Certified Approximation Algorithms for the Fermat Point

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18/03/2020 - Würzburg, Germany - EuroCG 2020

Fermat point

- **Foci:** set of points $A = \{\boldsymbol{a}_1, \dots, \boldsymbol{a}_k\} \subseteq \mathbb{R}^d$
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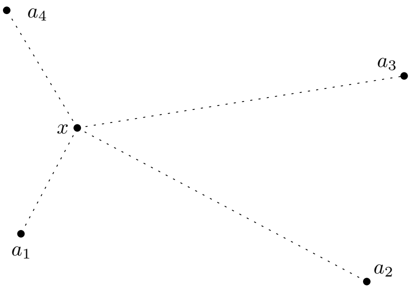
$$\varphi(\boldsymbol{x}) := \sum_{\boldsymbol{a} \in A} w(\boldsymbol{a}) \|\boldsymbol{x} - \boldsymbol{a}\|$$



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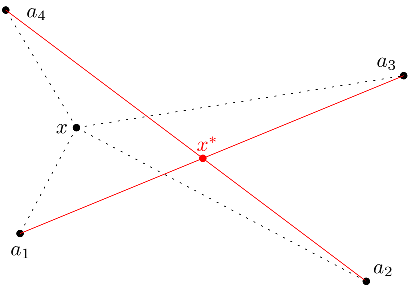
- **Fermat point** x^* minimizes φ



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- The foci are not collinear. (⇒ The Fermat point is unique.) [Mello and dos Santos, 2018, Morrison, 2010]
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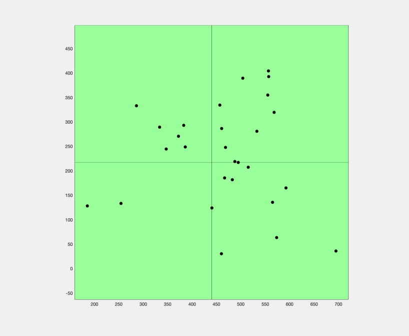
Other notions in literature:

- $\varphi(\widetilde{\boldsymbol{x}}^*) \leq \varphi(\boldsymbol{x}^*) + \varepsilon$
- $\varphi(\widetilde{\boldsymbol{x}}^*) \leq (1+\varepsilon)\varphi(\boldsymbol{x}^*)$

Our contribution

- Designed 2 approximation algorithms for the Fermat point
  - based on box subdivision



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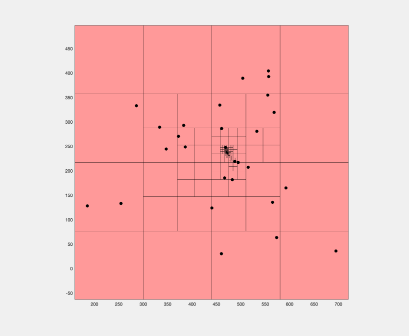

  - augmented the Weiszfeld point sequence with a guarantee
- Experimentally evaluated our algorithms

Box subdivision idea

Algorithm outline:

- Start with an axis-aligned box, which contains the Fermat point
- Depending on some tests on a box either
  - split the box into 2^d many subboxes or
  - discard the box
- Stop, if all the non-discarded boxes fit into a ball of radius ε

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We want predicates, that can decide if the Fermat point is contained in a box. However, the Fermat point is the solution of a polynomial of exponentially high degree in the number of foci.

Tests for predicates

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We conclude nothing if $T(B) = \texttt{false}$.

Soft predicates

Definition

Let T be a test for a predicate P. We call T a **soft predicate** of P if it is convergent in this sense: if $(B_n)_{n\in\mathbb{N}}$ is a monotone sequence of boxes $B_{n+1} \subseteq B_n$ that converges to a point $\boldsymbol{a}$, then $P(\boldsymbol{a}) = T(B_n)$ for n large enough.

