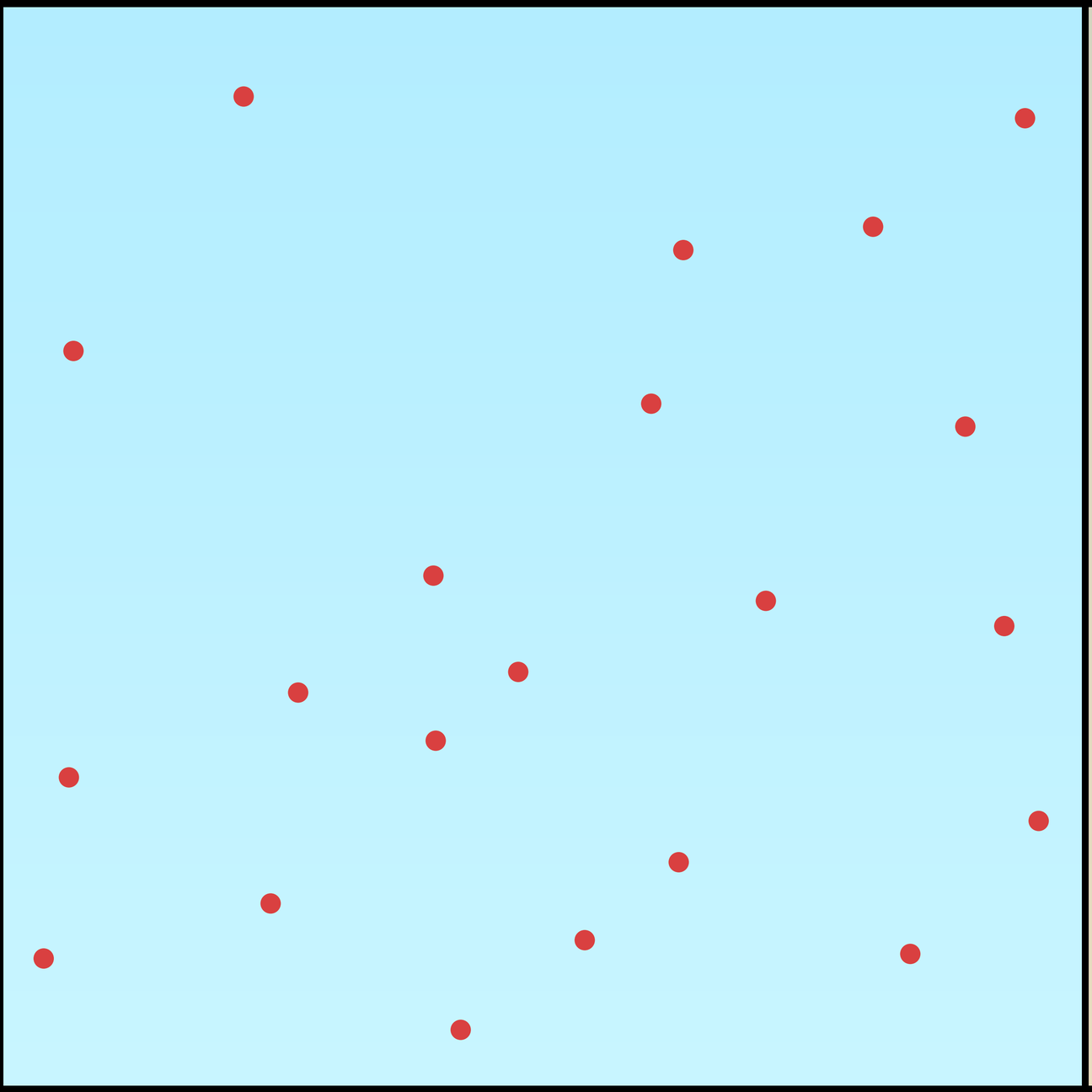
Uncertain locations

Quantifying information

State of the art in 1D

State of the art in 2D

Let $P$ be a set of $n$ points in the plane.



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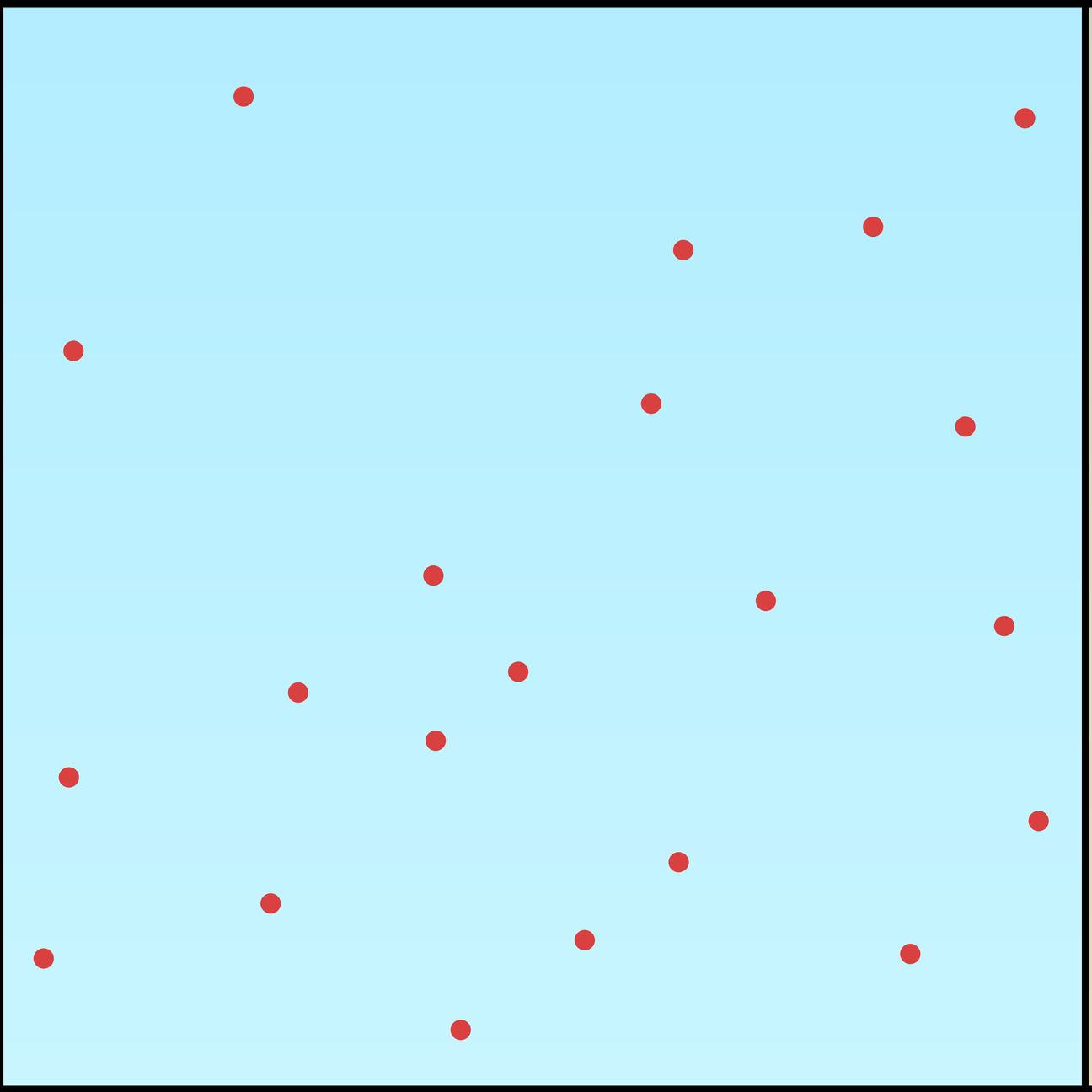

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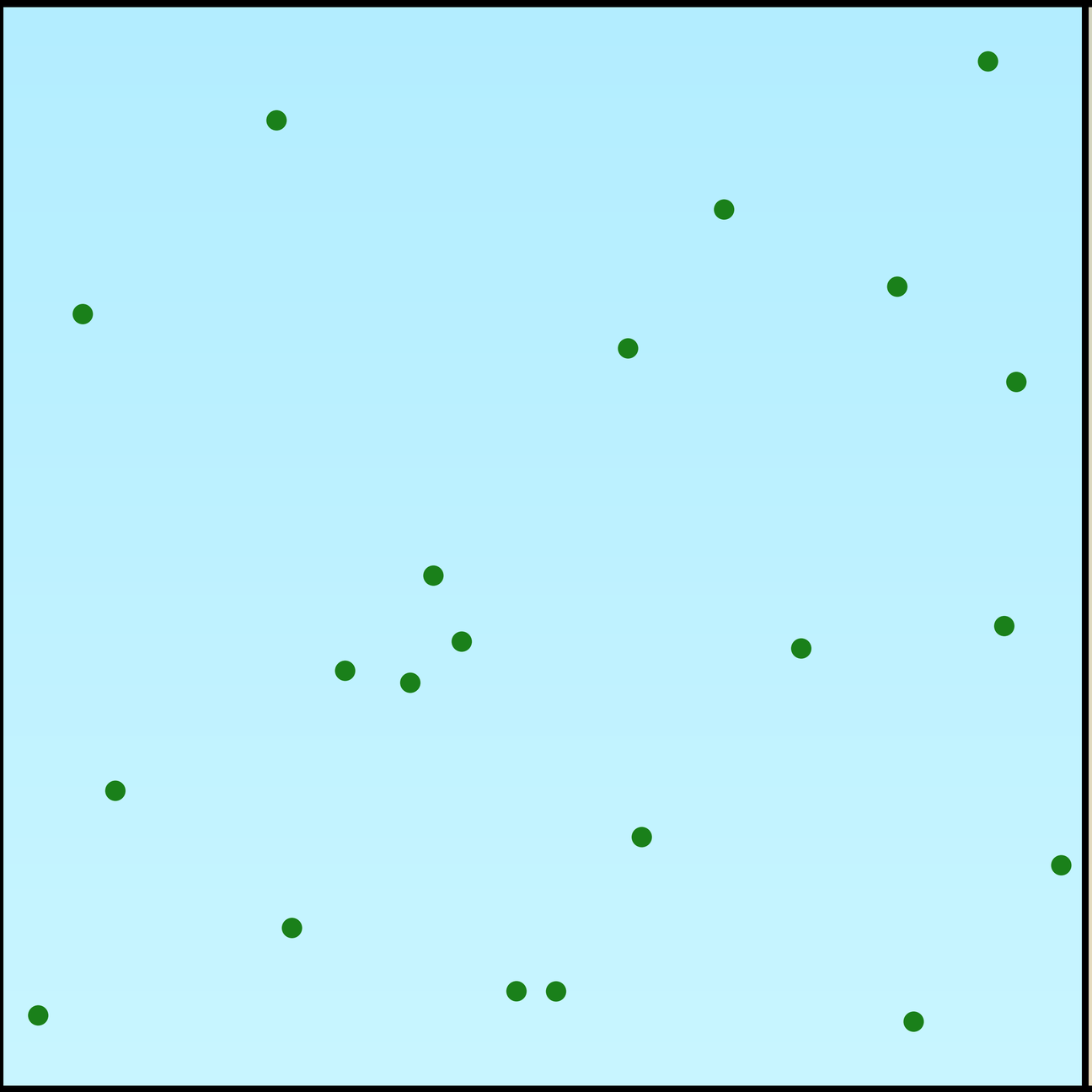

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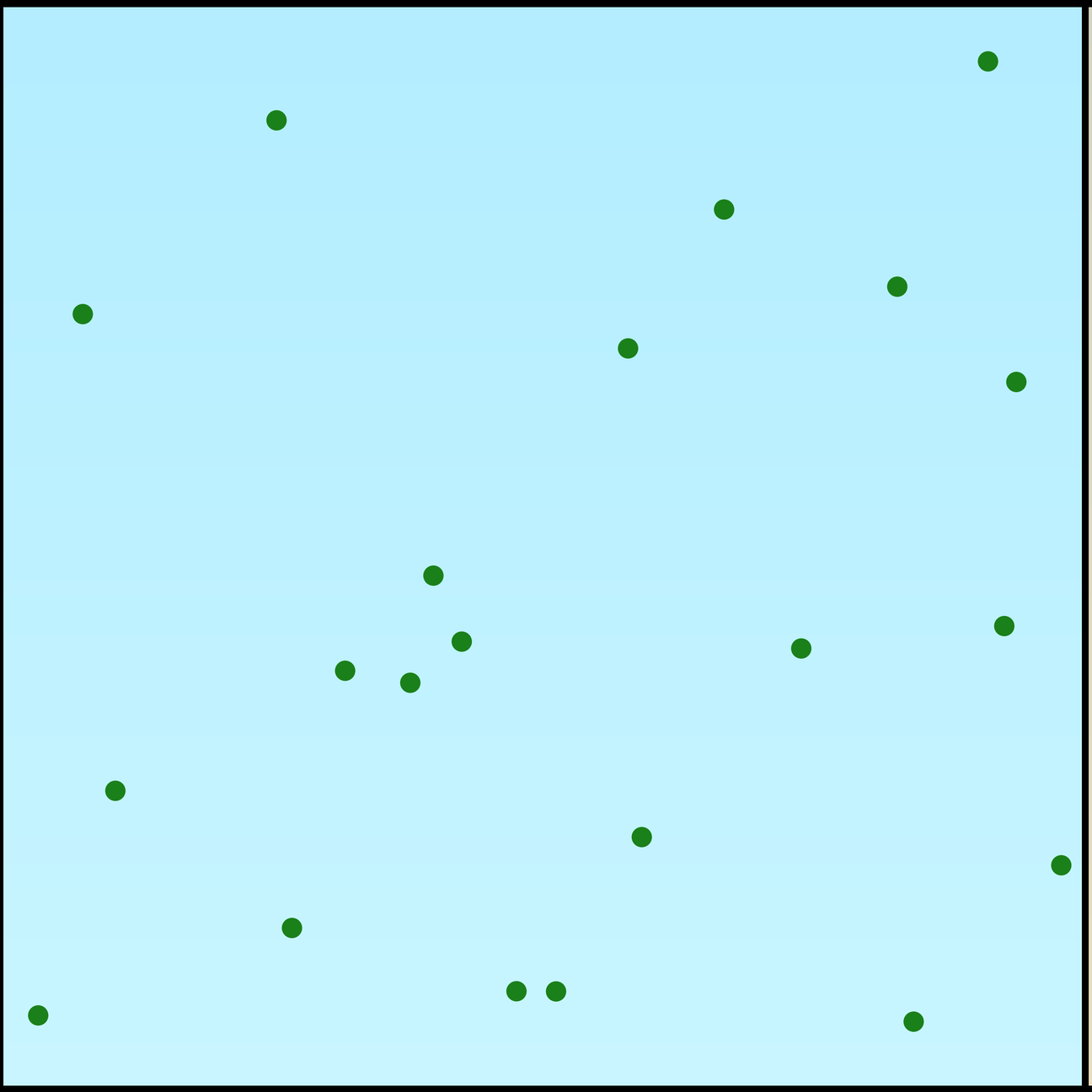

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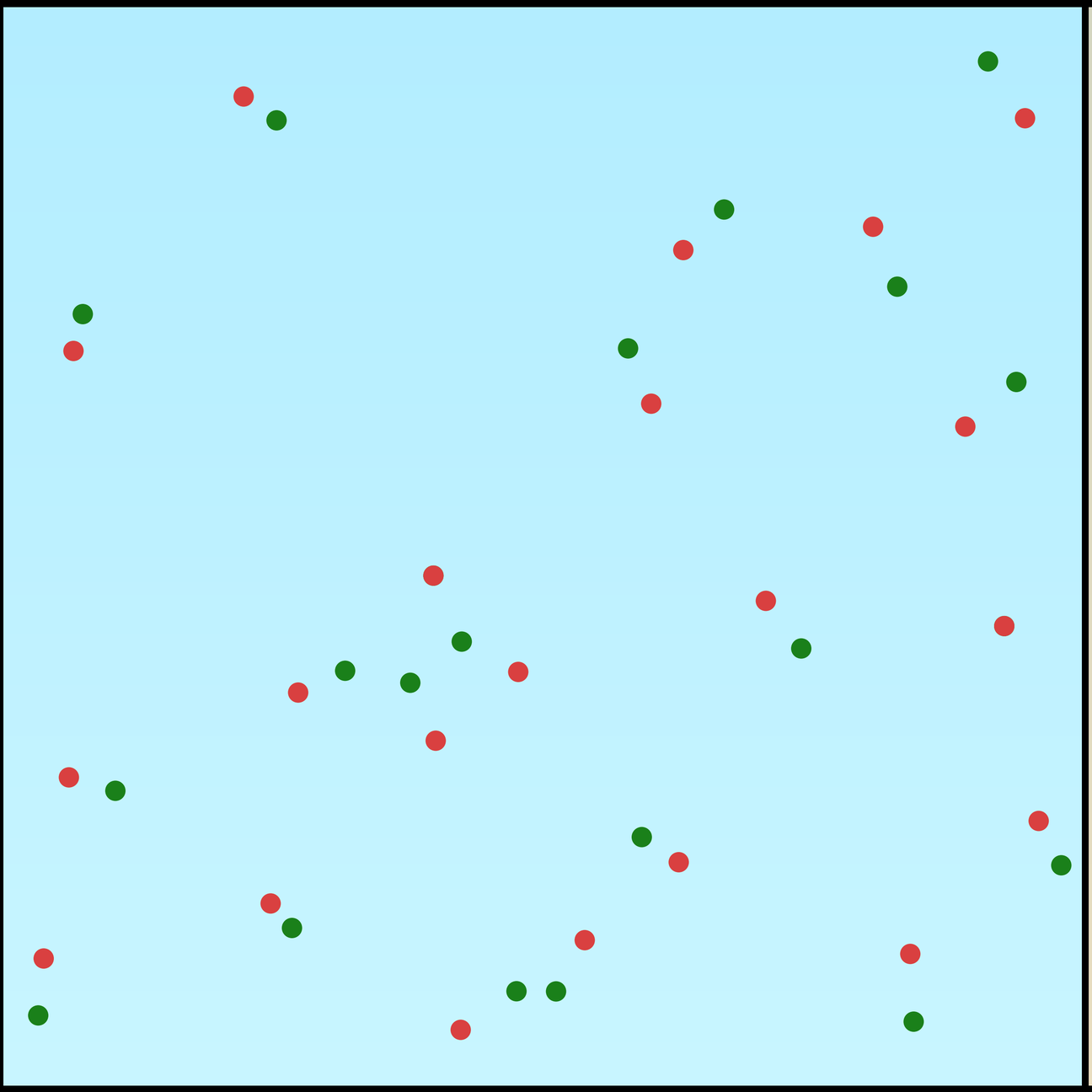

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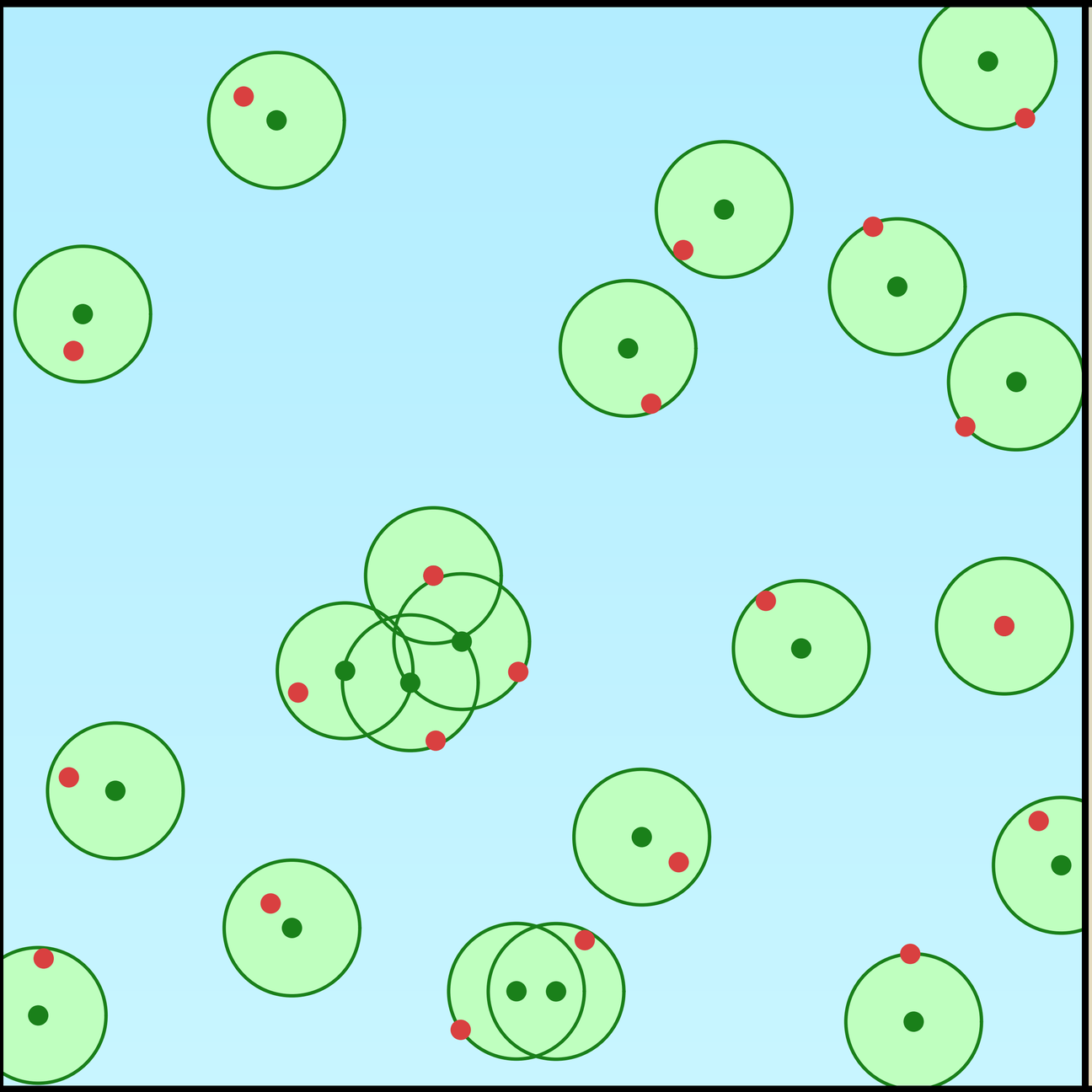

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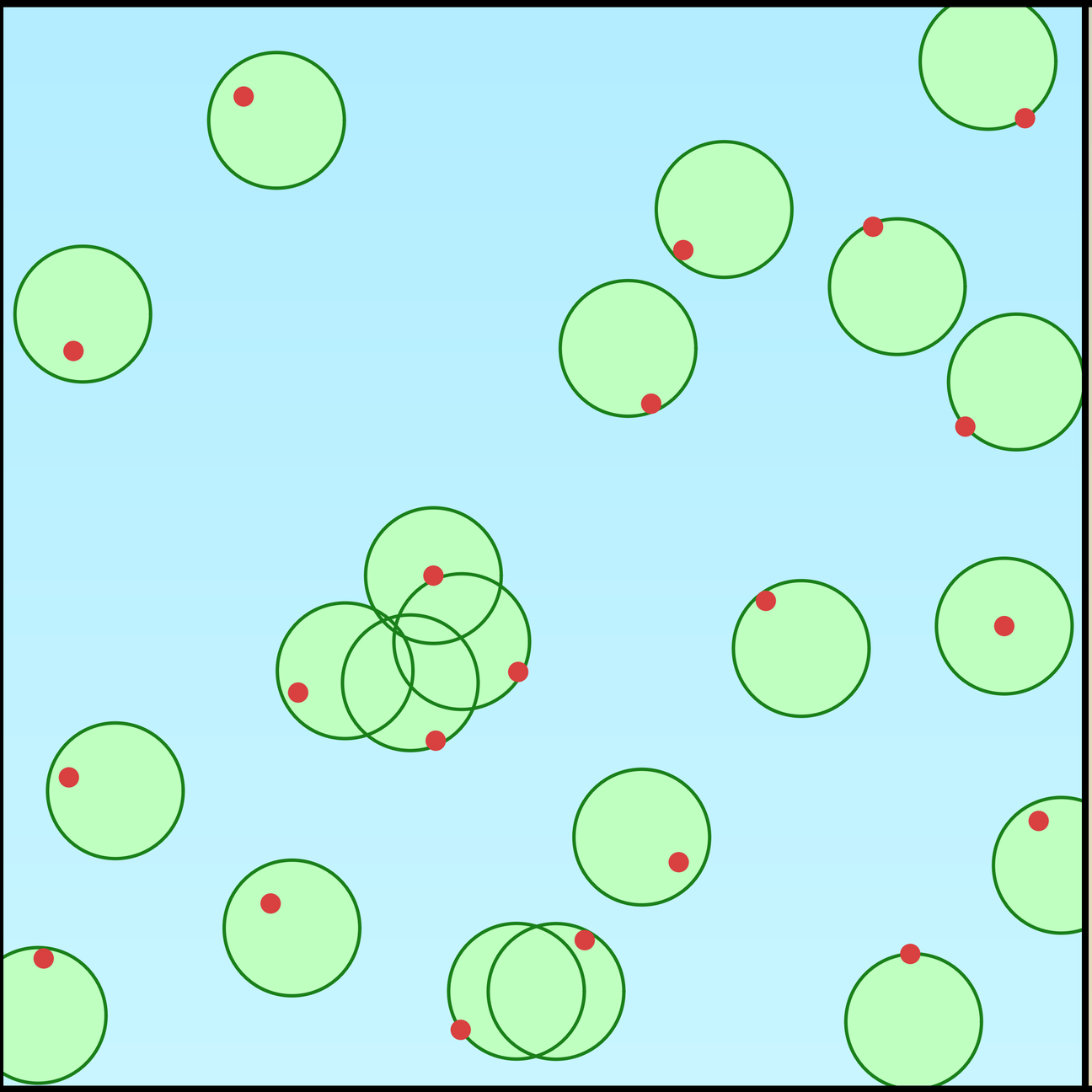

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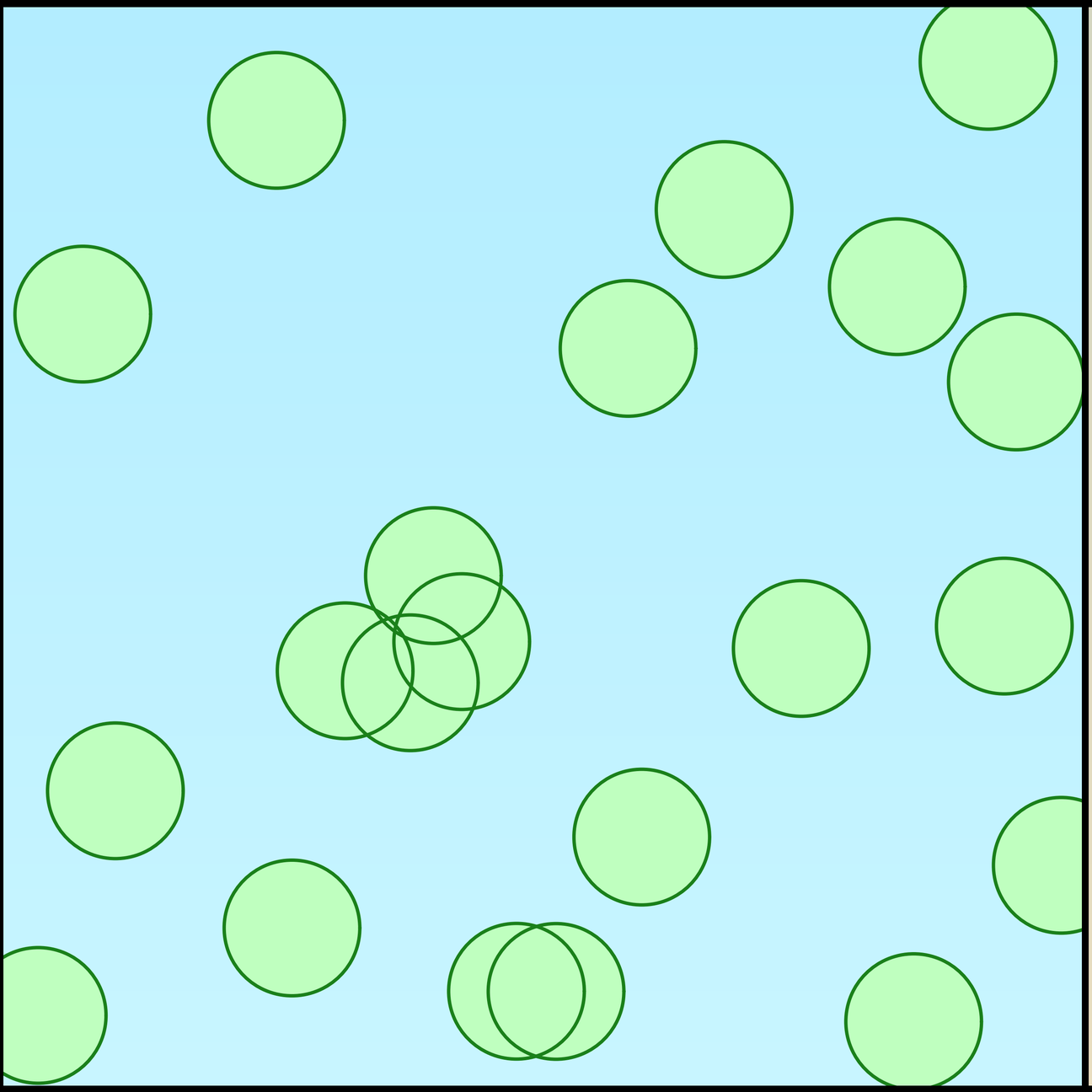
$\mathcal{R}$ is a set of *regions* which represent *partial information* about the *location* of a point set.



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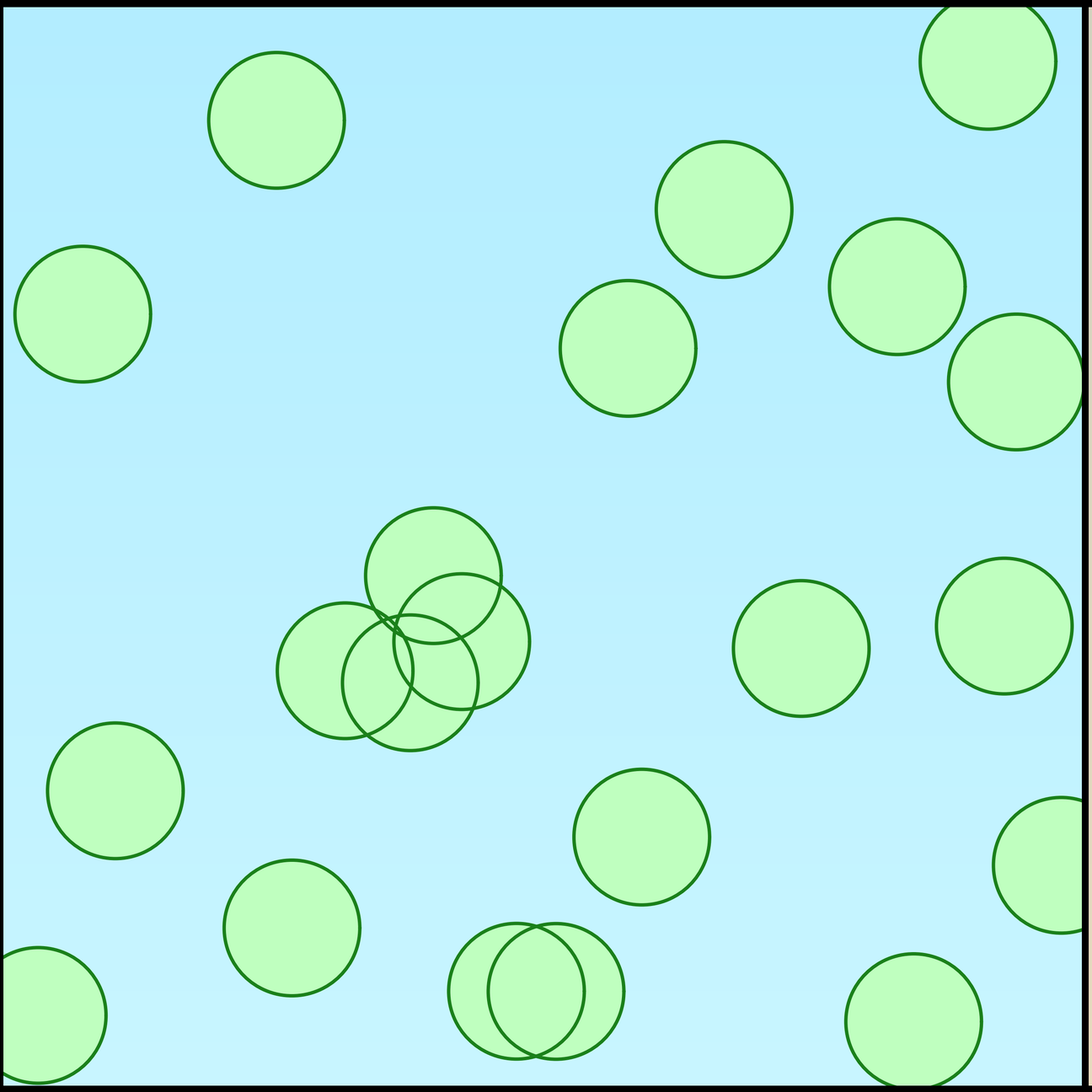

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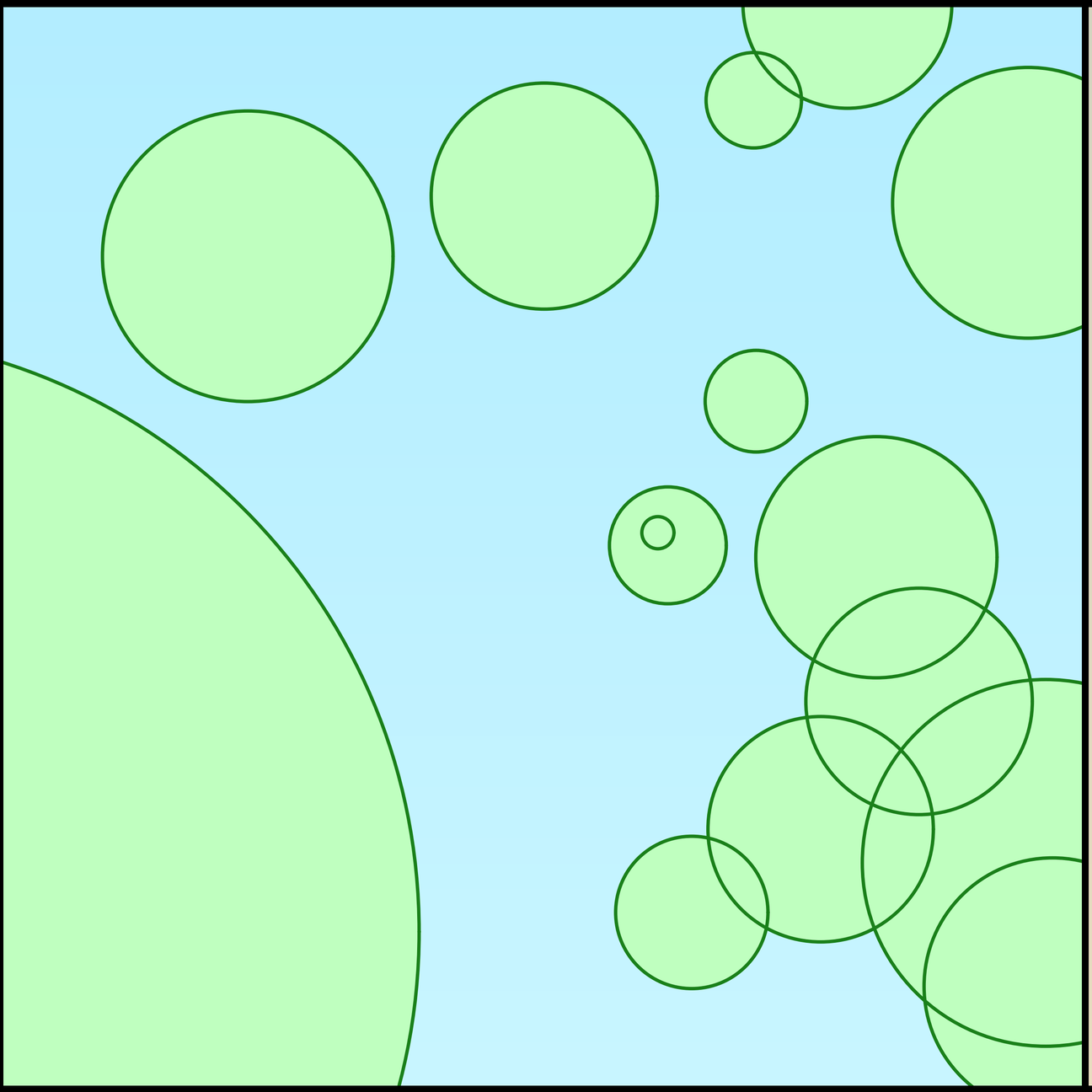
In principle, there is no reason to restrict $\mathcal{R}$ to a set of $\varepsilon$-disks.



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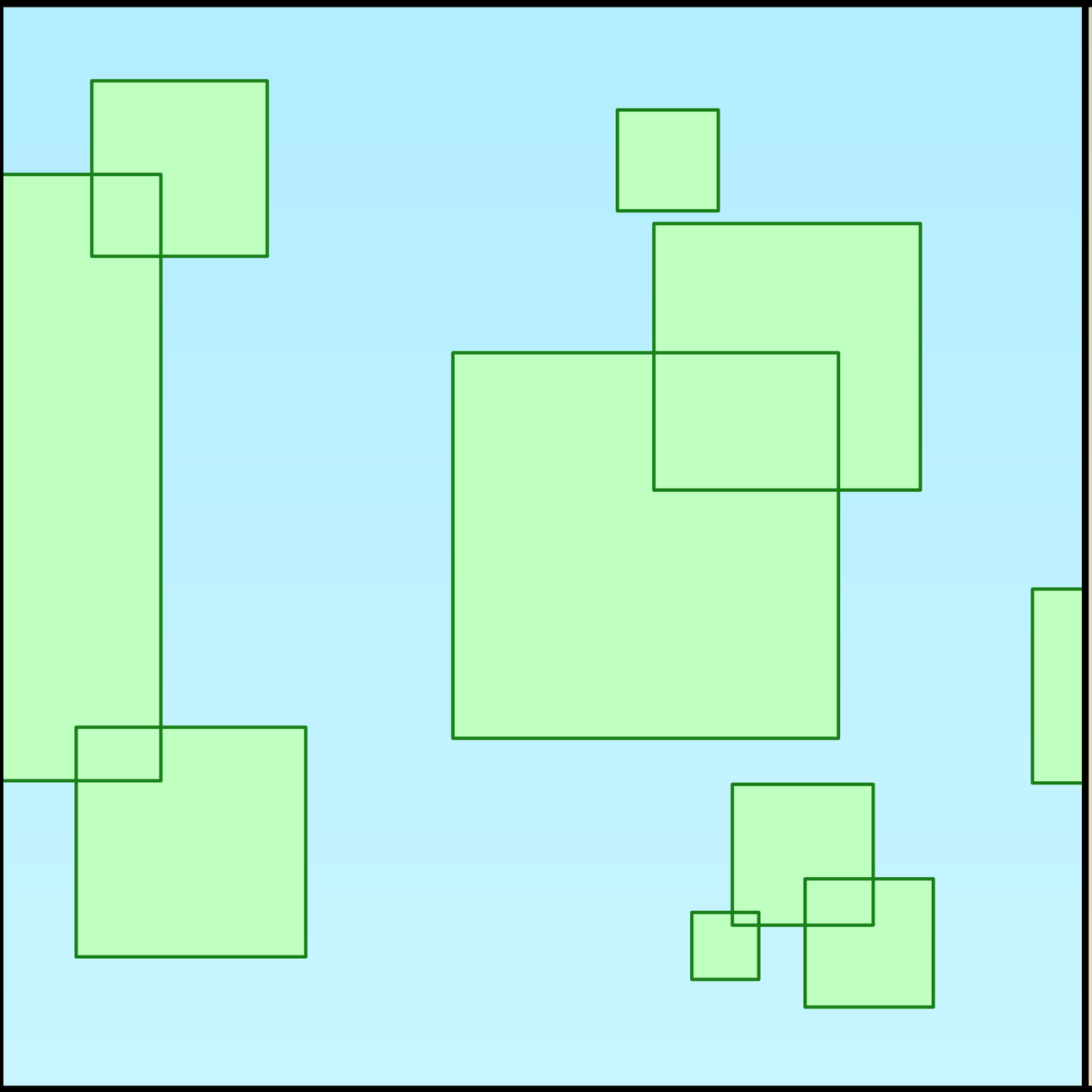

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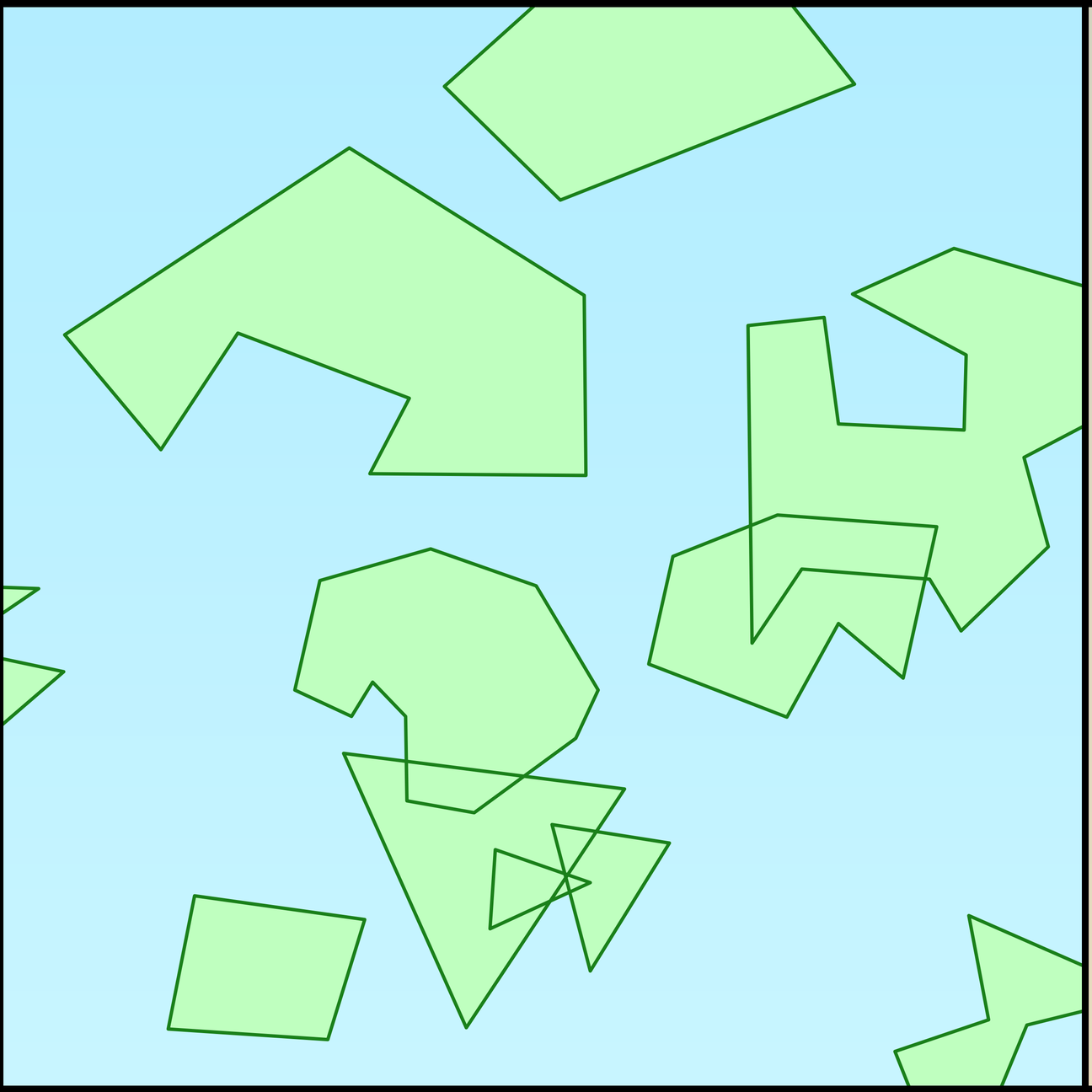

