Computational Complexity of the α-Ham-Sandwich Theorem

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Wednesday, 18th March 2020

European Workshop on Computational Geometry 2020

Input

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2. any vector $\alpha = (\alpha_1, \ldots, \alpha_d)$, where $1 \leq \alpha_i \leq n_i$ for all $i \in [d]$.

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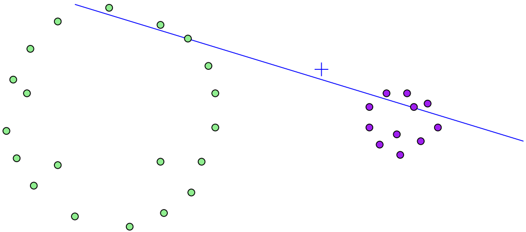

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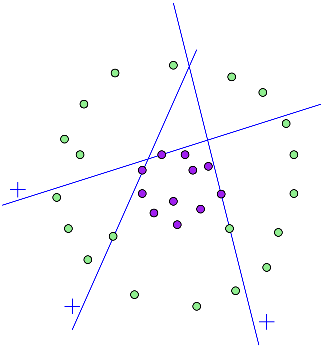

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- the answer is true under certain **conditions**...
- ...shown by [William Steiger, Jihui Zhao, 2010]
- continuous version (P_is are convex sets, $\alpha_i \in [0, 1]$) - [Imre Bárány, Alfredo Hubard, and Jésus Jerónimo, 2008]
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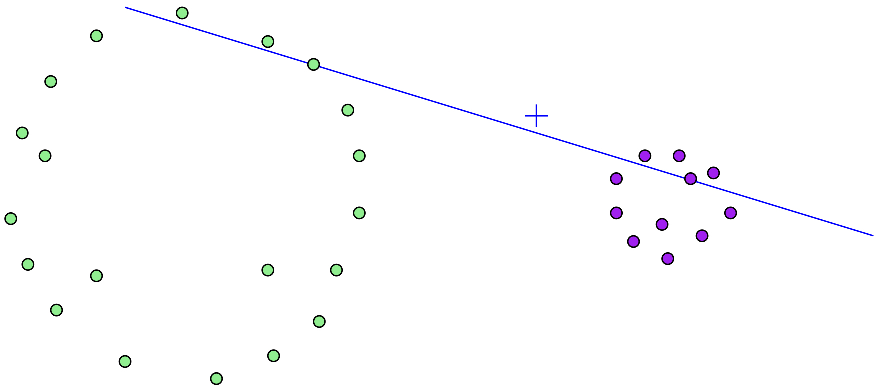
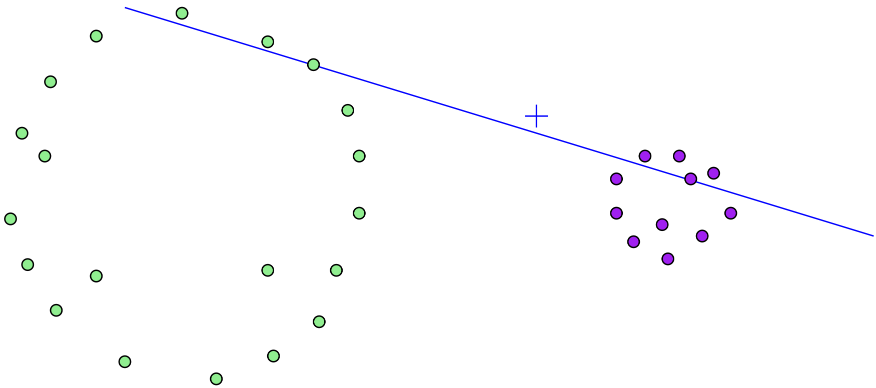
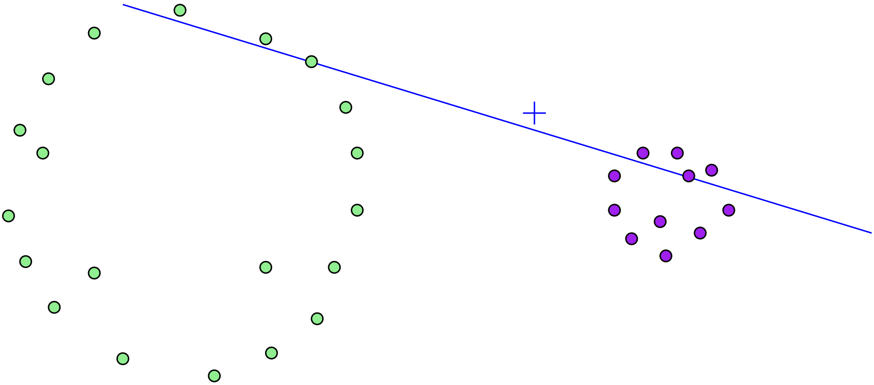


