On the maximum number of crossings in star-simple drawings of K_n with no empty lens

Stefan Felsner, Michael Hoffmann, **Kristin Knorr**, Irene Parada

EuroCG2020

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Simple Drawings vs. Star-simple Drawings

	Simple Drawings	Star-simple Drawings
Adjacent edges		
Not adjacent edges		

Simple Drawings vs. Star-simple Drawings

	Simple Drawings	Star-simple Drawings
Adjacent edges	no crossings	no crossings
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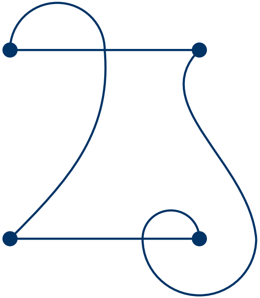

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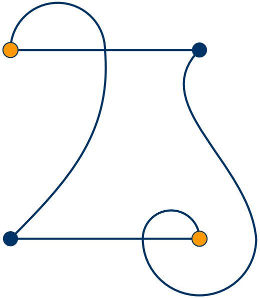

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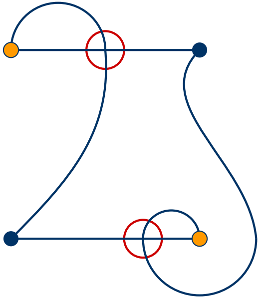

Simple Drawings vs. Star-simple Drawings

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Adjacent edges	no crossings	no crossings
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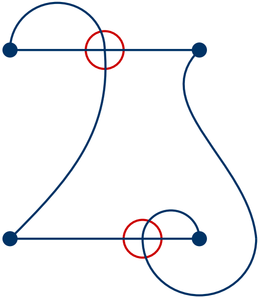

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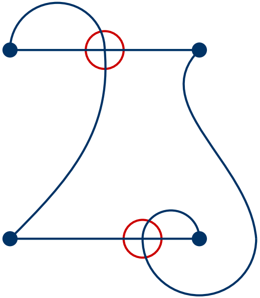

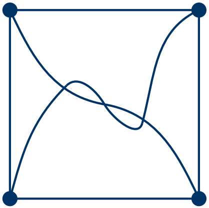

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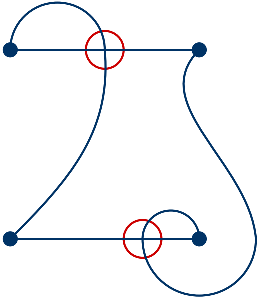

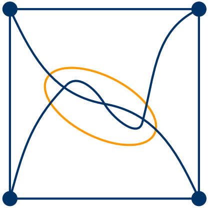

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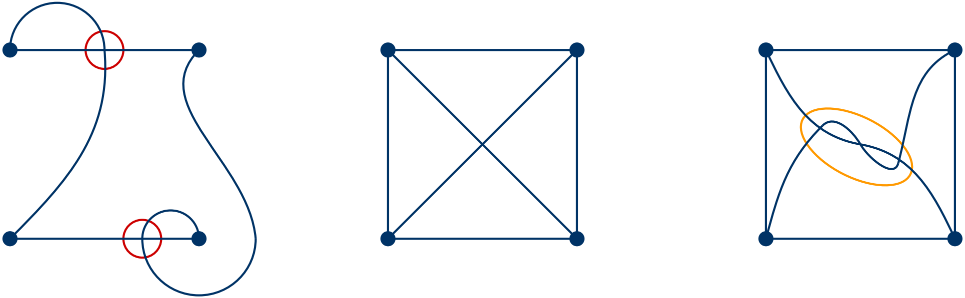

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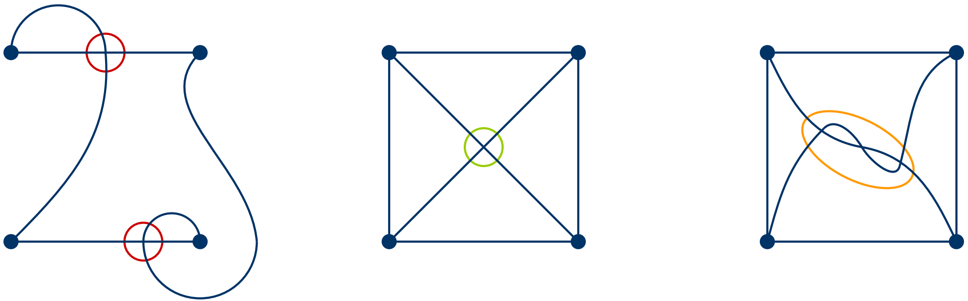