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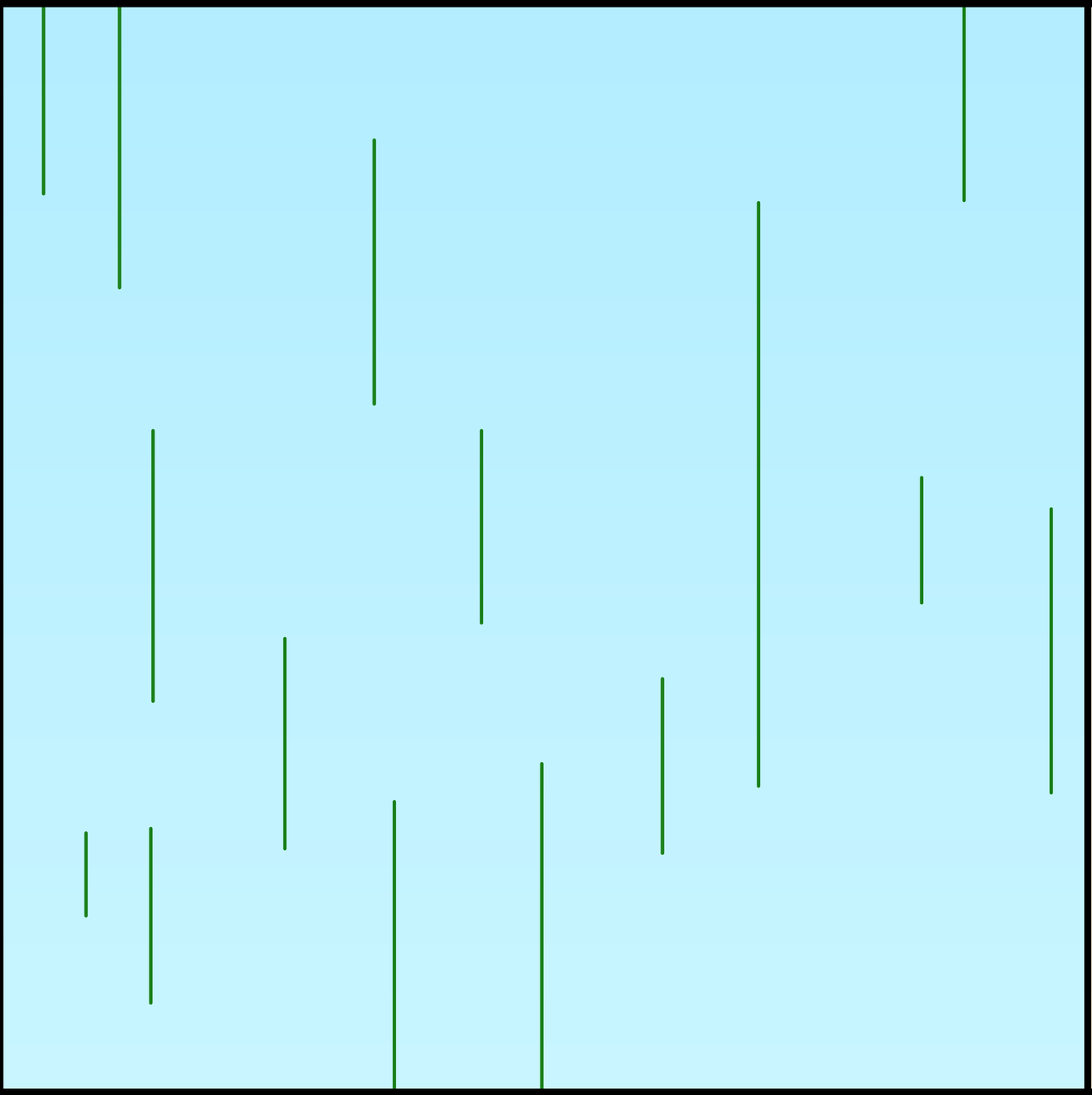

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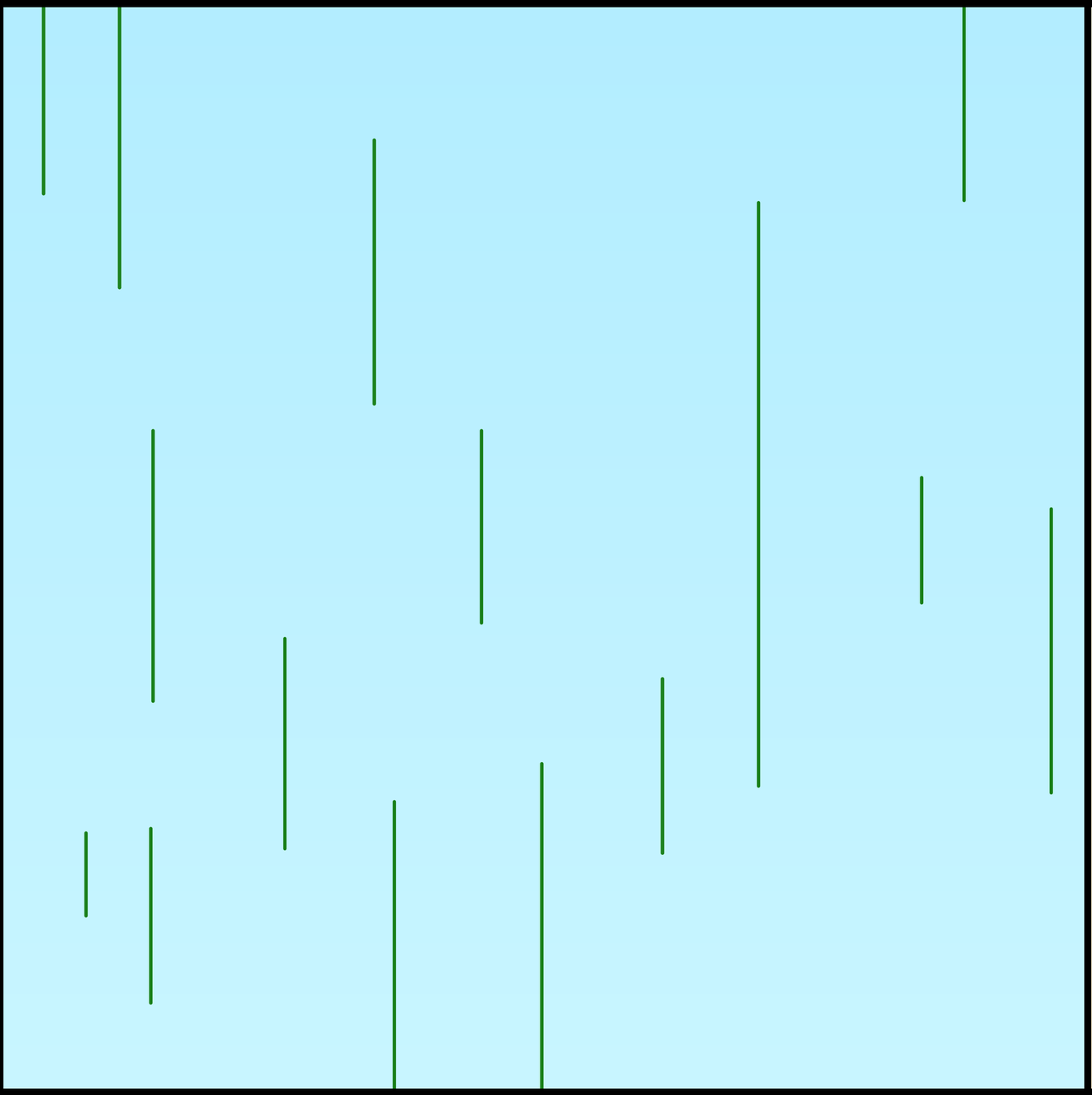

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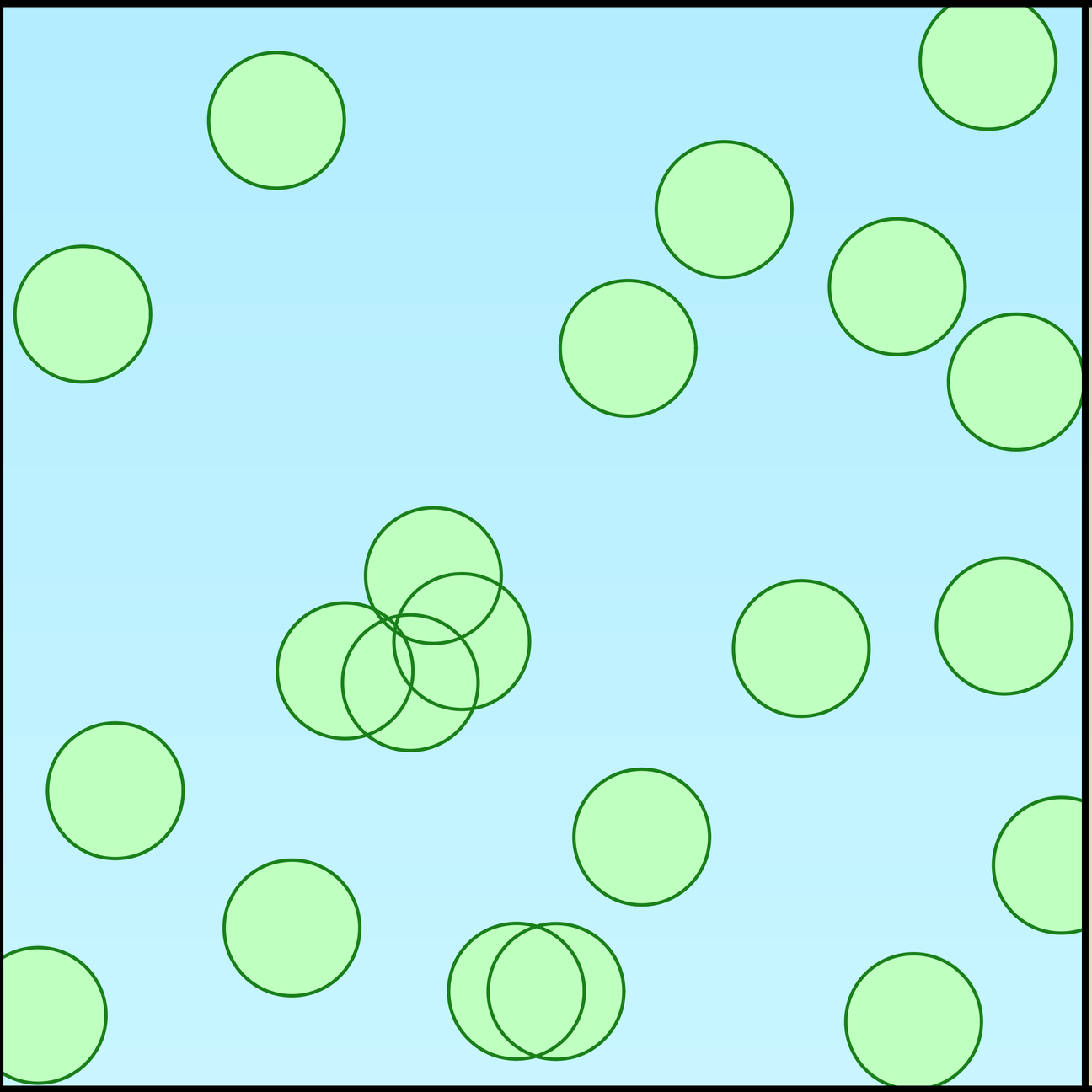
In principle, there is no reason to restrict $\mathcal{R}$ to a set of $\varepsilon$-disks.

$\mathcal{R}$ is a set of *imprecise* (or *uncertain*) points.

Let $\mathcal{R}$ be a set of regions representing an (unknown) point set $P$.



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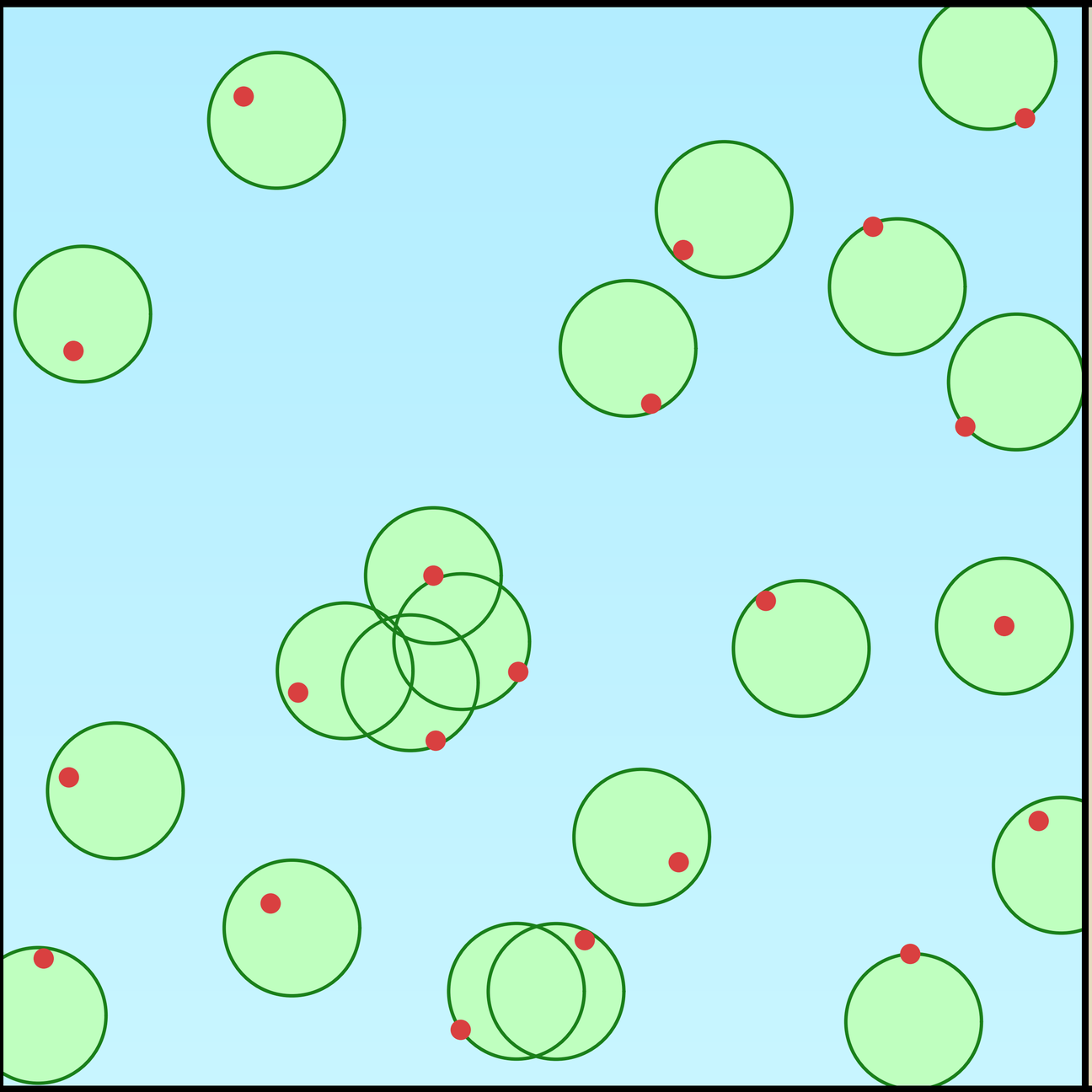

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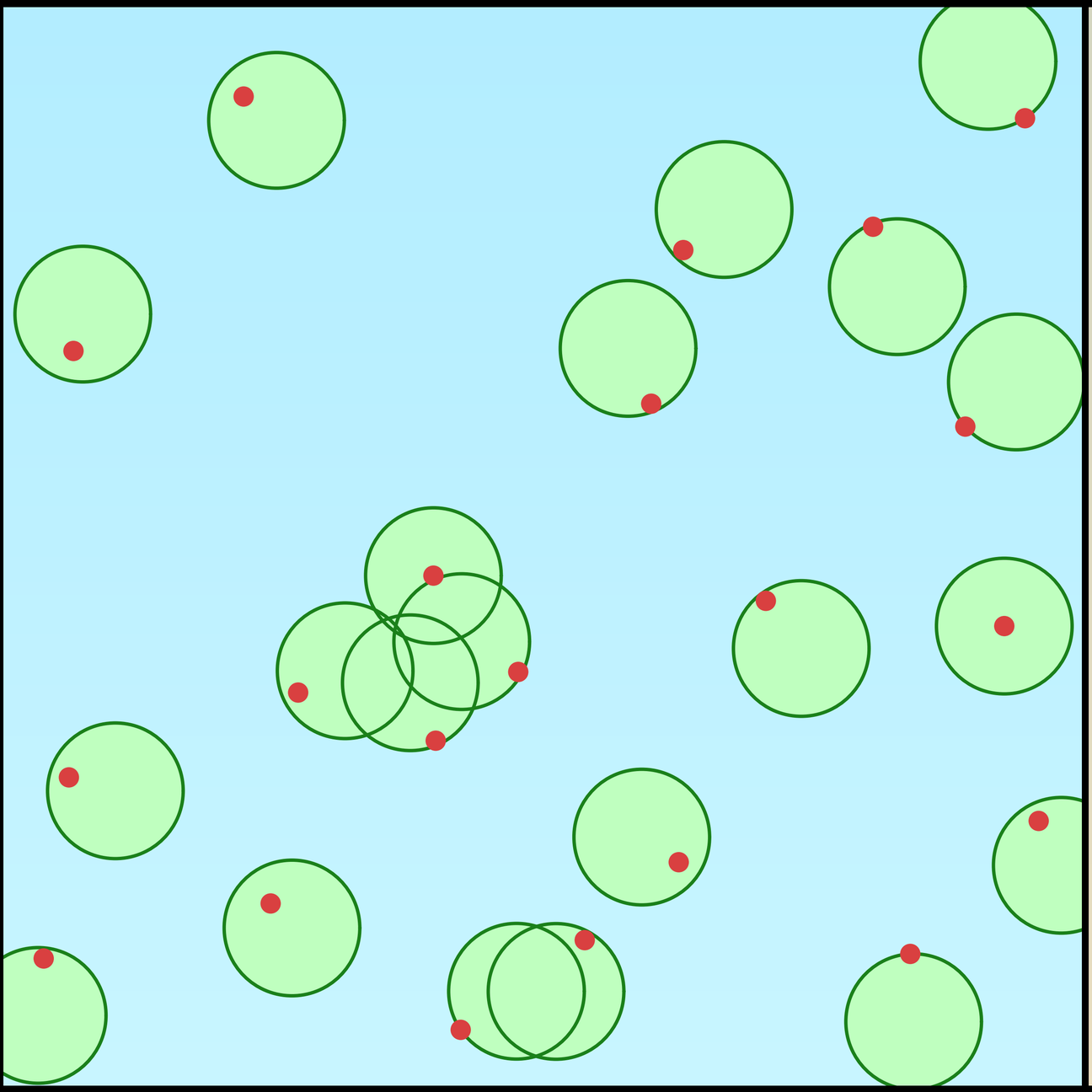

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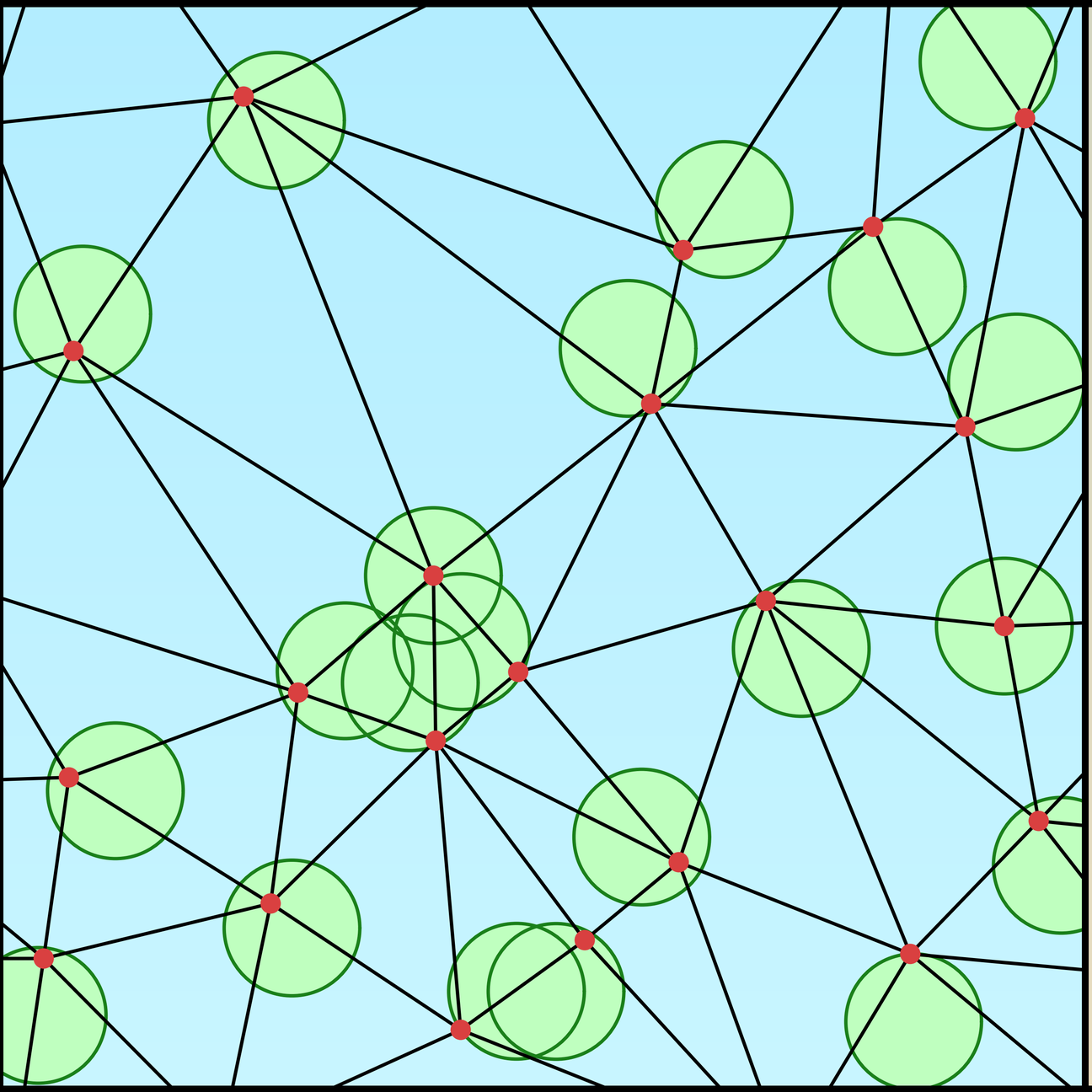

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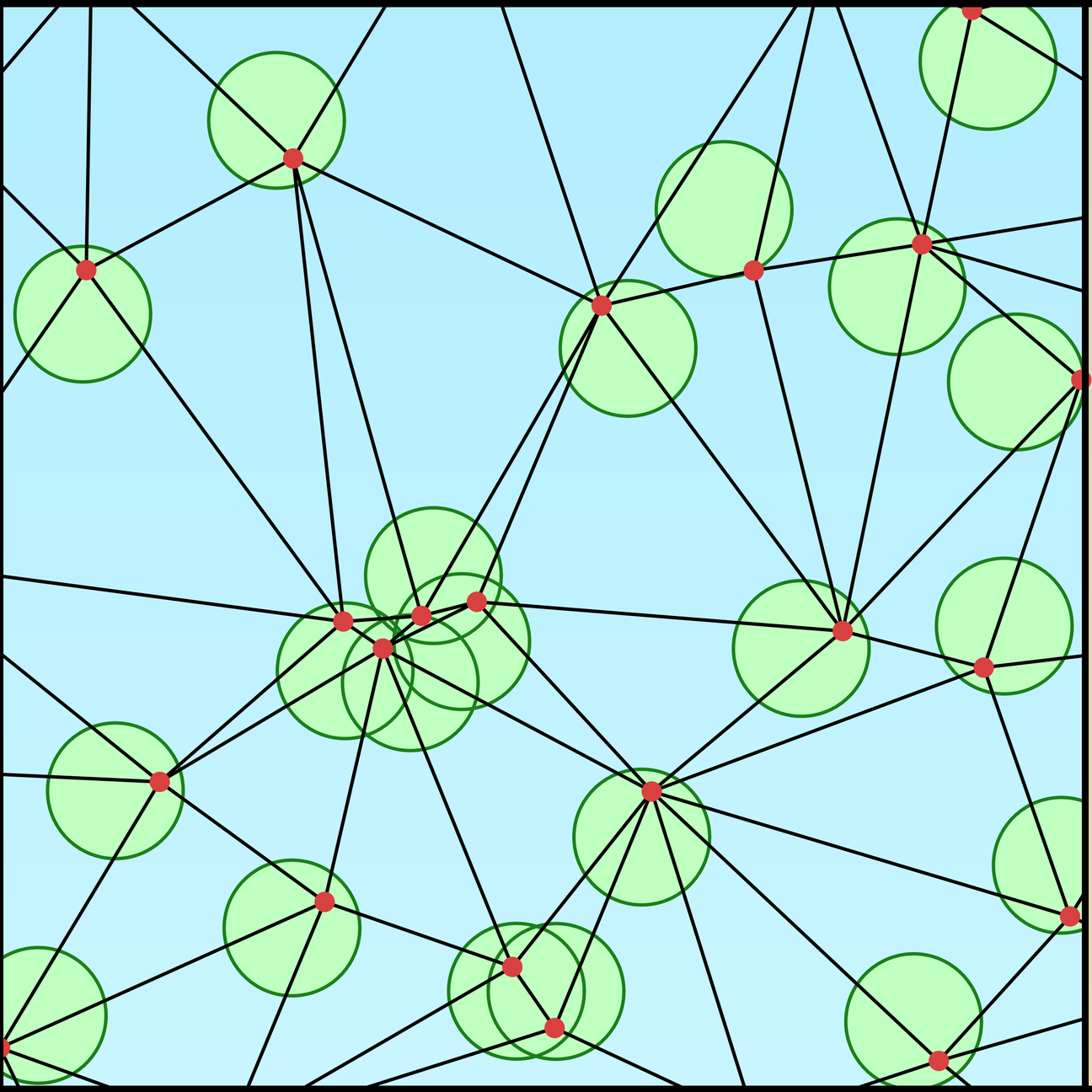

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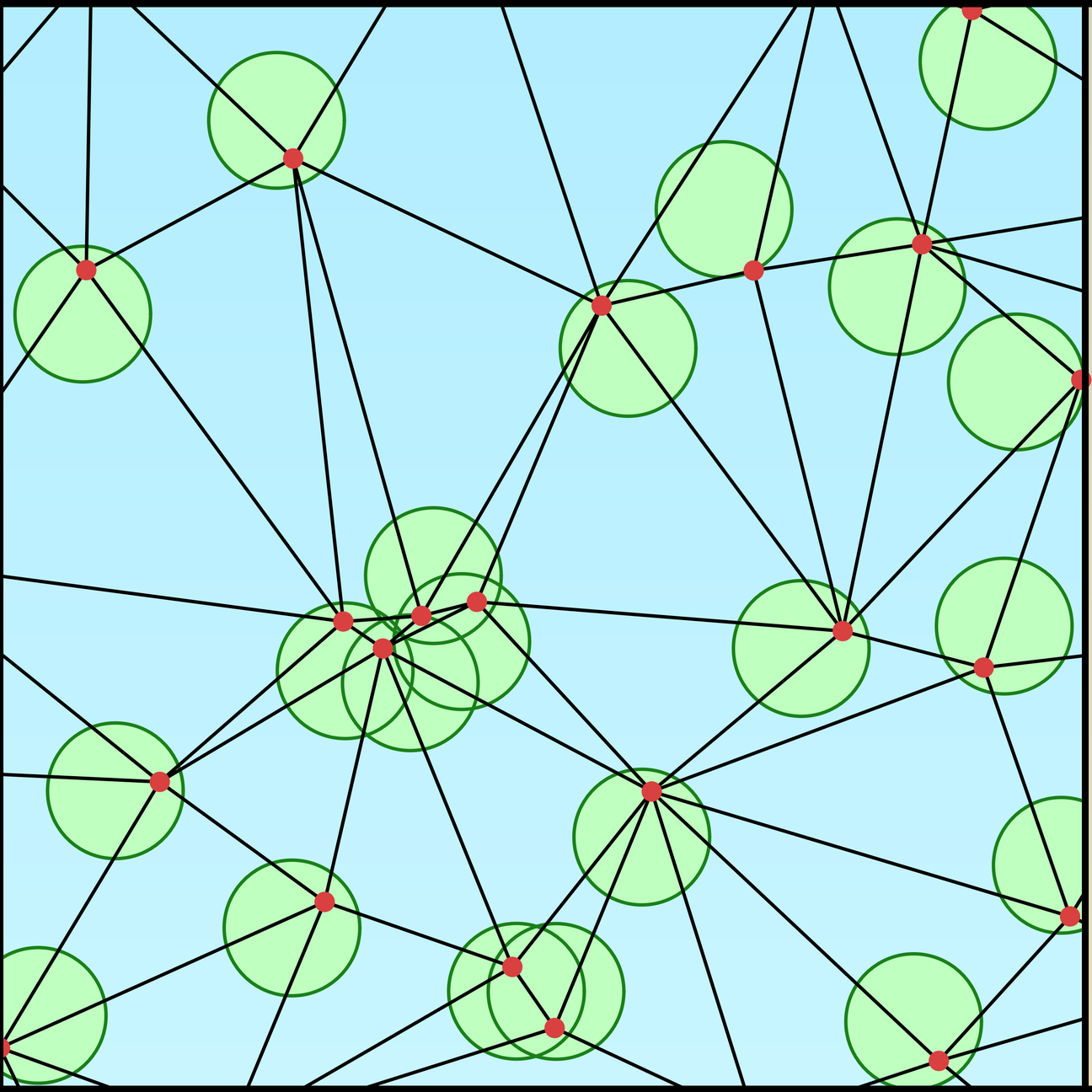

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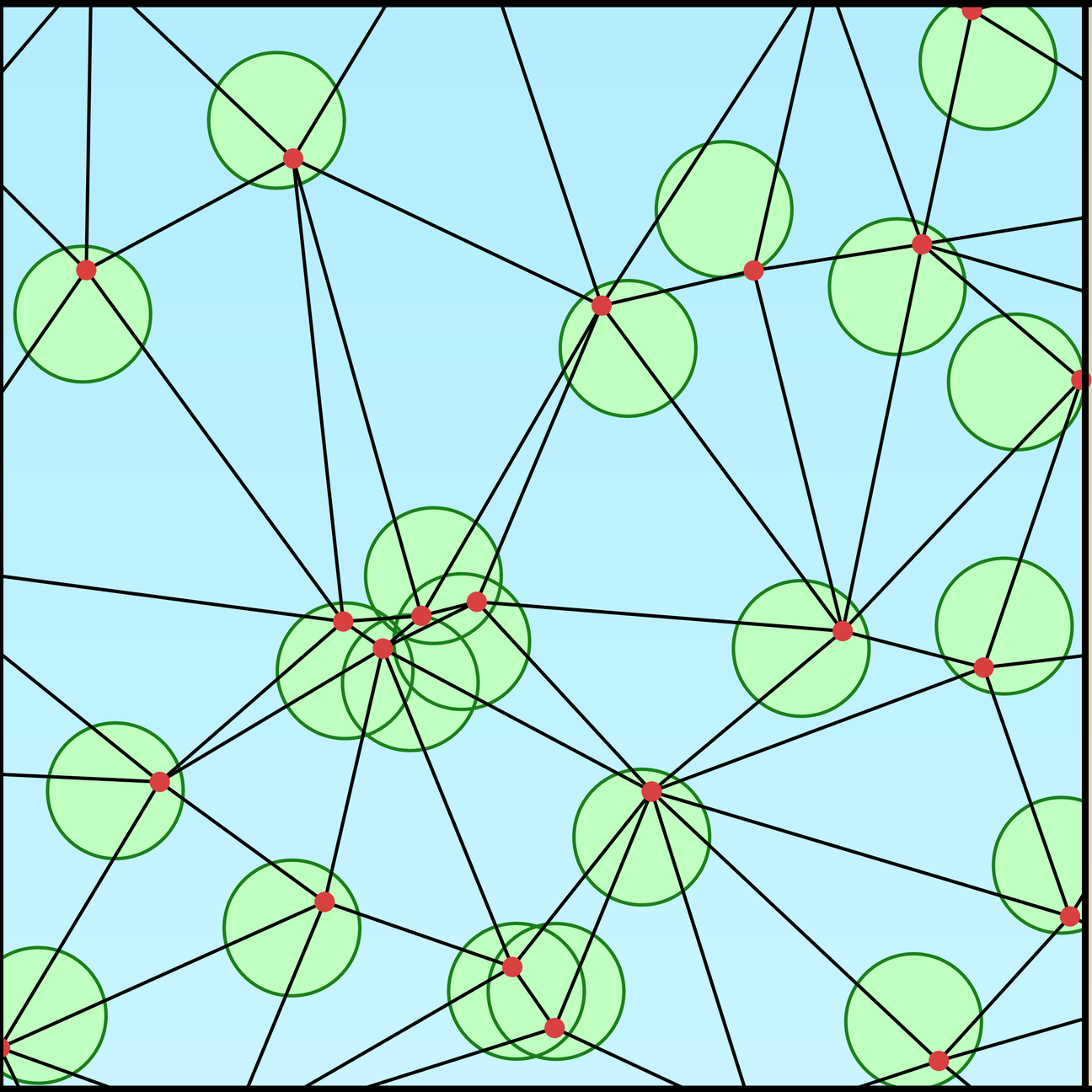

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But proximity is local!

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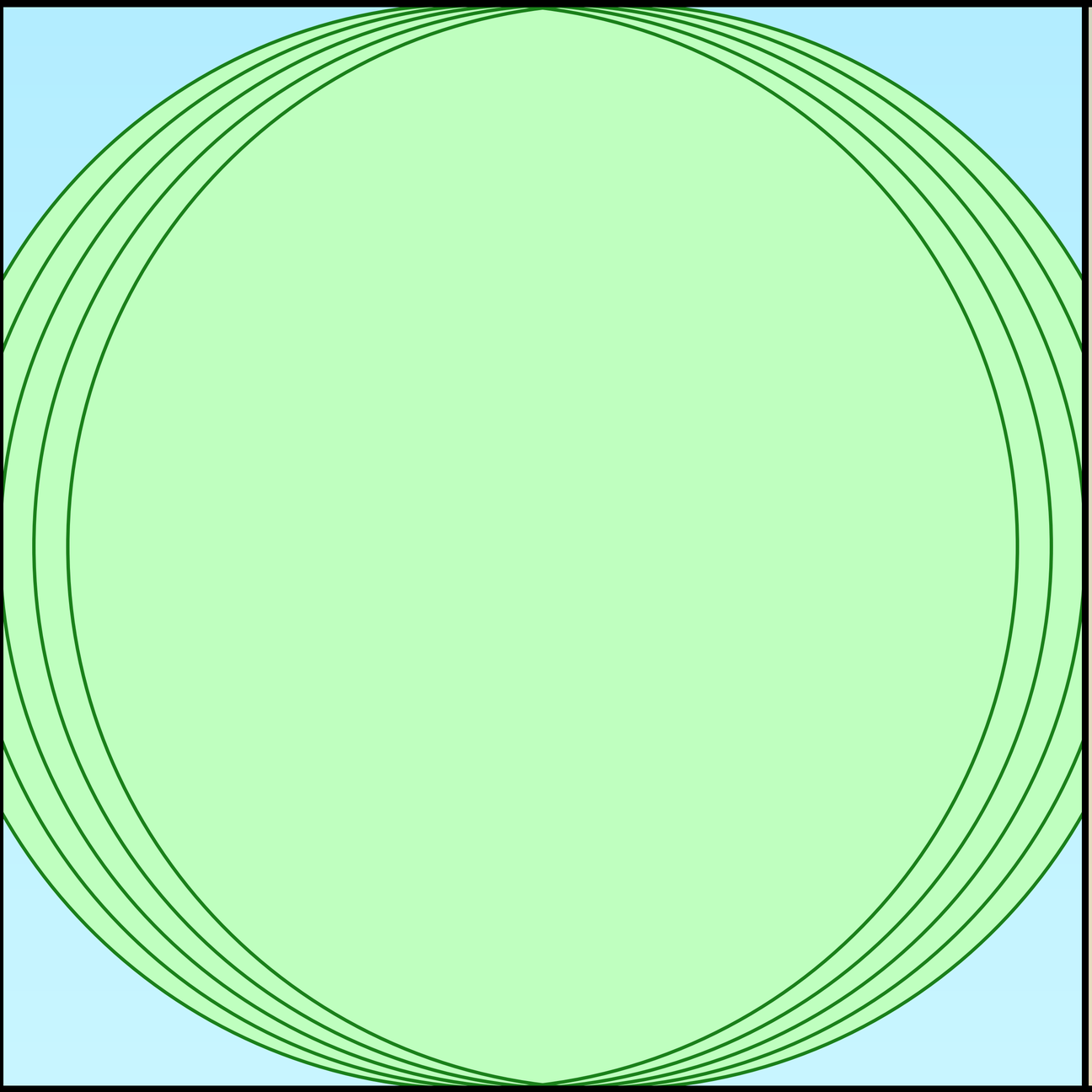

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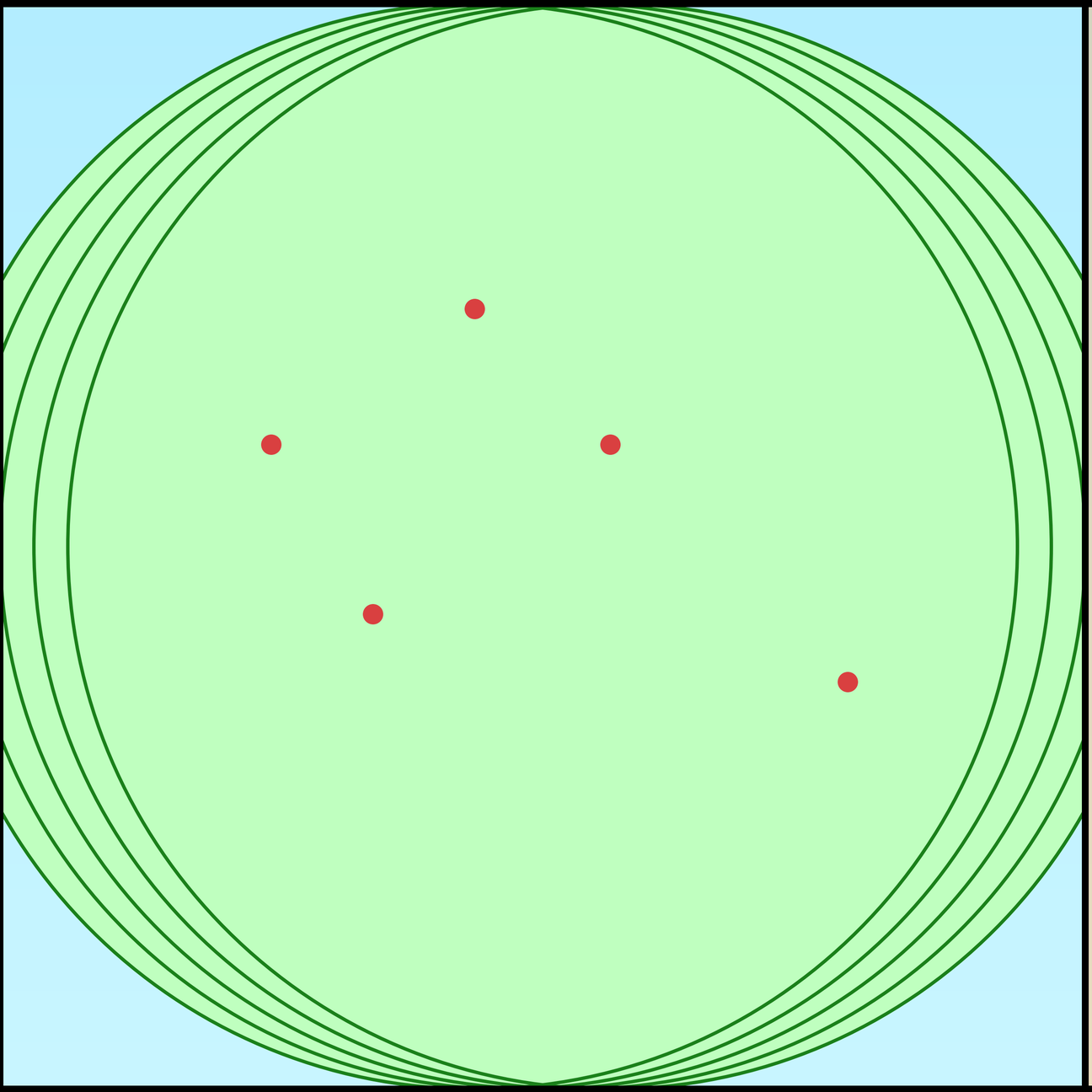

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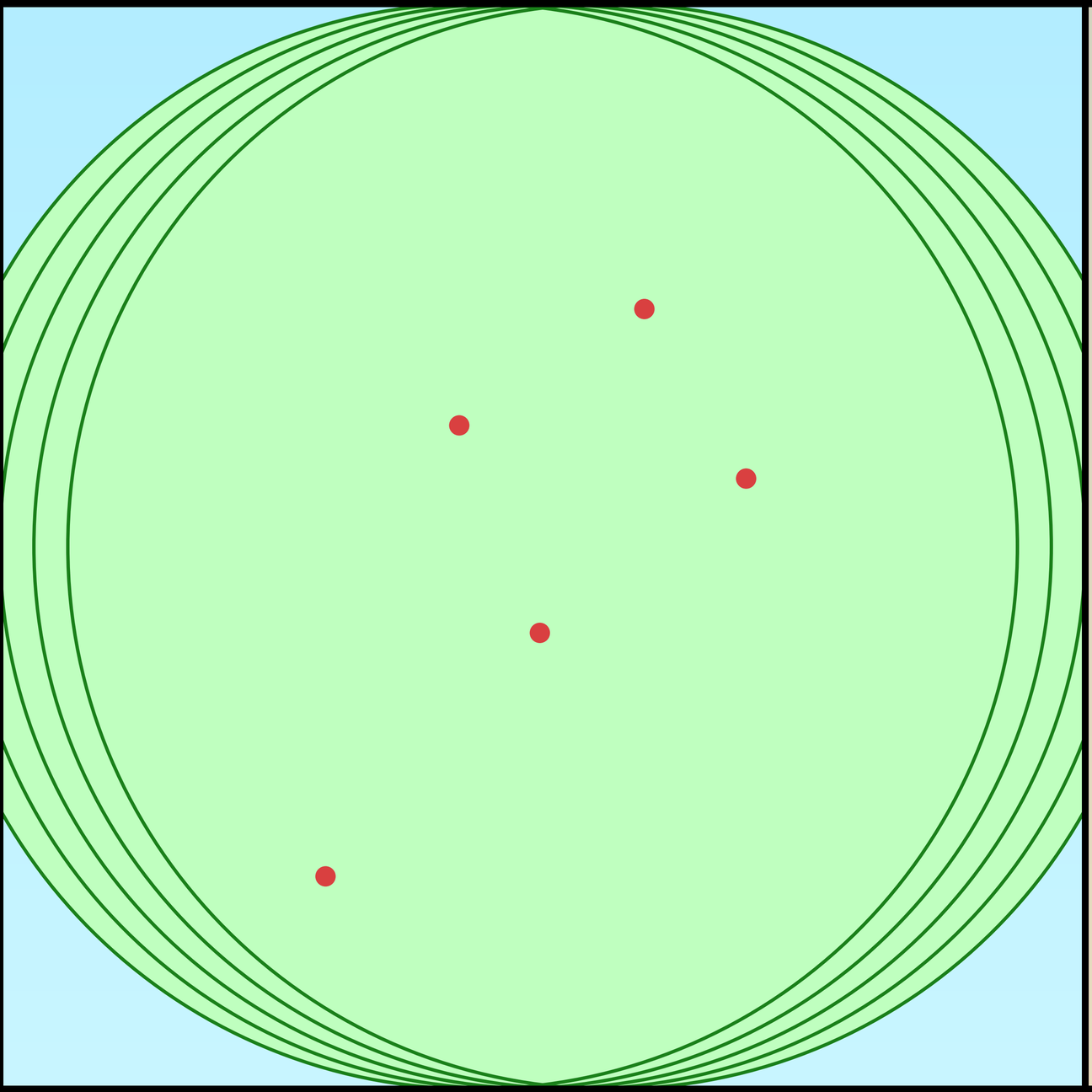