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Definition

Call g a **soft version** of function f if it is

(i) **conservative**, i.e. for all $B \in \square\mathbb{R}^d$, $g(B)$ contains $f(B) := \{f(p) : p \in B\}$, and

(ii) **convergent**, i.e. if for monotone sequence $(B_n)_{n\in\mathbb{N}}$ that converges to a point $\boldsymbol{a}$, the width of $g(B_n)$ converges to 0.

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Soft versions can be derived from analytic functions by evaluating them with interval arithmetic.

Tests

Gradient exclusion test $C^{\nabla}(B)$:
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$\Rightarrow$ If $C^{\varepsilon}(Q)$ returns true, then we have found an ε-approximation of the Fermat point.

Newton test

The Fermat point $\boldsymbol{x}^*$ is the root of $\nabla\varphi$.

This interval Newton operator $\square N(B) = m_B - \square J^{-1}_{\nabla\varphi}(B) \cdot \nabla\varphi(m_B)$ has the property:

$$\square N(B) \subset B \quad \Rightarrow \quad \boldsymbol{x}^* \in \square N(B)$$

[Brouwer, 1911, Nickel, 1971]

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Newton inclusion test $C^N(B)$:
Given a box B, it returns true if and only if $\square N(2B) \subset 2B$.

$\Rightarrow$ If $C^N(B)$ returns true, then the Fermat point is in $\square N(2B)$.

Box subdivision algorithm

Algorithm 1: Enhanced subdivision for Fermat Point (*E-SUB*)

Input : Foci set A, $\varepsilon > 0$. **Output:** Point $\boldsymbol{p}$.

$B_0 \leftarrow \textsc{Initial-Box}(A)$; $Q \leftarrow \textsc{Queue}()$; $Q.\textsc{push}(B_0)$;
while *not* $C^{\varepsilon}(Q)$ **do**
  $B \leftarrow Q.\textsc{pop}()$;
  if $C^{\nabla}(B)$ **then**
    if $C^{N}(B)$ **then**
      $Q \leftarrow \textsc{Queue}()$; $Q.\textsc{push}(\textsc{split}_2(\square N(2B))$;
    else $Q.\textsc{push}(\textsc{split}(B))$;
return $\boldsymbol{p} \leftarrow$ *Center of the bounding box of* Q;

Theorem

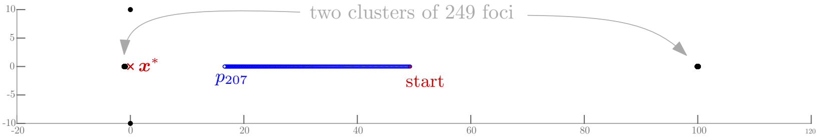
The subdivision algorithm returns an ε-approximation of the Fermat point.

Weiszfeld point sequence

The Weiszfeld point sequence $p_{n+1} = T(p_n)$ converges to the Fermat point. [Weiszfeld, 1937, Kuhn, 1973, Ostresh, 1978]

$$T(\boldsymbol{x}) = \begin{cases} \dfrac{\sum_{i=1}^{k} w(\boldsymbol{a}) \frac{\boldsymbol{a}_i}{\|\boldsymbol{x}-\boldsymbol{a}_i\|}}{\sum_{i=1}^{k} w(\boldsymbol{a}) \frac{1}{\|\boldsymbol{x}-\boldsymbol{a}_i\|}} & \text{if } \boldsymbol{x} \notin A \\[2em] \dfrac{\sum_{\boldsymbol{a}\in A, \boldsymbol{a}\neq \boldsymbol{x}} w(\boldsymbol{a}) \frac{\boldsymbol{a}}{\|\boldsymbol{x}-\boldsymbol{a}\|}}{\sum_{\boldsymbol{a}\in A, \boldsymbol{a}\neq \boldsymbol{x}} w(\boldsymbol{a}) \frac{1}{\|\boldsymbol{x}-\boldsymbol{a}\|}} & \text{if } \boldsymbol{x} \in A \end{cases}$$

Problem: When to stop?