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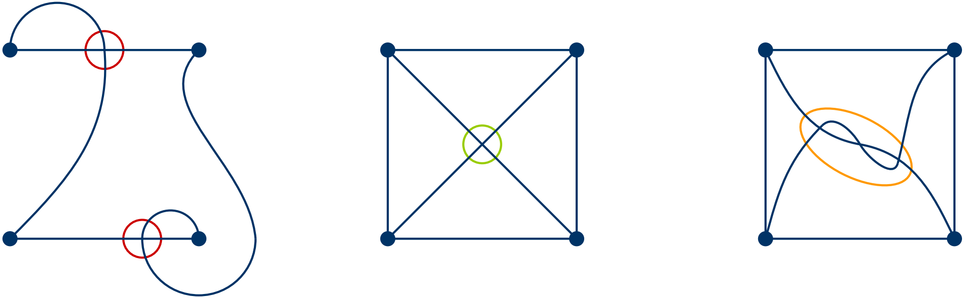

Simple Drawings vs. Star-simple Drawings

Definition (Maximum Crossing Number)

$$\textit{max-cr}(G) = \max_{\substack{D(G)\in \\ \text{Simple Drawings}}} (\#\ \text{Crossings in } D(G))$$

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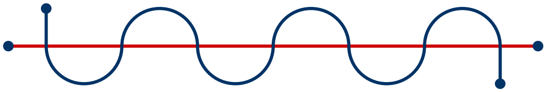

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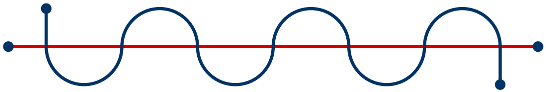

twisting

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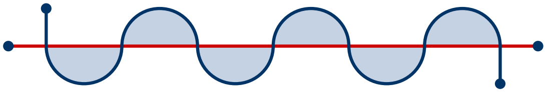

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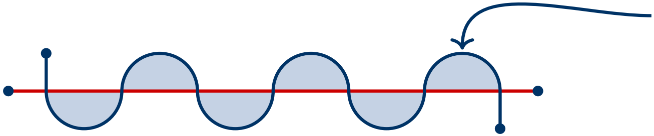

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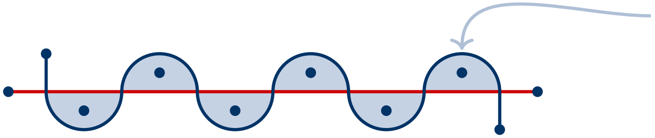

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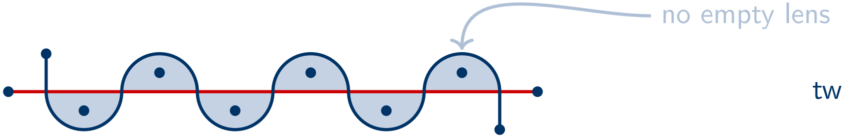

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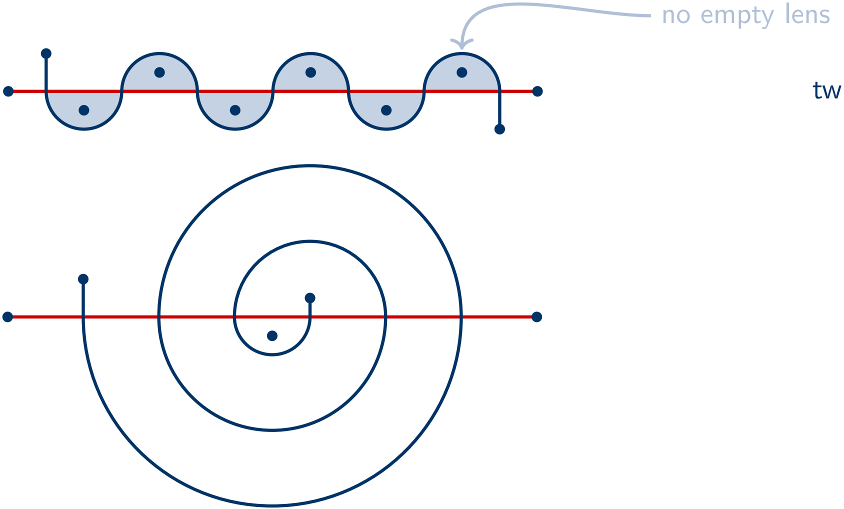