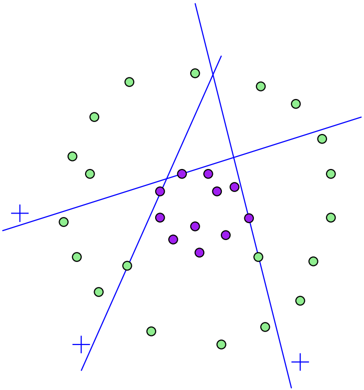


- taking one point from each color (point set) forms a **colorful** set
- a **hyperplane** is incident to a **set**



- taking one point from each color (point set) forms a **colorful** set
- a **hyperplane** is incident to a **set**
- only interested in **hyperplanes** (=**sets**) as solutions
- easy to define orientation

Conditions (i)



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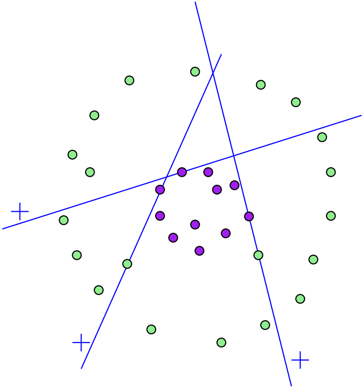

- the sets $P_1, \ldots, P_d$ must be *well-separated*
- disjoint index sets $I, J \subset [d]$, $\mathrm{conv}\left(\{\cup_{i \in I} P_i\}\right) \cap \mathrm{conv}\left(\{\cup_{j \in J} P_j\}\right) = \varnothing$

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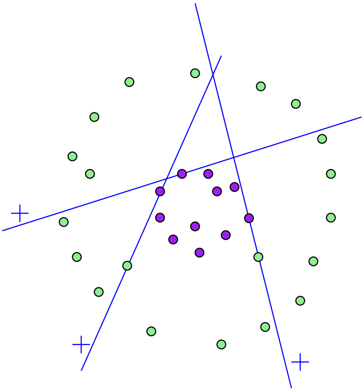

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- **short certificate** of violation: $I, J \subset [d] \ldots$

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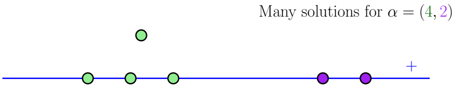

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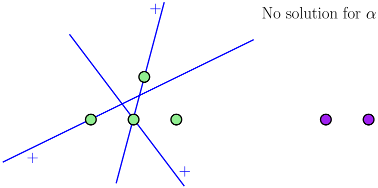

No solution for $\alpha = (3, 1)$

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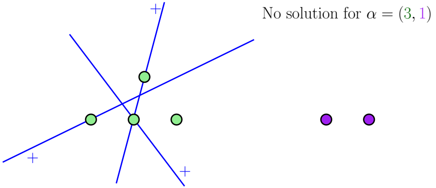

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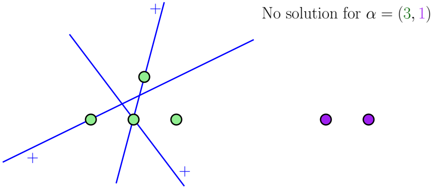

- $P_1, \ldots, P_d$ must be in *general position* (weak)
- weak general position is also expensive to verify
- **short certificate** of violation: a set of $(d+1)$ points (of d colors) of affine dimension $d-1$

Theorem (α-Ham-Sandwich Theorem, [Steiger and Zhao, 2010])

Let $P_1, \ldots, P_d \subset \mathbb{R}^d$ be well-separated point sets in weak general position. Let $\alpha = (\alpha_1, \ldots, \alpha_d)$ be any vector, where $\alpha_i \in [n_i]$ for all $i \in [d]$. Then there is a unique oriented ***hyperplane*** *H such that $|H^+ \cap P_i| = \alpha_i$ for all i.*

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If well-separation or weak general position fails,

- for some α there is no solution, and
- for some α there are multiple solutions

An anomaly

An actual sandwich[1] is not well-separated... all cuts are not guaranteed

[1] source:wikiHow

Complexity

- Steiger and Zhao gave a prune-and-search algorithm taking time $O\left(n(\log n)^{d-3}\right)$
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We study the computational complexity of this problem

- premise: "promise" of well-separation and weak general position
- effect: a solution exists for each α-vector
- convert it to a non-promise, total problem

Definition (Alpha-HS)

$\boldsymbol{P} = P_1 \cup \ldots \cup P_d$ in $\mathbb{R}^d$

$\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$ such that $\alpha_i \leq |P_i| \; \forall i \in [d]$