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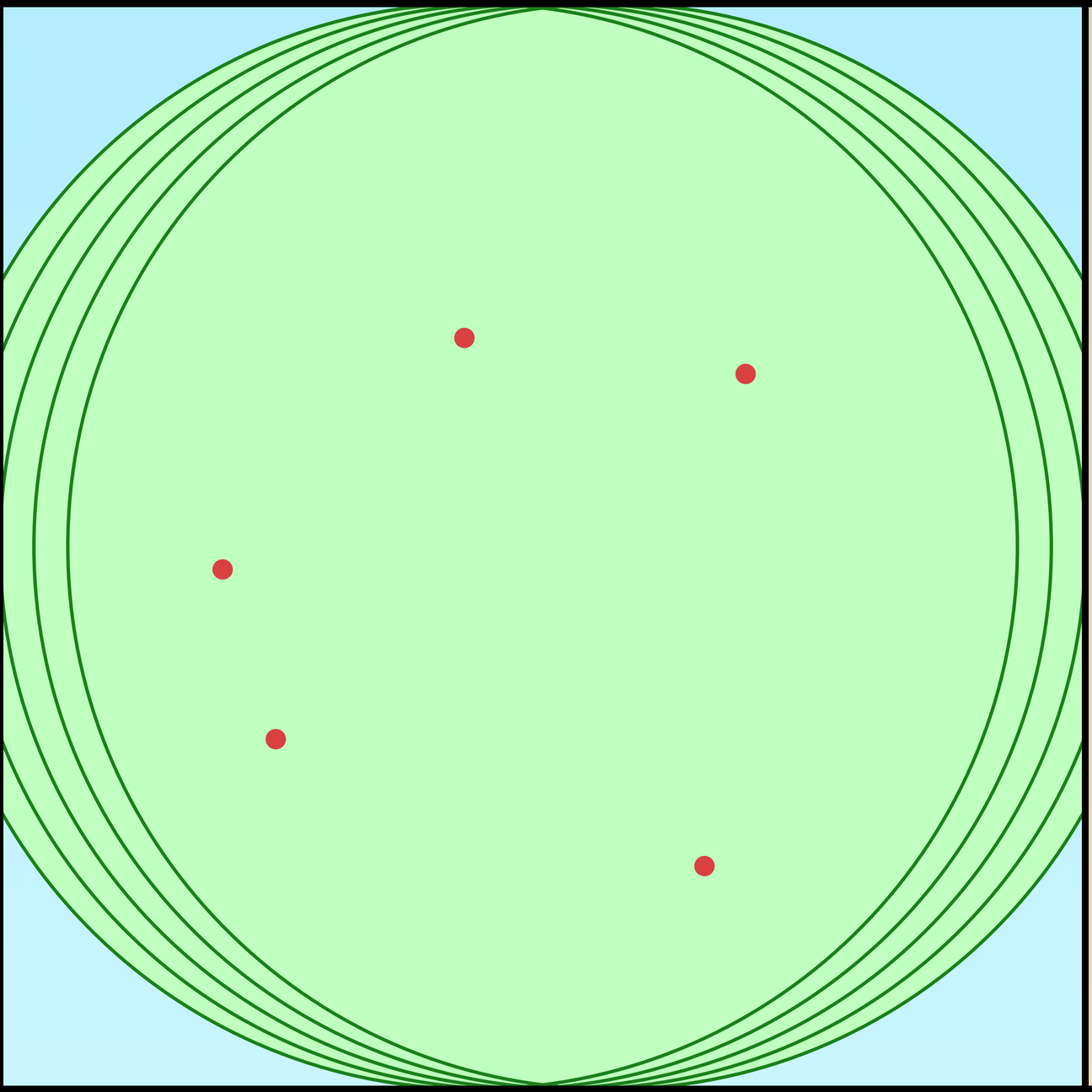

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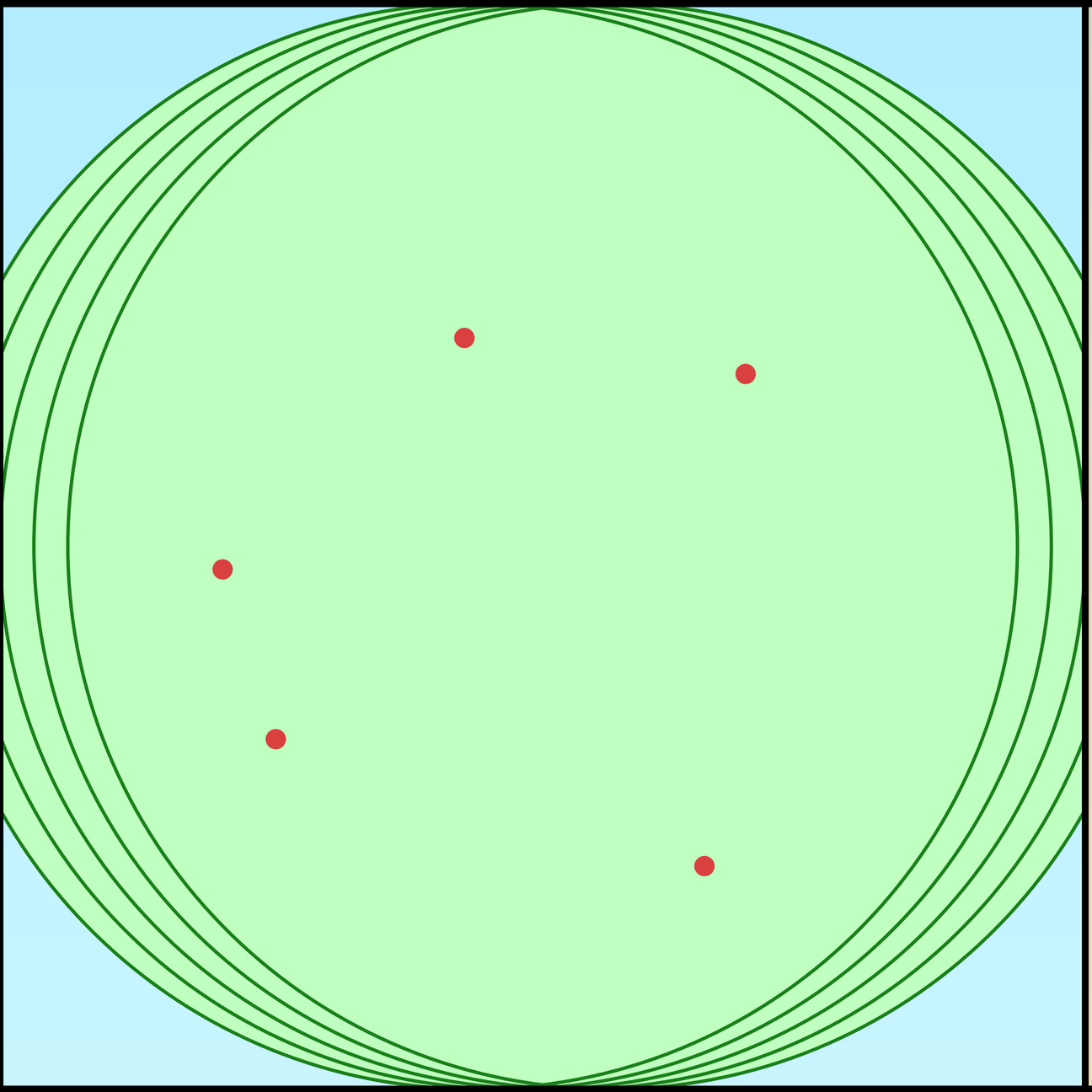

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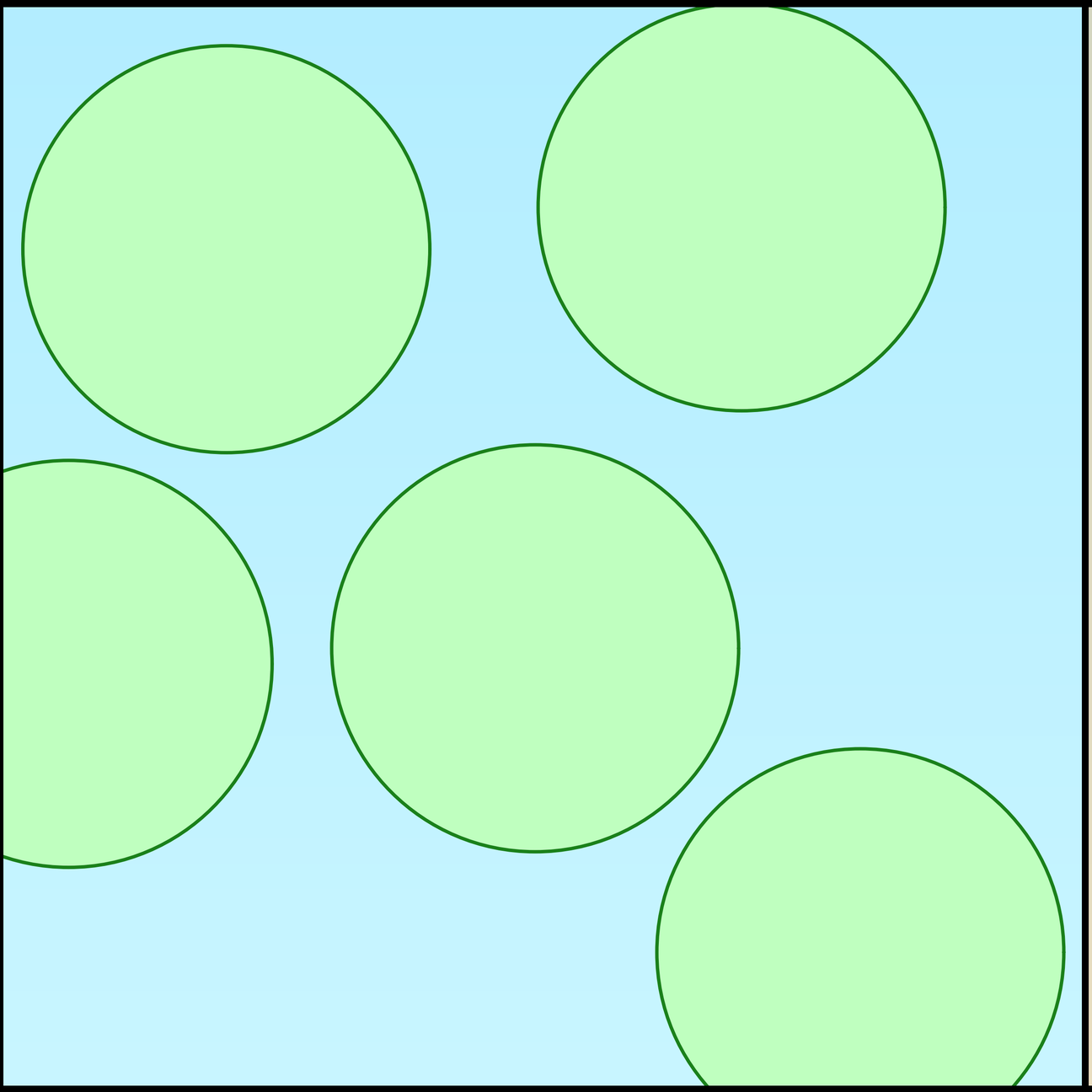

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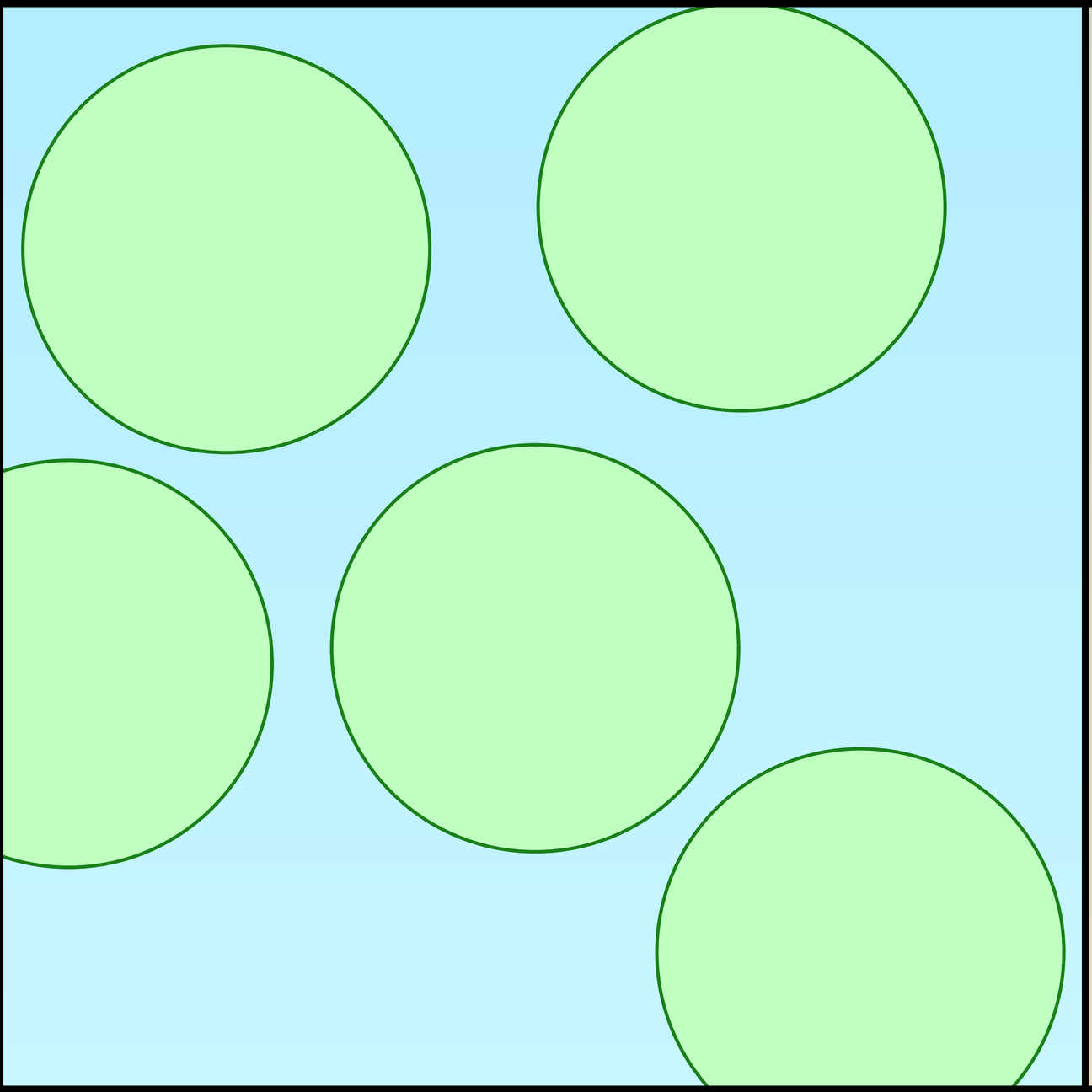

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What if the regions are *disjoint*?

How can we even formalize this question?

# Chapter 2
*PARTIAL INFORMATION*

Uncertain locations

Quantifying information

State of the art in 1D

State of the art in 2D

We can capture *information* in terms of *computation*.

Set of regions $\mathcal{R}$

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Proximity structure $S(P)$

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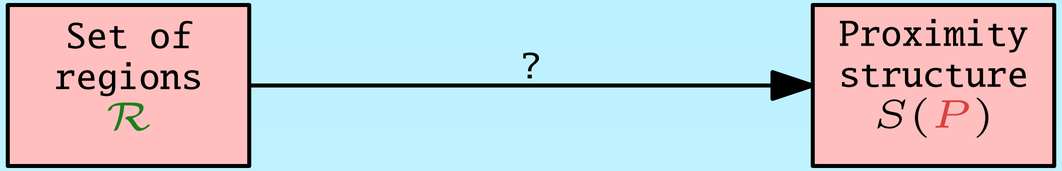


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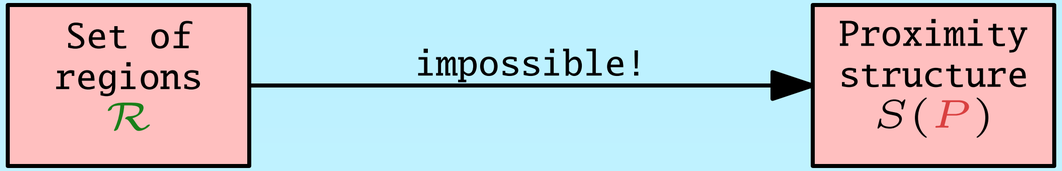


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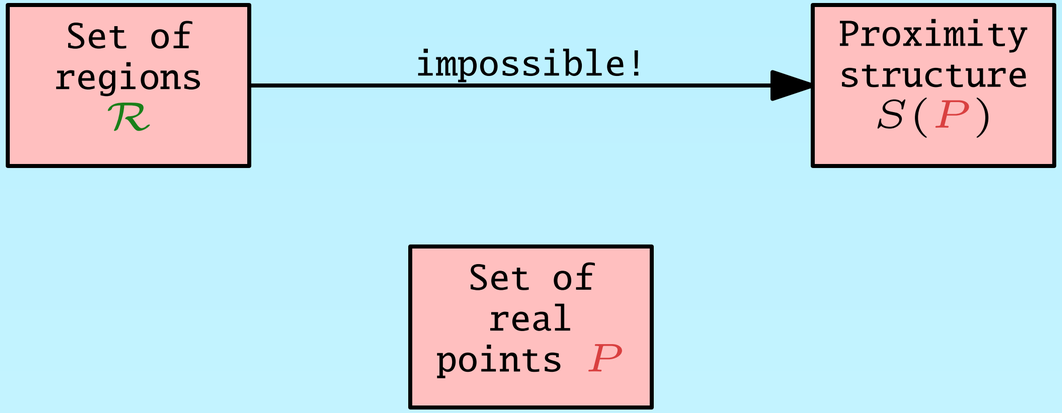


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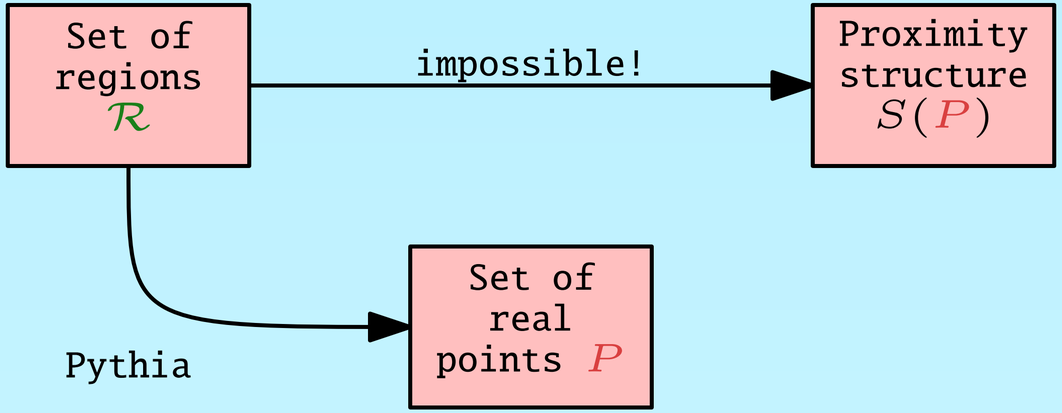


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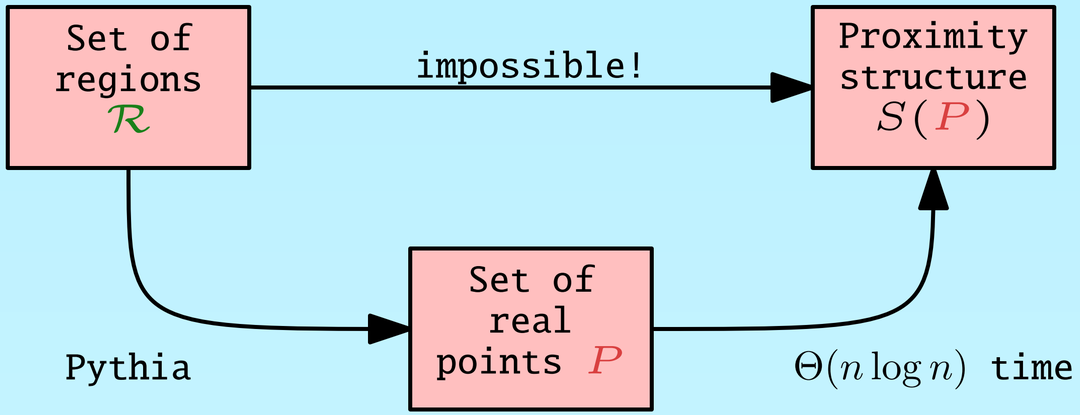


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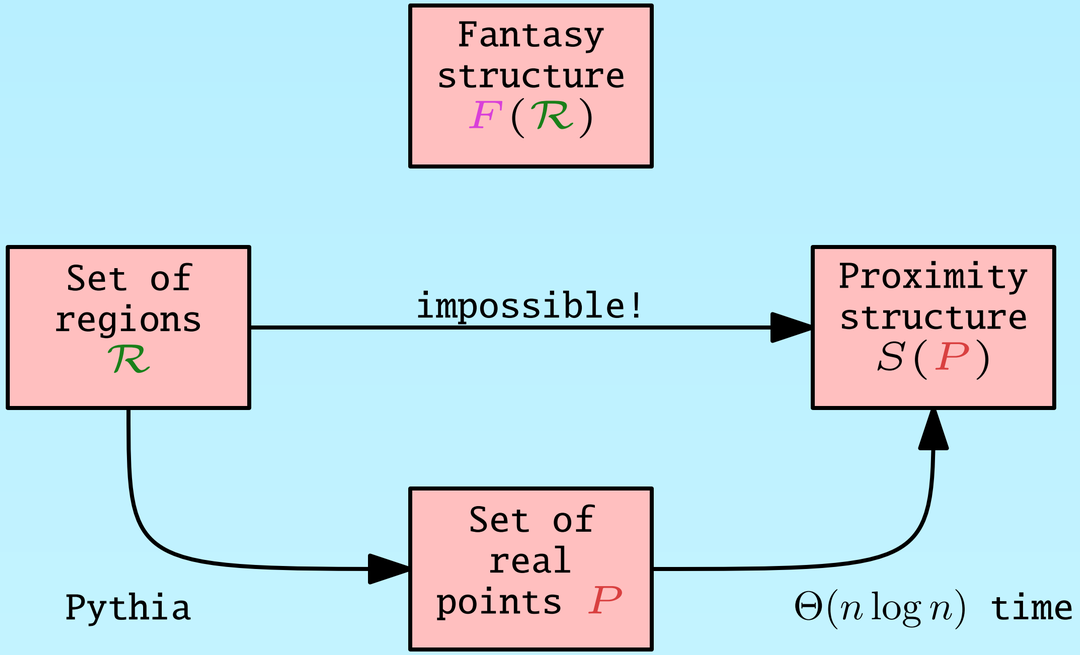


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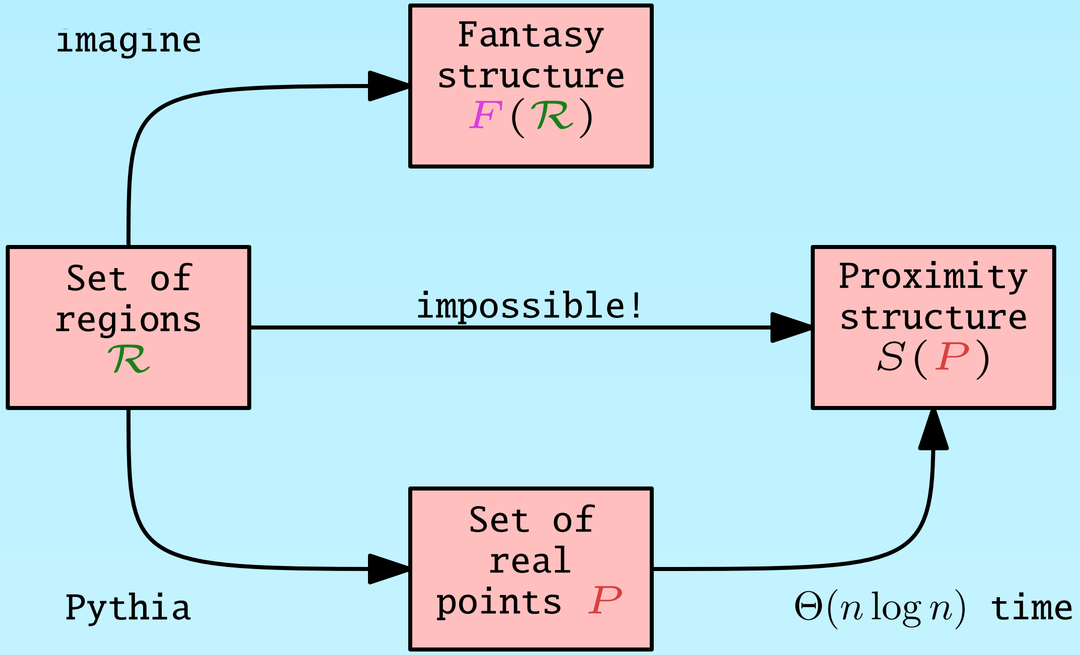


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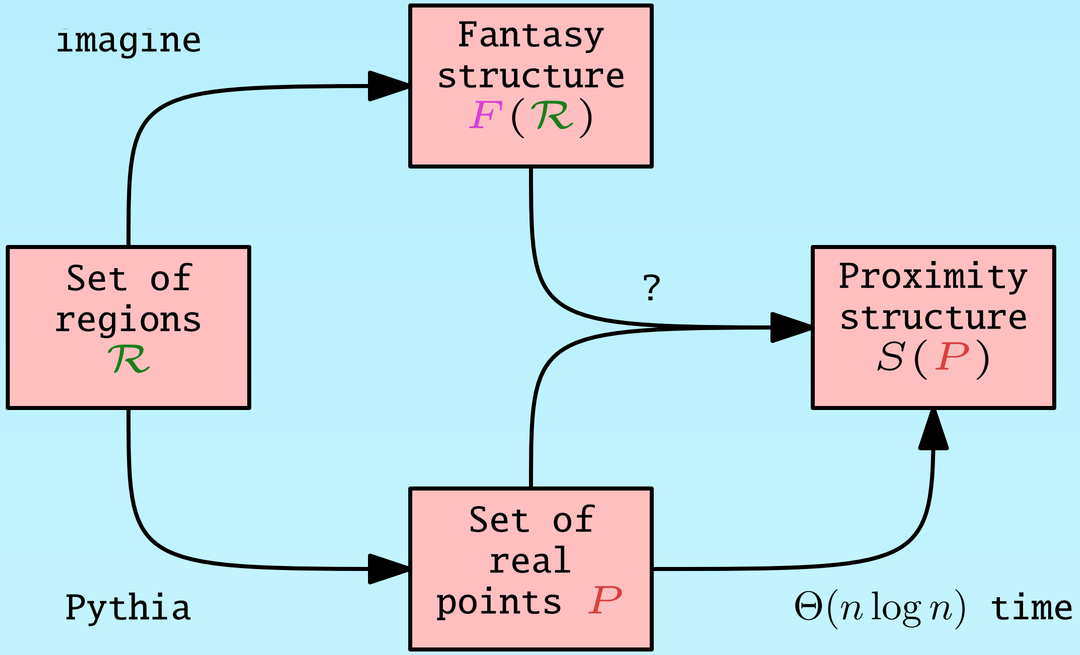


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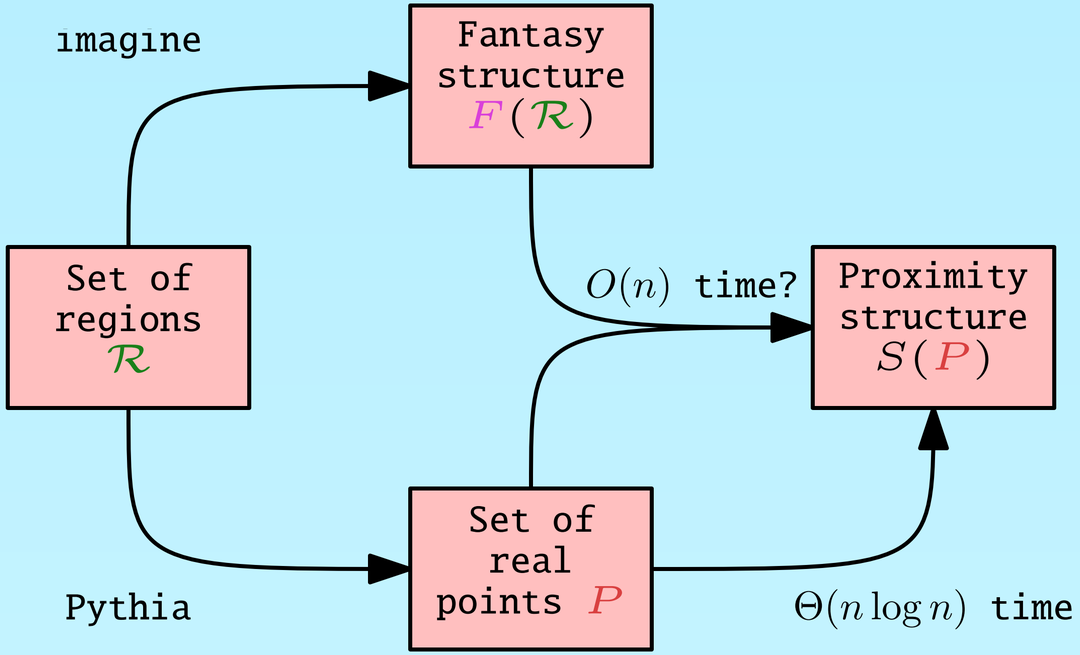


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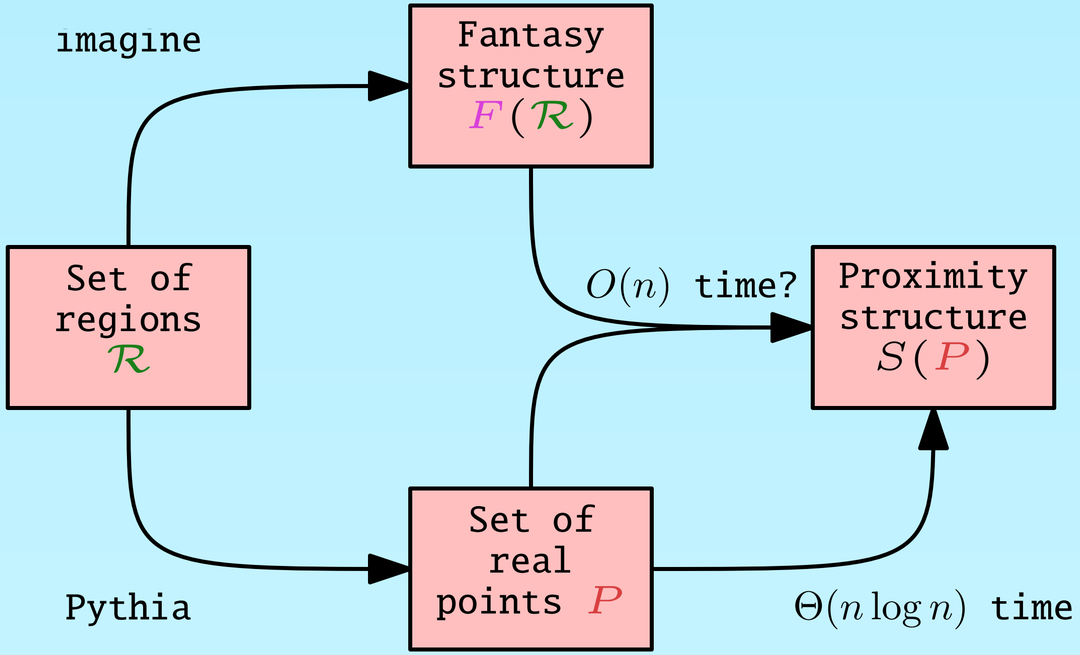


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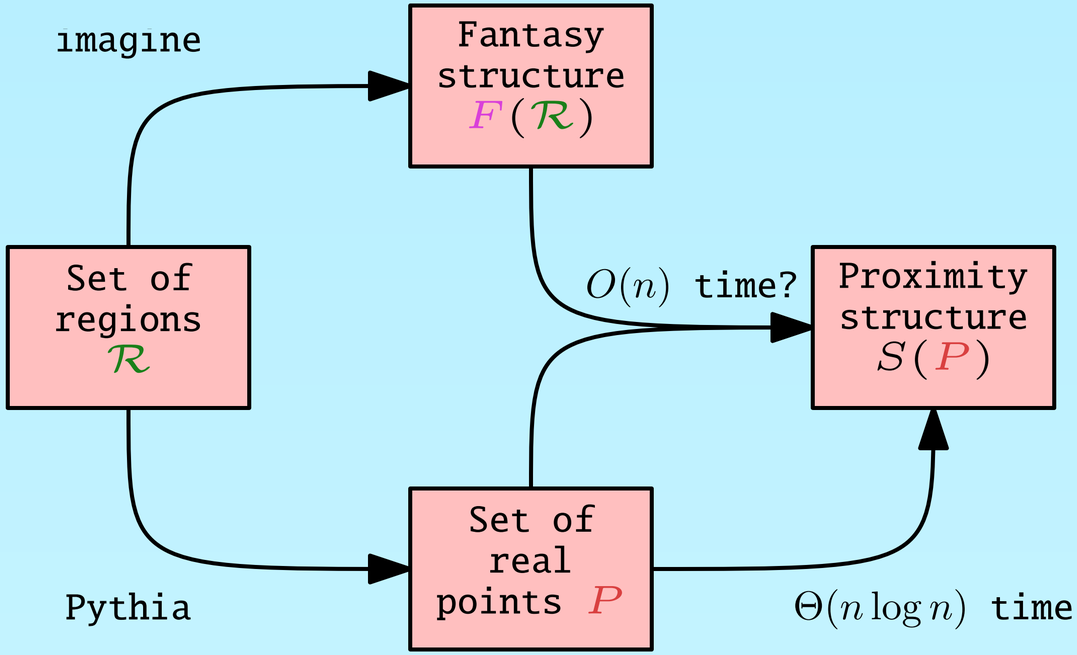


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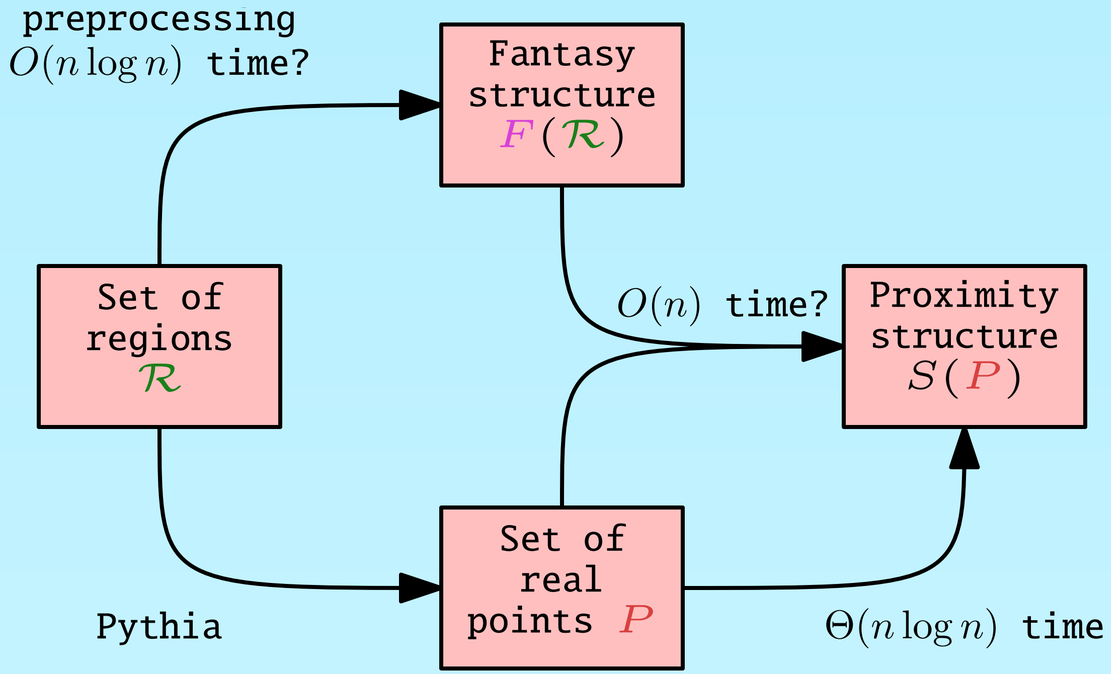


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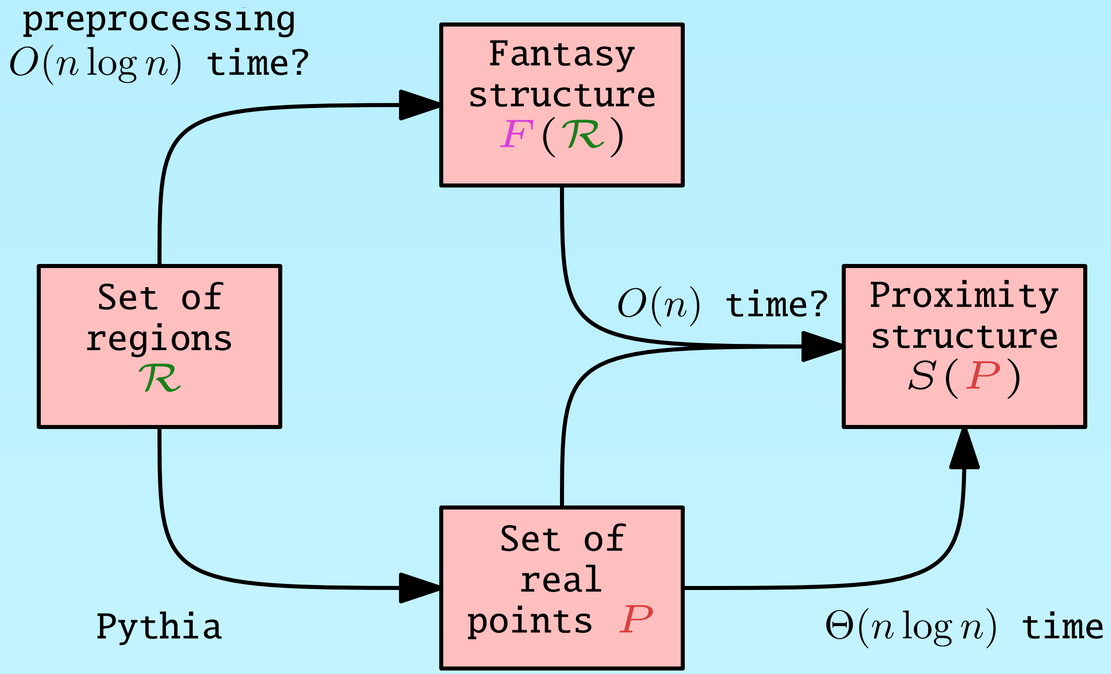


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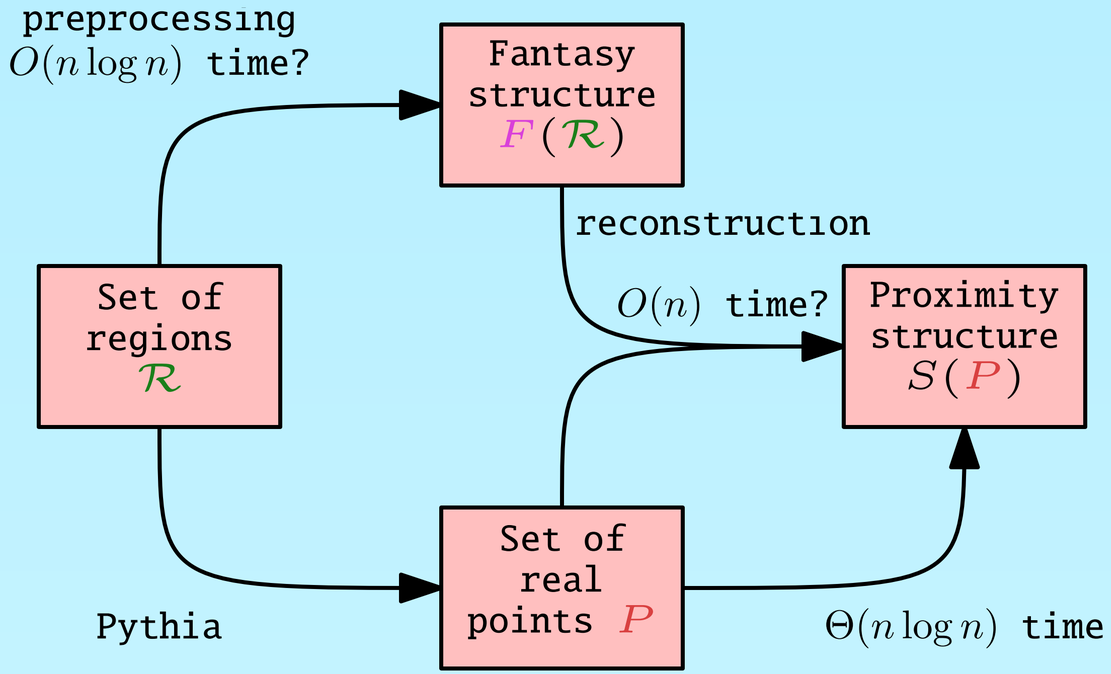