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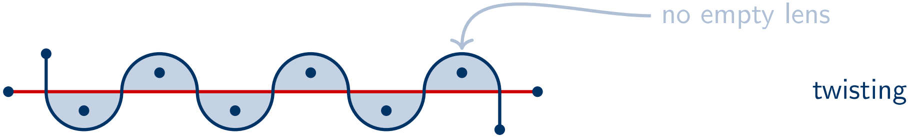

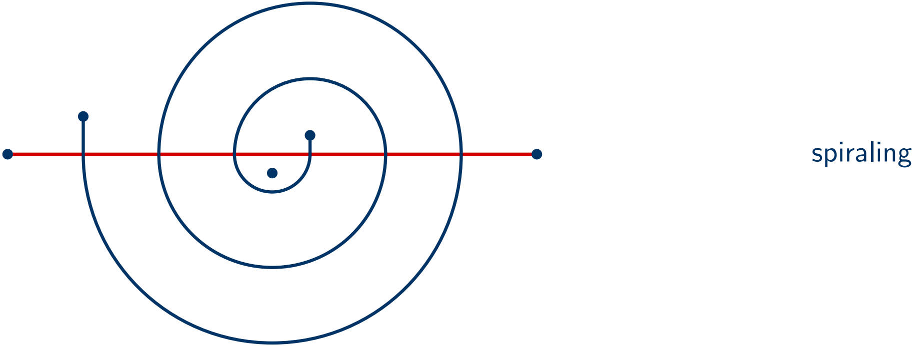

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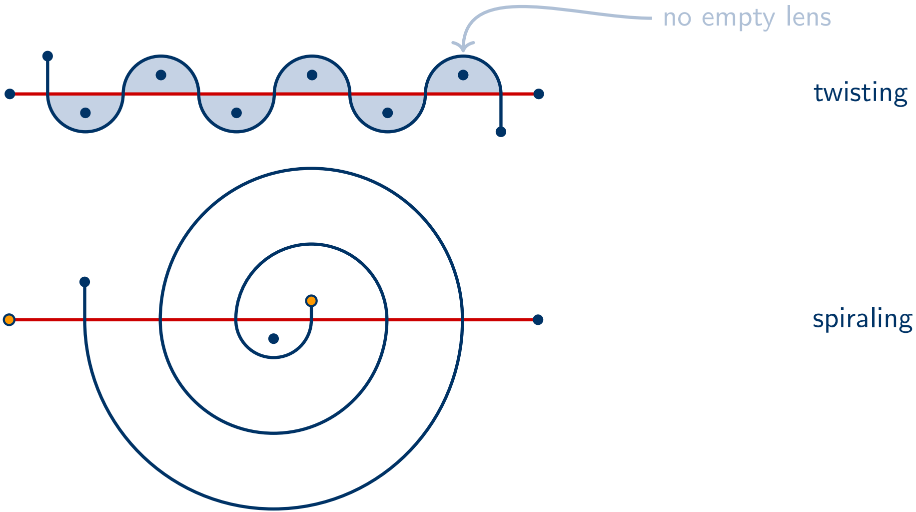

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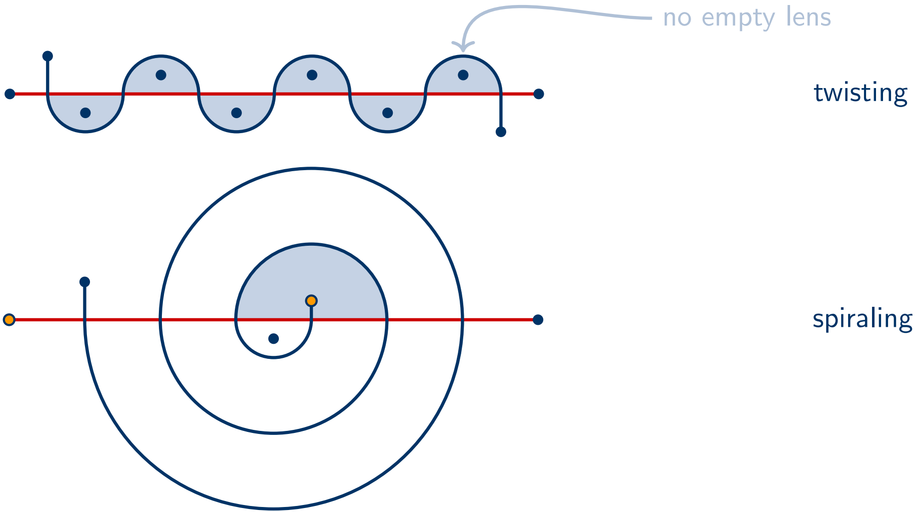

