Colouring bottomless rectangles and arborescences

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Motivation

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Given a finite family of intervals that cover a point set on the line, can we k-colour them so that each colour class covers m_k-fold covered points?
Yes! In fact,

Theorem
$m_k = k$ for colouring intervals with respect to points.

History

In general, given a collection $\mathcal{F}$ of sets in $\mathbb{R}^d$, can we always find a constant $m_{k,\mathcal{F}}$ such that any finite subcollection $\mathcal{F}$ has a k-colouring with the property that any $m_{k,\mathcal{F}}$-fold covered point is covered by all k colours?

History

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Conjecture (Pach, 1980)

$m_{k,\mathcal{F}}$ is finite when $\mathcal{F}$ consists of all translates of a fixed convex set in the plane.

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(Pálvölgyi, Tóth, 2010)

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What does this have to do with bottomless rectangles and arborescences?

Back to intervals

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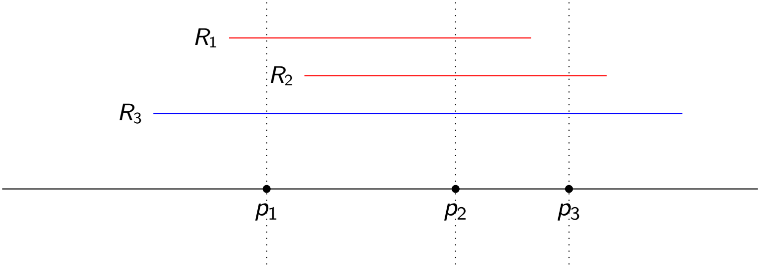

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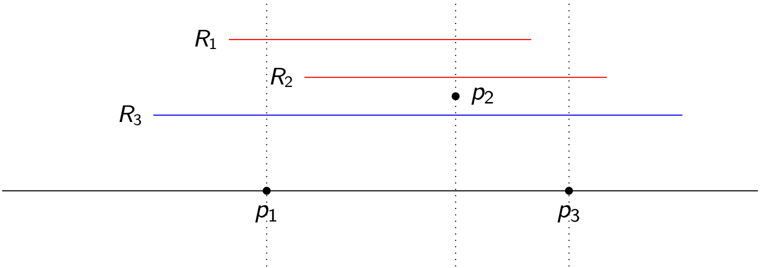

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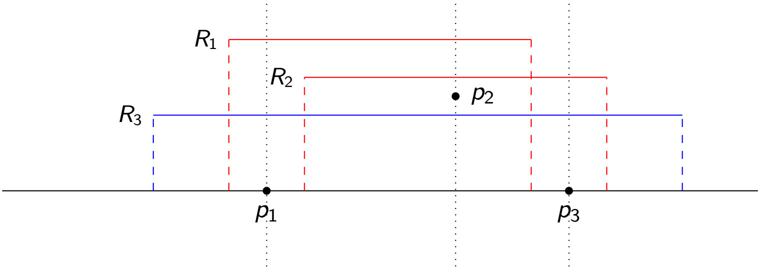

The bottomless rectangle problem

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What is the minimum integer m_k such that any finite family $\mathcal{F}$ of bottomless rectangles can be k-coloured so that any m_k-fold covered point is covered by all k colours?

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Best known results

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Best known results

$m_2 = 3$ [Keszegh, 2011]
$m_k = O(k^{5.09})$ [Cardinal, Knauer, Micek, Ueckerdt, 2013]

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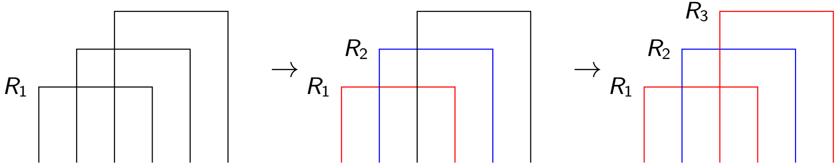


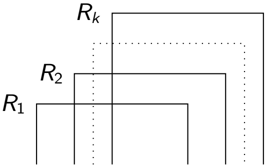
Figure: A semi-online algorithm that colours the rectangles from the left. When R_1 is presented, it does not colour it. R_1 is coloured only when R_2 appears.

Erdős-Szekeres configurations

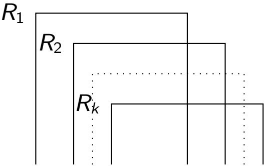
Another idea is to exploit some structure of bottomless rectangles.

Erdős-Szekeres configurations

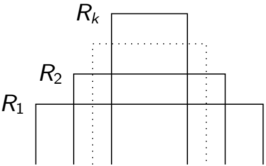
Any point contained in $O(k^4)$ rectangles is contained in one of the following Erdős-Szekeres configurations.



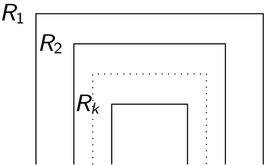
increasing k-steps



decreasing k-steps



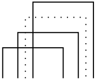
a *k-tower*



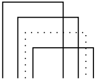
a *k-nested set*

Erdős-Szekeres configurations

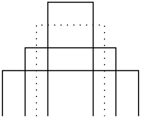
We restrict the colouring problem to each fixed configuration. For example, can we k-colour any family of bottomless rectangles so that any point covered by $m_{k,\text{inc.}}$-increasing steps is covered by all k colours?



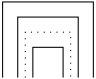
inc. steps



dec. steps



a tower



a nested set

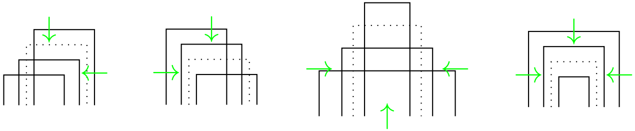
Erdős-Szekeres configurations

For example, can we k-colour any family of bottomless rectangles so that any point covered by $m_{k,\text{inc.}}$-increasing steps is covered by all k colours?