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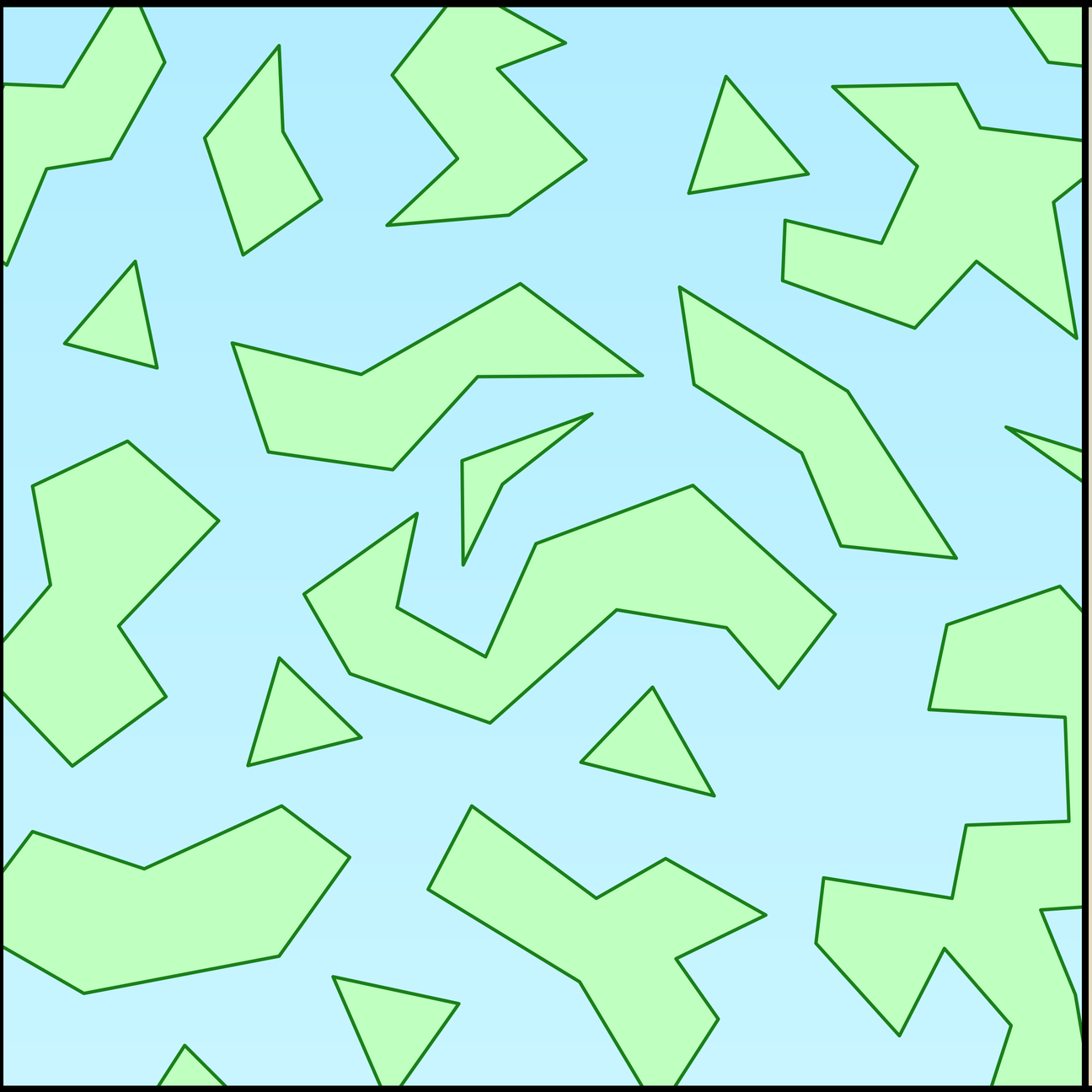

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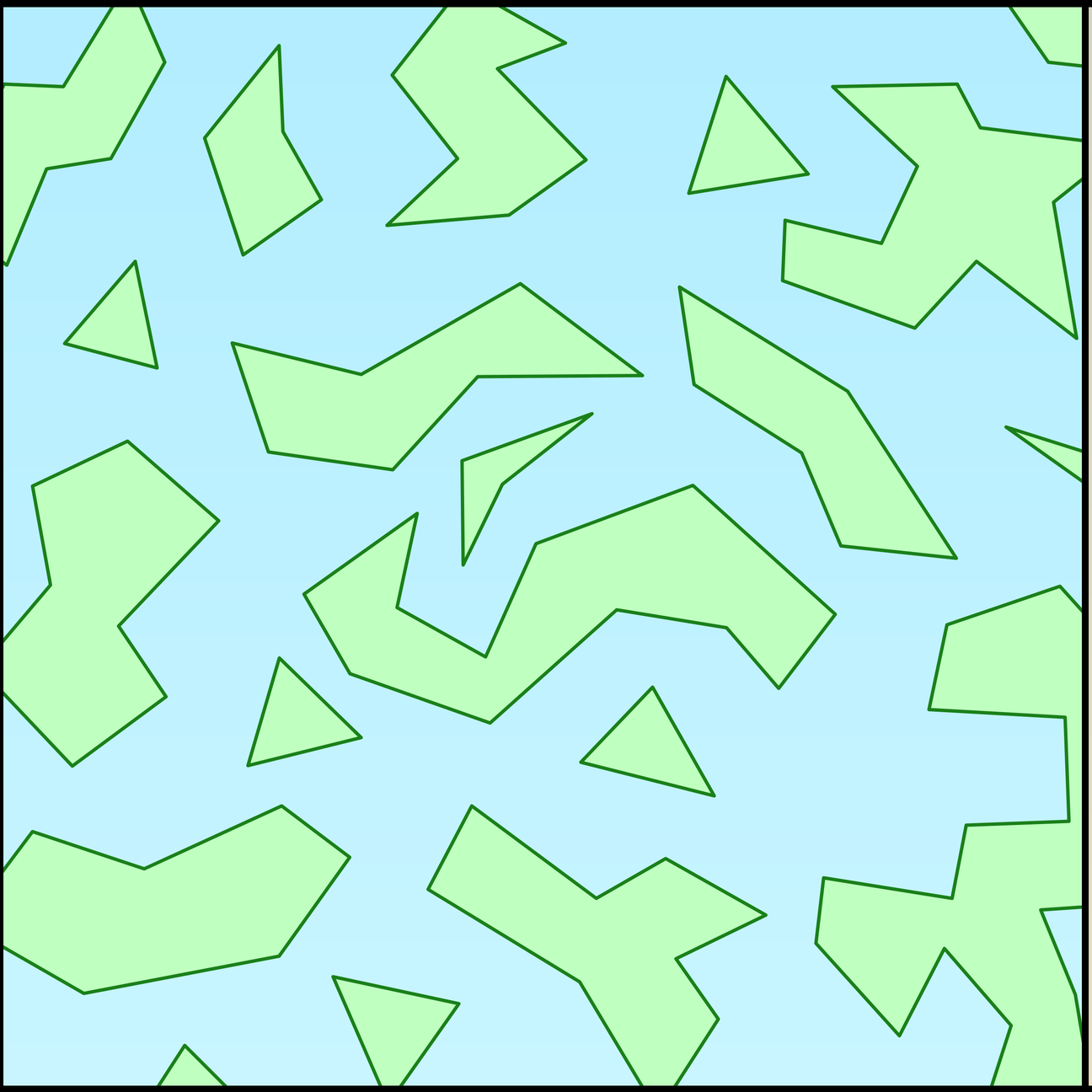

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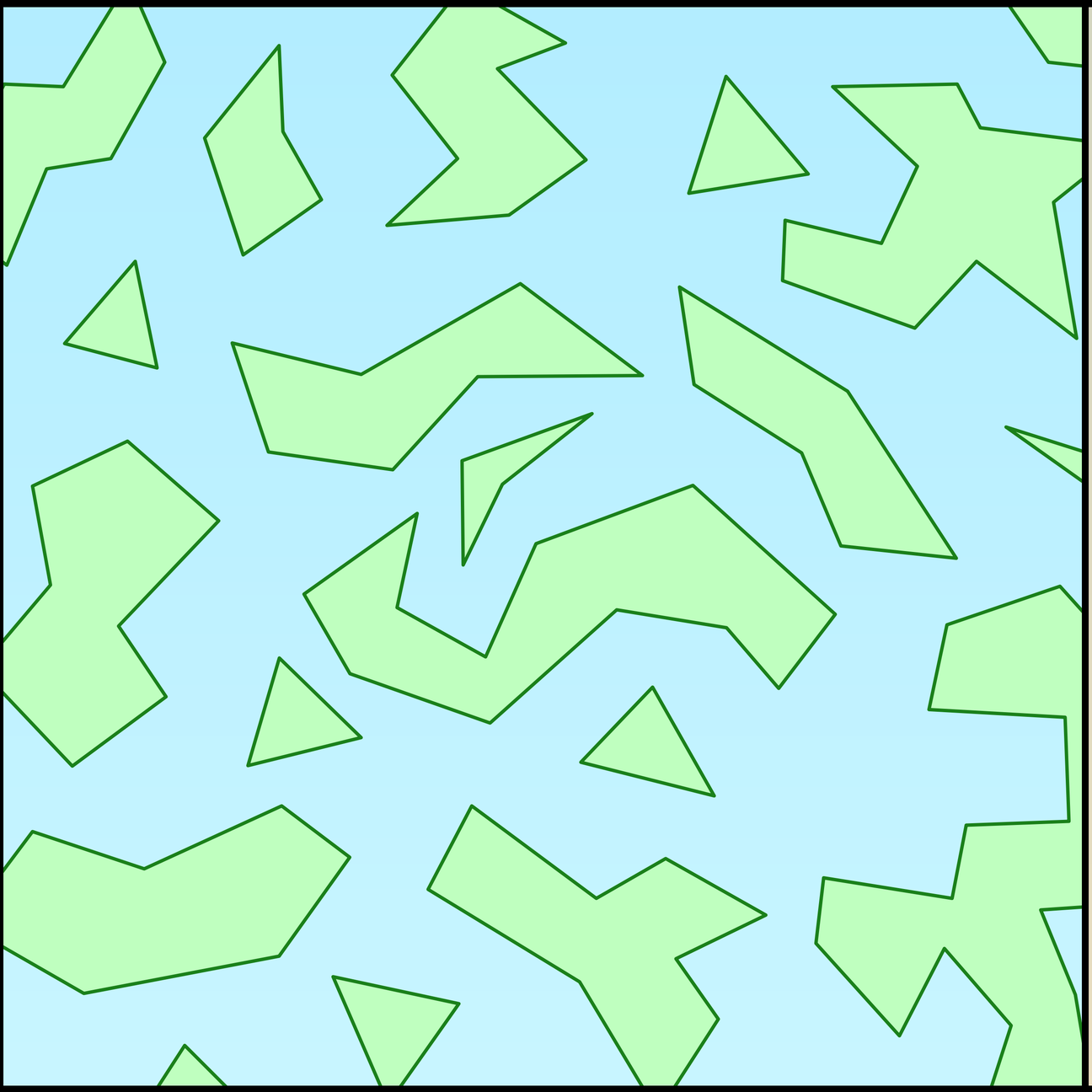

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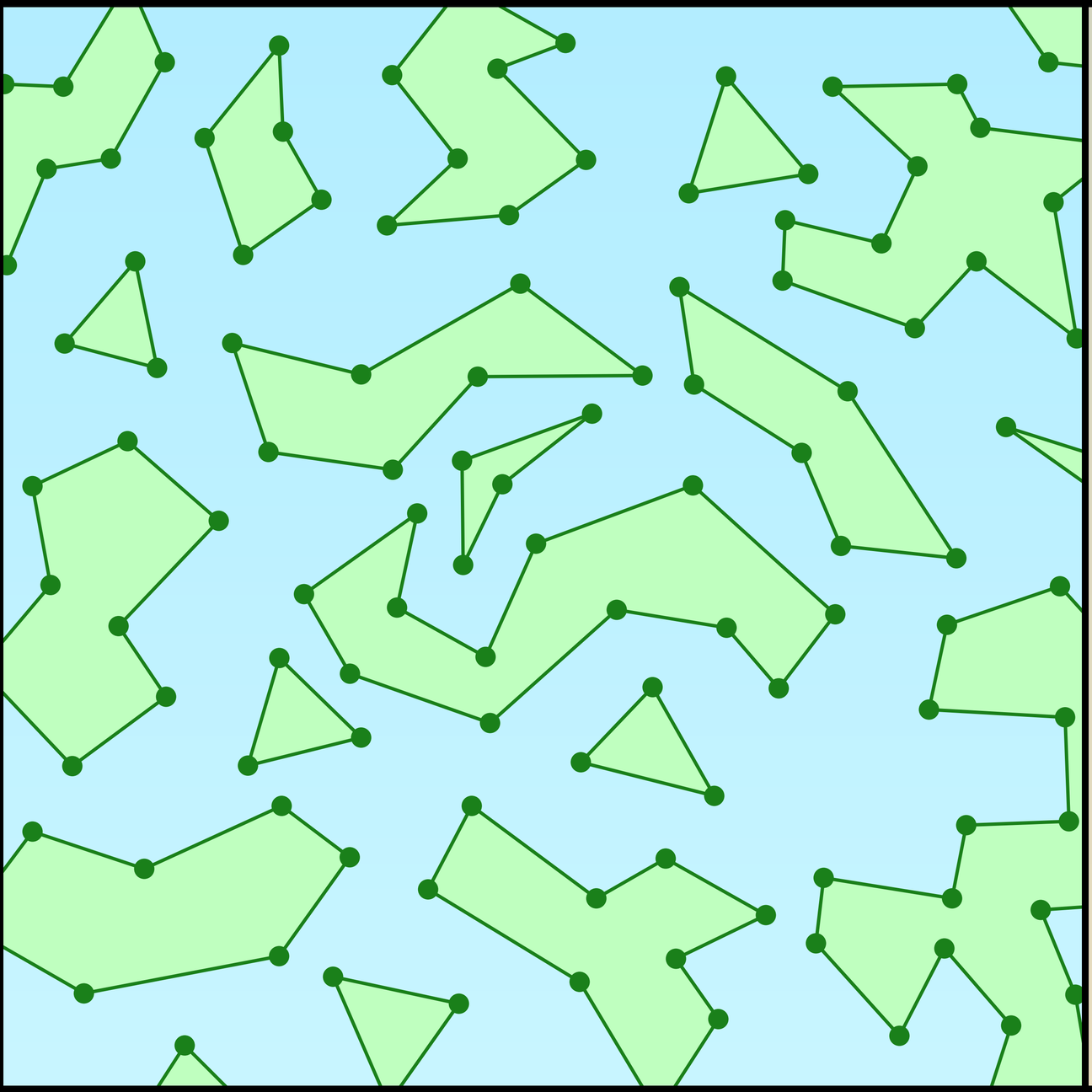

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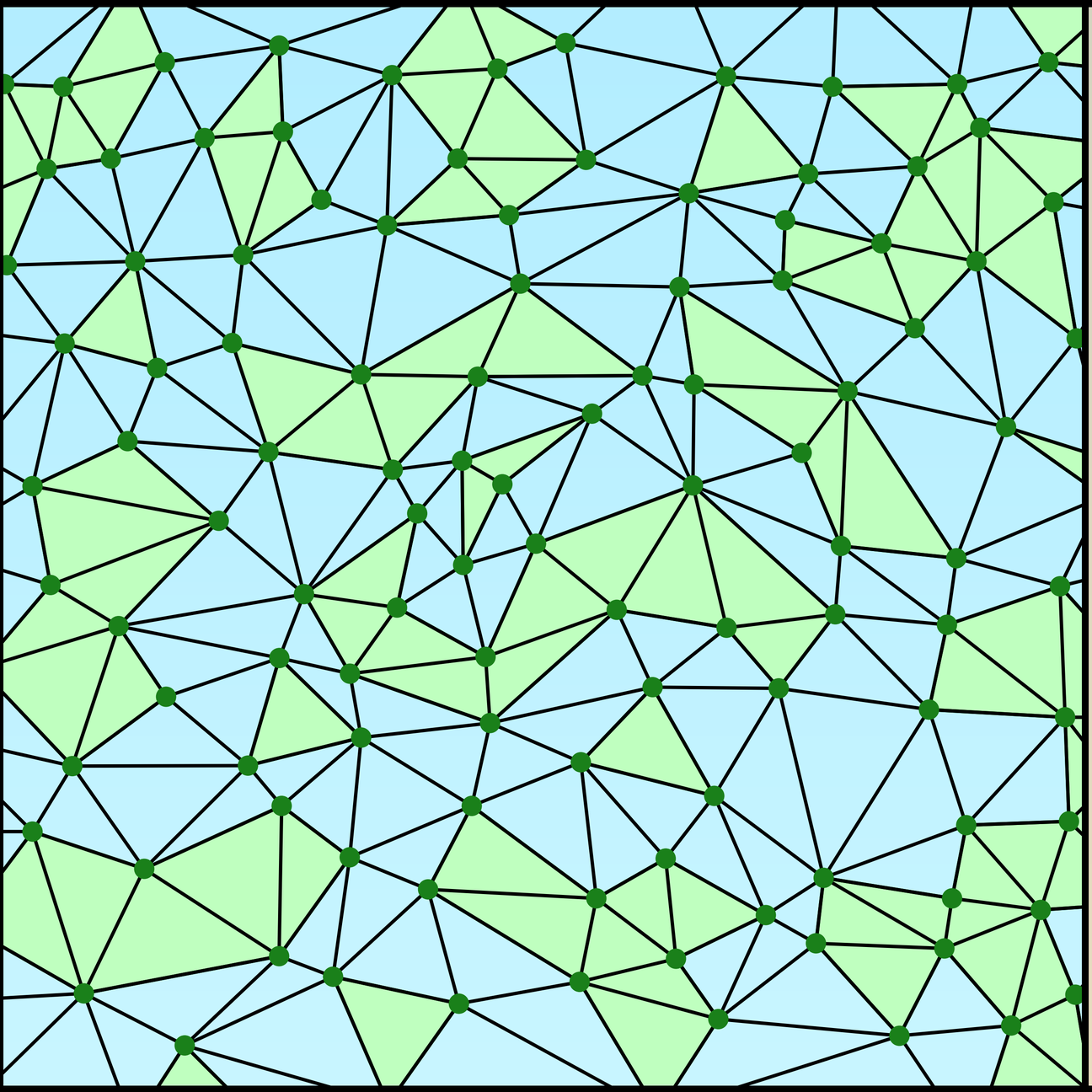

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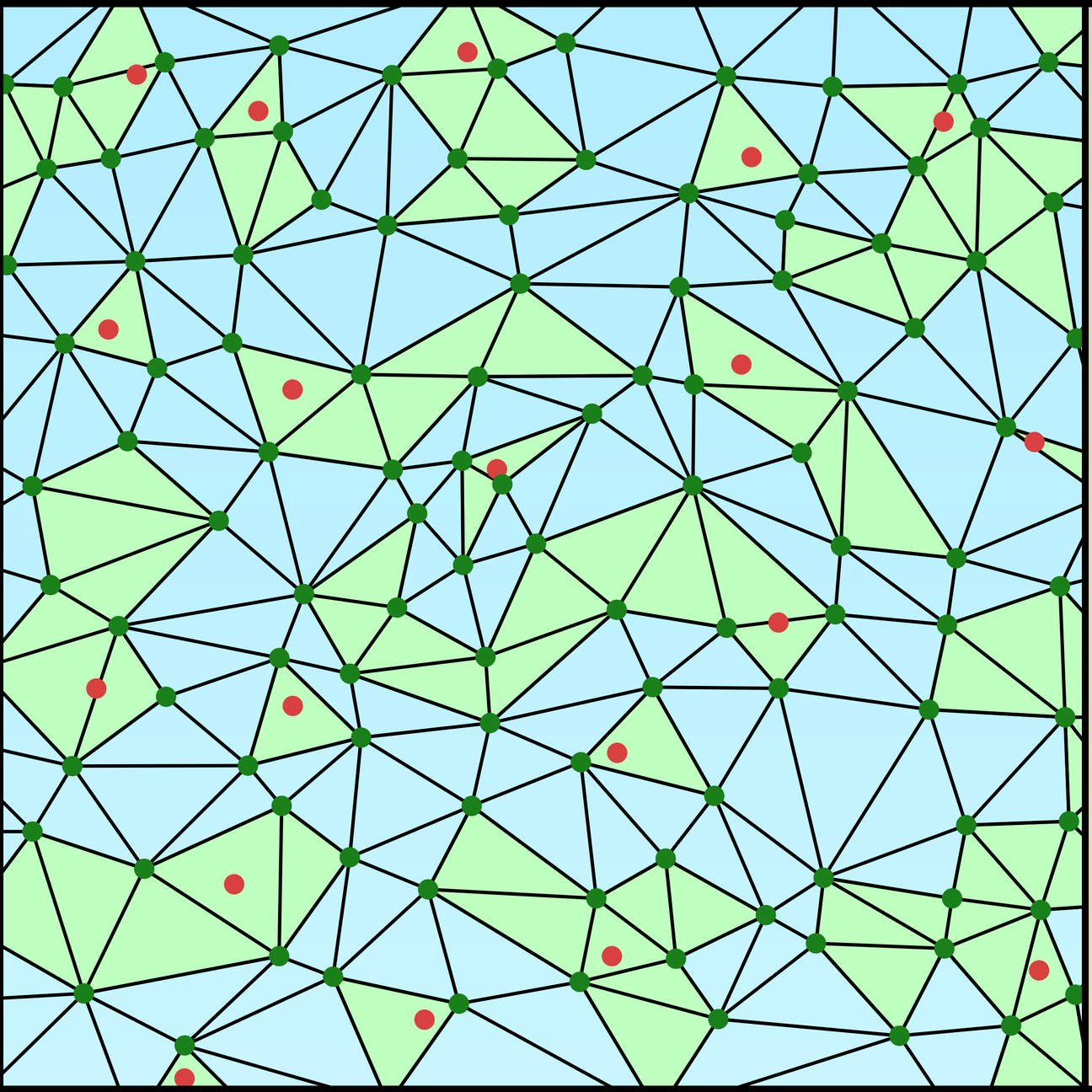

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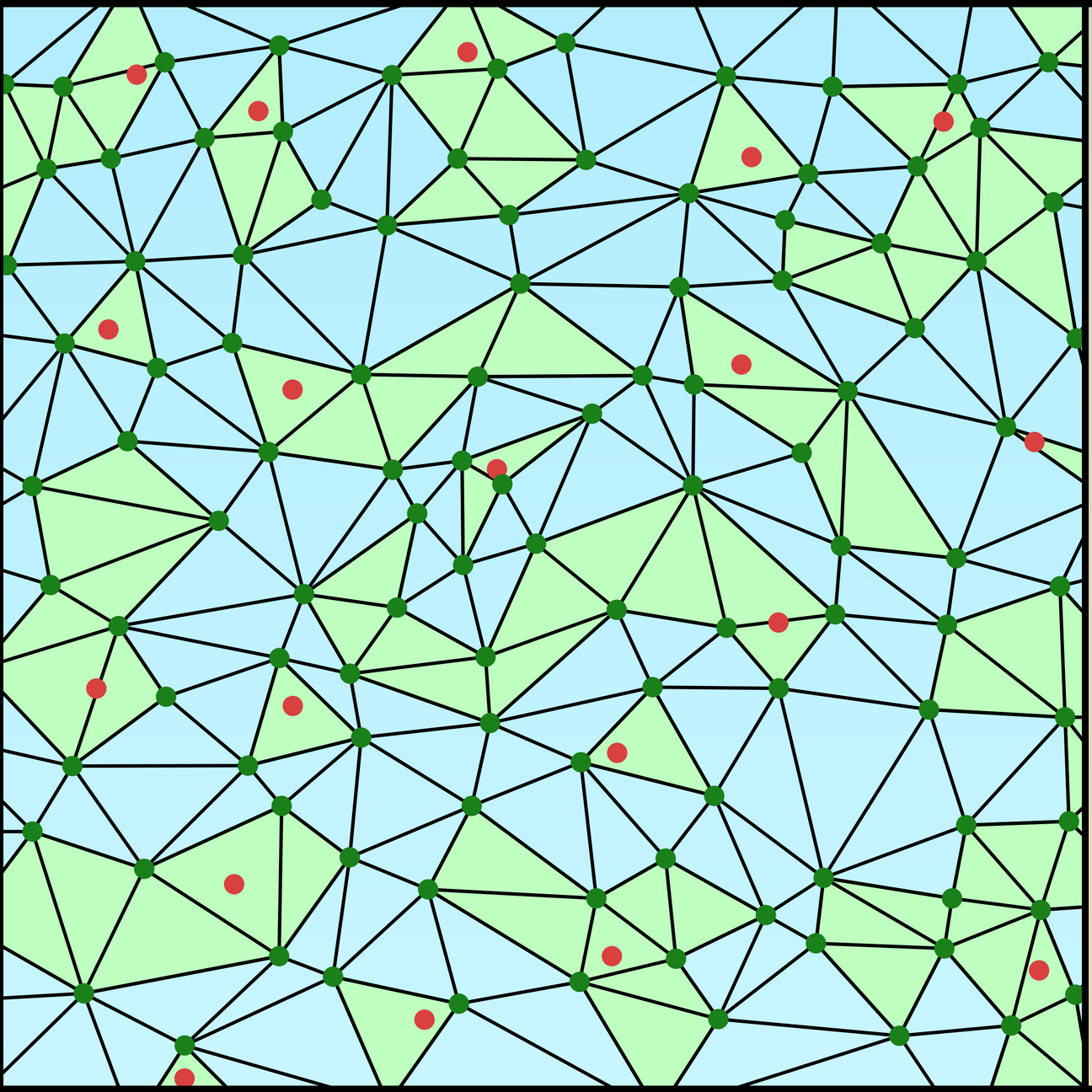

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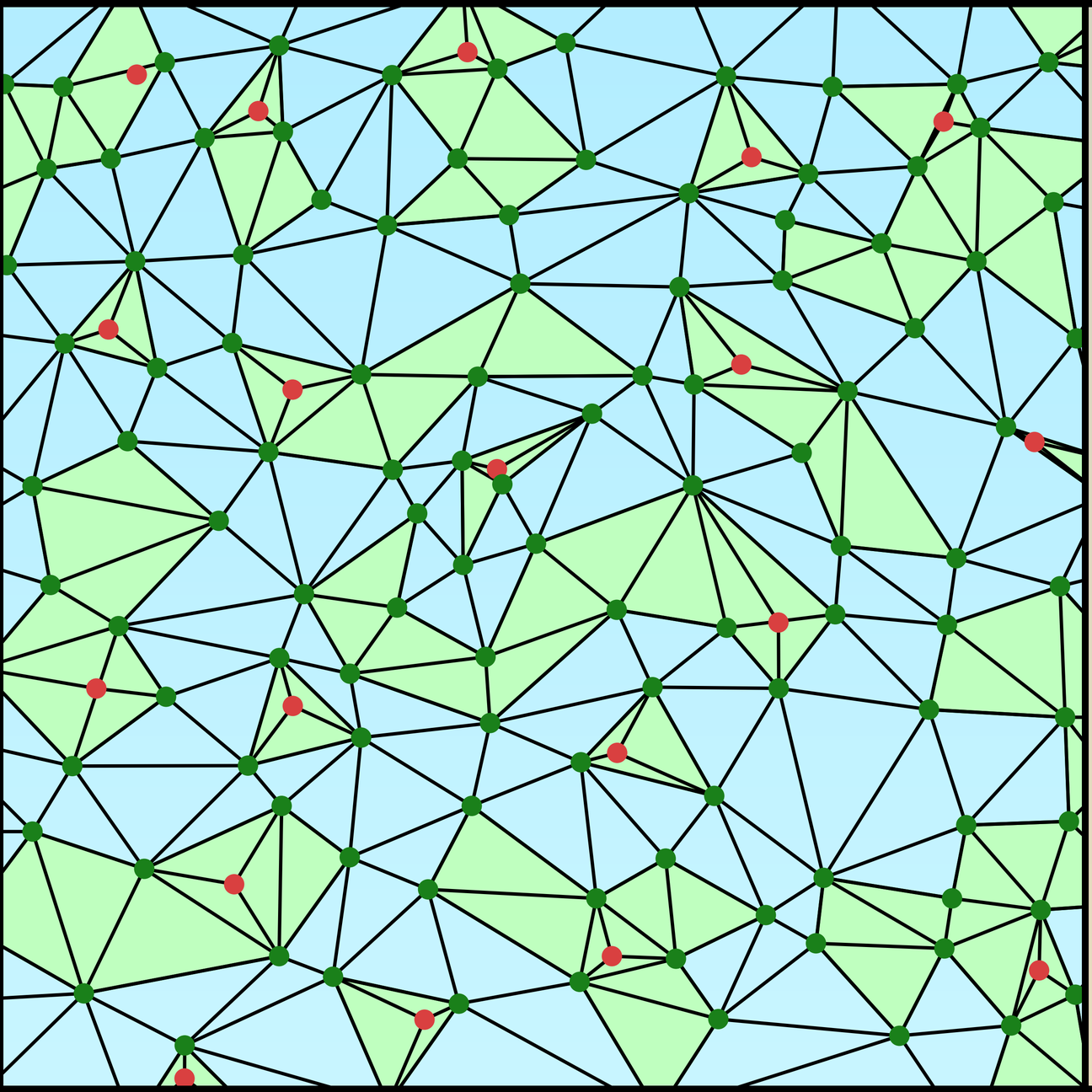
During reconstruction, we insert $P$ into the triangulation, and then remove the old vertices.



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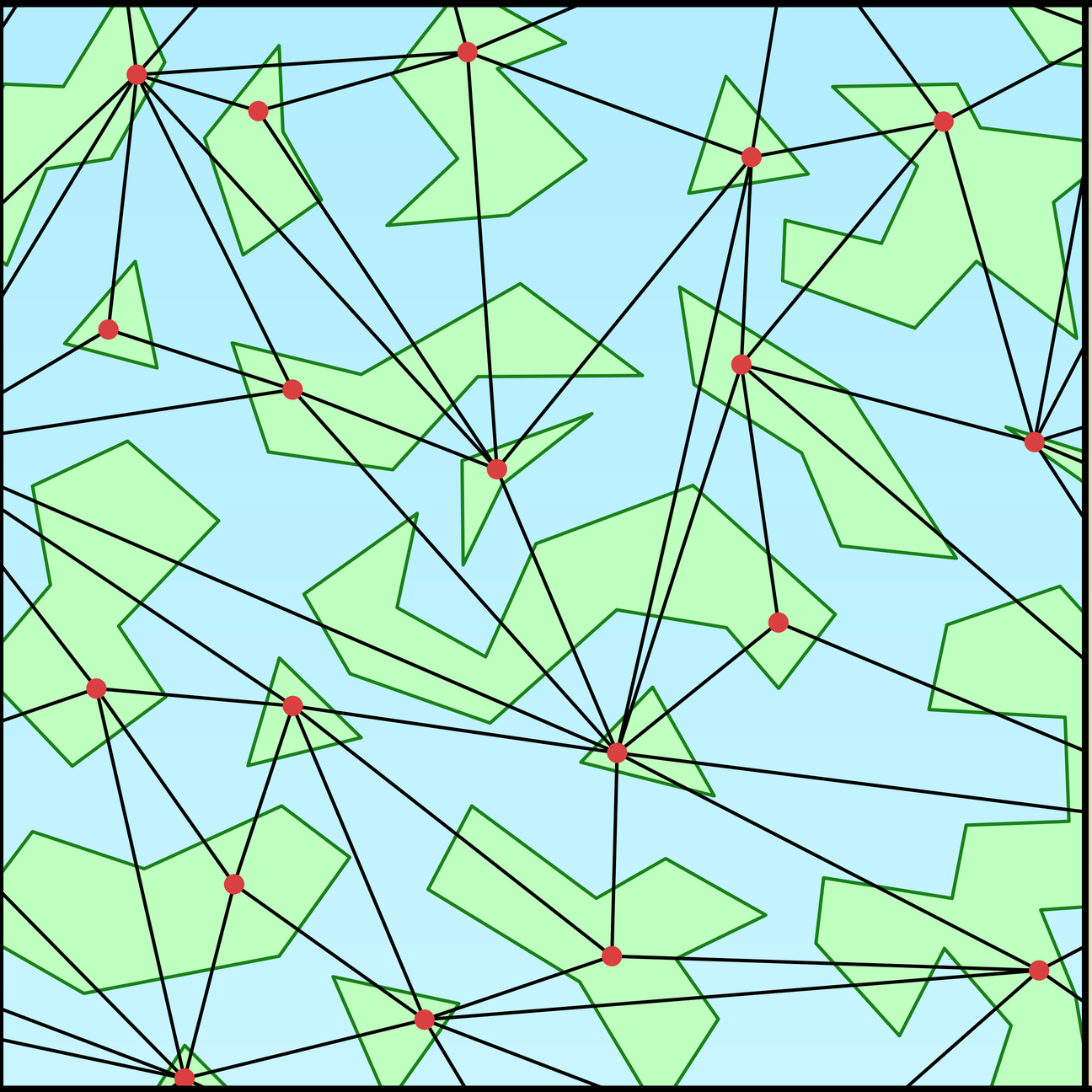

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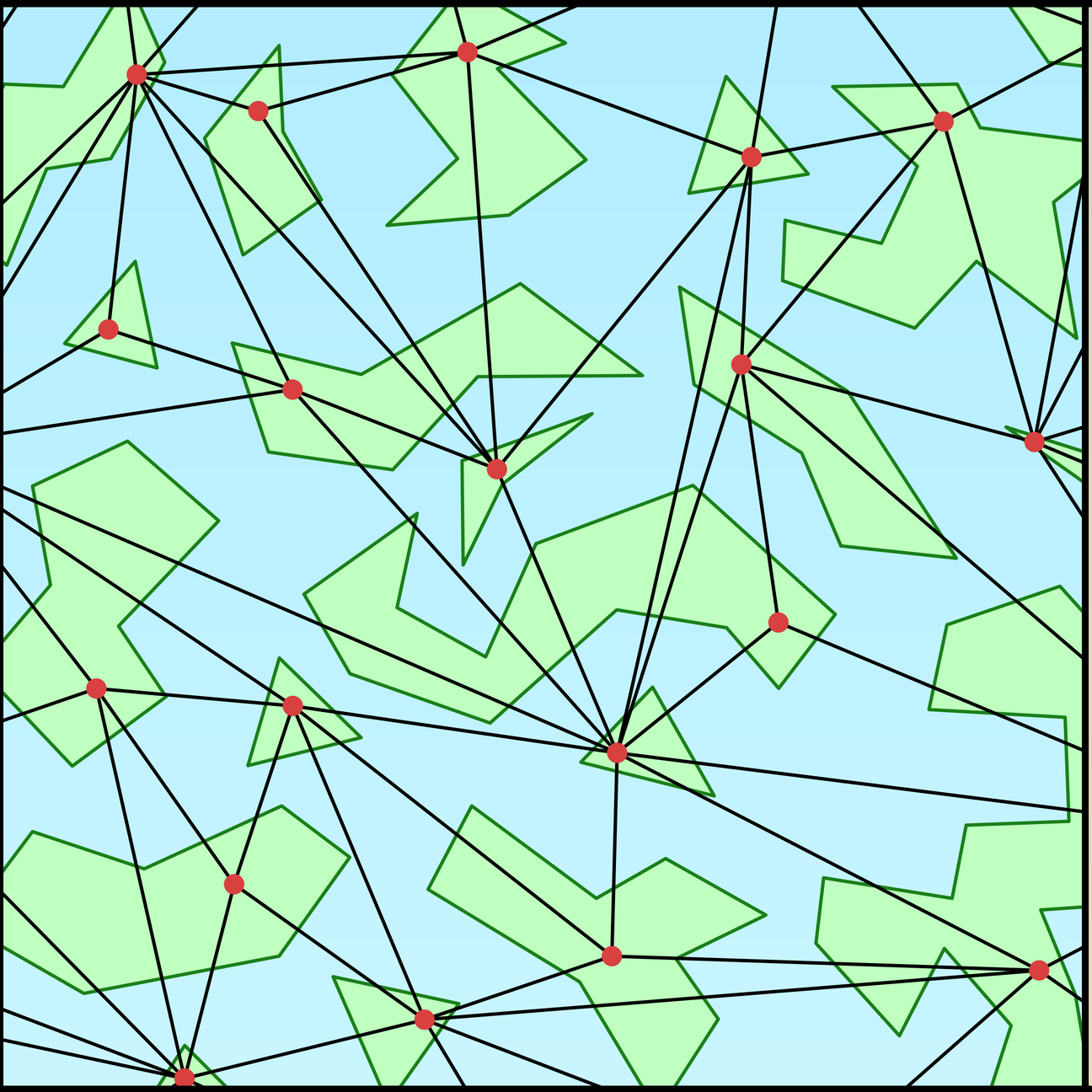

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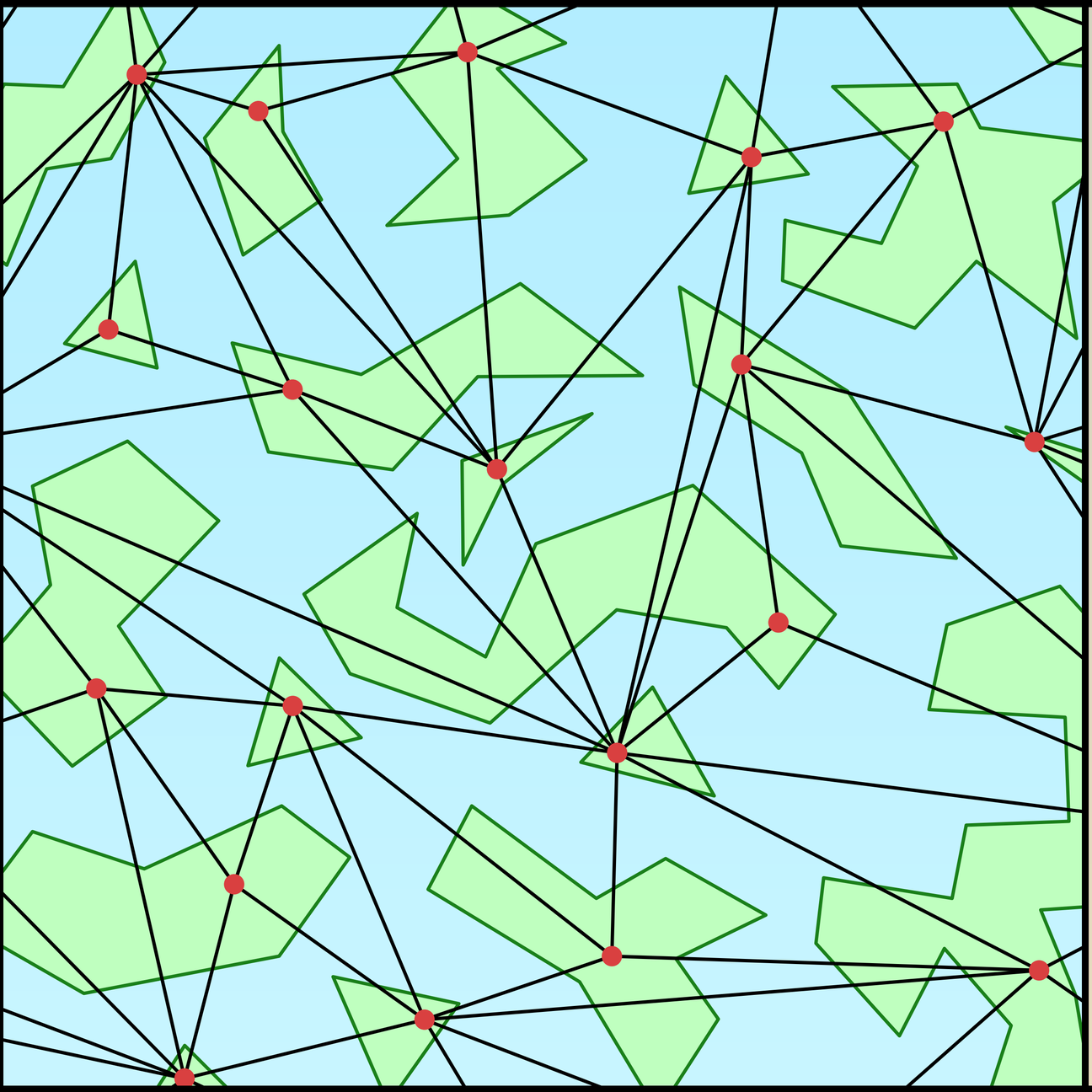

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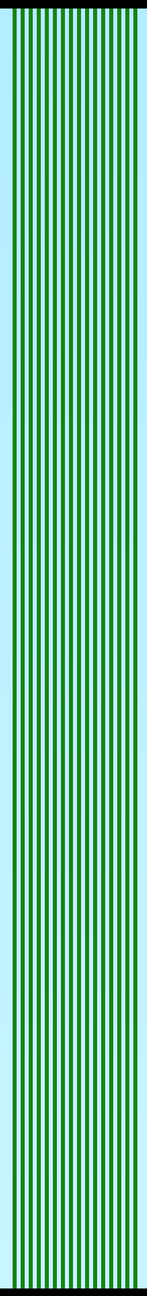

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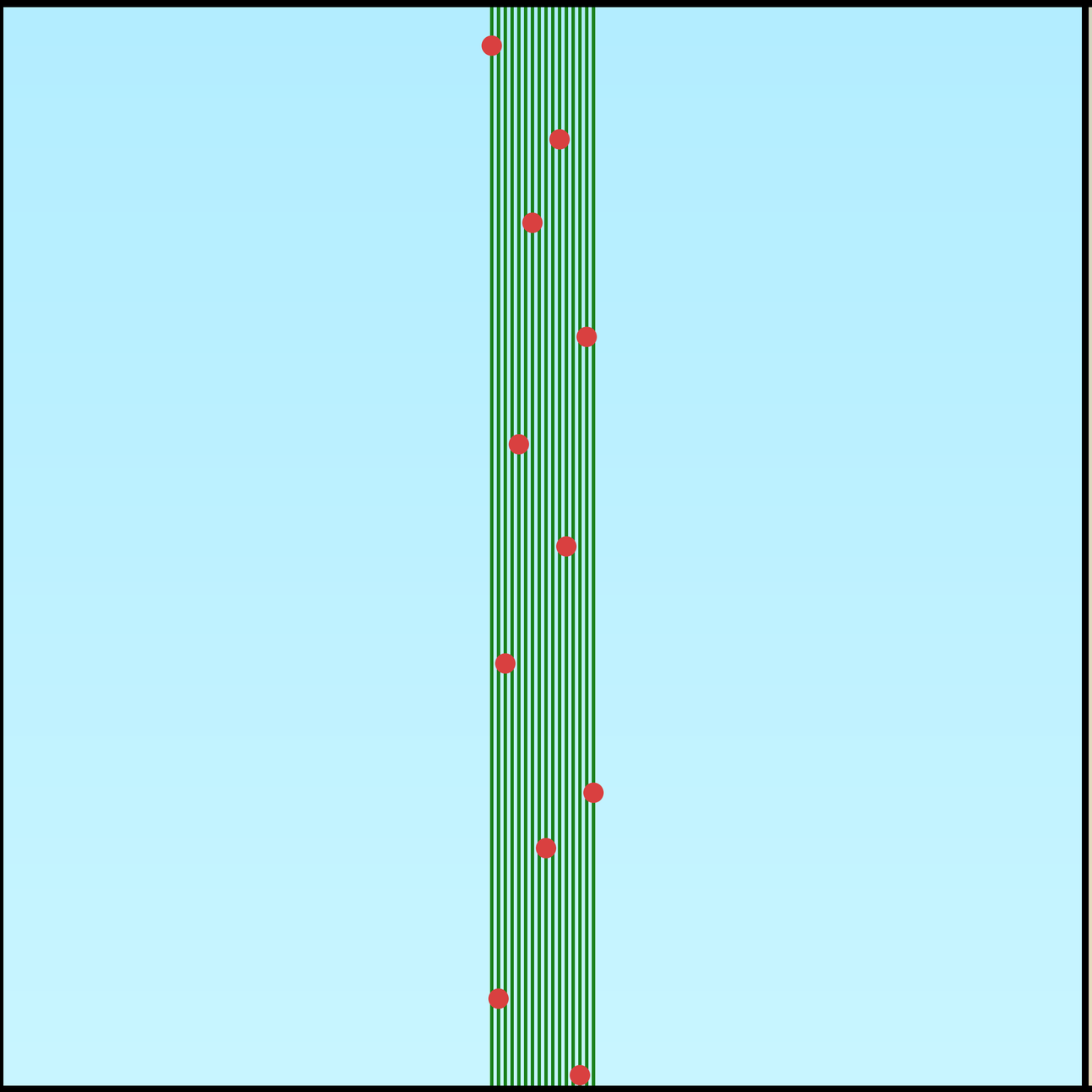