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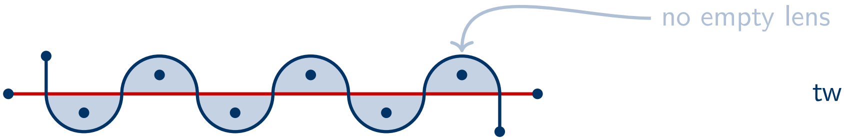

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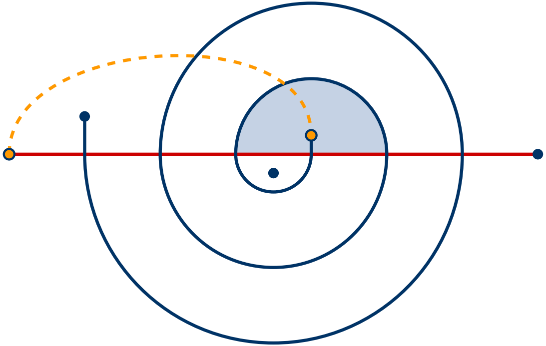

spiraling

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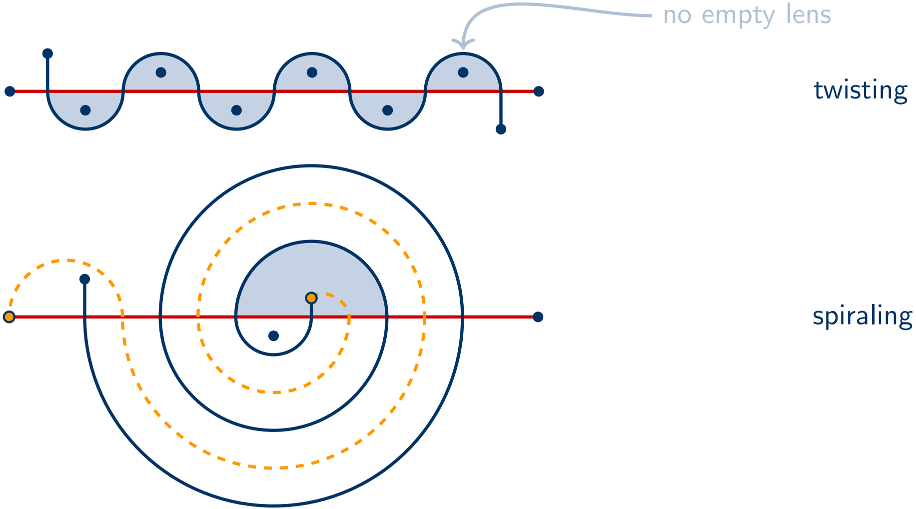

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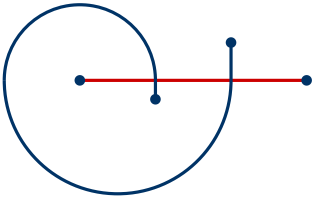

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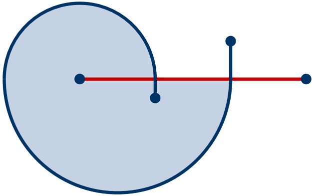

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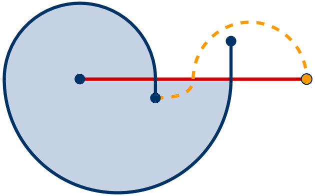

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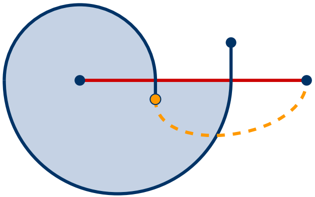