Output: any of

- **G1** an $(\alpha_1, \ldots, \alpha_d)$-cut
- **GV1** a certificate of violation of weak general position
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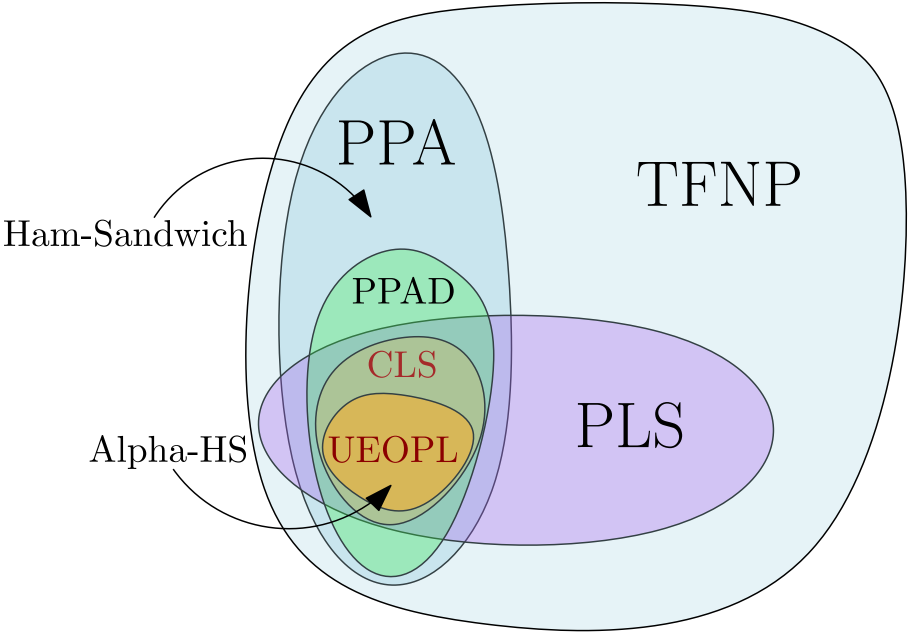
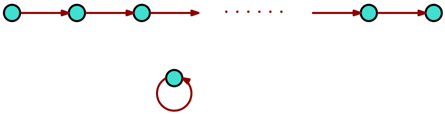
- **GV1** and **GV2** are violations
- **G1** is guaranteed if no violations are presented
- so ALPHA-HS is a total search problem, lying in TFNP
- we can do better...

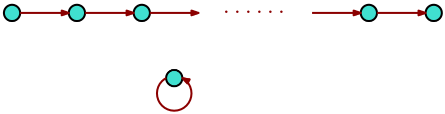

PPA
TFNP
Ham-Sandwich
PPAD
CLS
PLS
UEOPL
Alpha-HS



UniqueEOPL

Input: a description of a graph **G**, satisfying

Line



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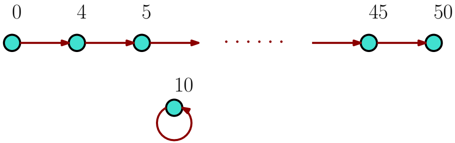

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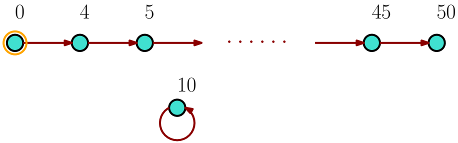

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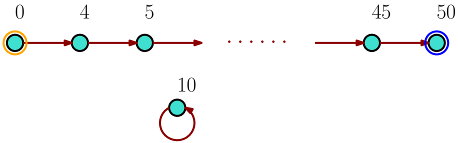

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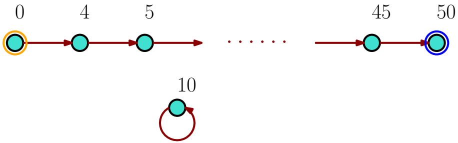

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- introduced by [Fearley, Gordon, Mehta, and Savani [ICALP 2019]]

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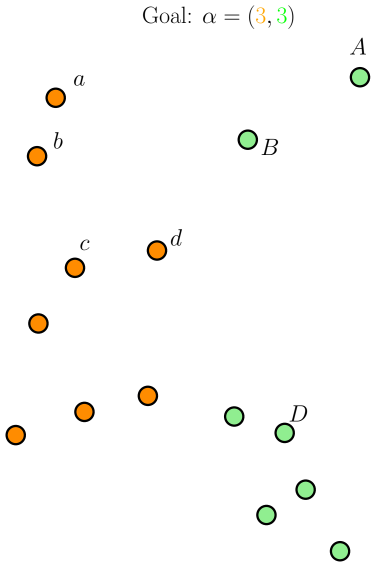
- this formulation gives a total search problem
- defines class UEOPL $\subset$ CLS (Continuous Local Search)
- fixed point of contraction map, P-matrix complementarity problem, unique sink orientation