$\|p_{207} - p_{206}\| < \frac{1}{10}$ but p_{207} is still far from the Fermat point $\boldsymbol{x}^*$.

Point sequence algorithm

Algorithm 2: Point sequence algorithm (*P-SEQ*)

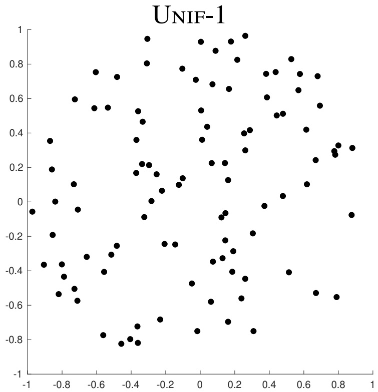
Input : Foci set A, $\varepsilon > 0$. **Output:** Point $\boldsymbol{p}$.

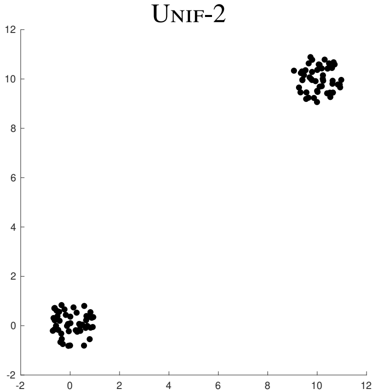
$\boldsymbol{p} \leftarrow$ center of mass; $\quad w \leftarrow \varepsilon$;
while $\textit{True}$ **do**
- $B \leftarrow \text{Box } B(m_B = p,\ \omega(B) = w)$;
- **if** $\square N(B) \subseteq B$ **then** **return** $\boldsymbol{p}$;
- **else if** $\square N\left(\frac{B}{10}\right) \cap \frac{B}{10} = \emptyset$ **then** $w \leftarrow \min\{\varepsilon, w \cdot 10\}$;
- **else** $w \leftarrow \frac{w}{10}$;
- $\boldsymbol{p} \leftarrow T(\boldsymbol{p})$

Theorem

The point sequence algorithm returns an ε-approximation of the Fermat point.

Experiments





Points in UNIF-1 are uniformly sampled from a disk and in UNIF-2 are uniformly sampled from two disjoint disks.

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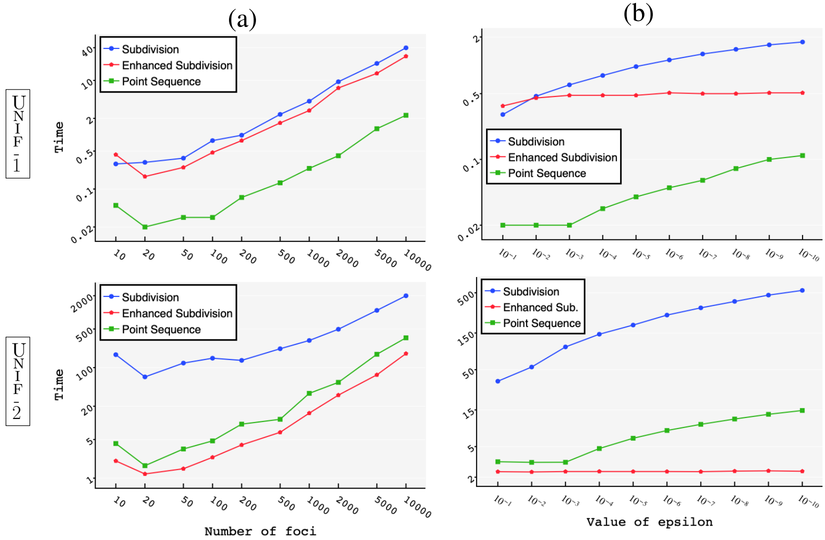

Finding the Fermat point with times as function of
(a) k, with $\varepsilon = 10^{-4}$ and
(b) ε, with $k = 100$.

Thank you for your attention!

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