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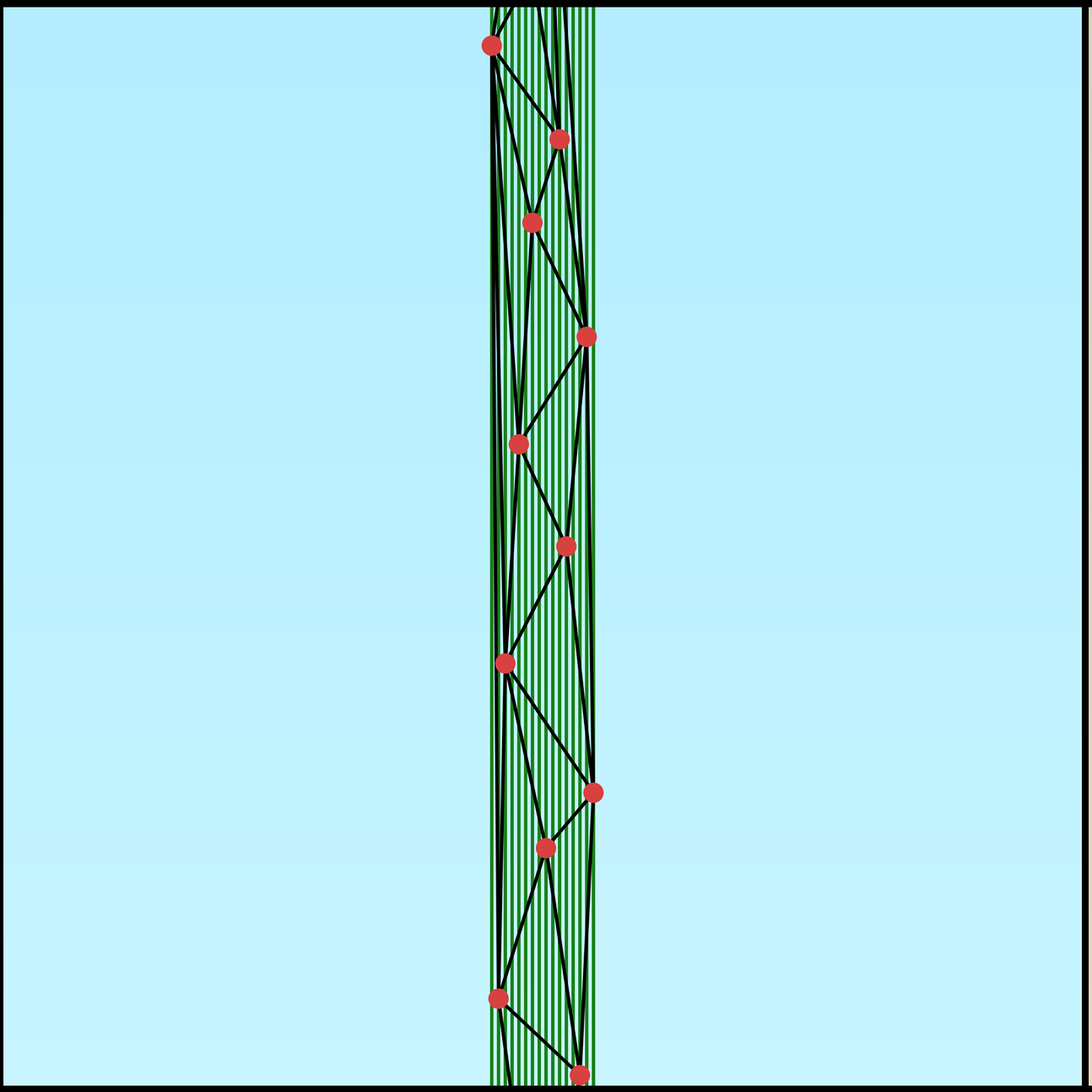

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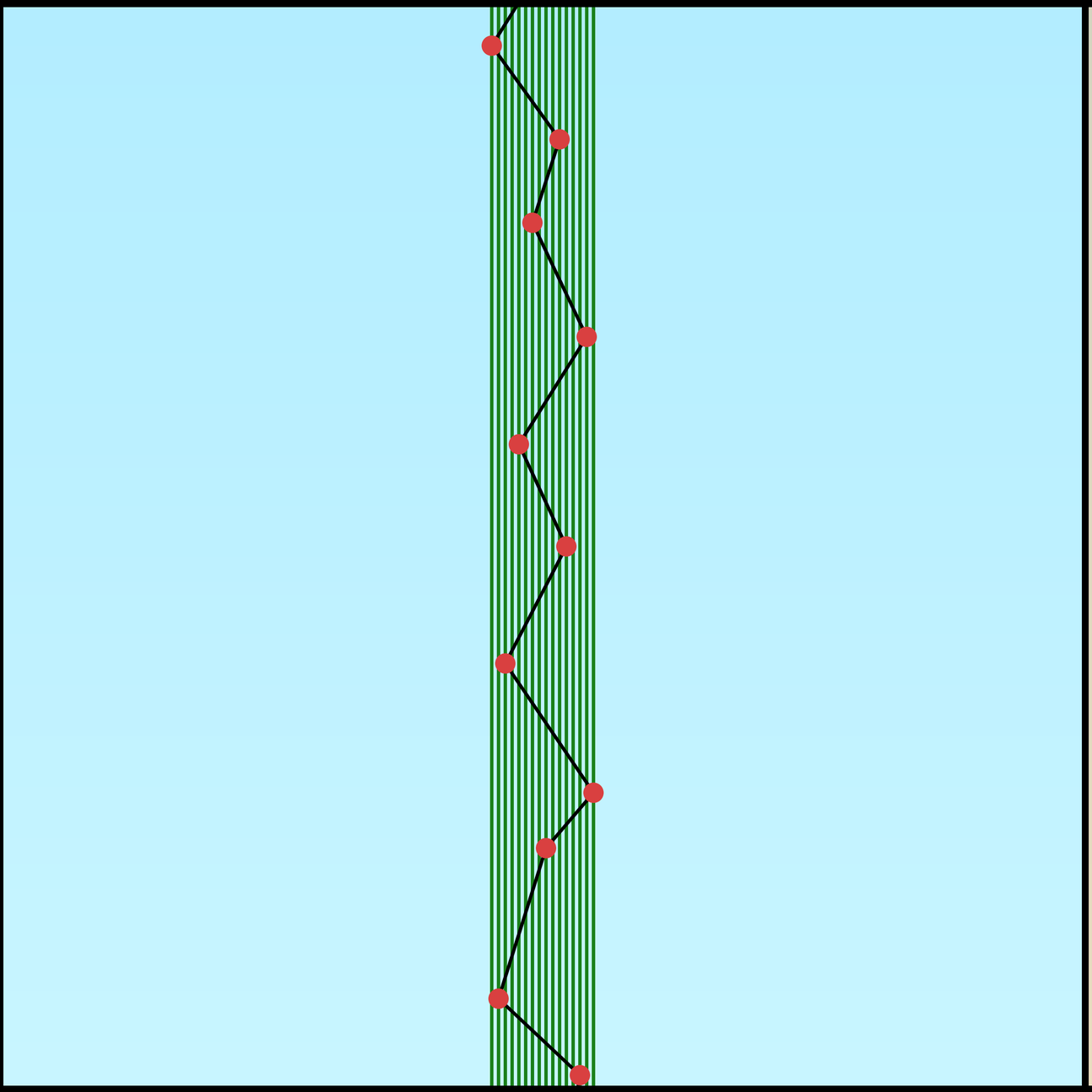

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# Chapter 2
*PARTIAL INFORMATION*

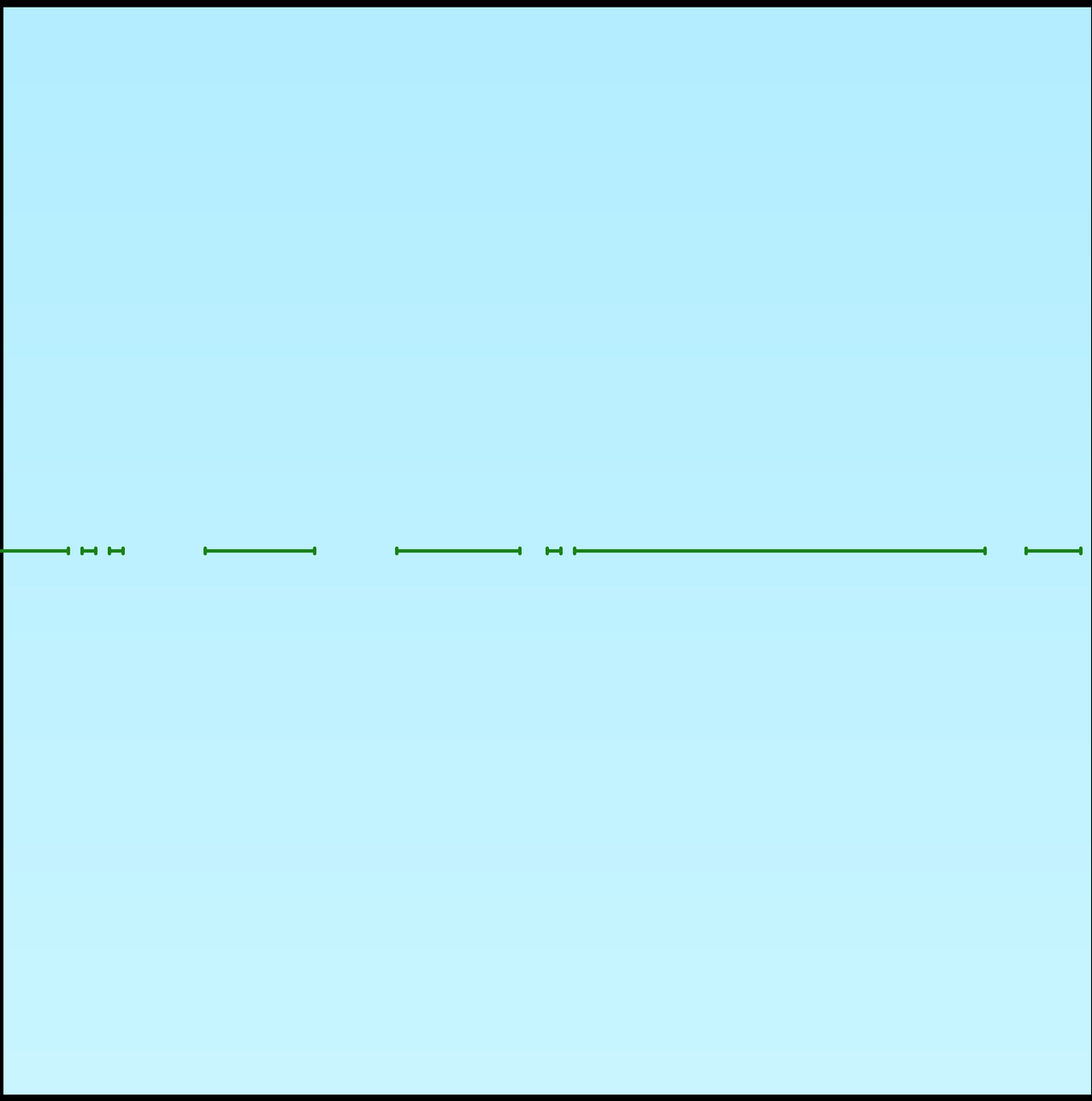
Uncertain locations

Quantifying information

State of the art in 1D

State of the art in 2D

In $\mathbb{R}^1$, this is question: can we presort a set of disjoint intervals?



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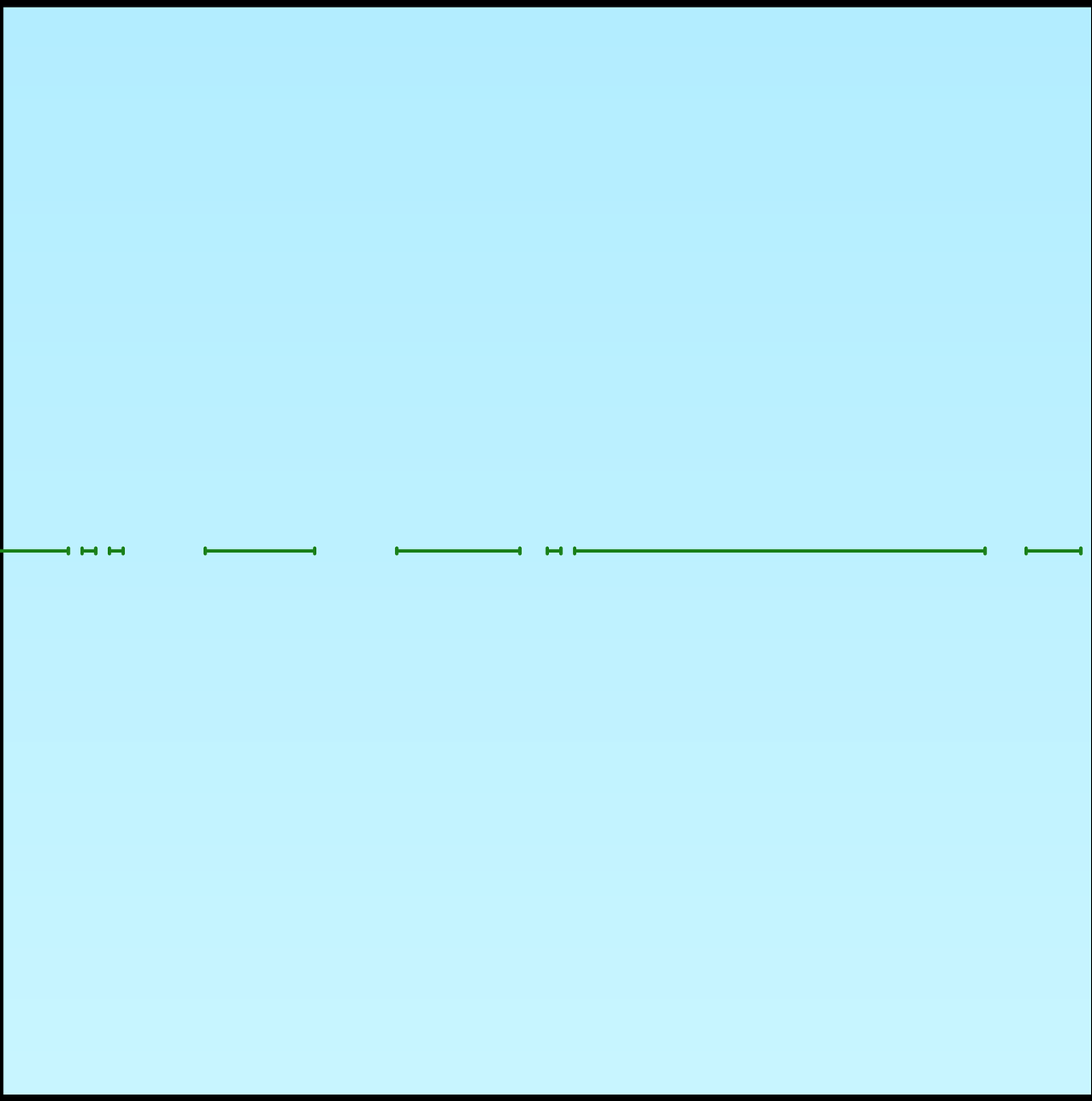

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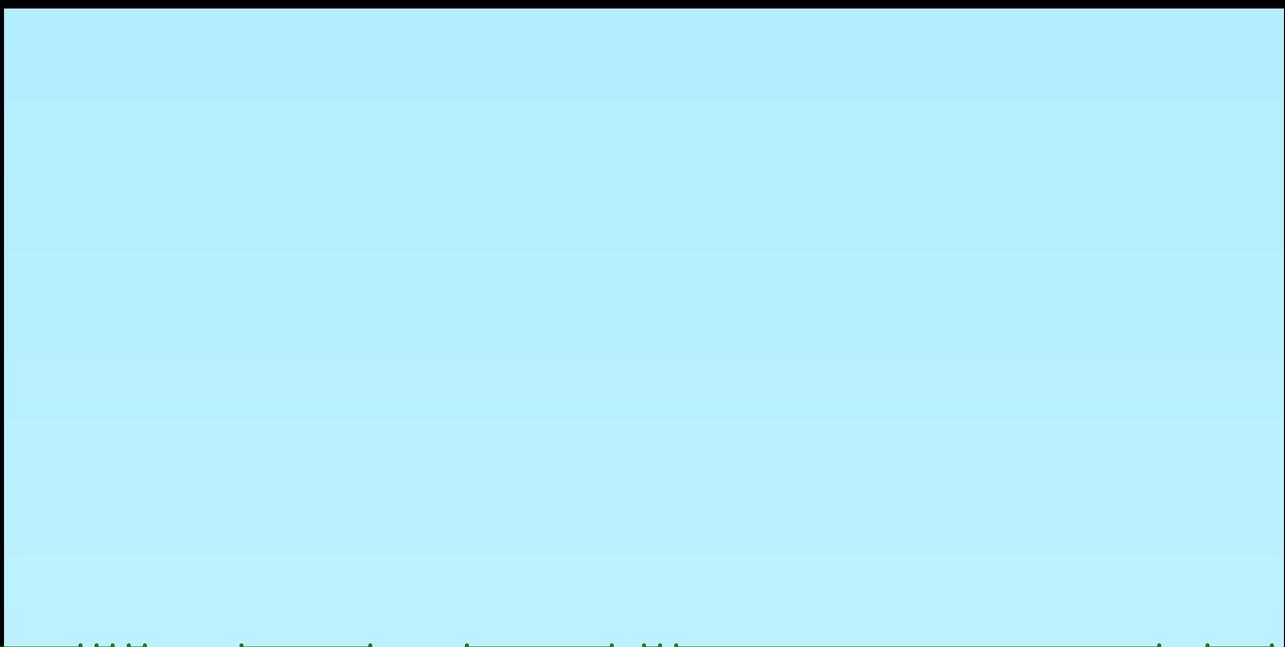

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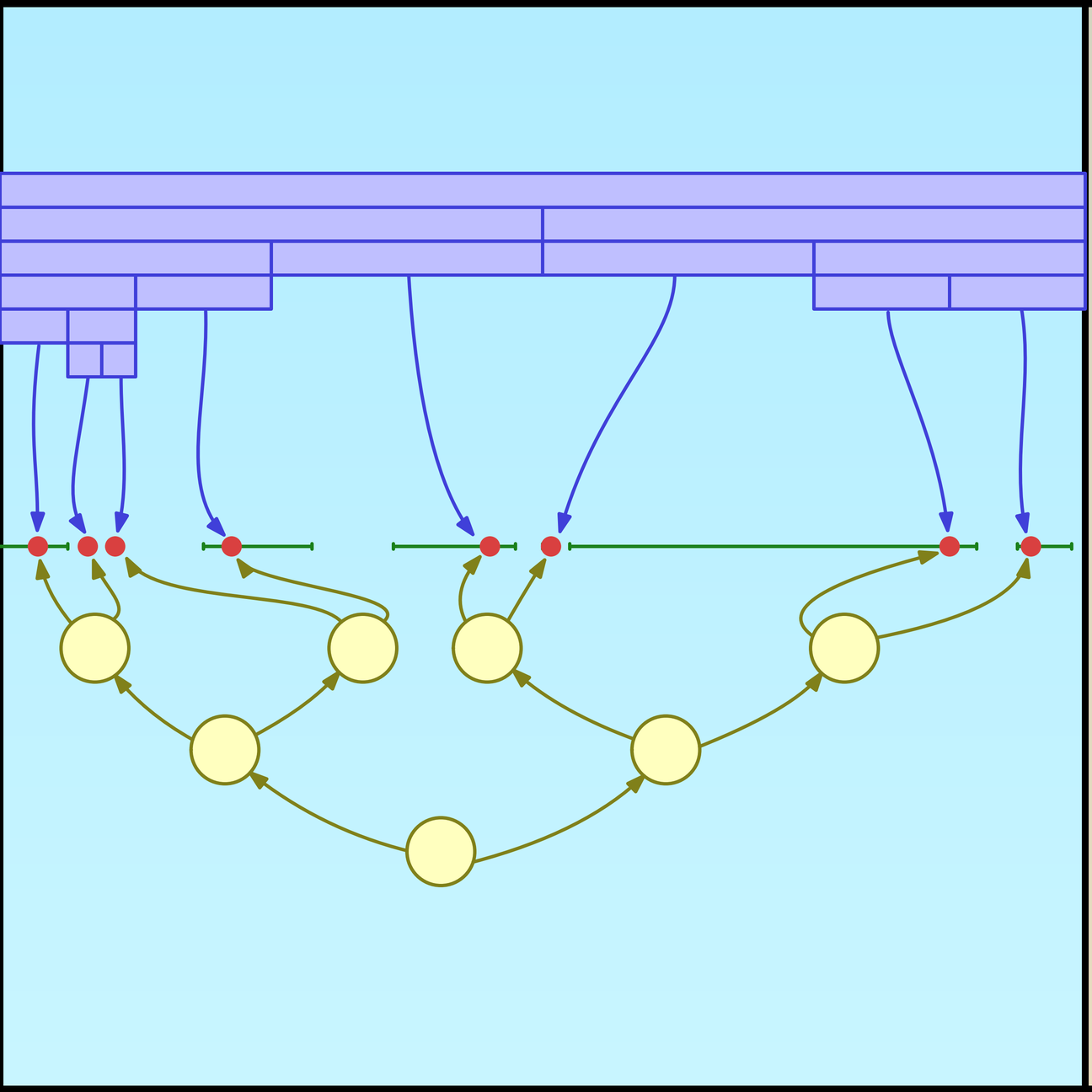

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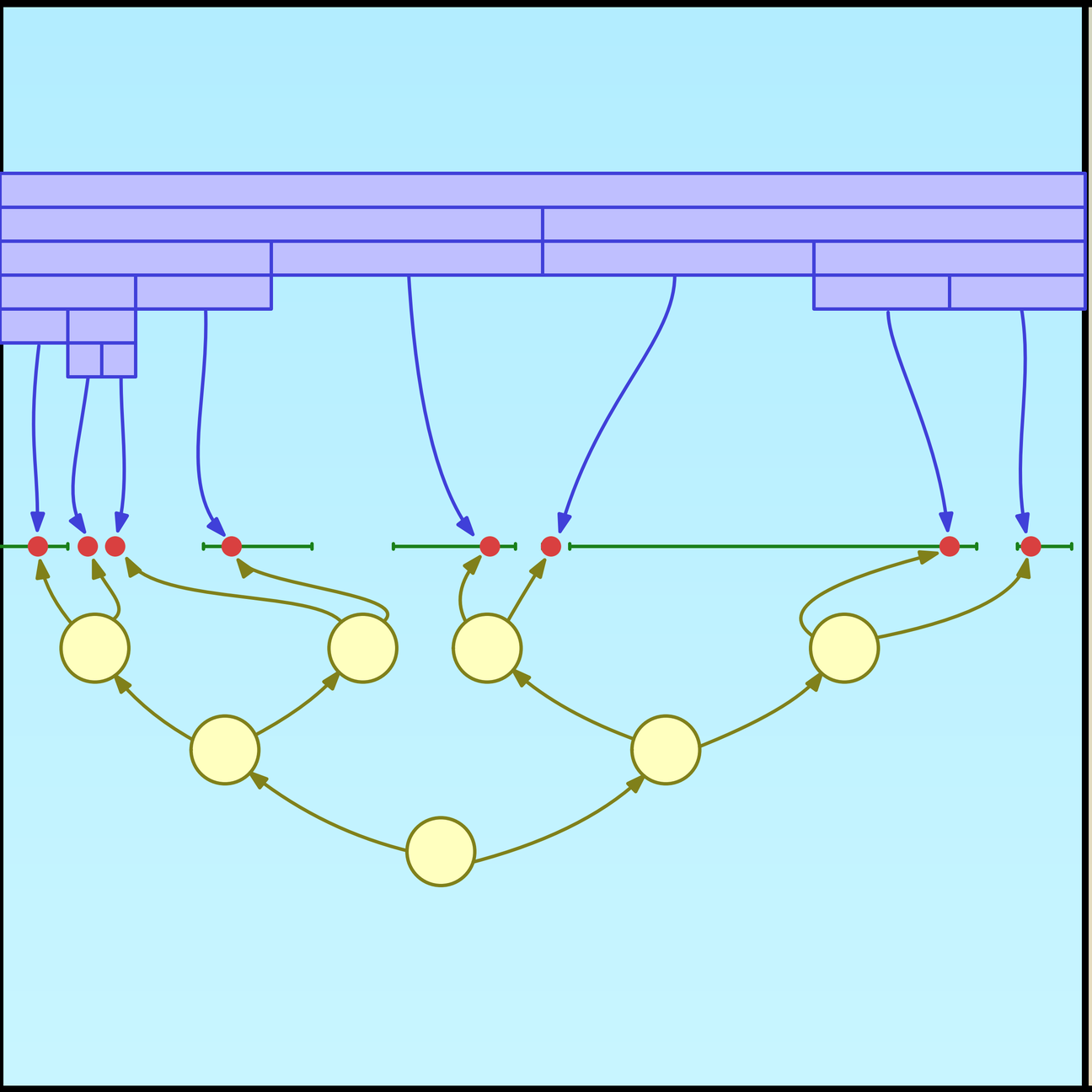

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...everything is easy.

# Chapter 2

*PARTIAL INFORMATION*

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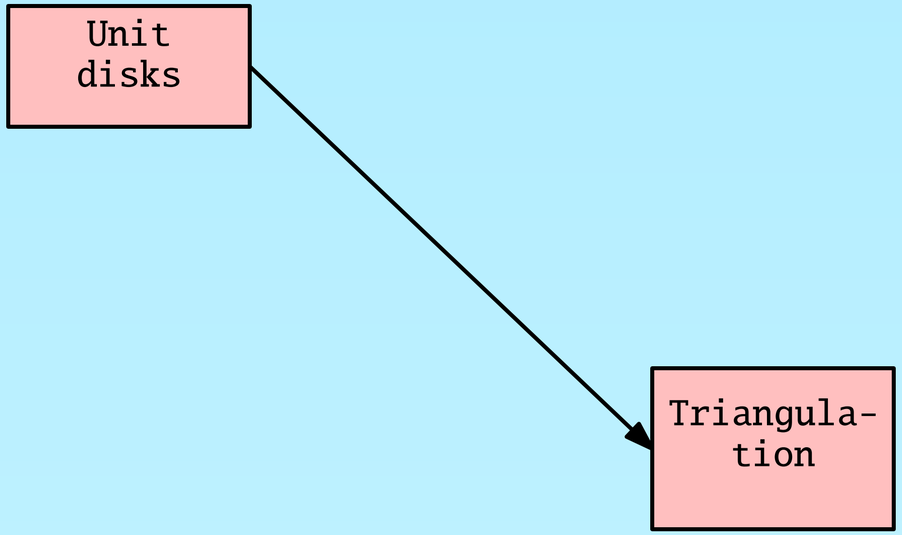


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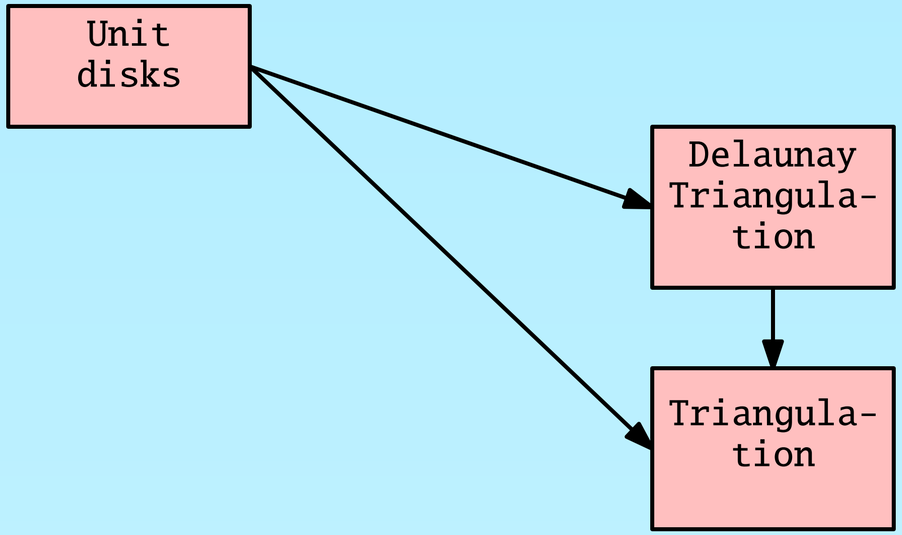
⟶ Deterministic linear time

[Held & Mitchell, 2008]

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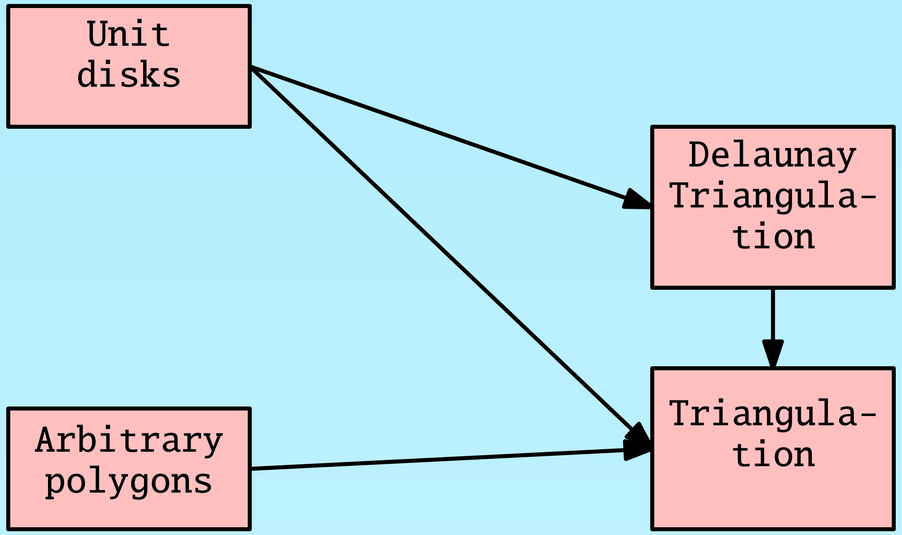


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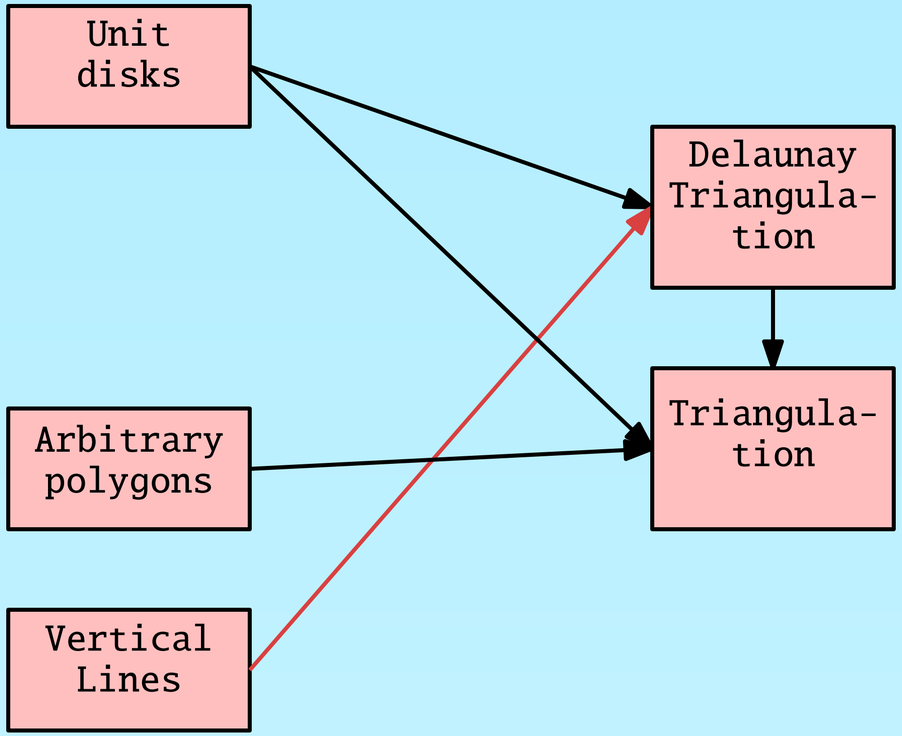


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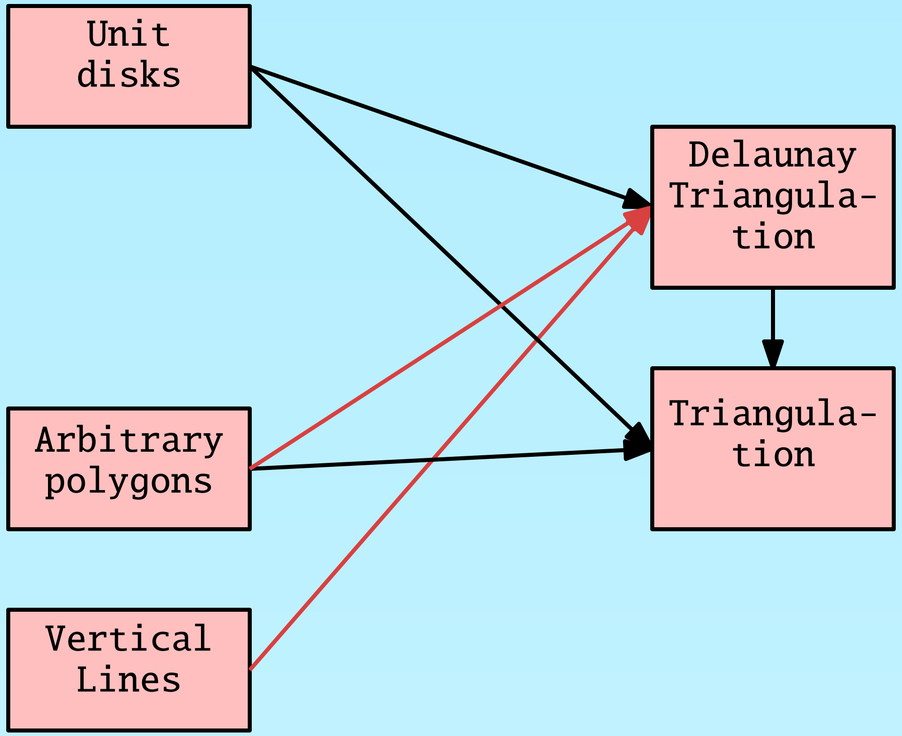


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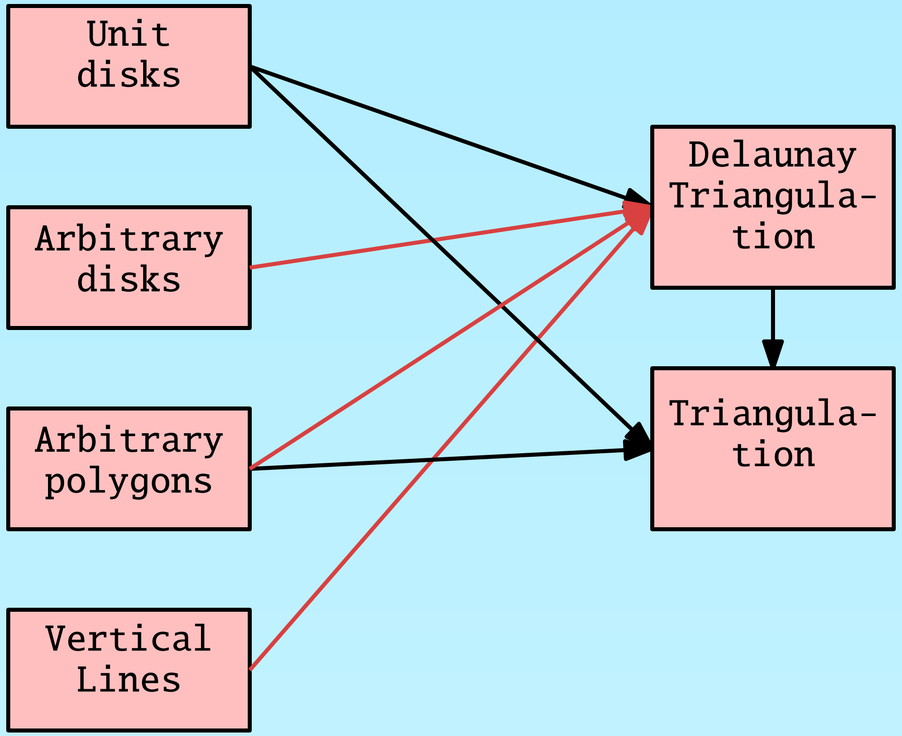


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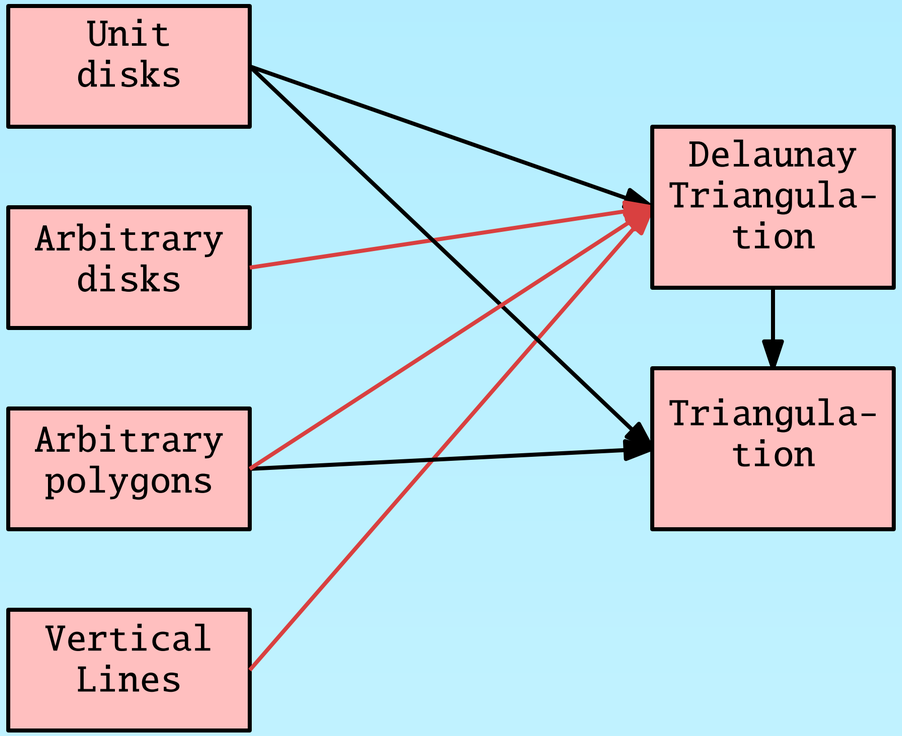


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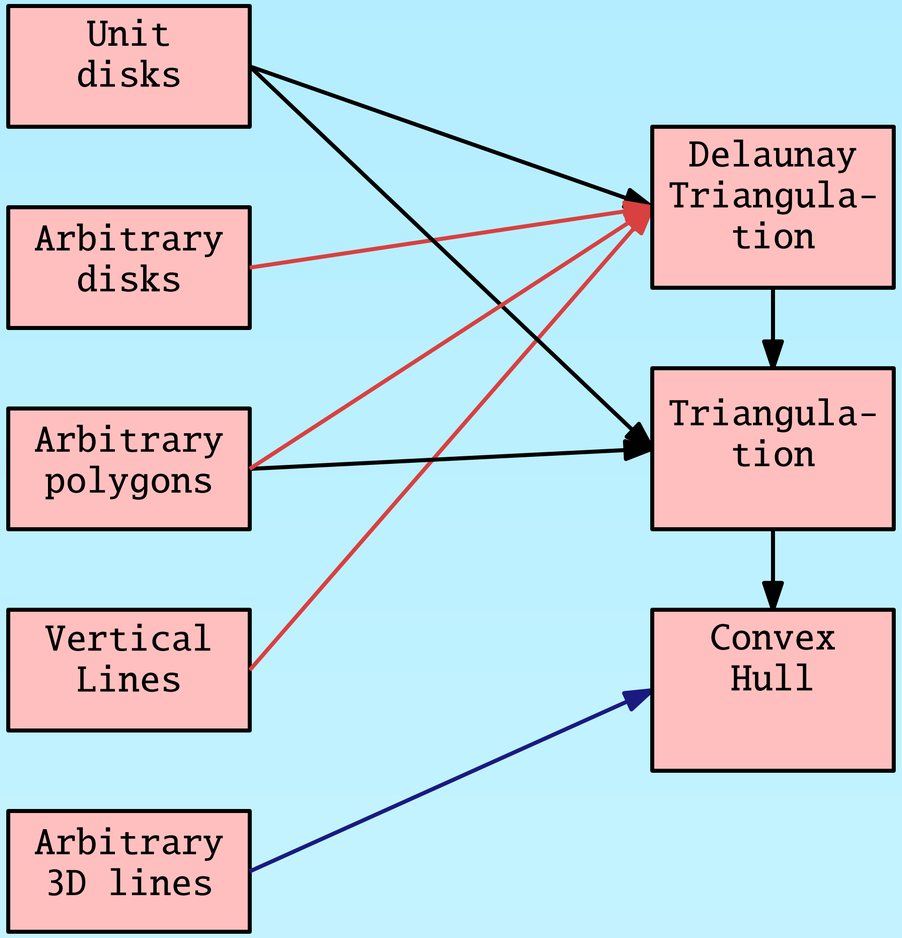


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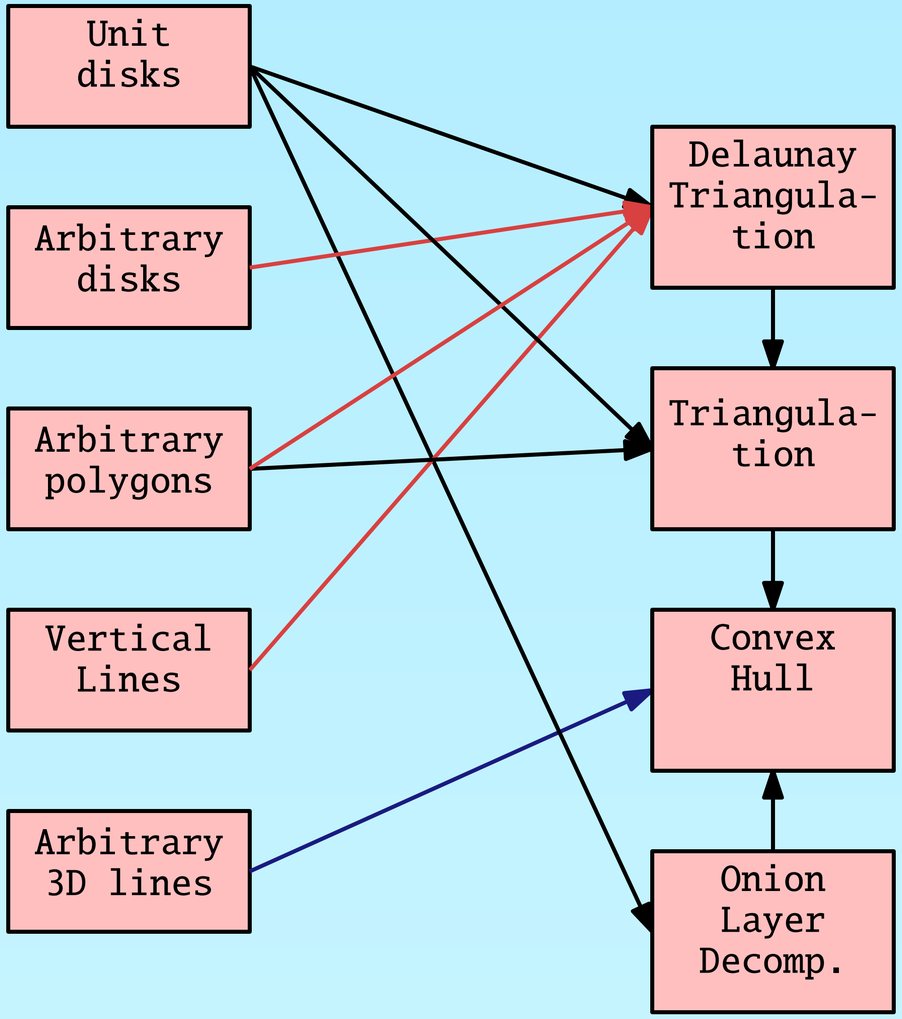