Intuition behind ALPHA-HS $\leq_p$ UNIQUEEOPL



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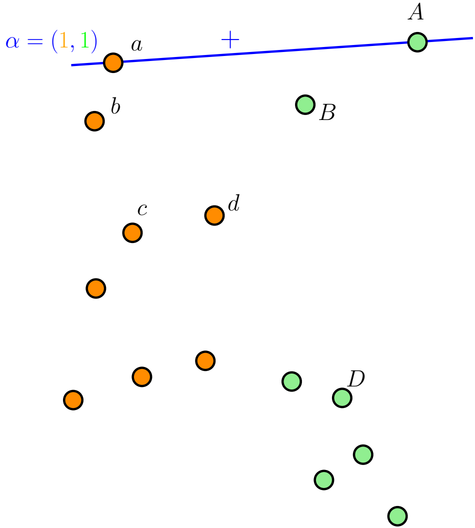

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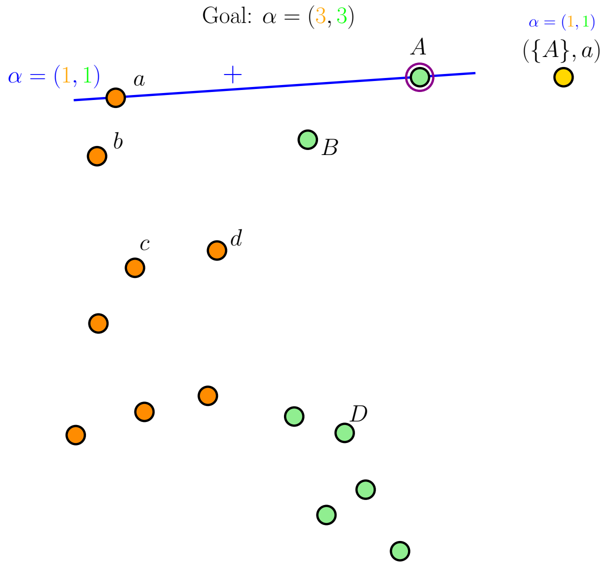

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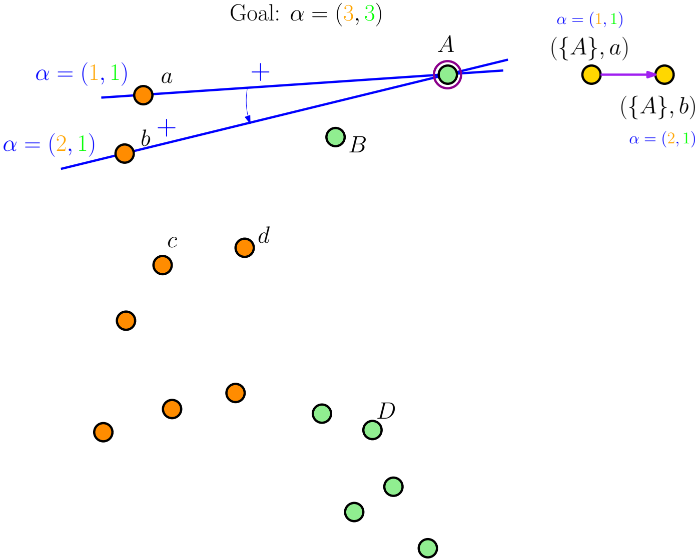

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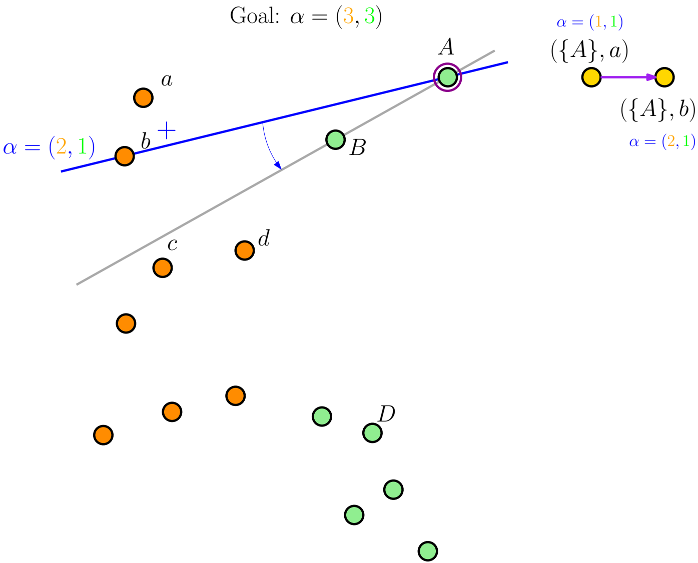