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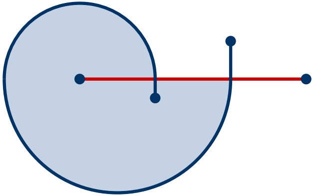

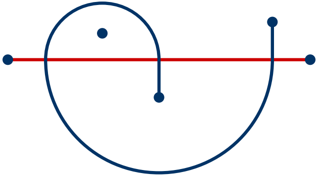

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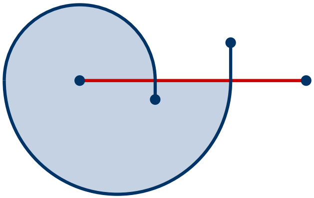

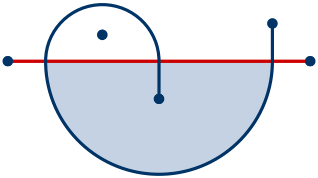

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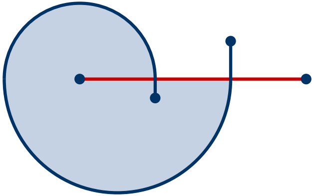

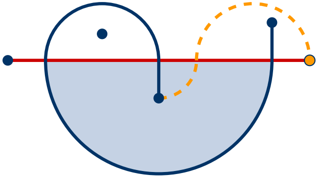

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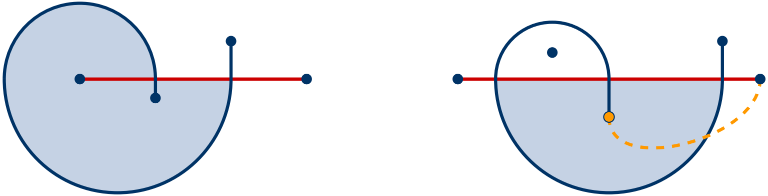

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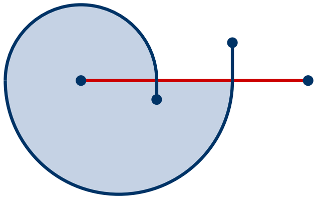

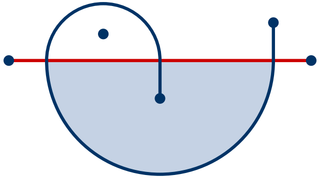

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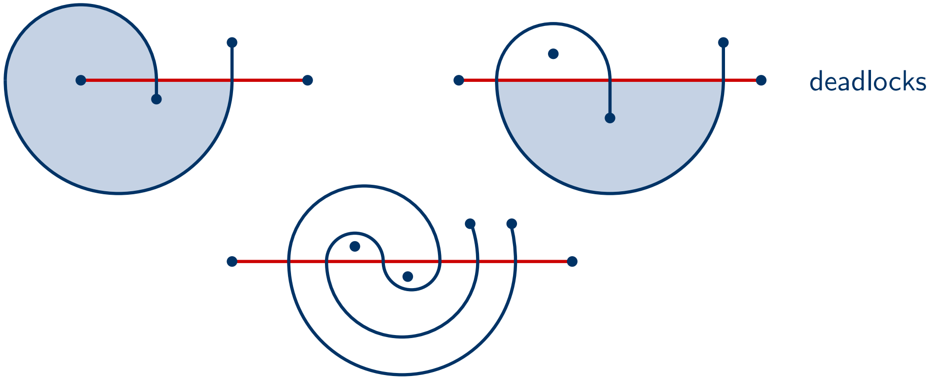

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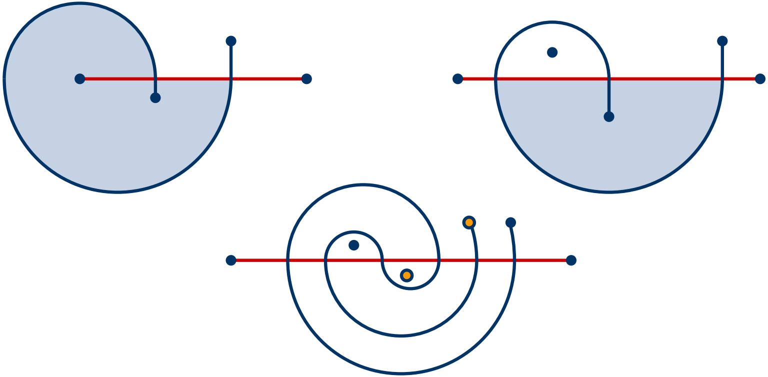