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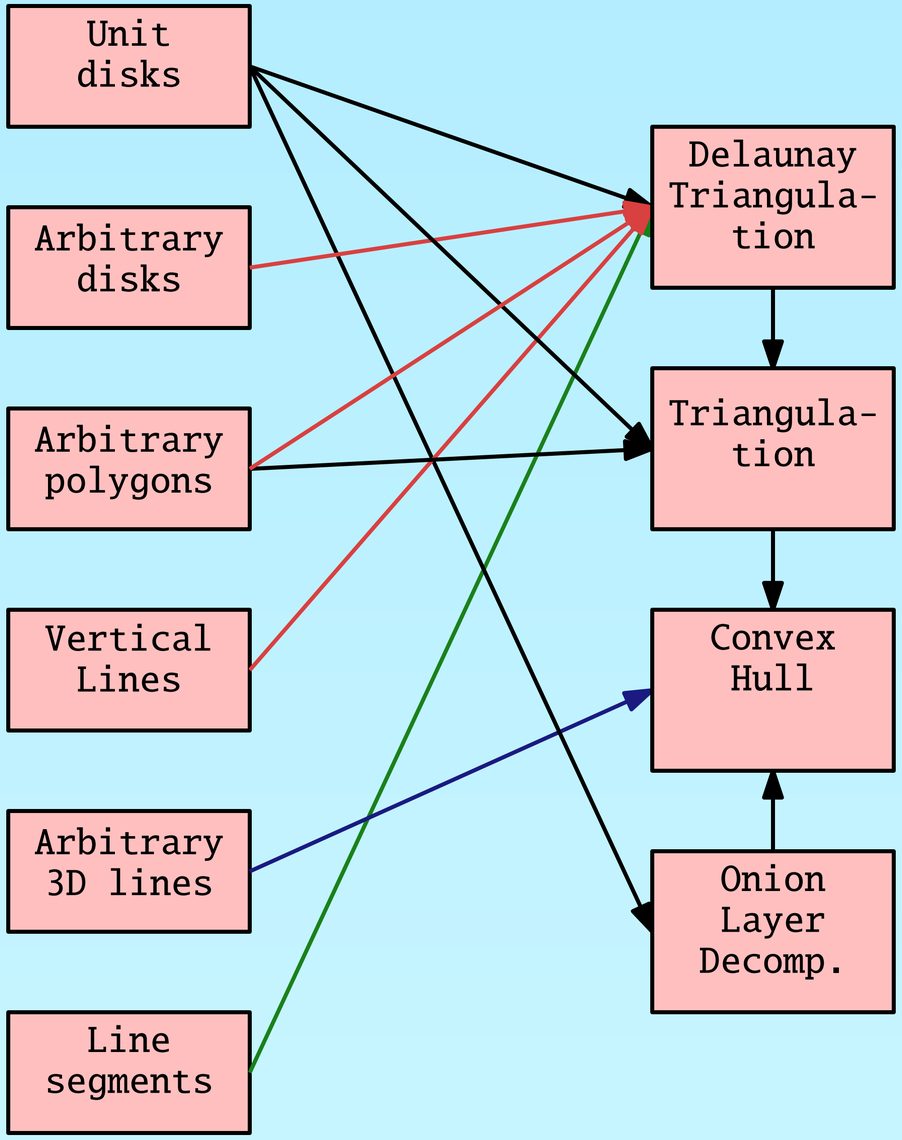


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→ Deterministic linear time
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→ expected linear time (expectation over $P$)

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[Ezra & Mulzer, 2013]
[Löffler & Mulzer, 2014]
[Keikha, Mohades & Davoodi, 2016]

# Chapter 3
## *FINE-GRAINED MEASURES*

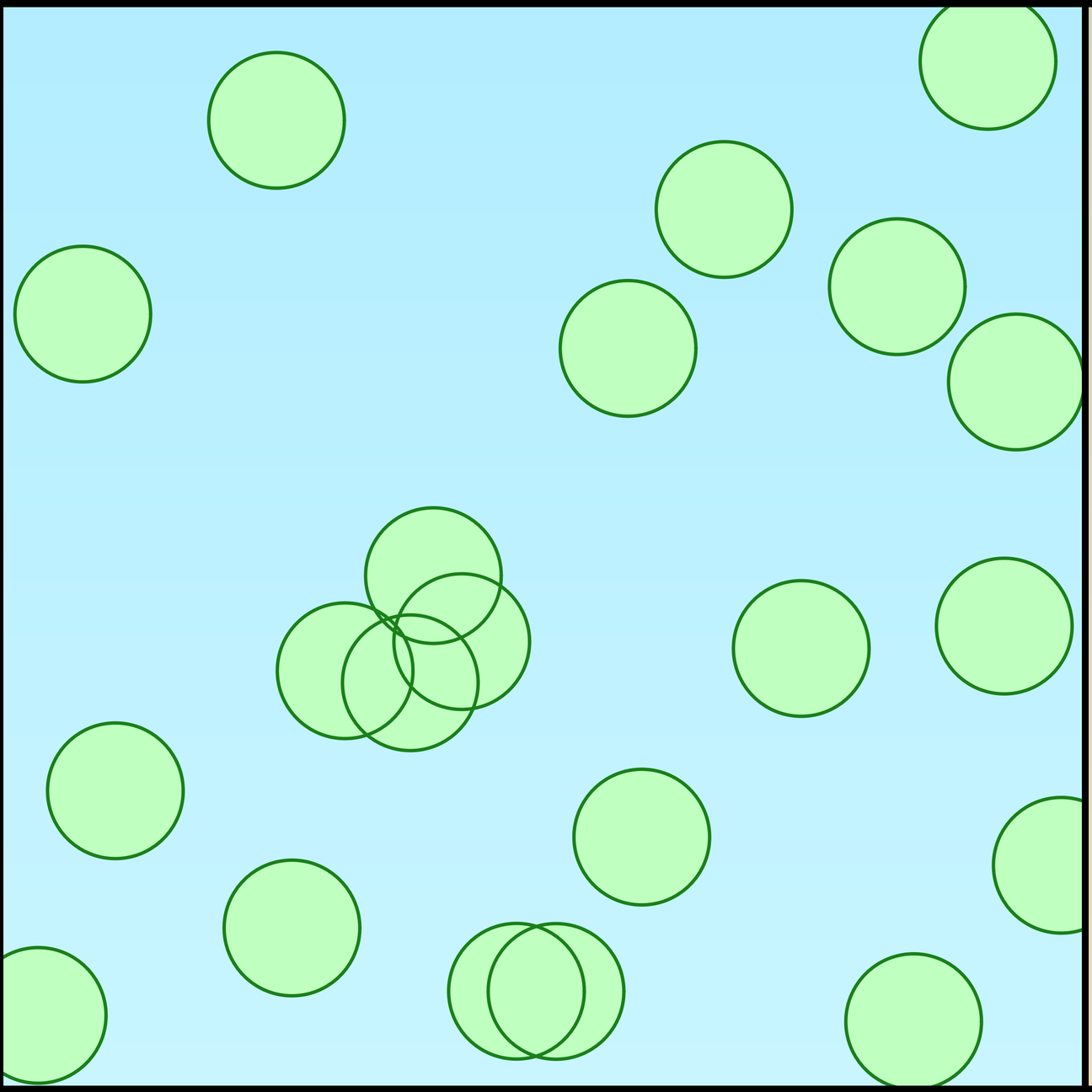
Ambiguity of regions

Information theory

State of the art in 1D

State of the art in 2D

What if the regions in $\mathcal{R}$ are not disjoint, but only overlap a little bit?



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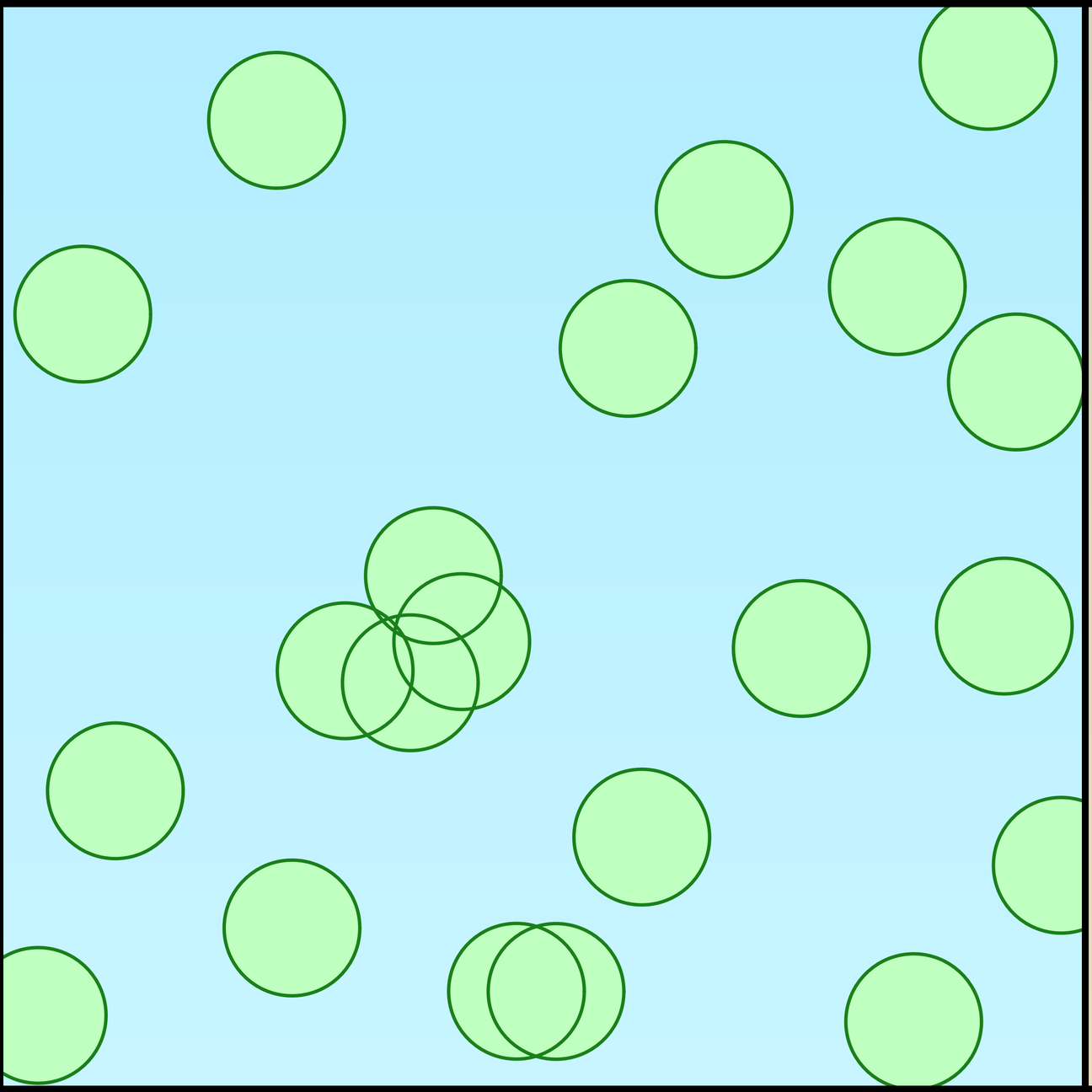

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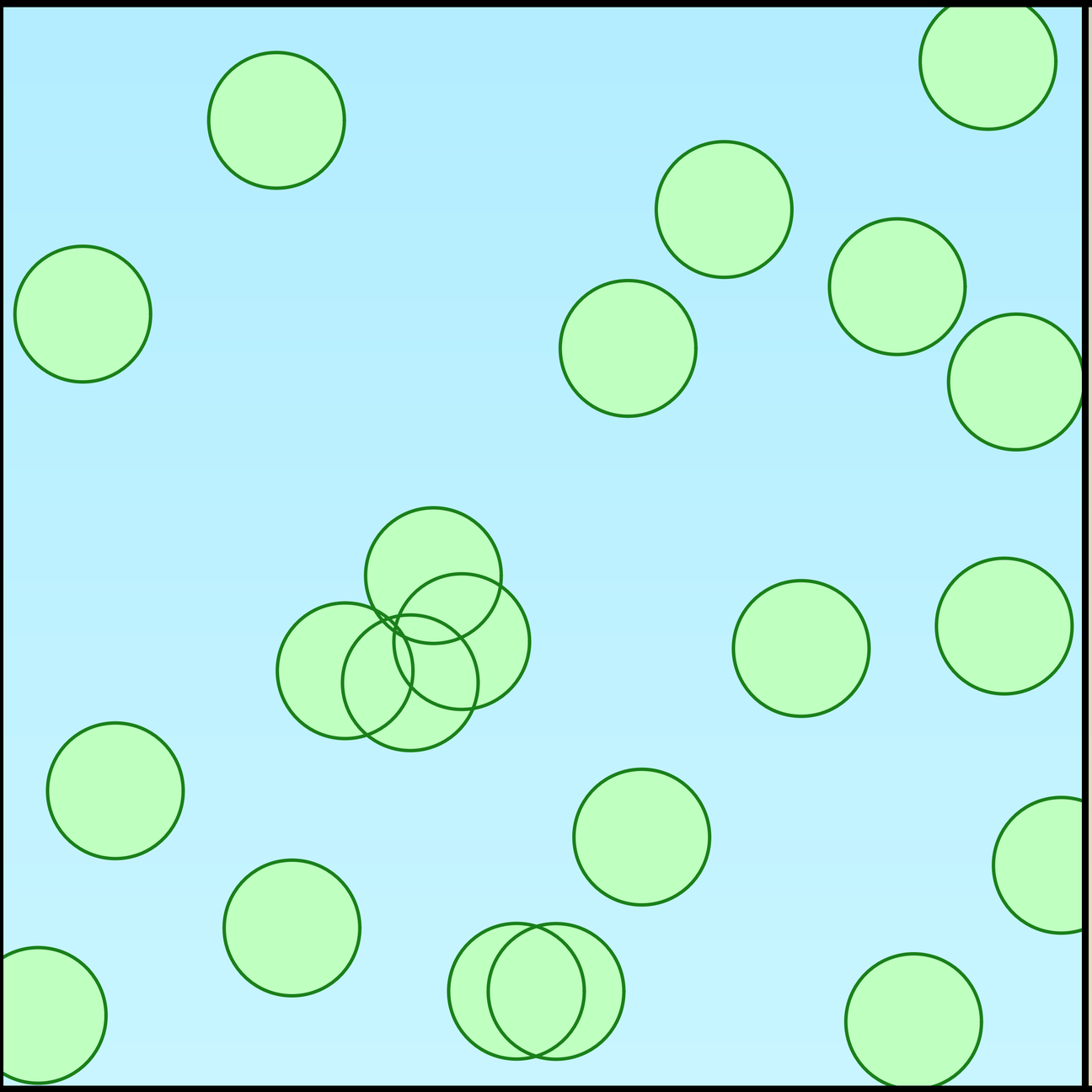

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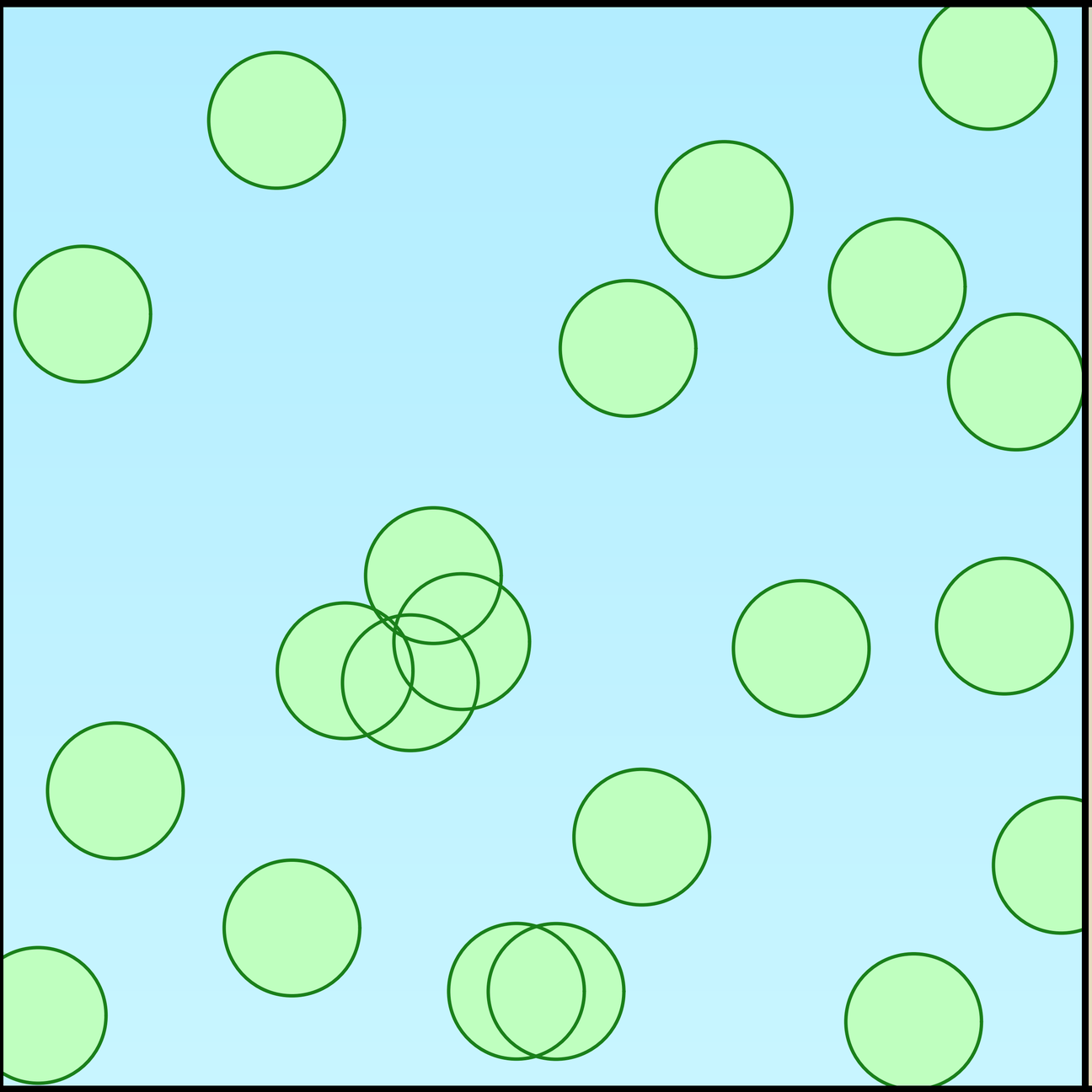

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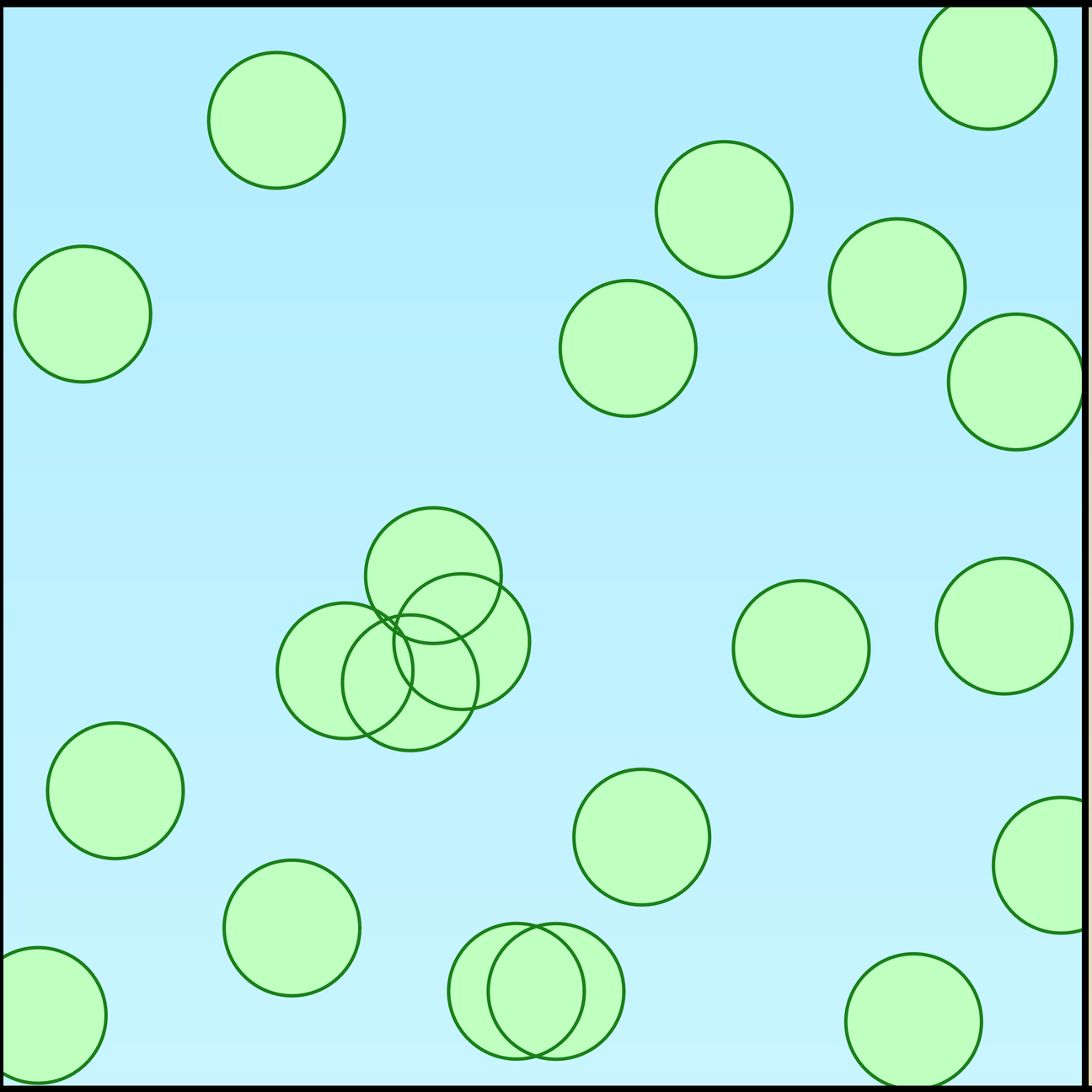
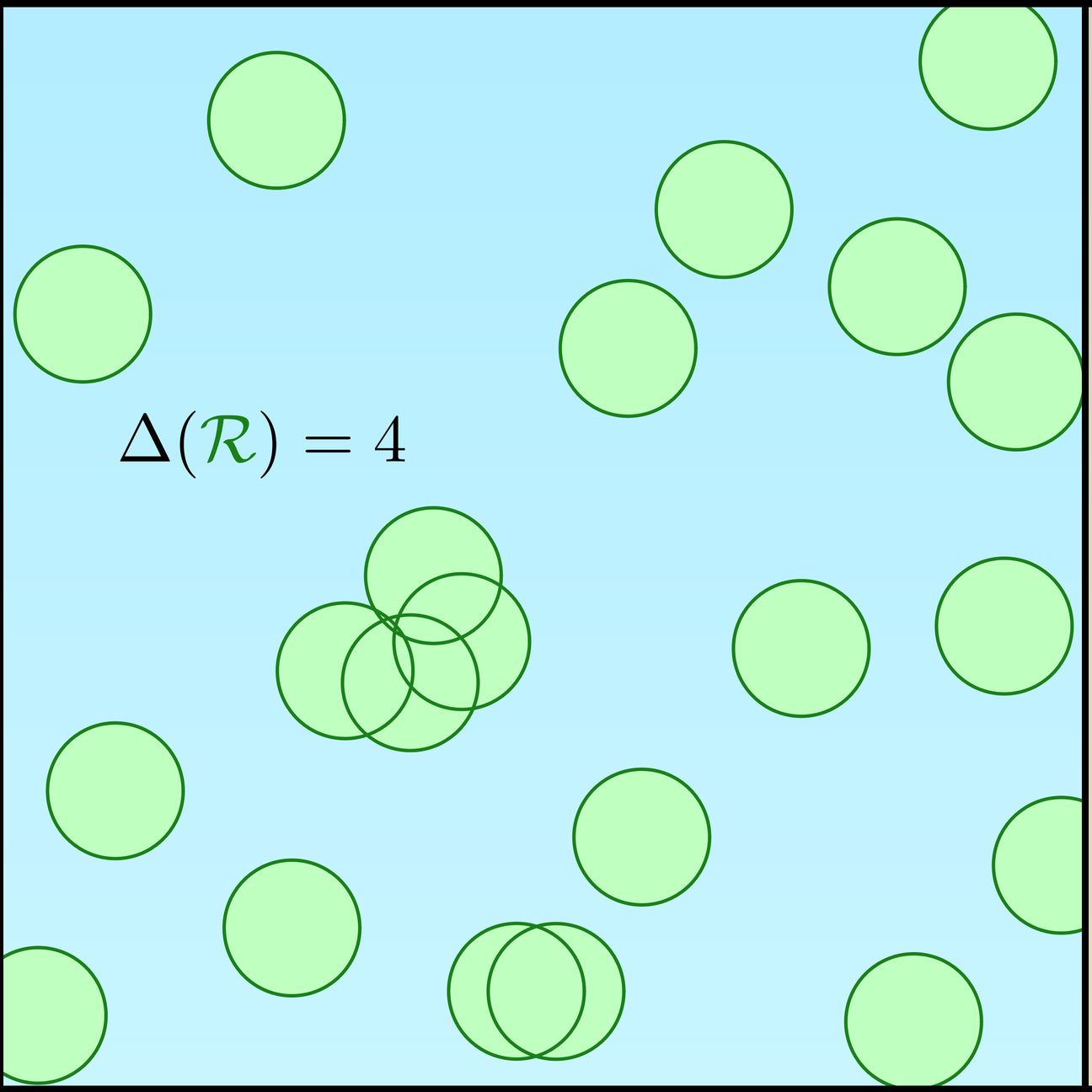
Can we measure the amount of overlap in $\mathcal{R}$?



The *ply* $\Delta(\mathcal{R})$ measures the maximum number of regions that contain a common point.

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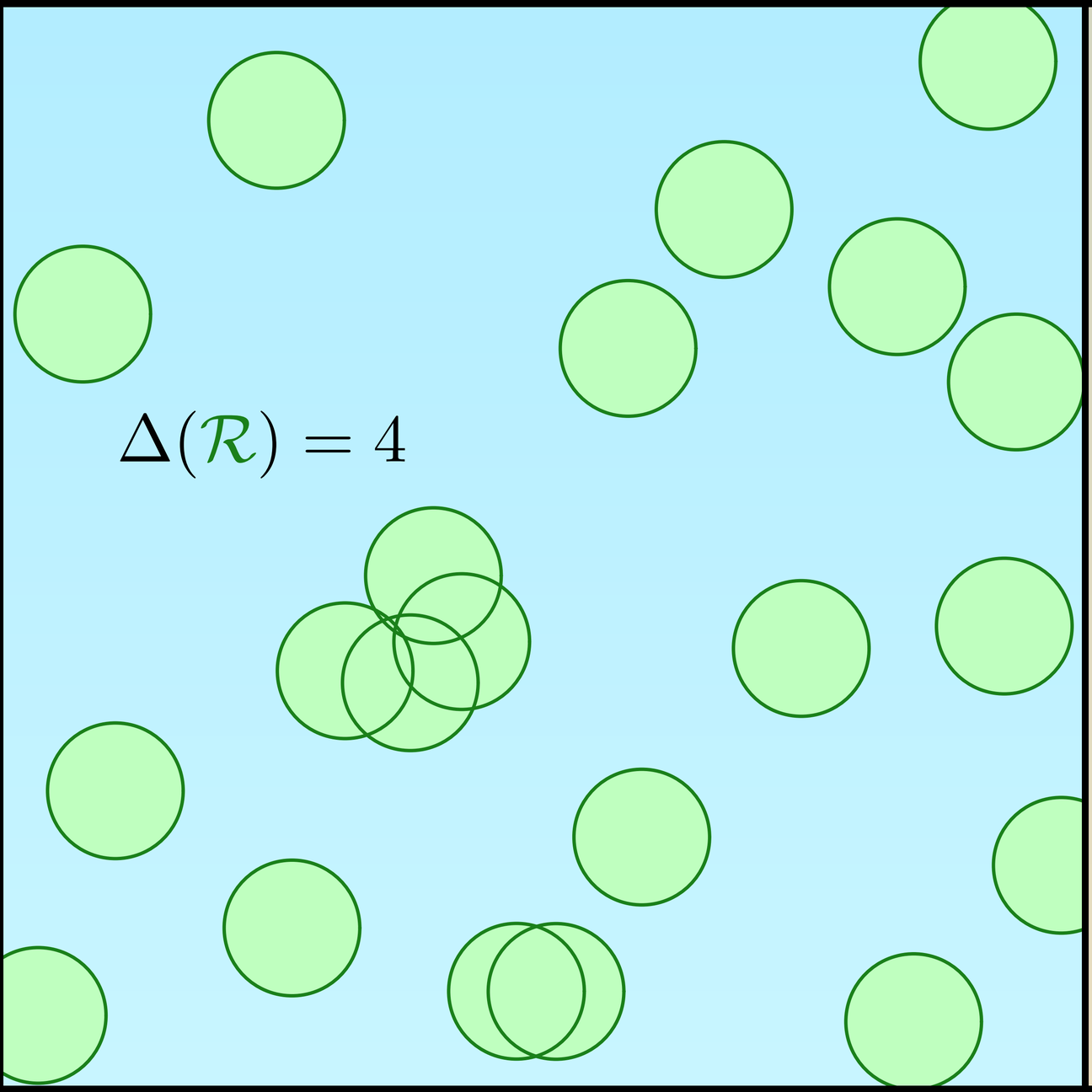


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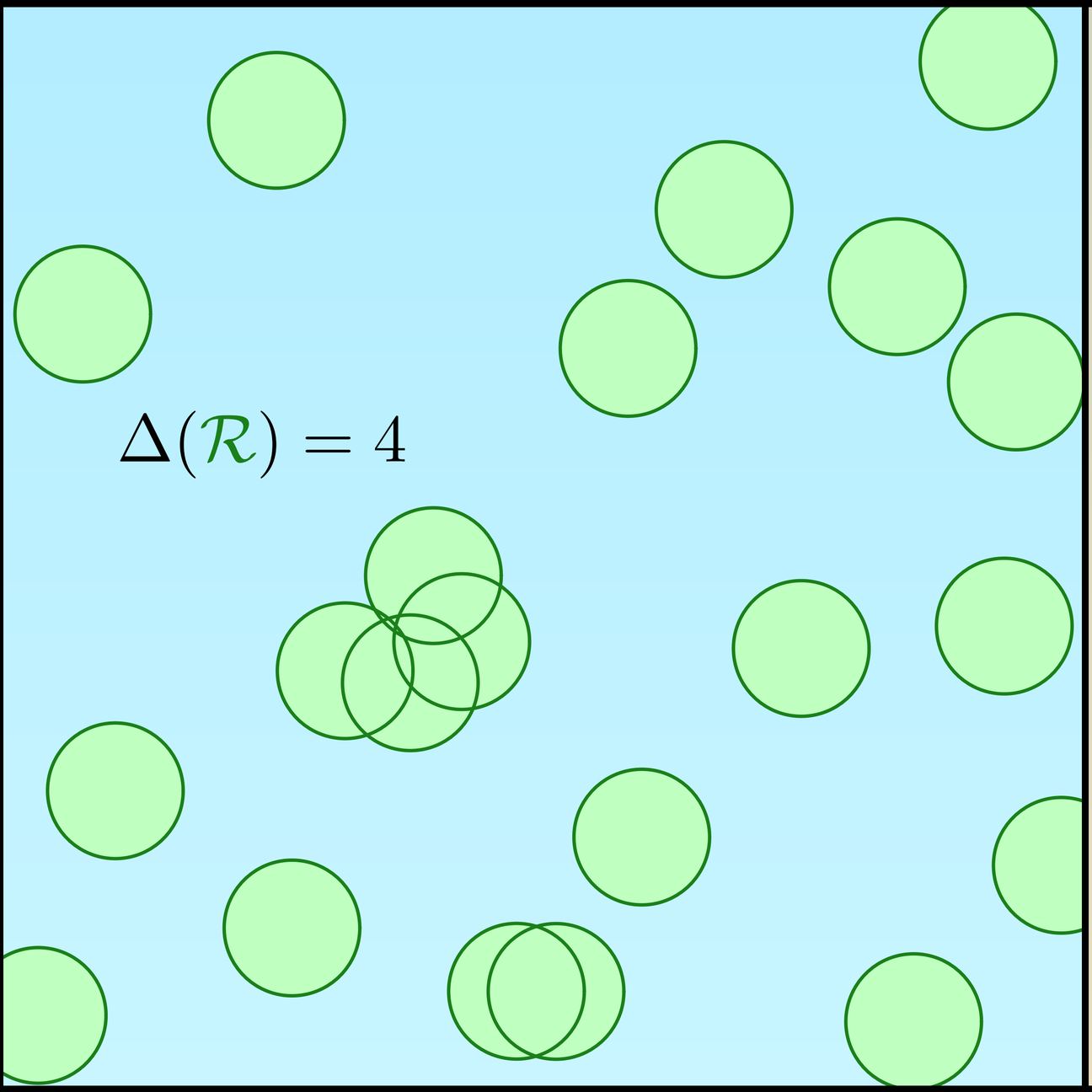
We may hope for reconstruction in time that depends on $\Delta$.

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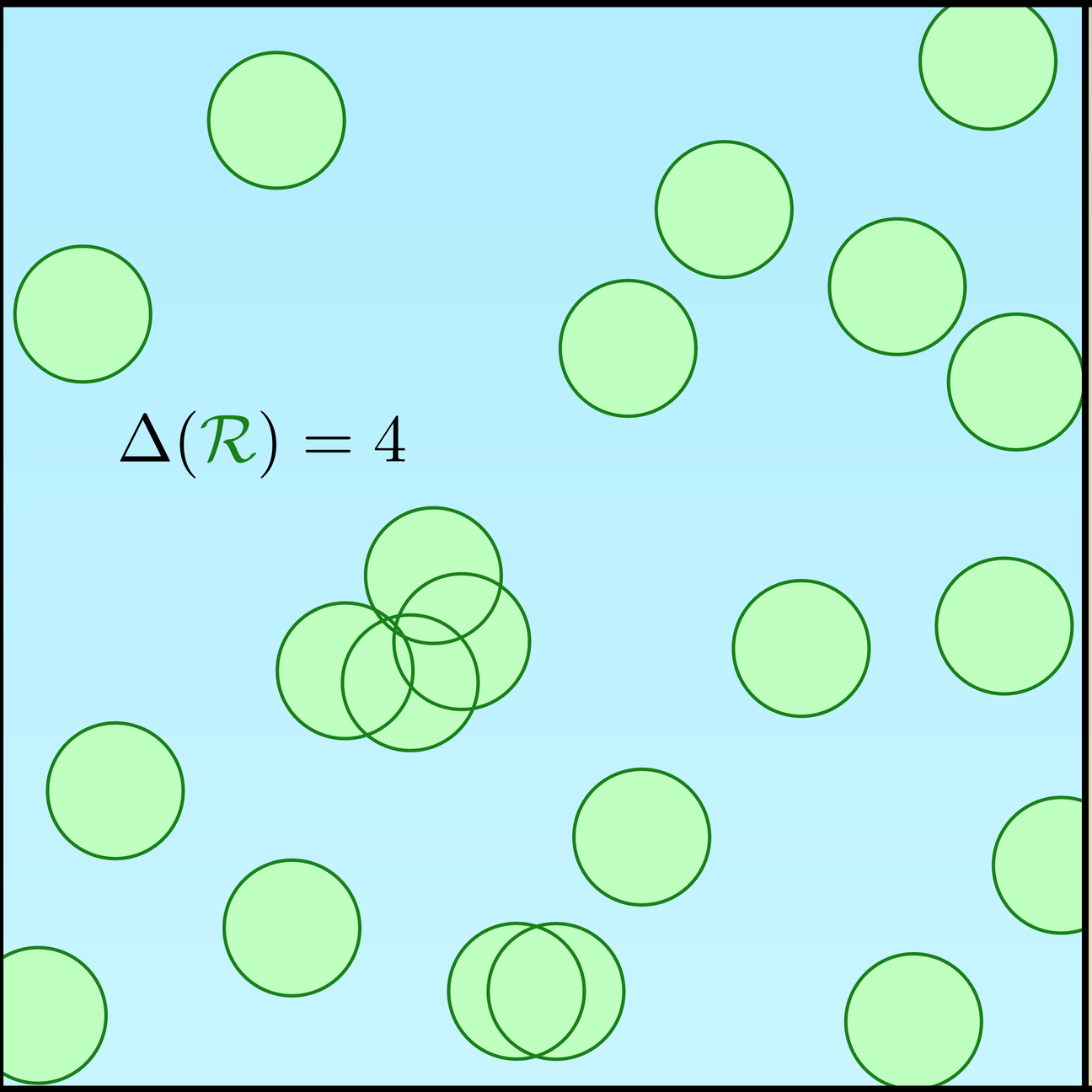
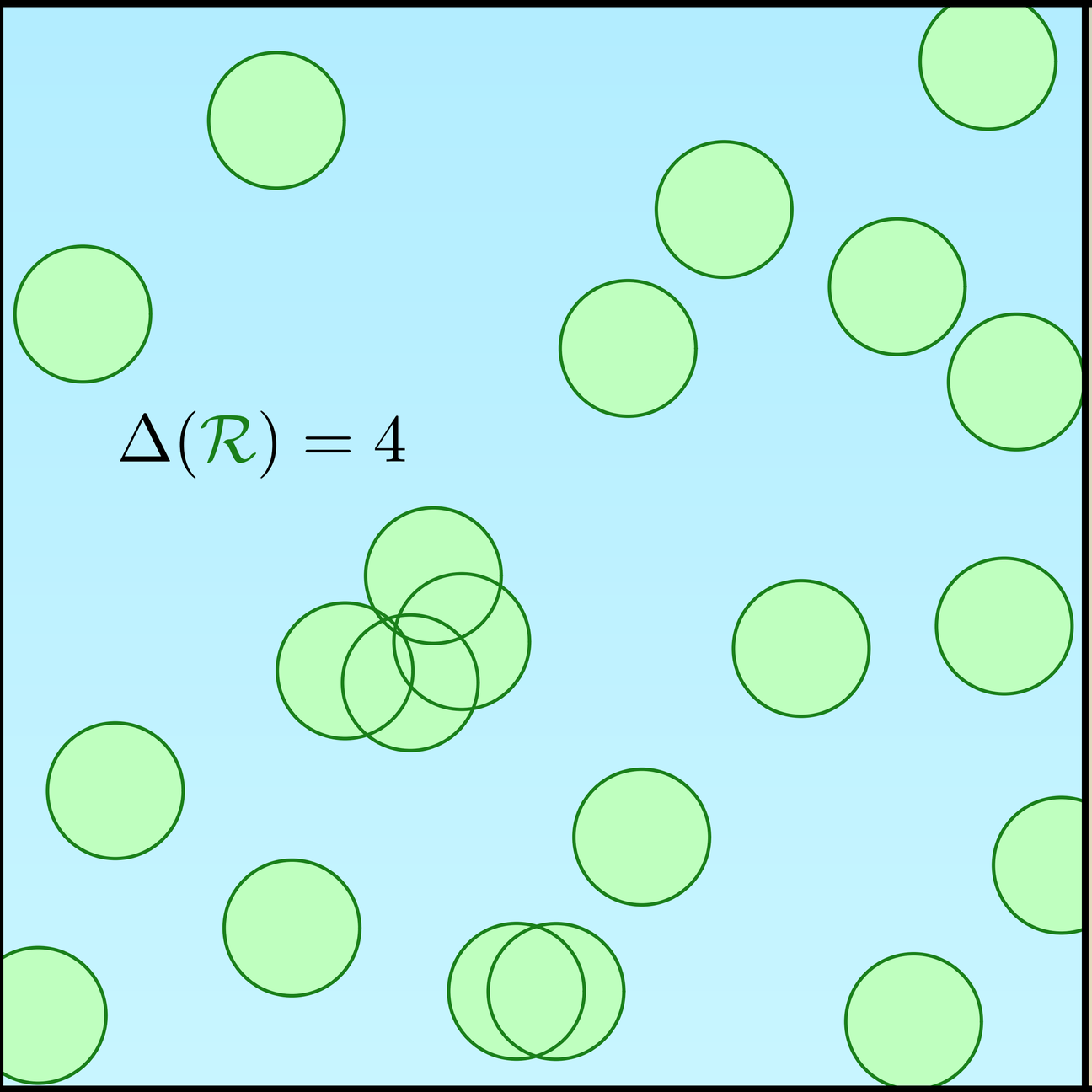
For example: the Delaunay triangulation can actually be reconstructed in $O(n \log \Delta)$ time.

[Buchin, Löffler, Morin & Mulzer, 2011]




The ply is a *global* measure of overlap.