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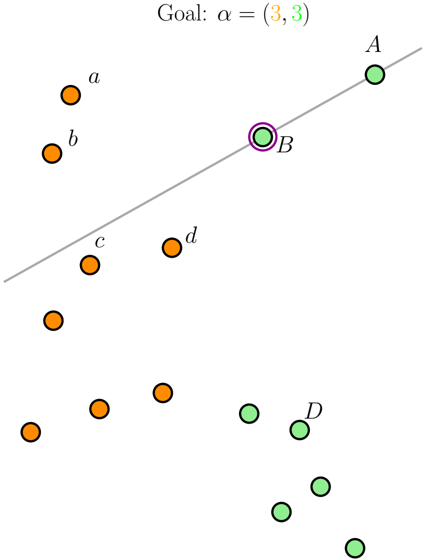

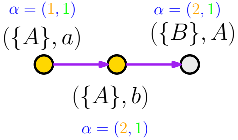

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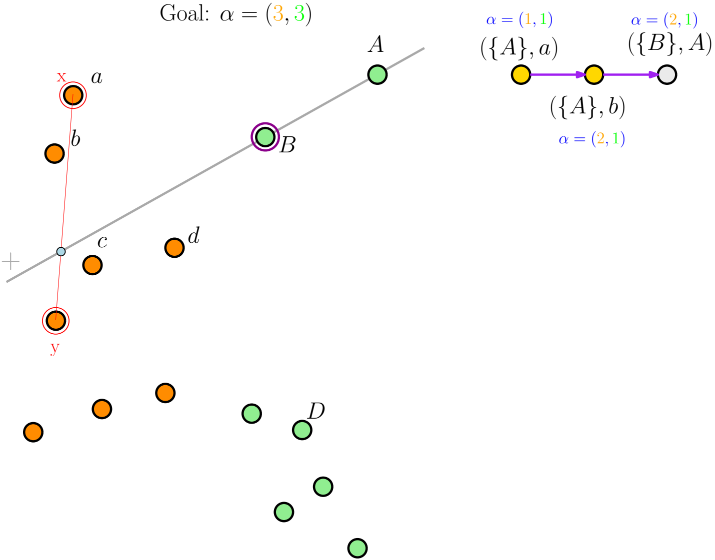

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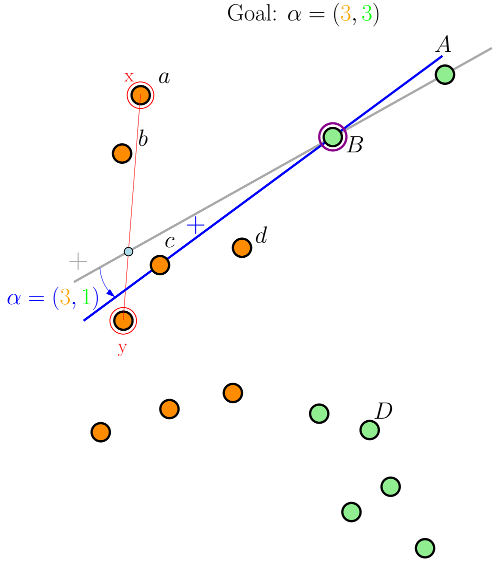

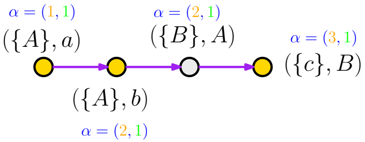

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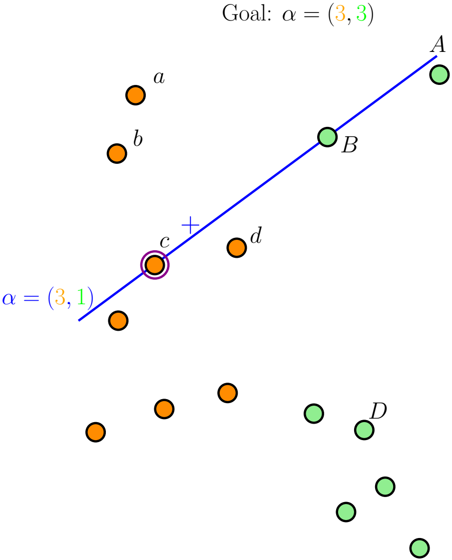