Result

$$m_{k,\text{inc.}} = m_{k,\text{dec.}} = m_{k,\text{towers}} = m_{k,\text{nested}} = k$$

In fact, for each m_k value the colouring is given by an online algorithm that colours the rectangles from a "good" direction (marked by green arrows in the figure below).



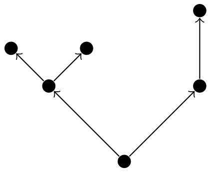
inc. steps　dec. steps　a tower　a nested set

What does this have to do with ~~bottomless rectangles and~~ arborescences?

Arborescences, and other nature-related terminology

Definition

An arborescence is a directed rooted tree with all edges directed away from the root



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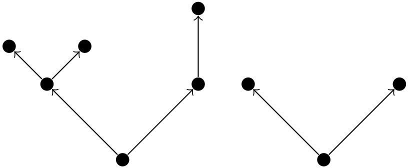
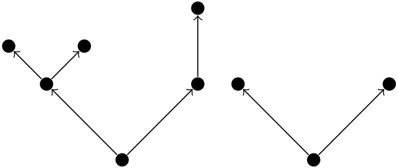
A branching is a disjoint union of arborescences.



An arborescence colouring problem

Definition

A k-colouring of the vertices of a branching is m-polychromatic if every directed path of length m contains all k colours.



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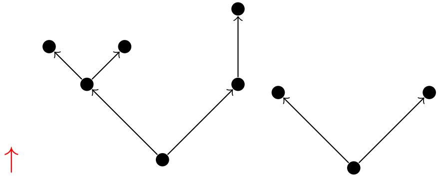
A k-colouring of the vertices of a branching is m-polychromatic if every directed path of length m contains all k colours.

Definition

An ordering on the vertices of a branching $\mathcal{F}$ is root-to-leaf if each vertex is presented before its out-neighbours.



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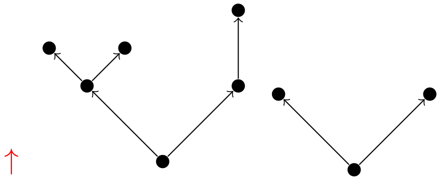
(Easy) claim
The vertices of a branching can be k-coloured in a root-to-leaf order so that the colouring is k-polychromatic.



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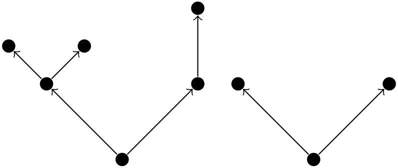
Definition

A k-colouring of the vertices of a branching is *m-polychromatic* if every directed path of length m contains all k colours.

Definition

A k-colouring of the vertices of a branching is *m-proper* if every directed path of length m contains at least 2 colours.



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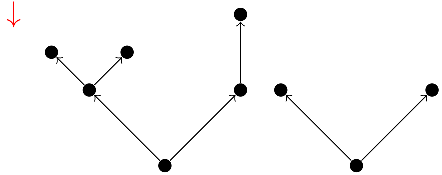
An ordering on the vertices of a branching $\mathcal{F}$ is leaf-to-root if each vertex is presented before its in-neighbours.



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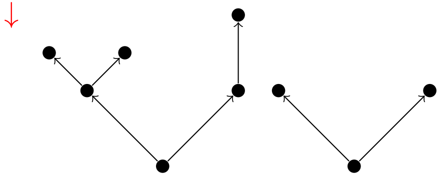
An ordering on the vertices of a branching $\mathcal{F}$ is leaf-to-root if each vertex is presented before its in-neighbours.

(Not easy) result

For any integer m, there is no semi-online k-colouring algorithm that receives the vertices in a leaf-to-root order, and maintains that the colouring is m-proper at each step.



An arborescence colouring problem

(Not easy) result

For any integer m, there is no semi-online k-colouring algorithm that receives the vertices in a leaf-to-root order, and maintains that the colouring is m-proper at each step.

Corollary

For any k and m, and a semi-online k-colouring algorithm that colours the rectangles from the left, the right, above, or below, there will be an m-fold covered point that is covered by at most 1 colour.

An arborescence colouring problem

Corollary

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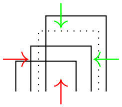
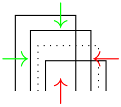
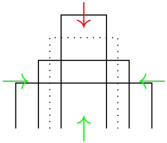
The proof of the corollary proceeds by showing that for each configuration, an algorithm that colours the rectangles along one of the red arrows can be realised as an algorithm that colours a branching in a leaf-to-root order.



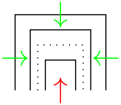
inc. steps



dec. steps



a tower



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An arborescence colouring problem

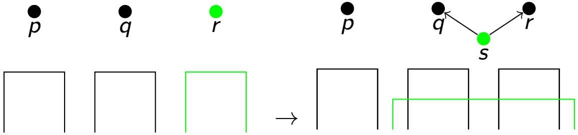
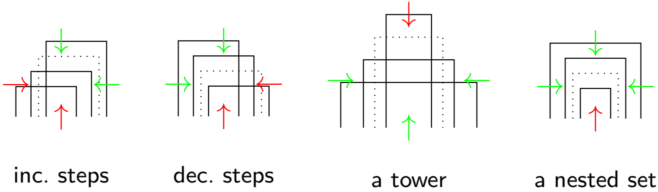


Figure: Realising a leaf-to-root order as a tower from above: each time a disjoint element (such as r) is presented, we realise it as a disjoint rectangle to the right. We are then able to realise s as a minimal element that forms a tower with q and r.



Thank you for listening!

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