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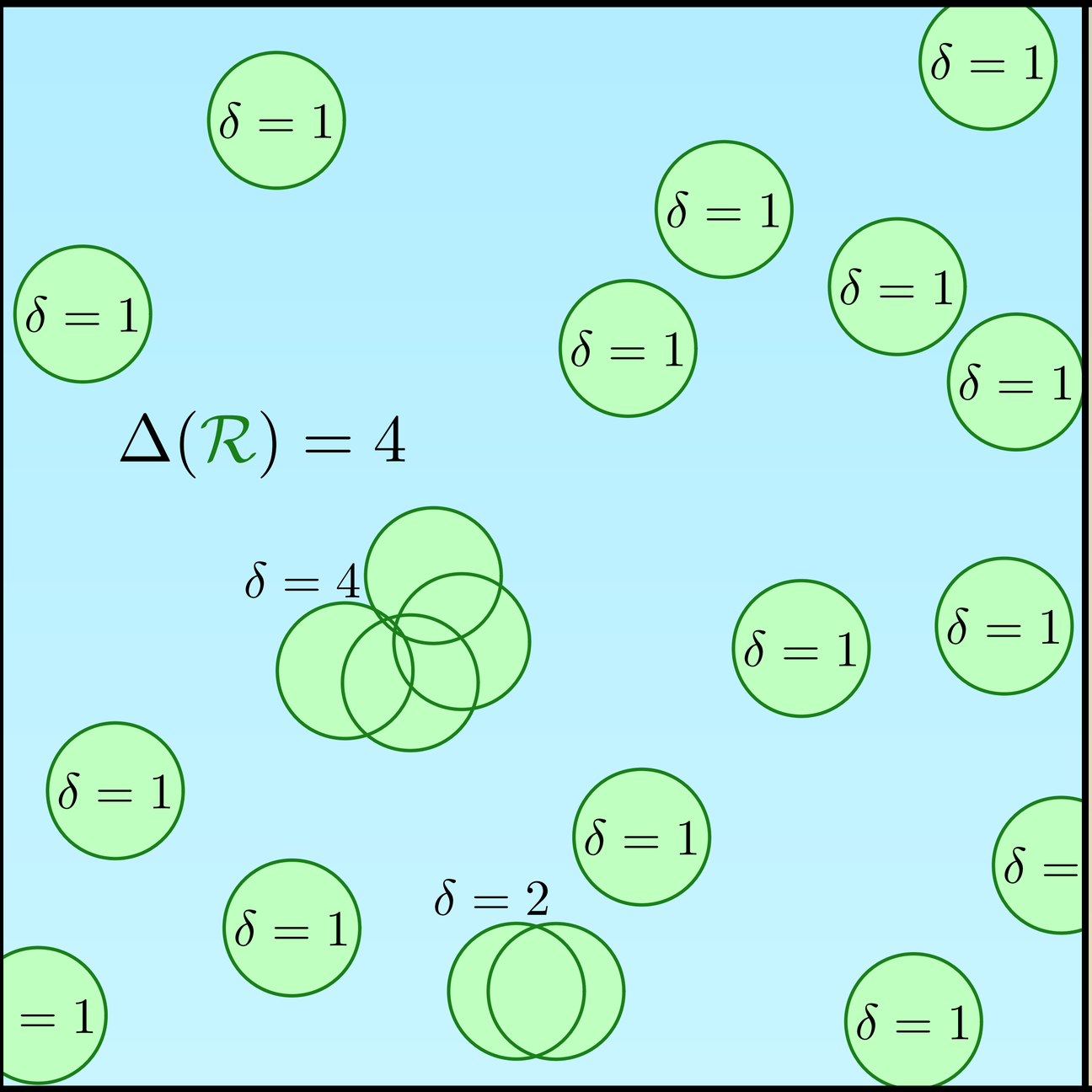


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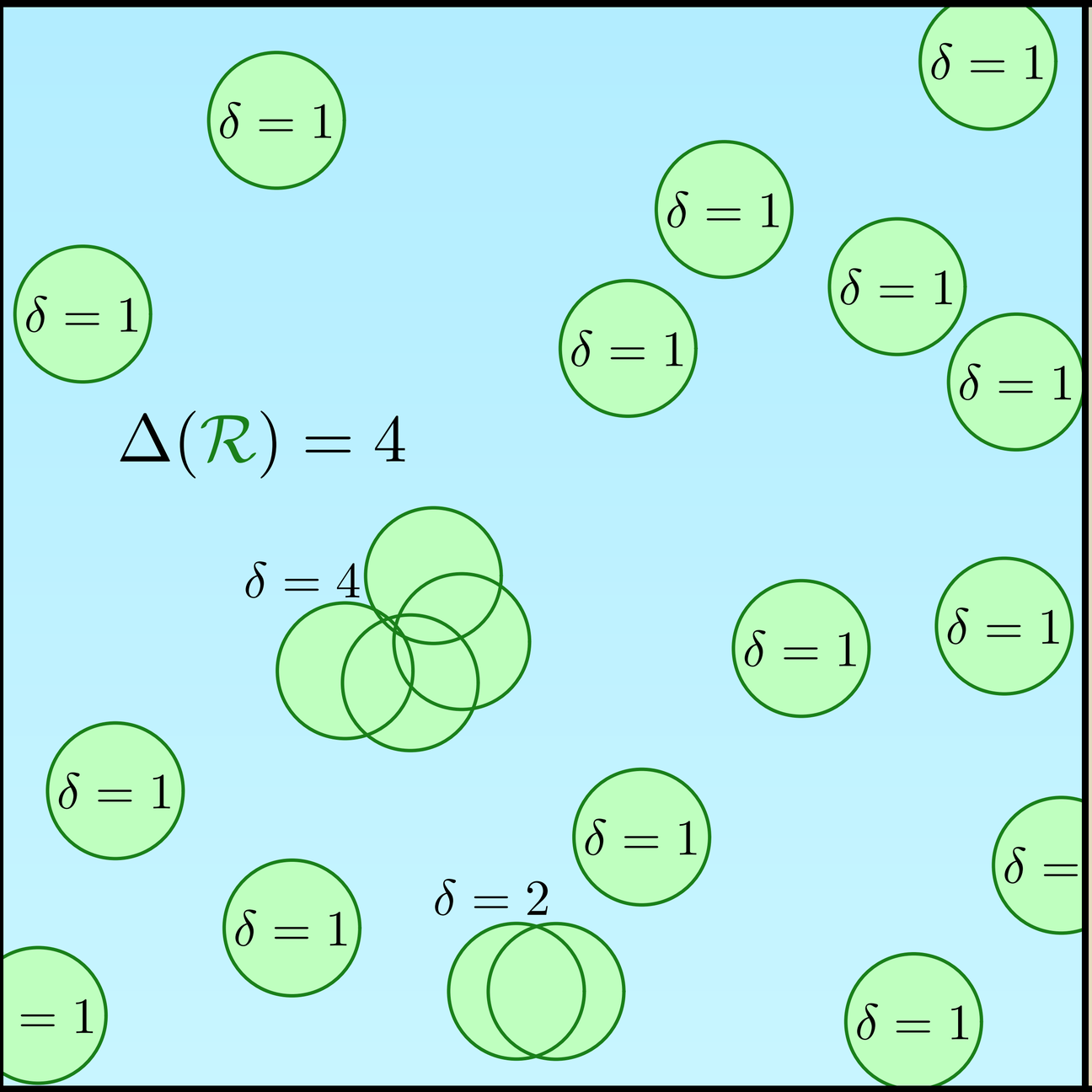


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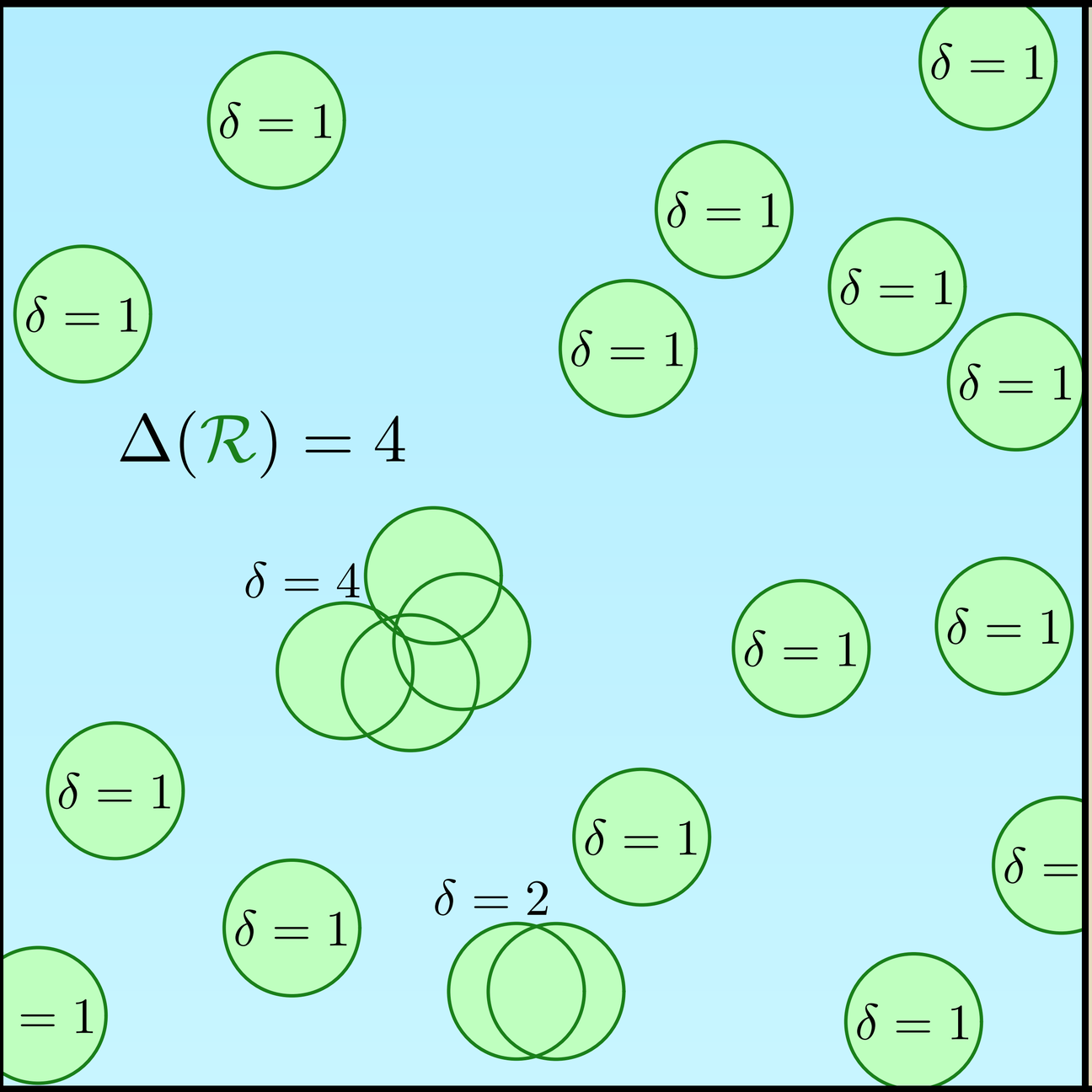
We want reconstruction time to only "locally" depend on $\delta(R)$.




The ply is a *global* measure of overlap.

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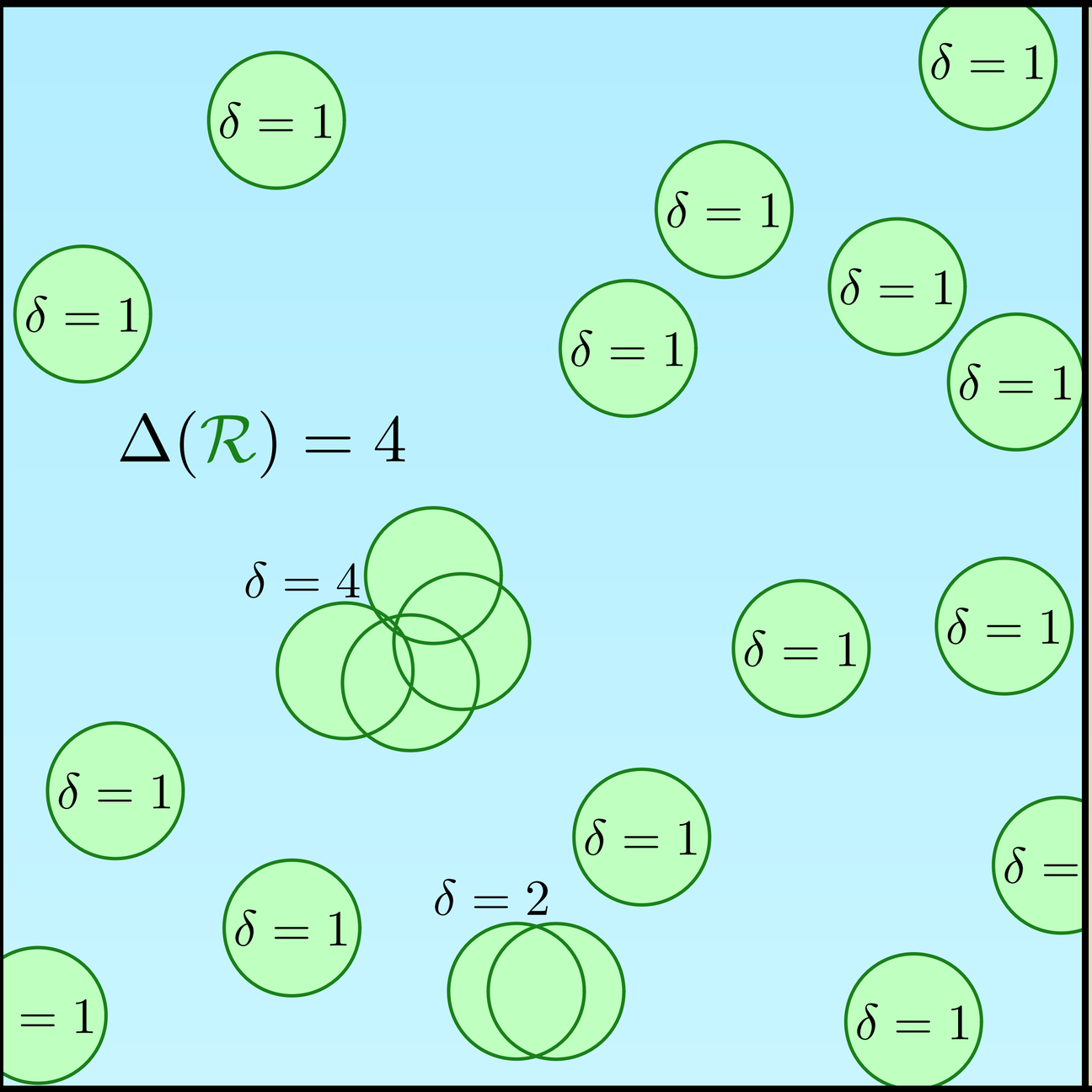


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Is $O(A(\mathcal{R}))$ time possible?

# Chapter 3

*FINE-GRAINED MEASURES*

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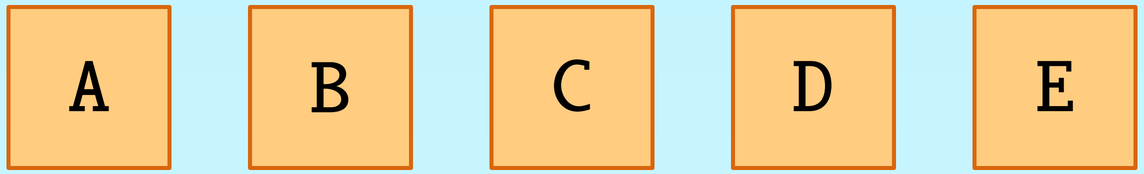
Consider all $N$ possible outputs of a computational problem.

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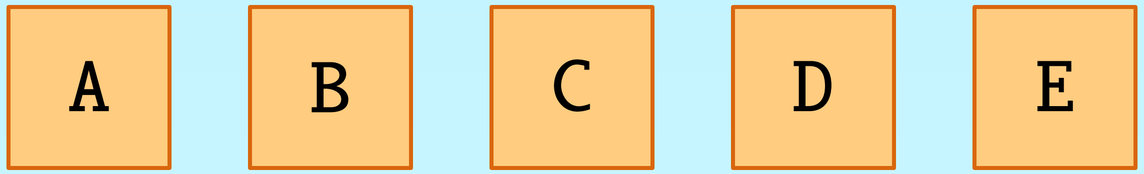


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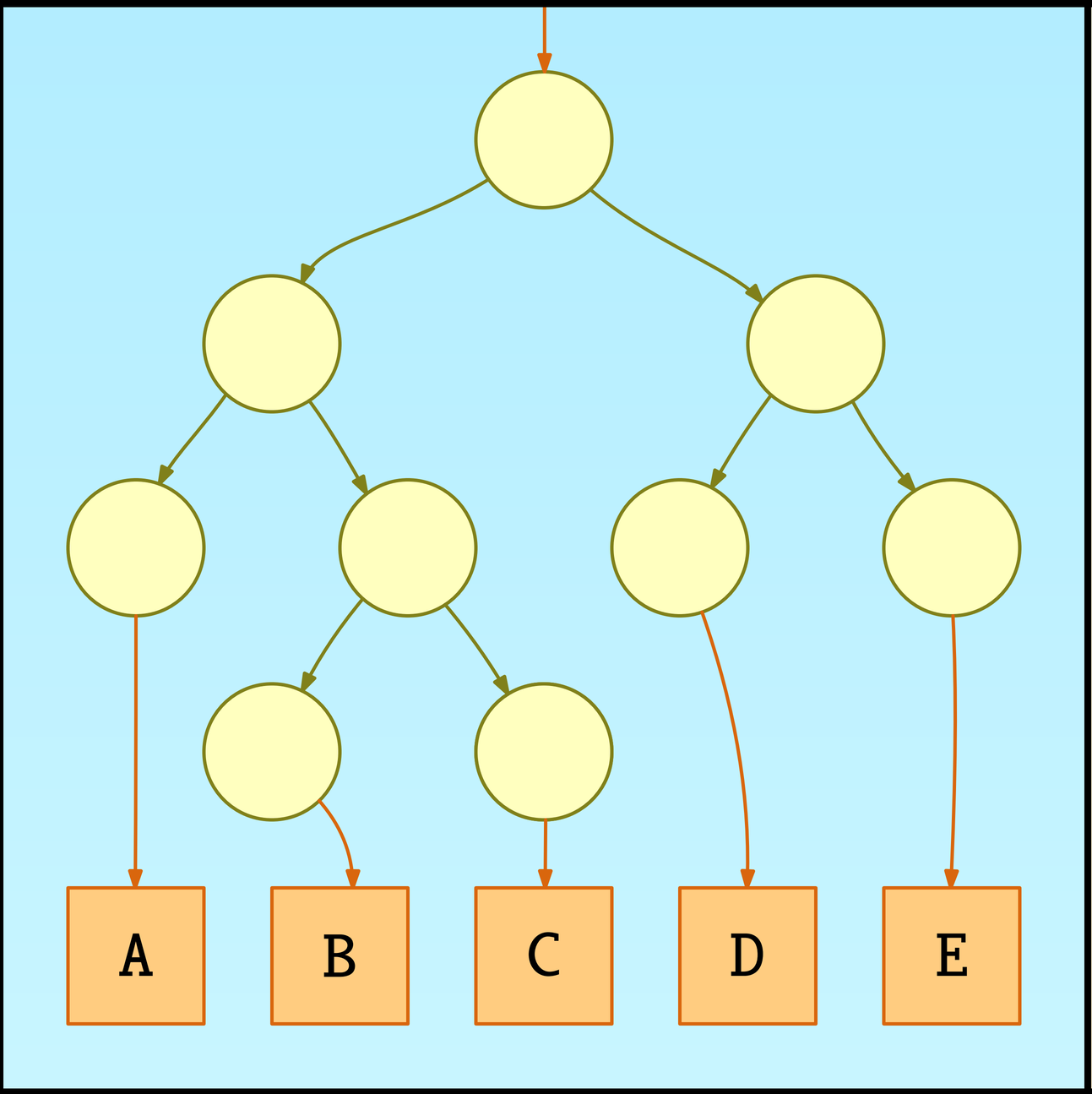
Now consider the decisions an algorithm (on a pointer machine) must make to reach these outputs.

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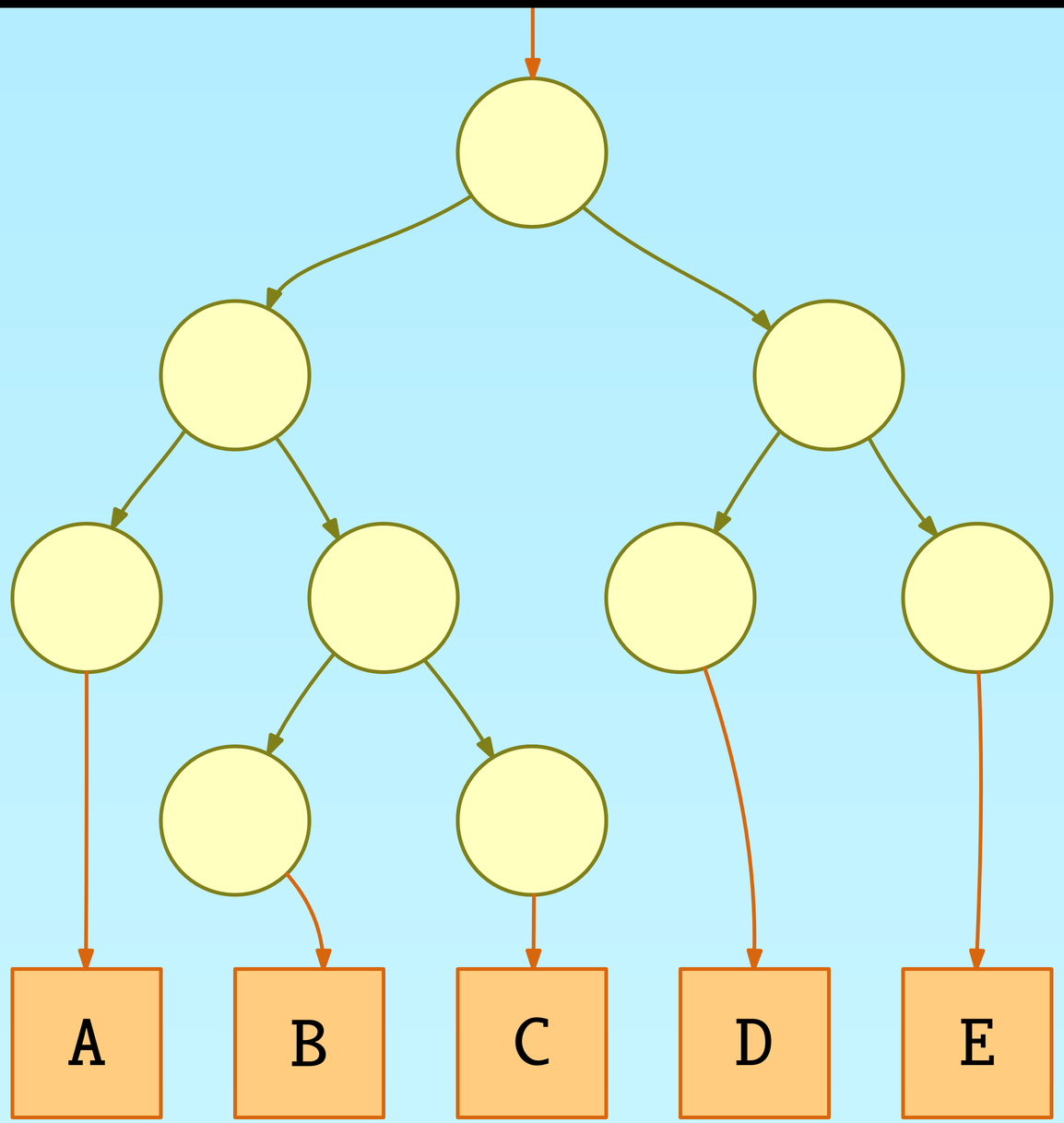


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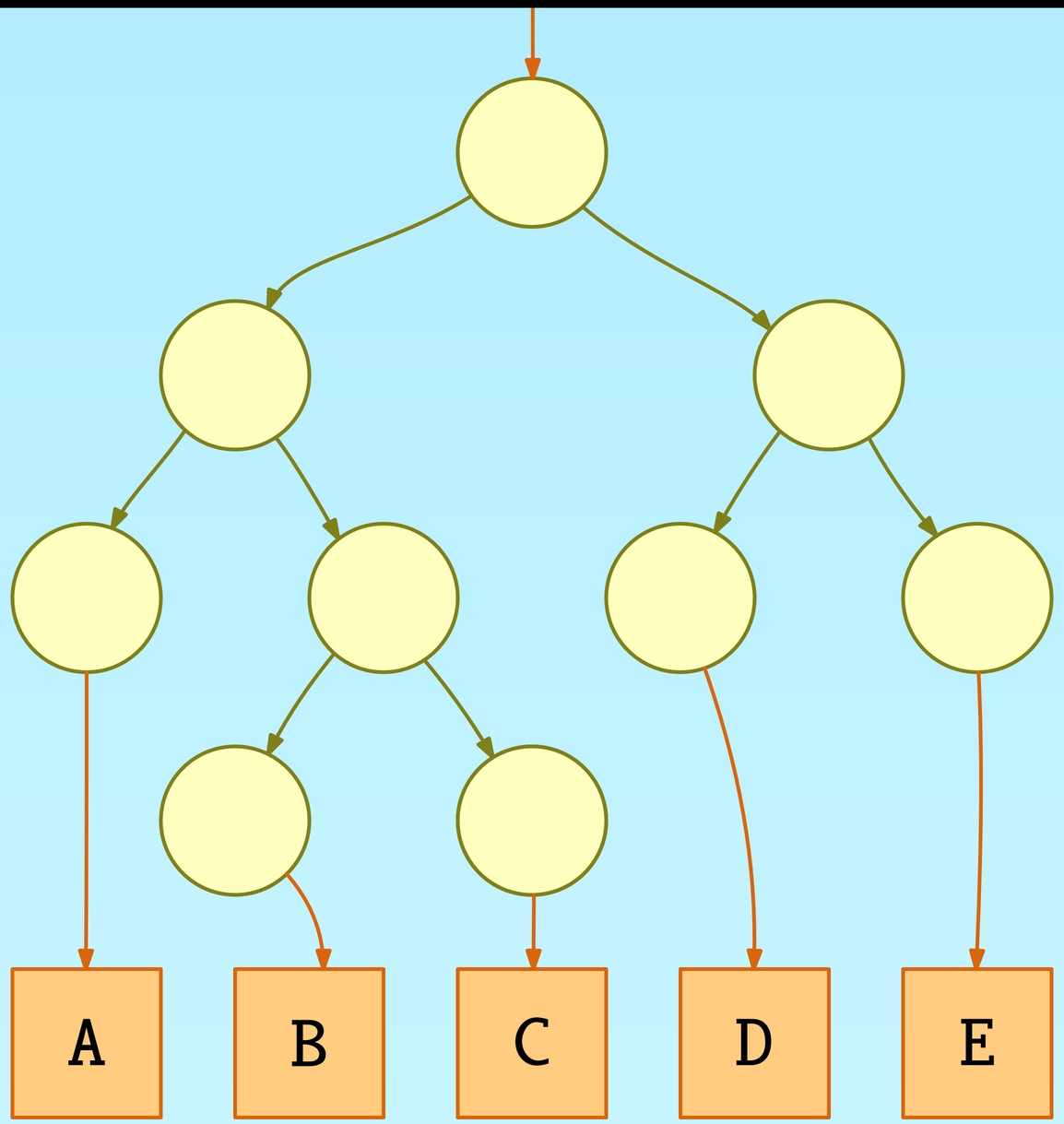
The *height* of this tree is at least $\log N$.

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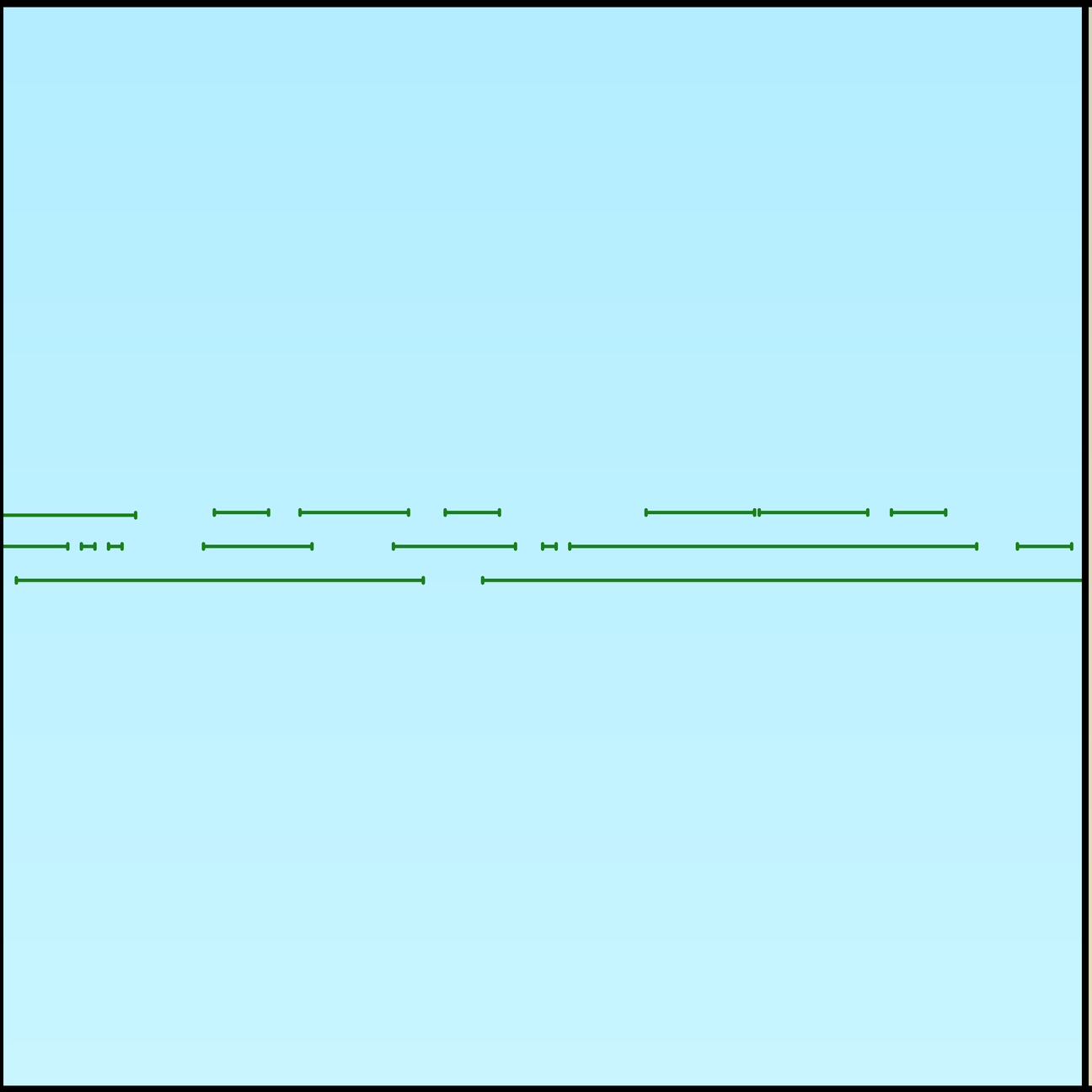
The *height* of this tree is at least $\log N$.

So, the worst-case running time is at least $\log N$.

Now, consider a set of overlapping regions in $\mathbb{R}^1$.



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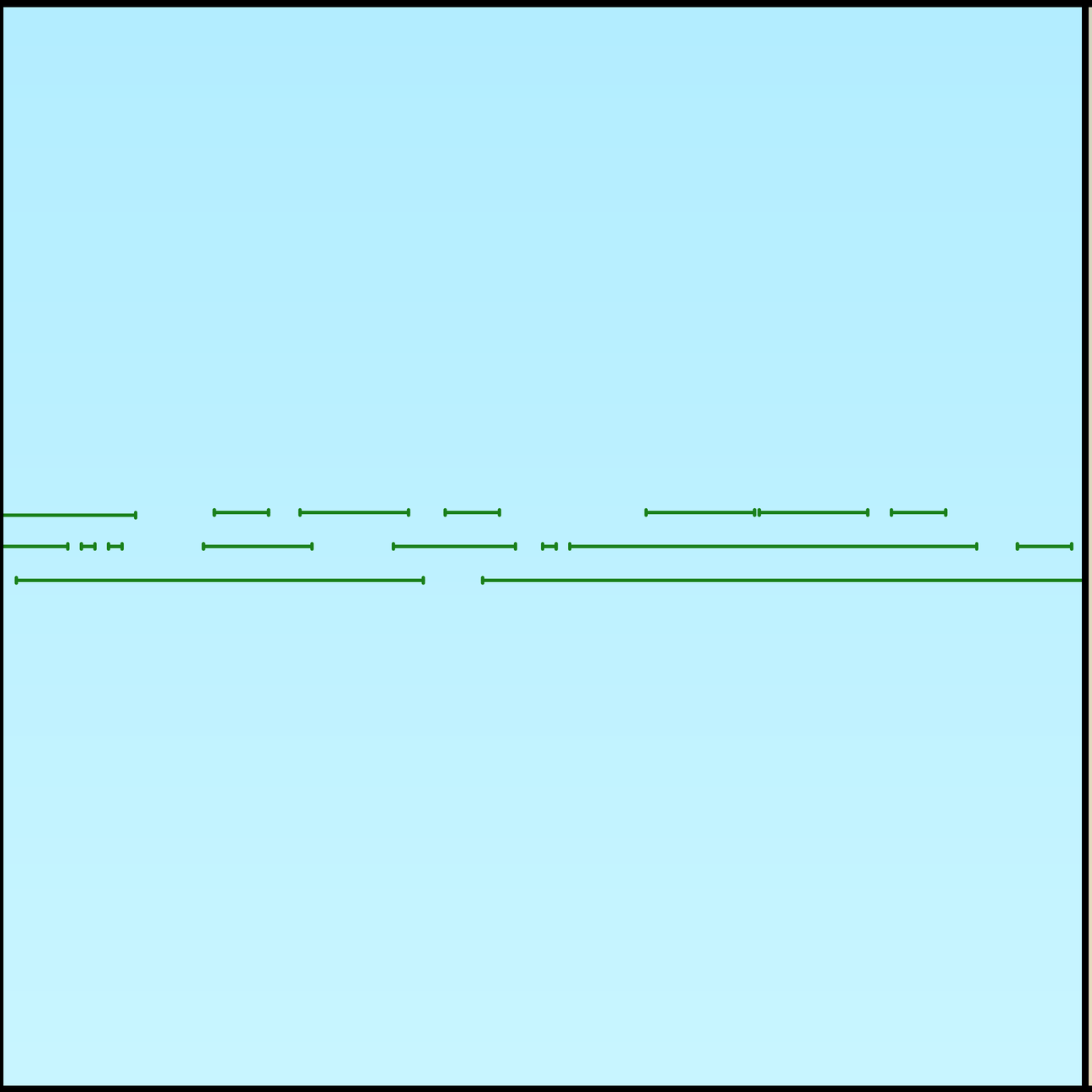

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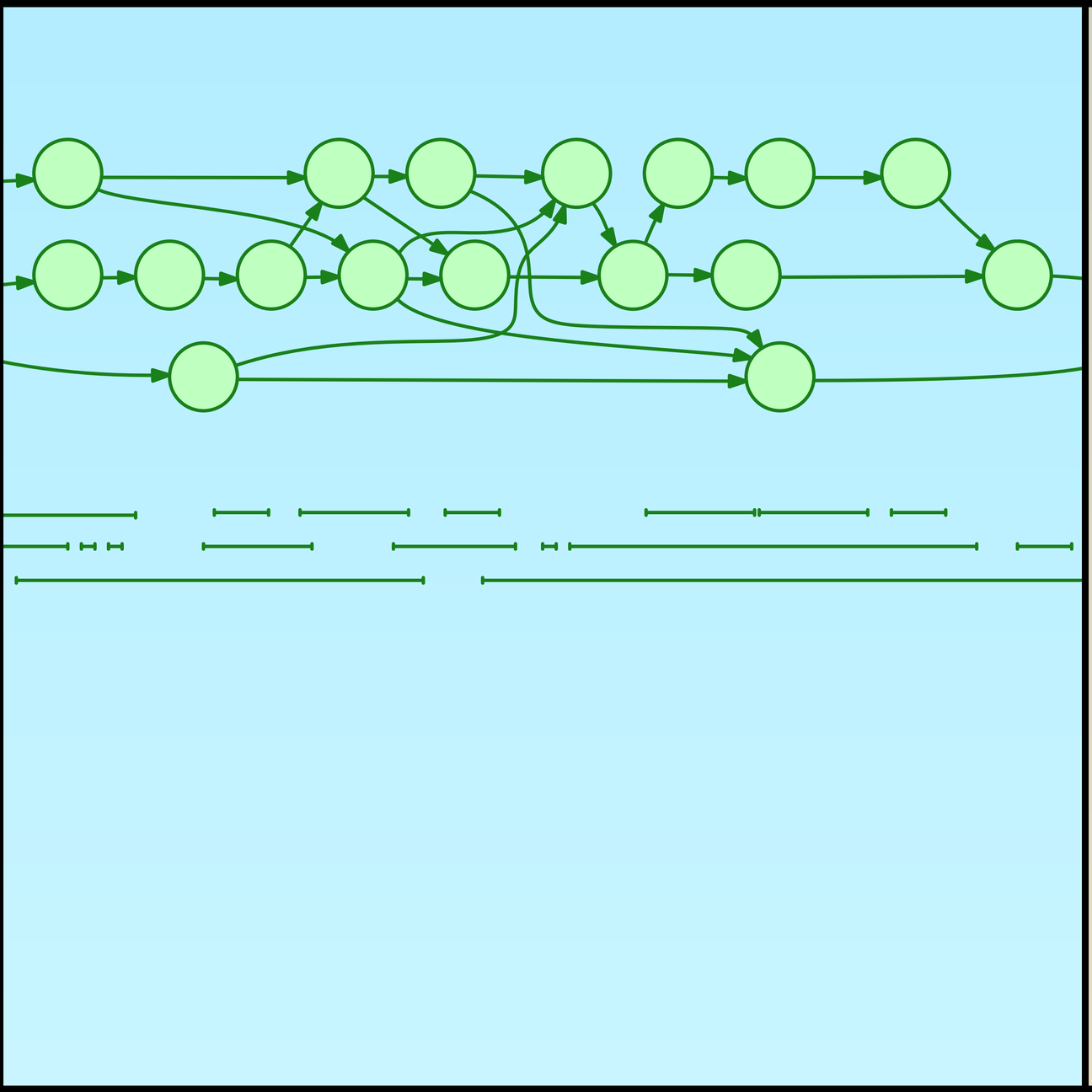
These intervals induce a *partial order*.



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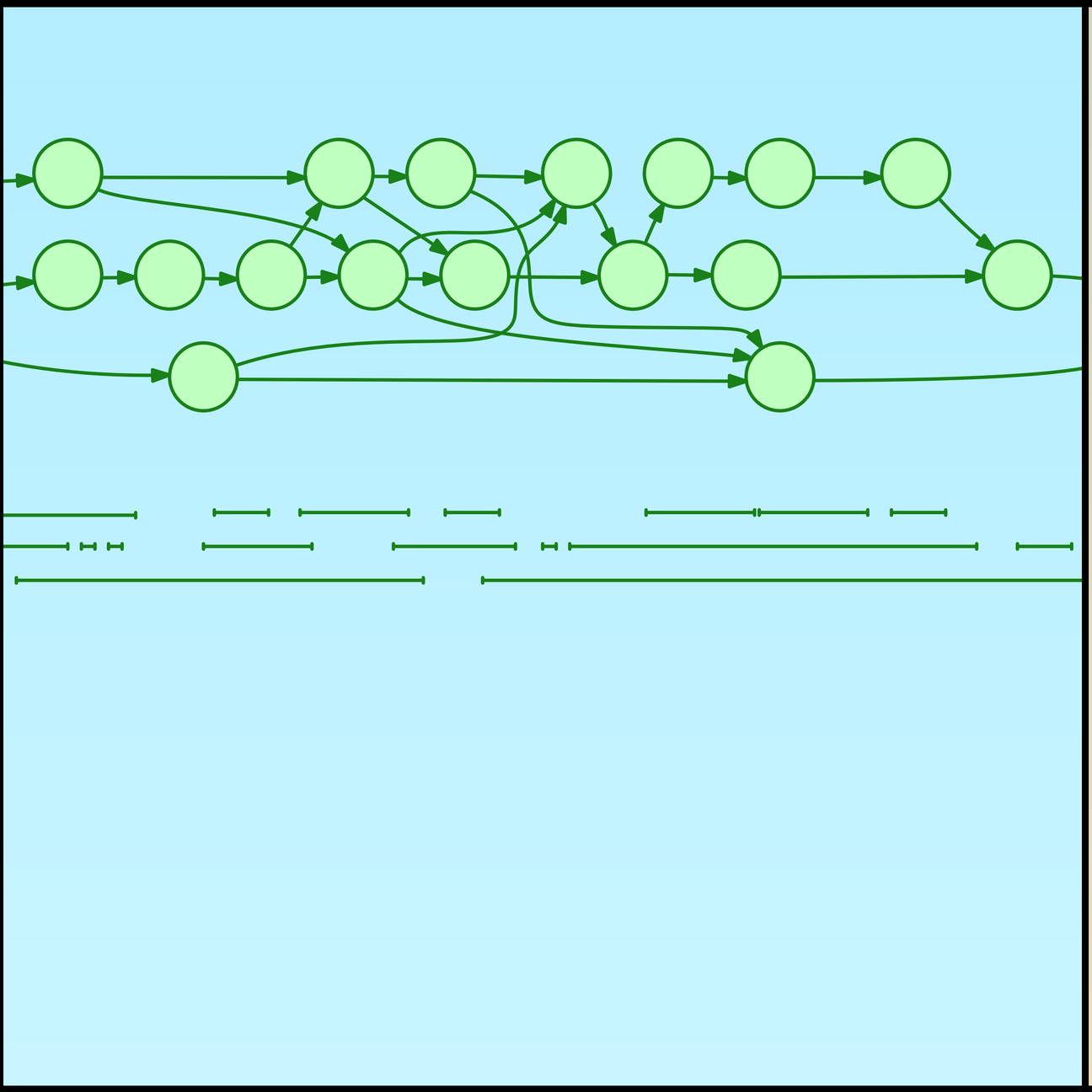

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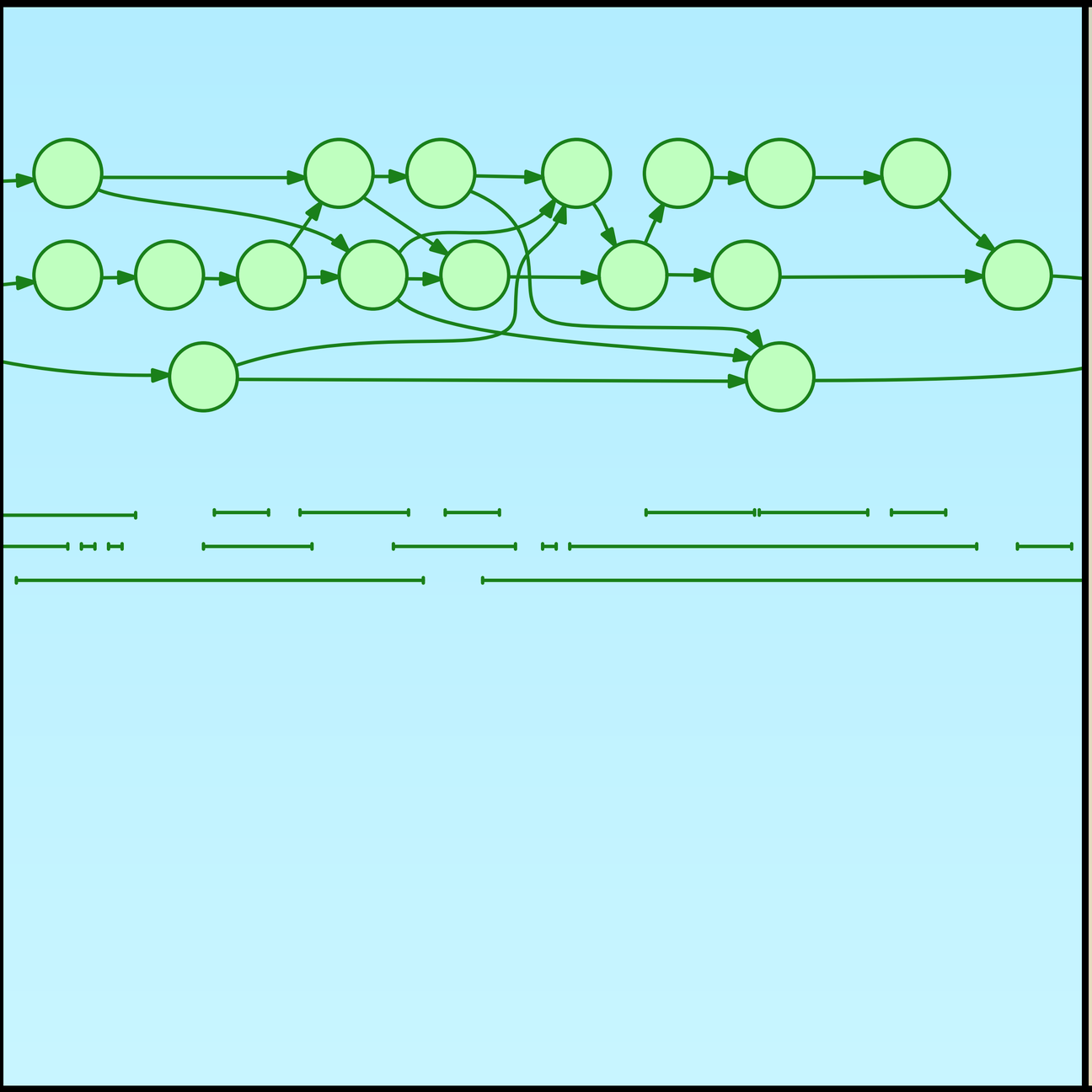