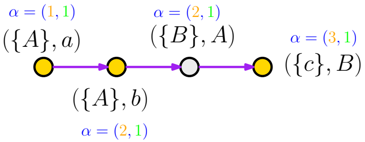

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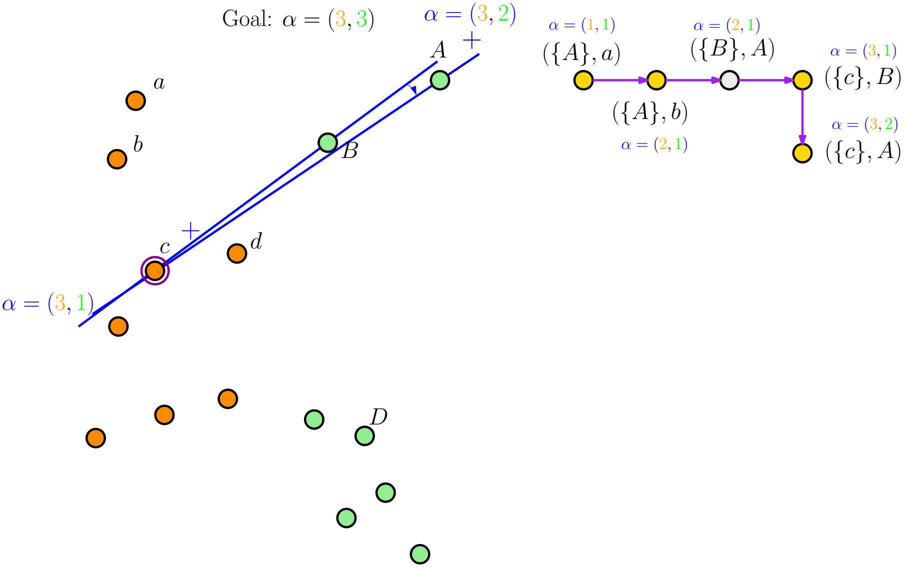

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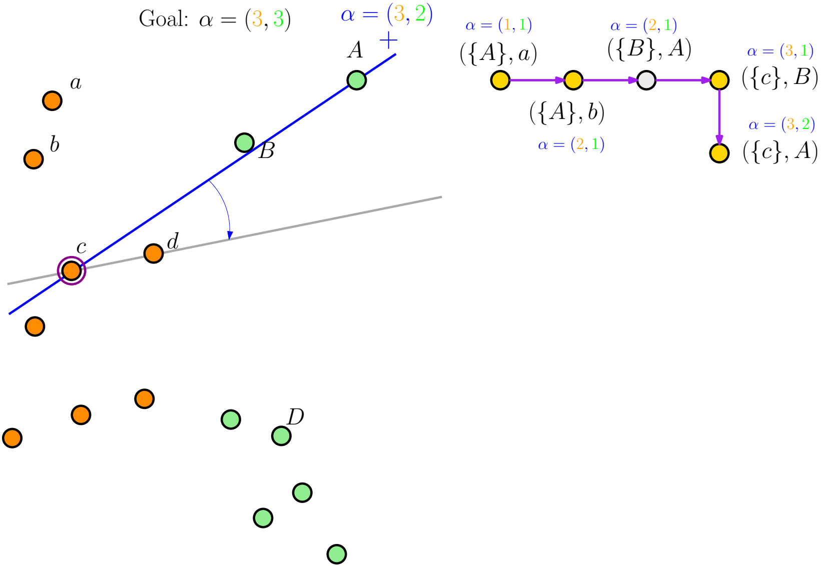

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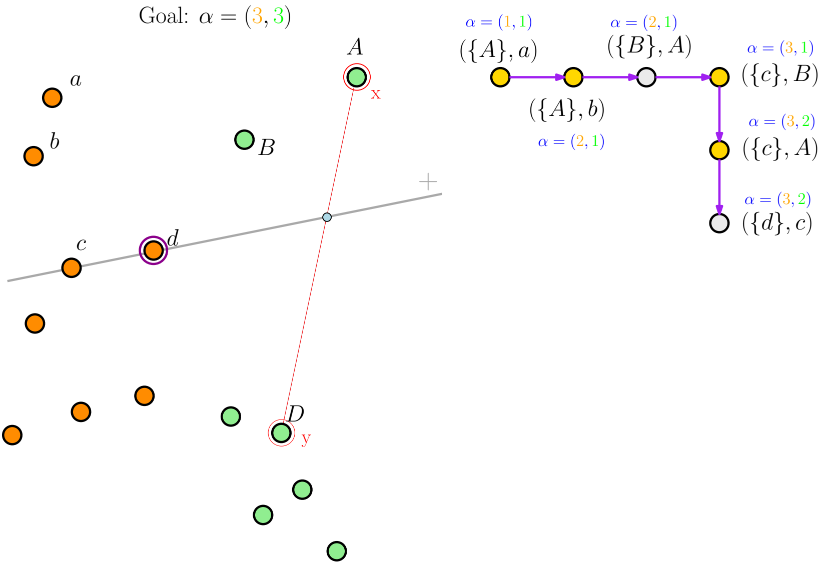

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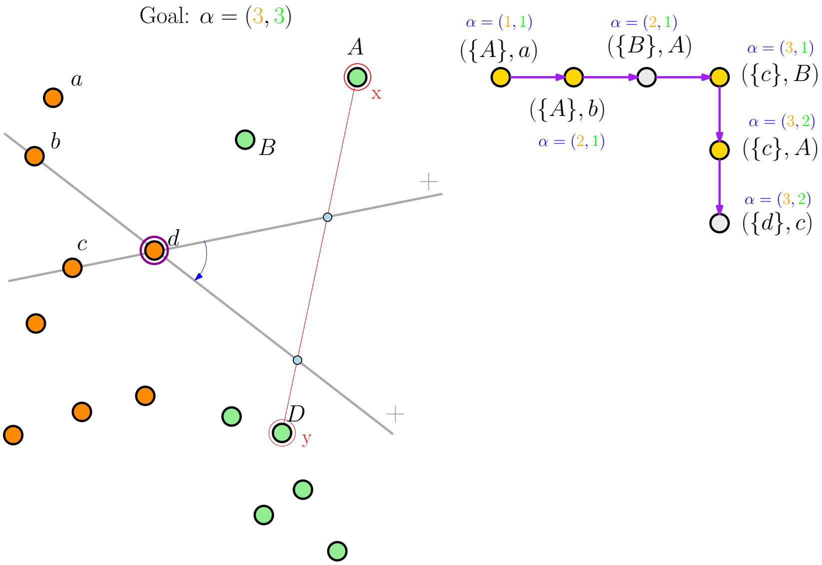

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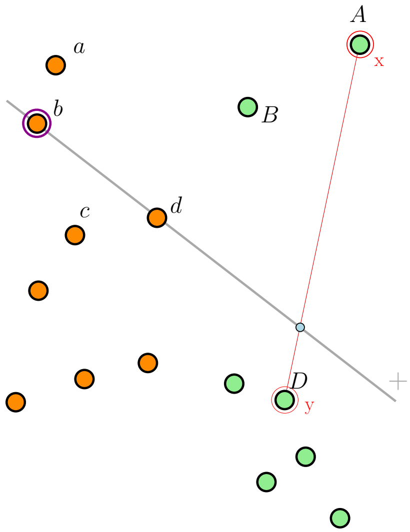

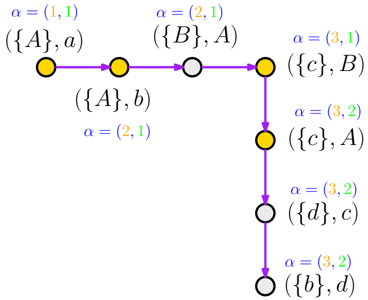

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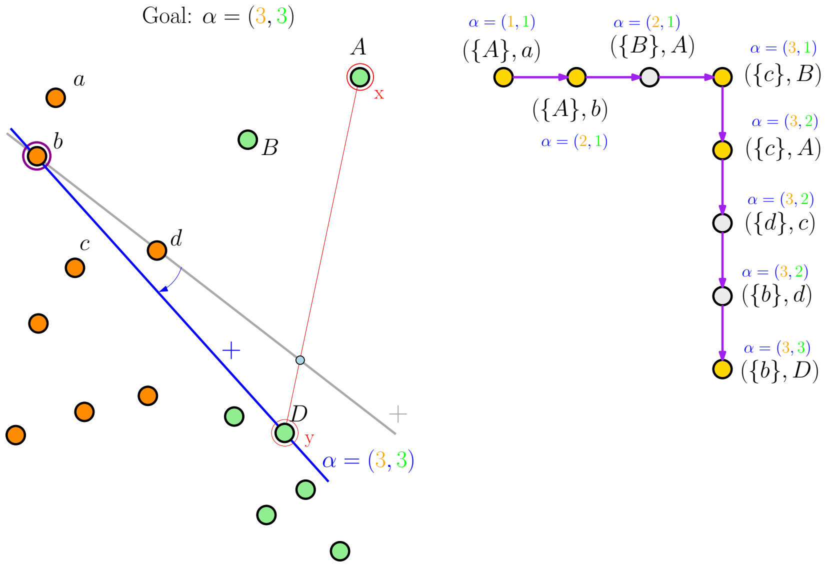