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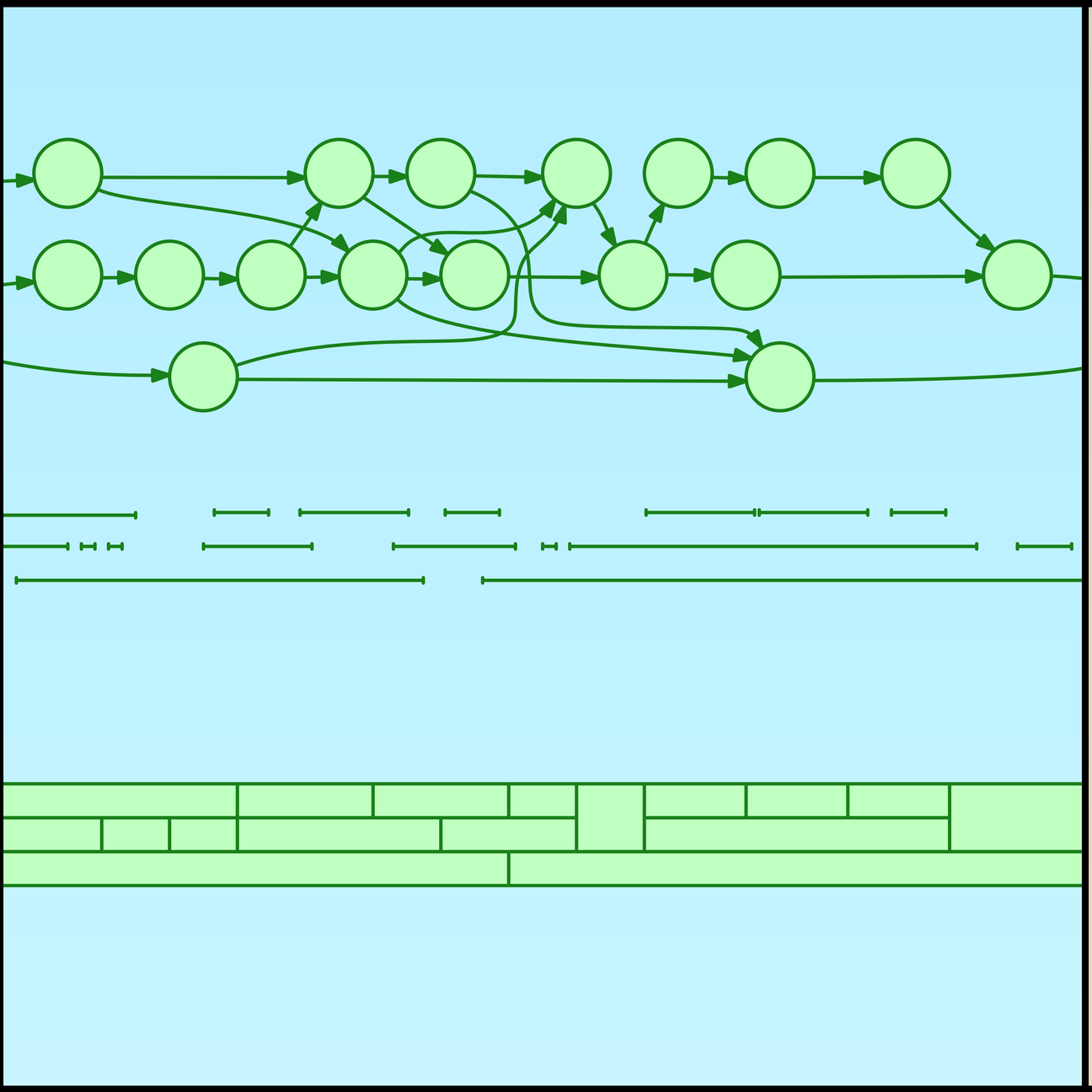

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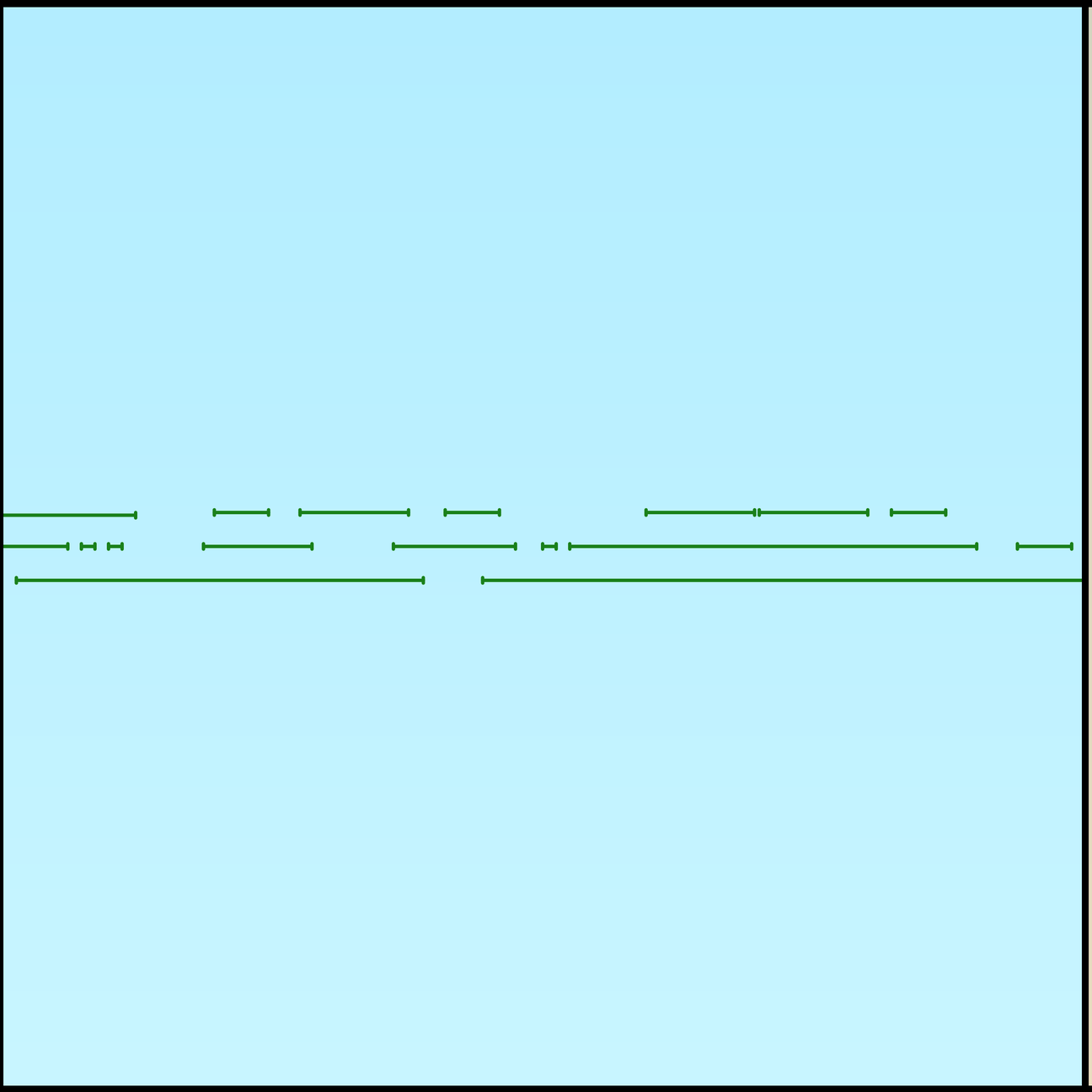

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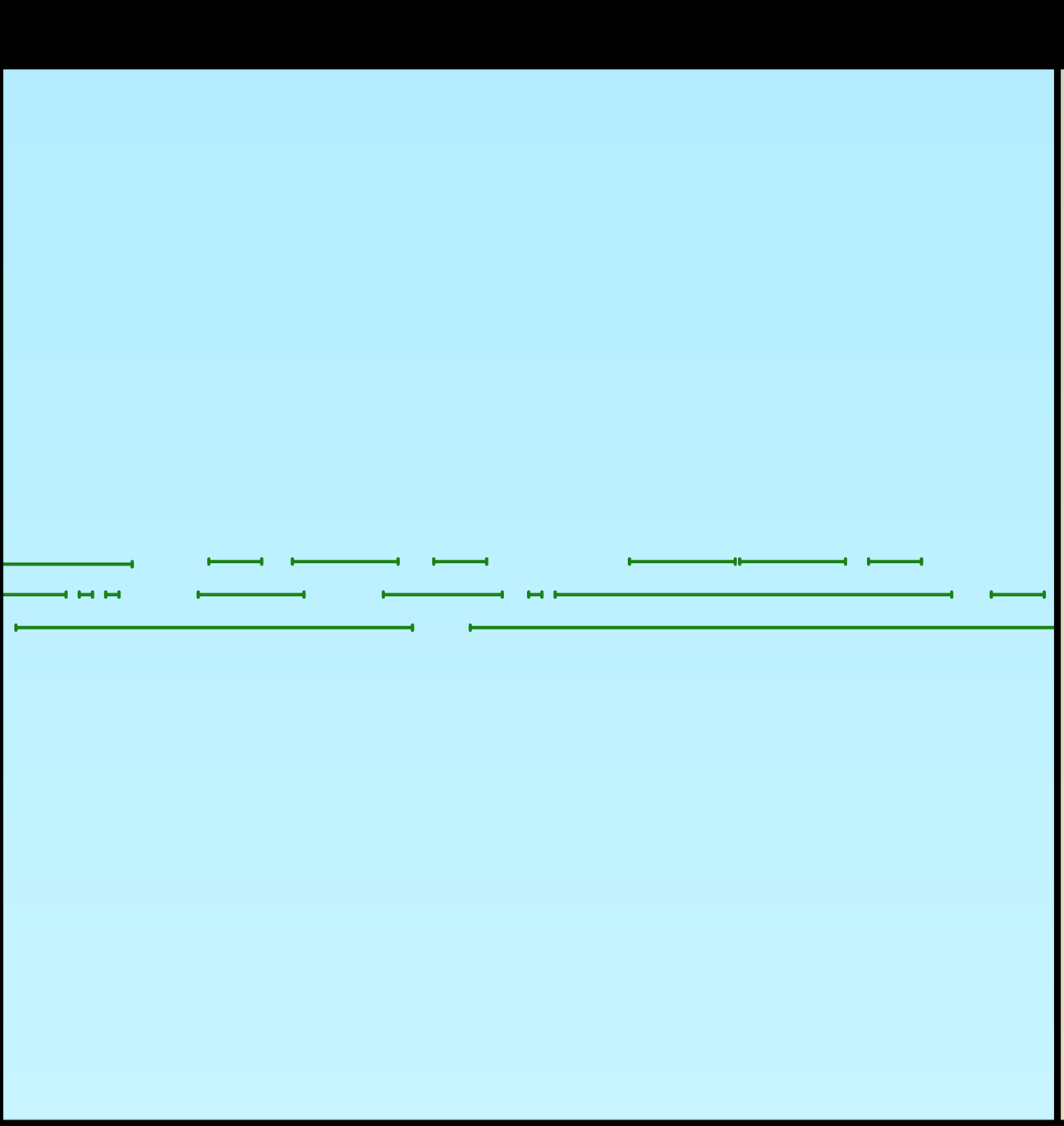

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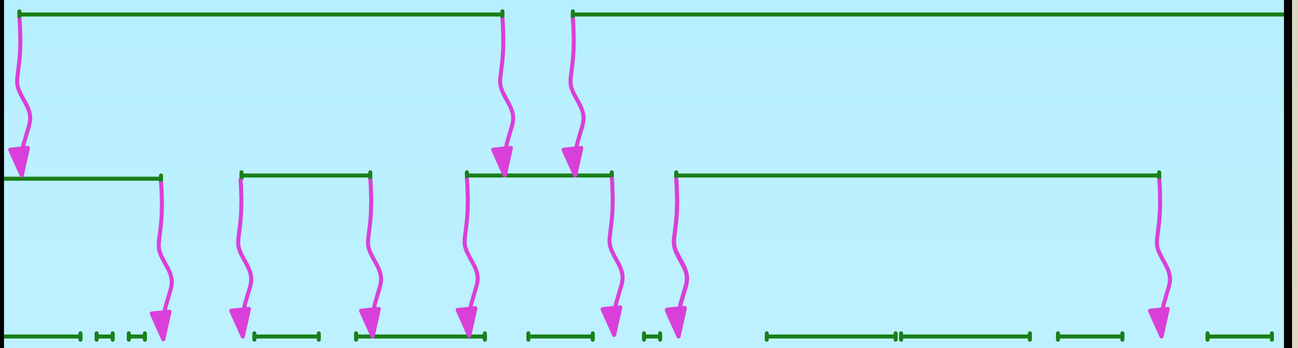

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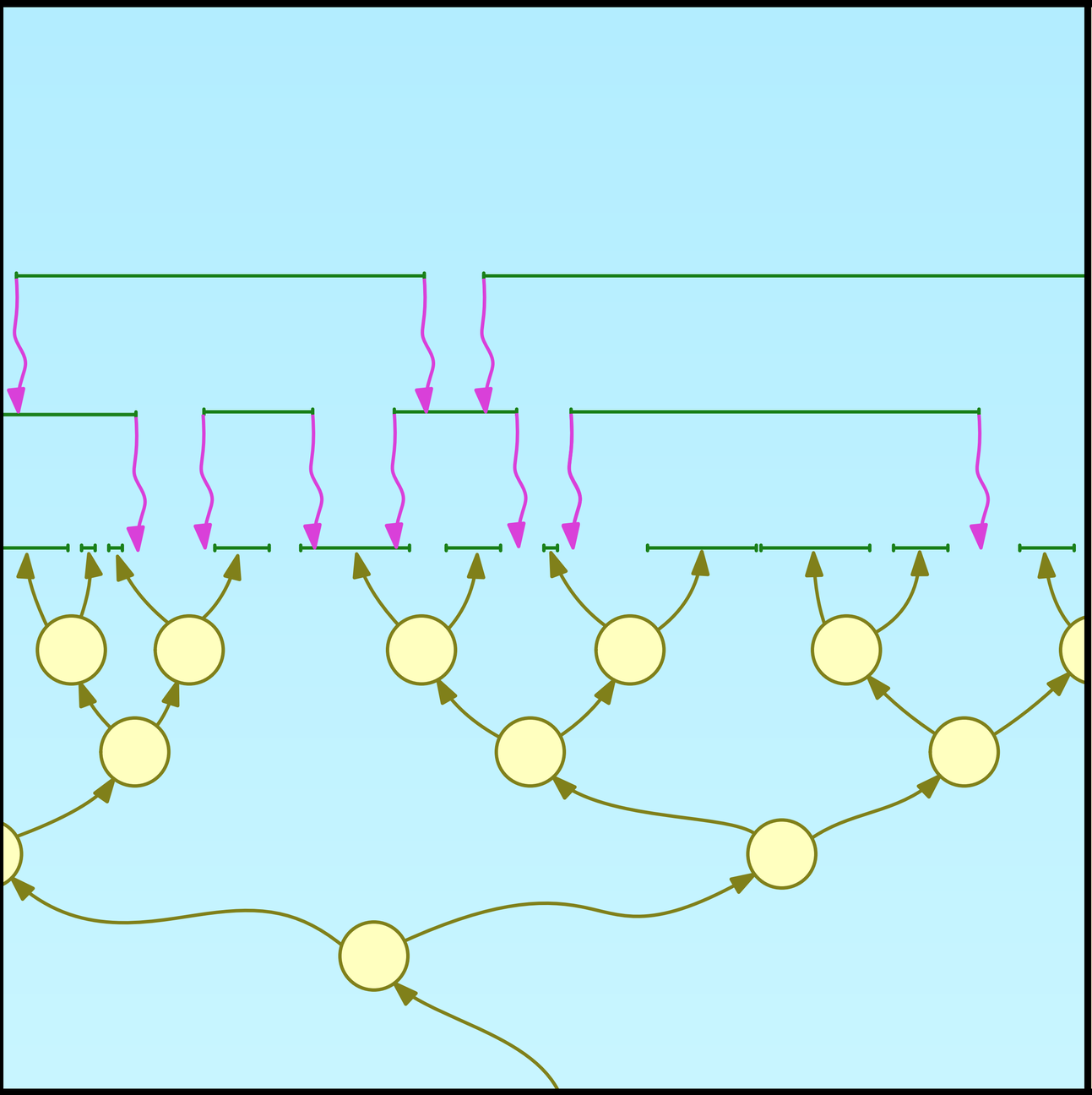

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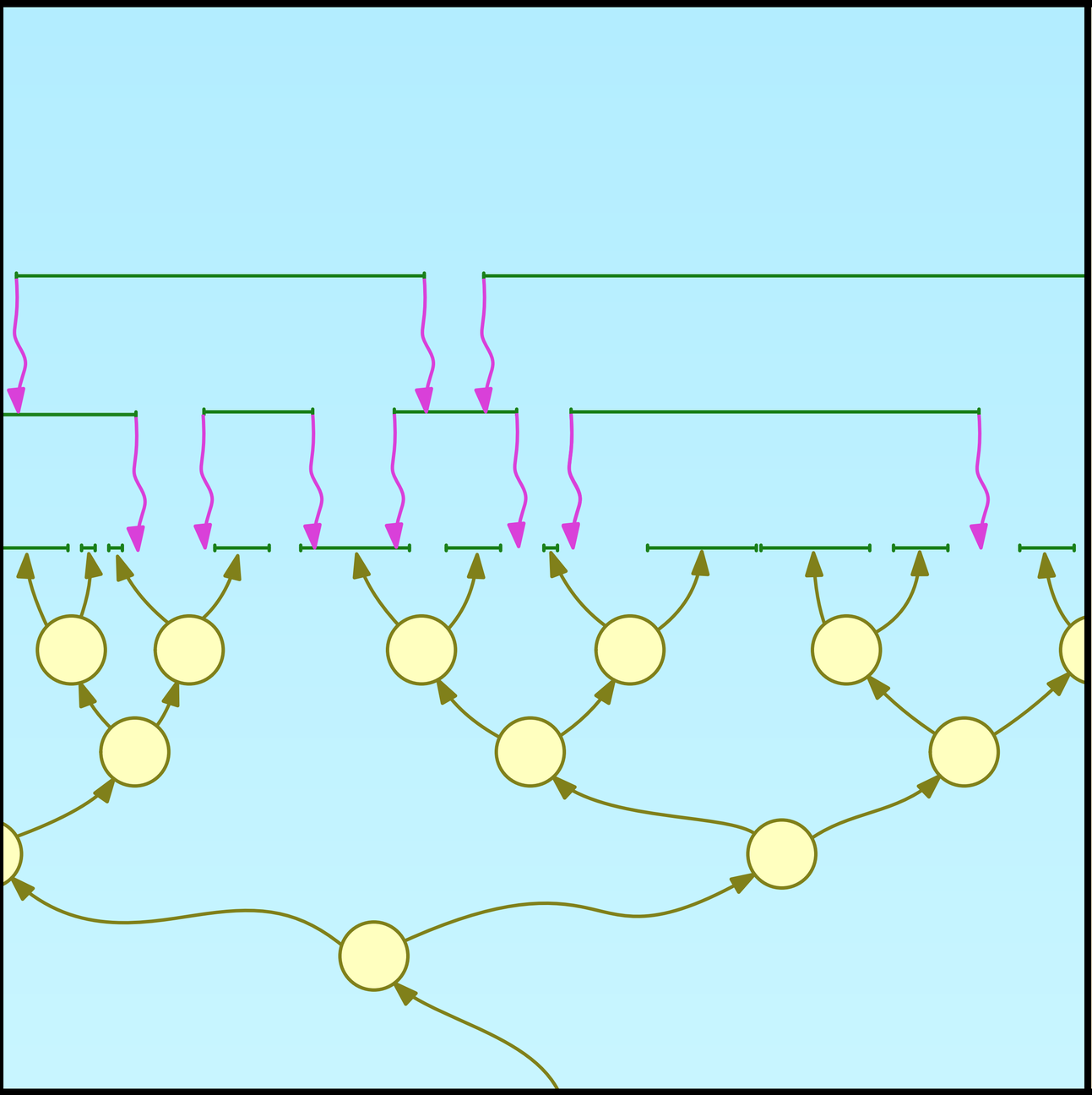

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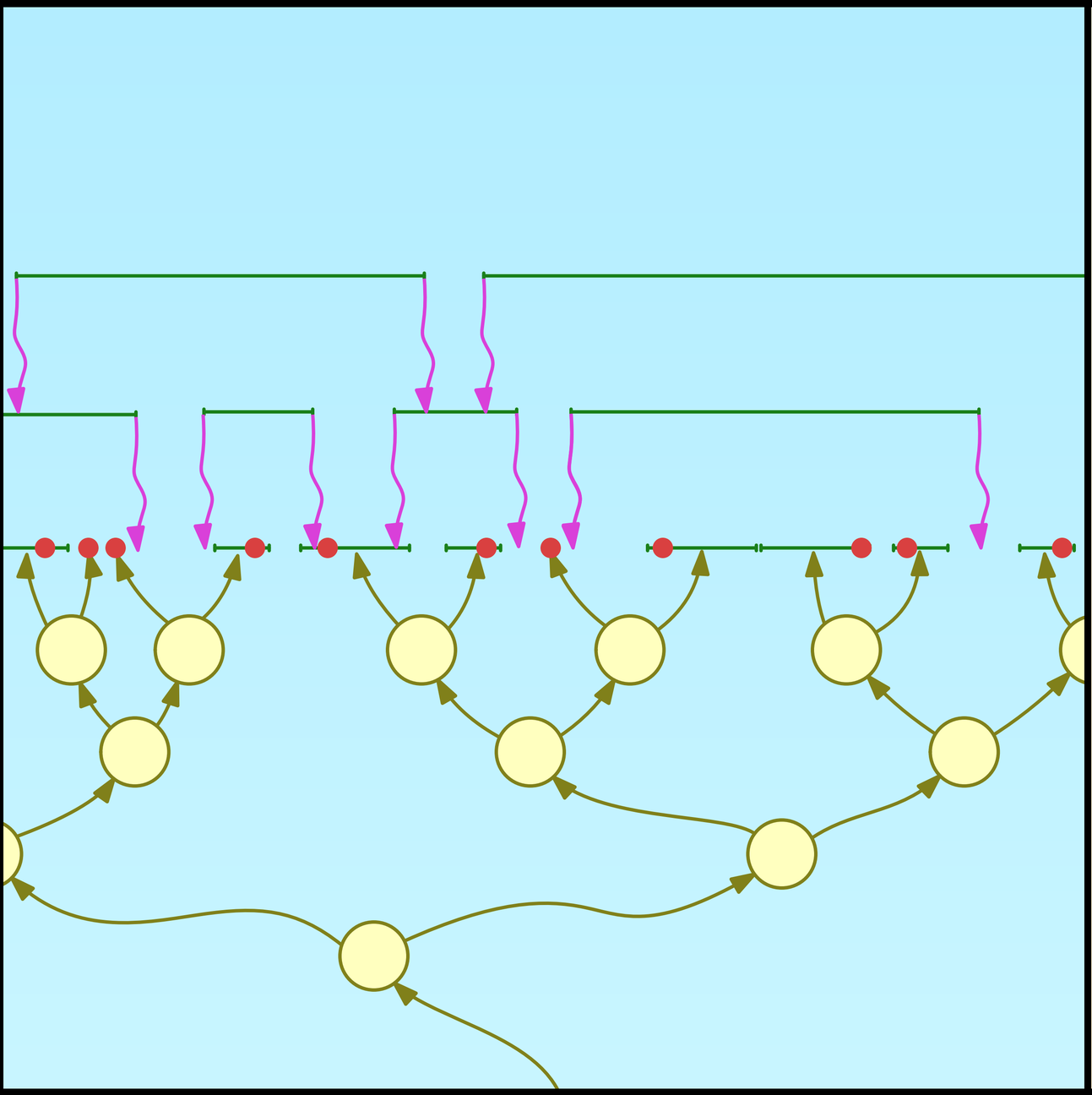
[van der Hoog, Kostitsyna, Löffler & Speckmann, 2019]



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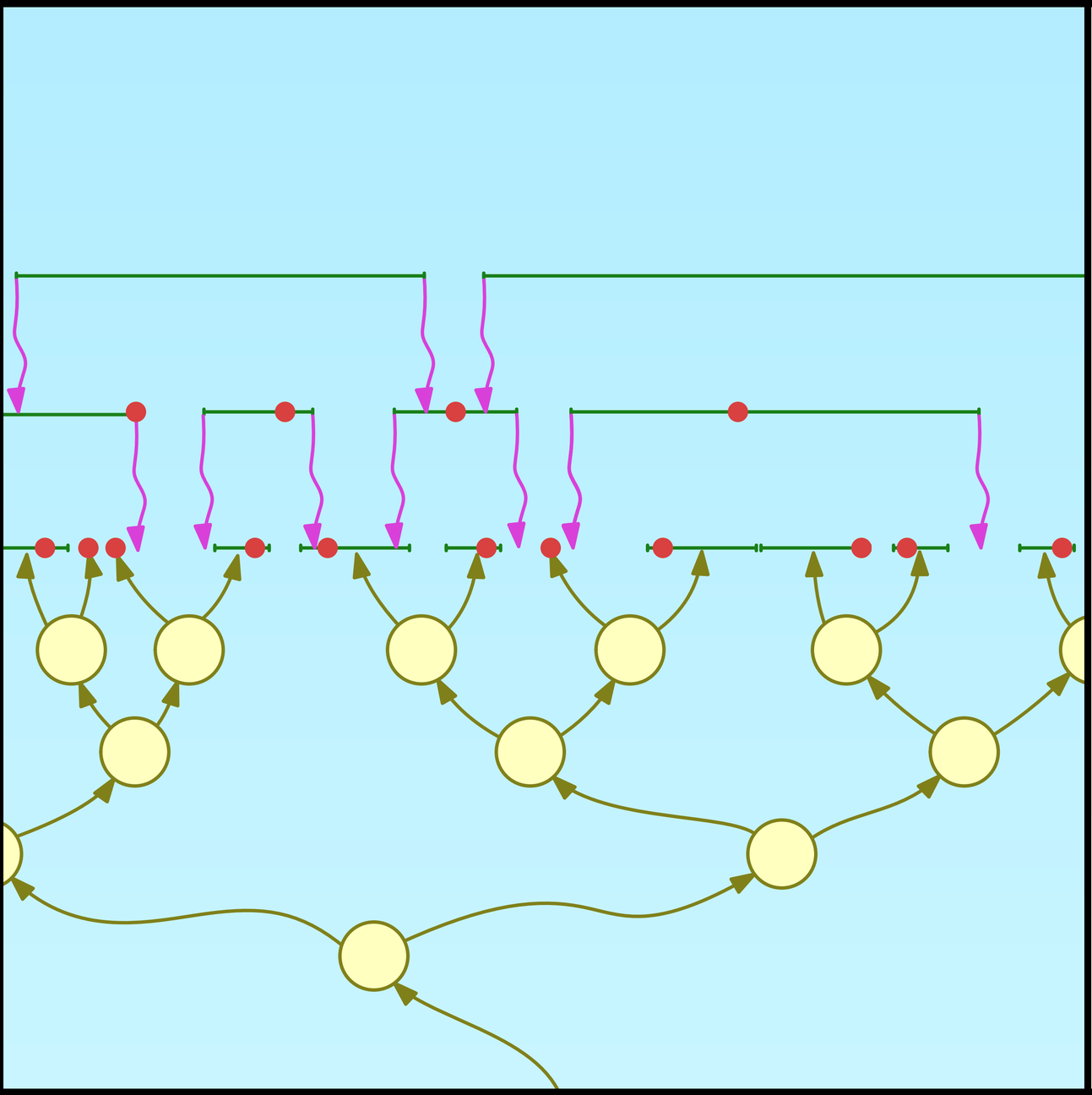

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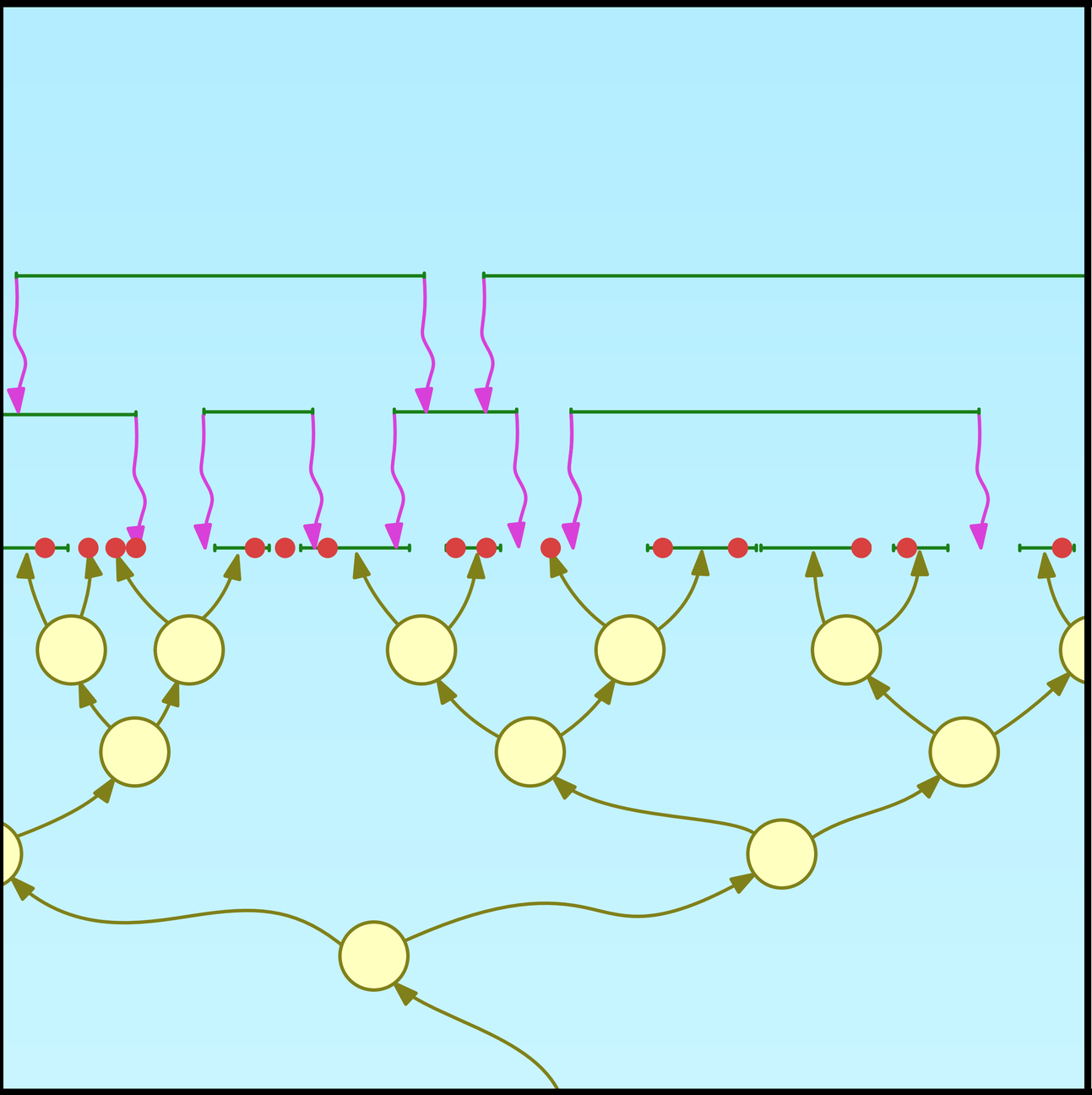

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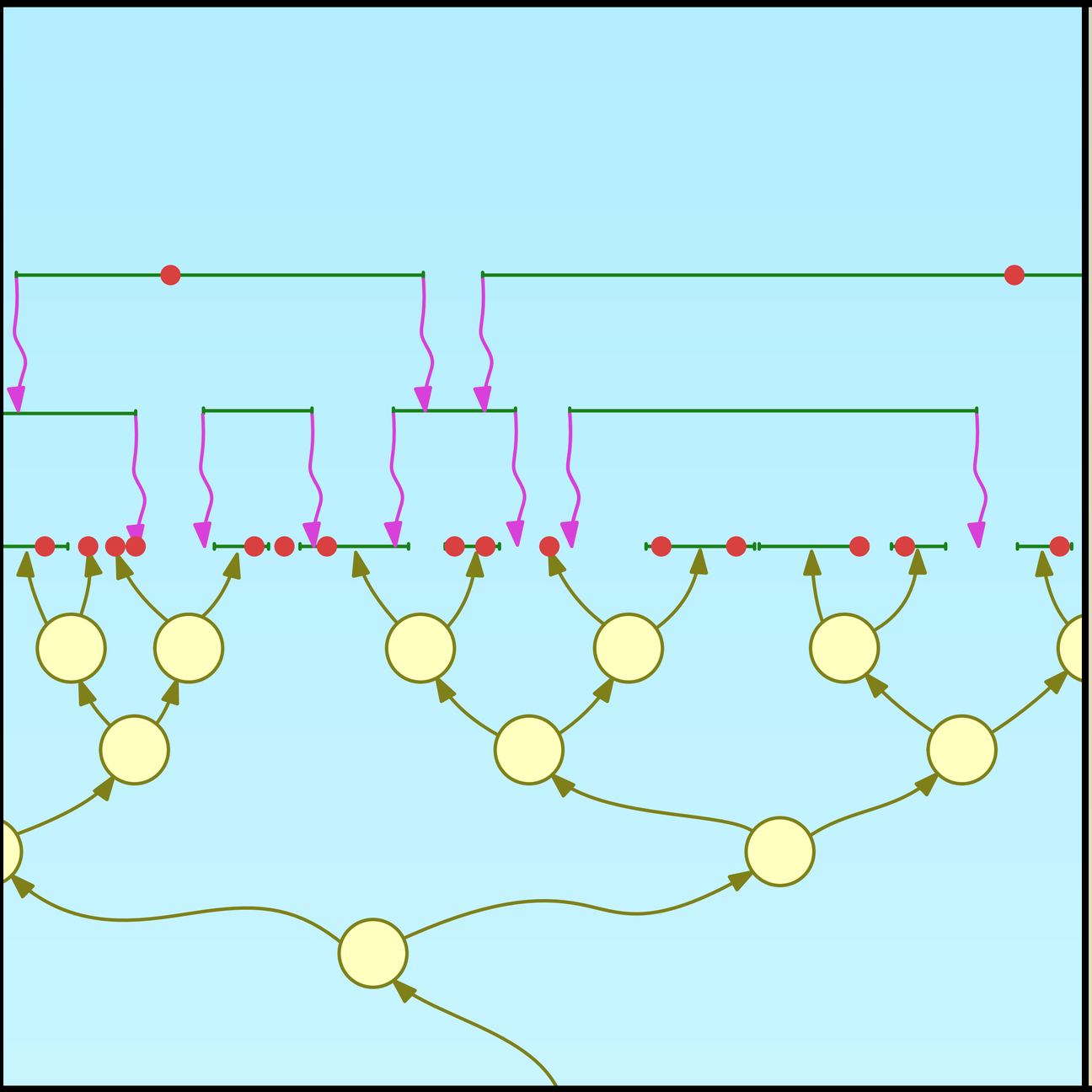

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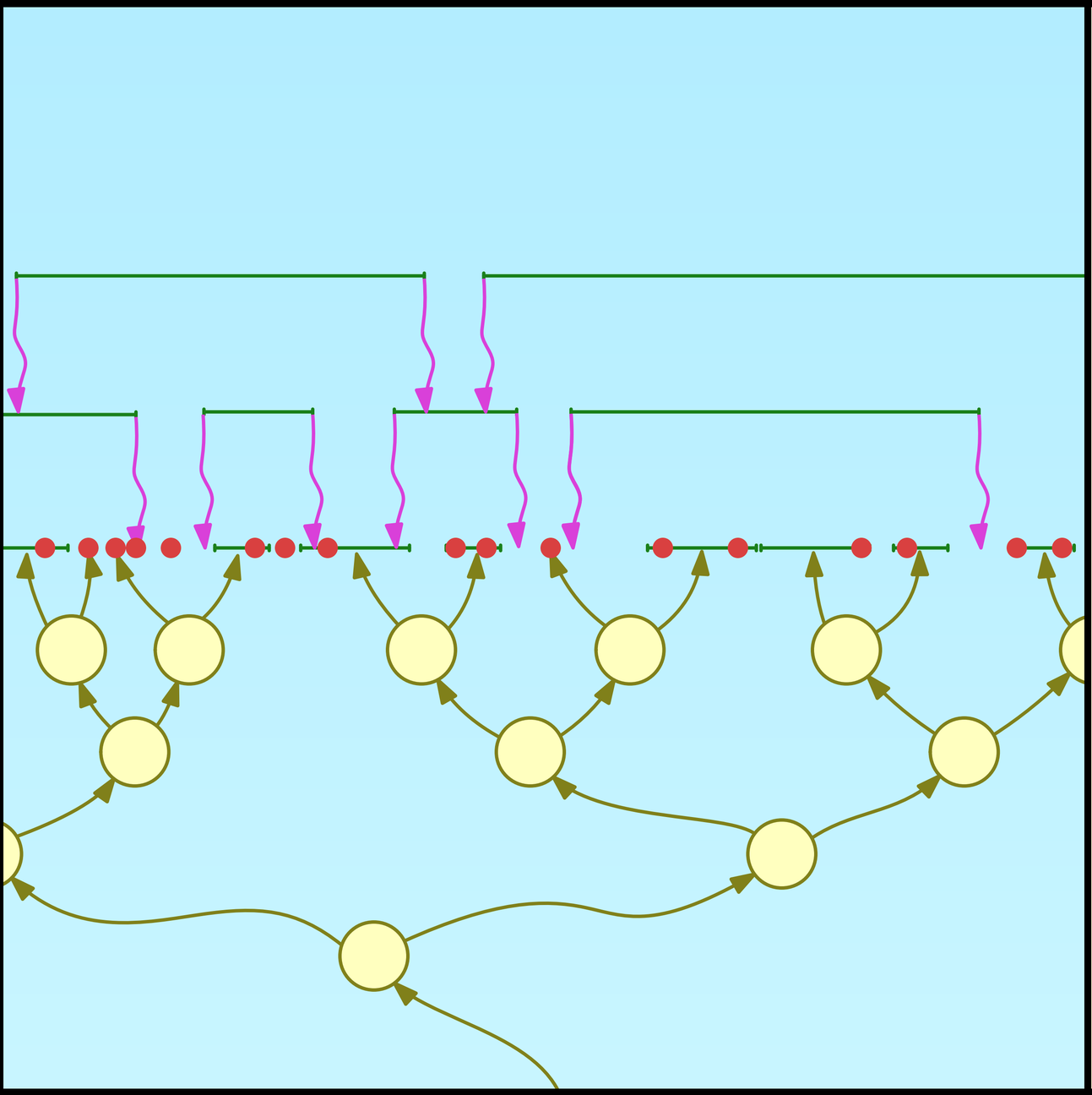

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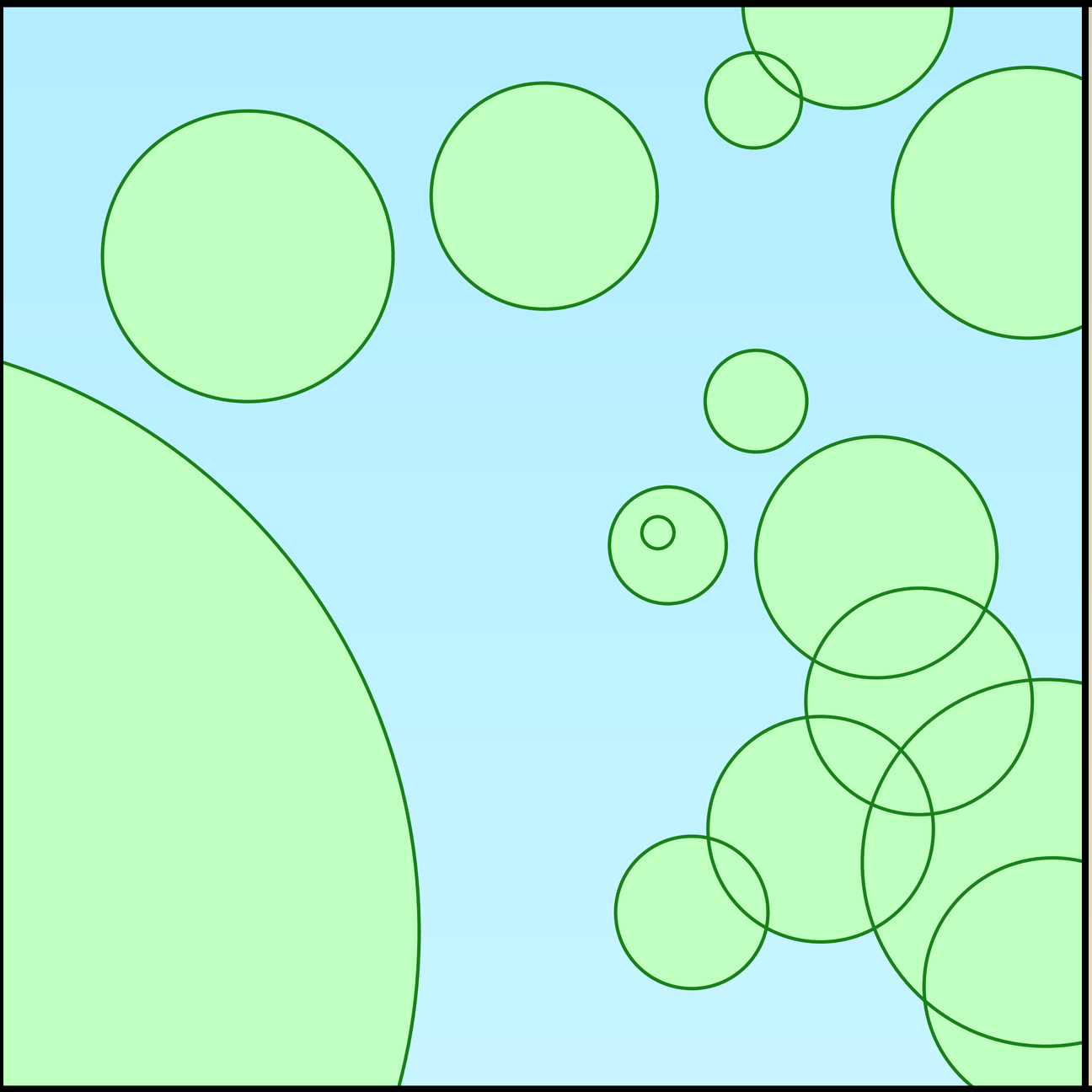

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# Thank You!

## Location & Information

Maarten Löffler
*Utrecht University*

*Based on joint work with*

Kevin Buchin
Ivor van der Hoog
Irina Kostitsyna
Marc van Kreveld
Joe Mitchell
Pat Morin
Wolfgang Mulzer
Jack Snoeyink
Bettina Speckmann