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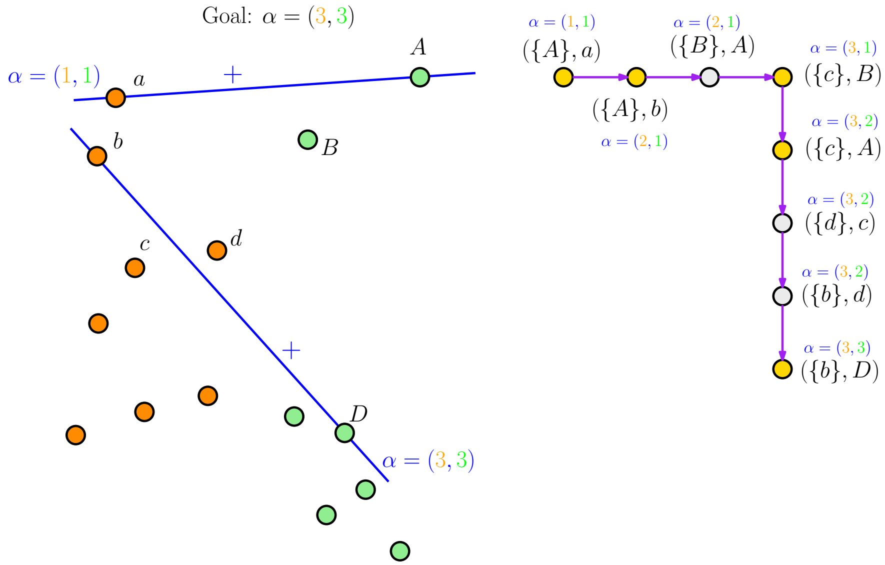
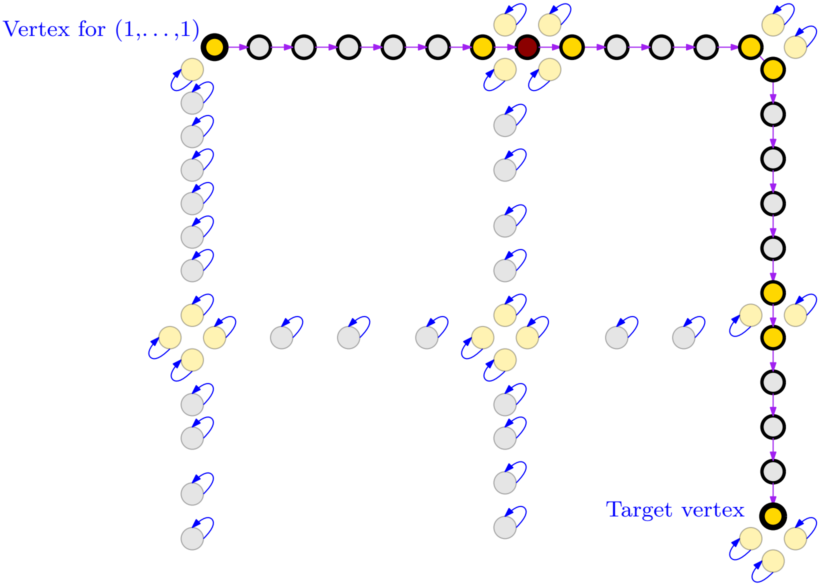
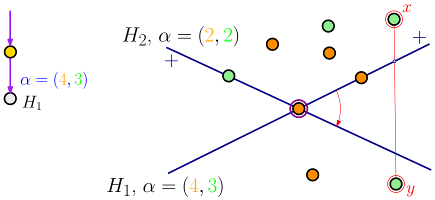


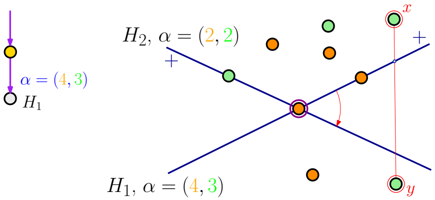


Graph without violations

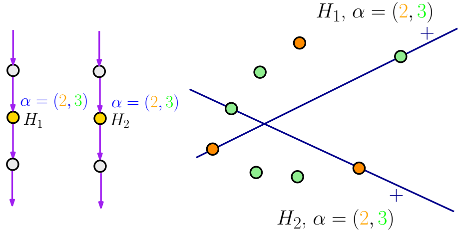
- **U1** with wrong α - a vertex on which the potential does not increase strictly



- **U1** with wrong α - a vertex on which the potential does not increase strictly



- **UV3**- certificate of multiple lines existing



Summary and outlook

- we show a poly-time reduction from $\textsc{Alpha-HS}$ to $\textsc{UniqueEOPL}$

$$\textsc{Alpha-HS} \in \mathsf{UEOPL}$$

Summary and outlook

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- ...is Alpha-HS hard for UEOPL?

Summary and outlook

- we show a poly-time reduction from ALPHA-HS to UNIQUEEOPL

  ALPHA-HS $\in$ UEOPL

- ALPHA-HS is weaker than HAM-SANDWICH
- HAM-SANDWICH is PPA-complete...
- ...is ALPHA-HS hard for UEOPL?
- ...is ALPHA-HS in P?