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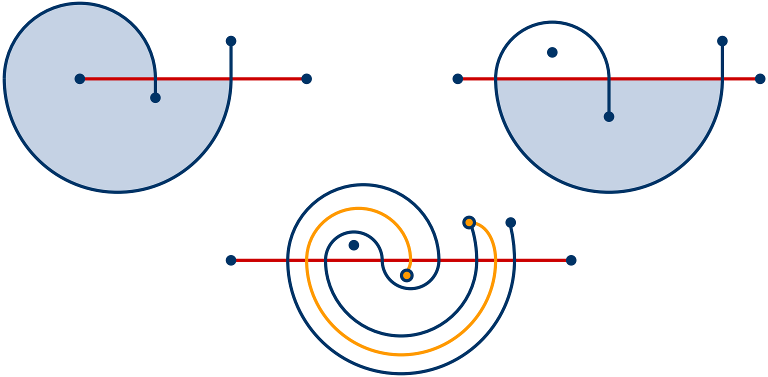

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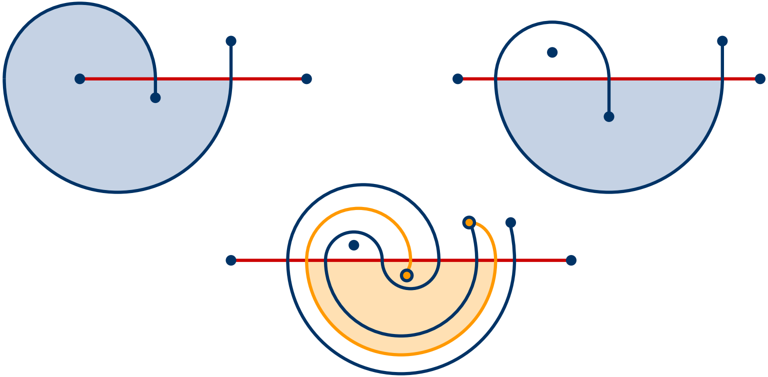

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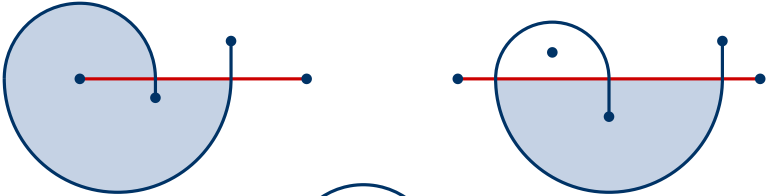

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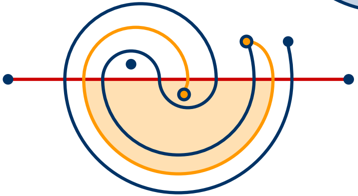

spirals

Lower Bound of maximum crossing number in K_n

	Simple Drawings $max\text{-}cr(K_n)$	Star-simple Drawings $max\text{-}cr^*_{neL}(K_n)$
Lower Bound	$\binom{n}{4}$ †	2^n ‡
Upper Bound	$\binom{n}{4}$ †	unbounded?

‡ Aichholzer, Oswin, et al. "On semi-simple drawings of the complete graph." XVII Spanish Meeting on Computational Geometry. 2017.

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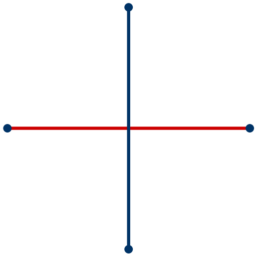

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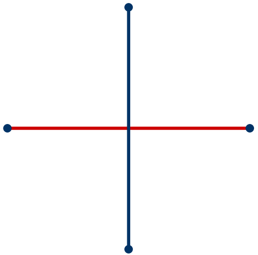

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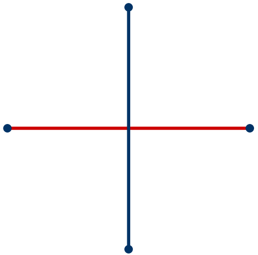

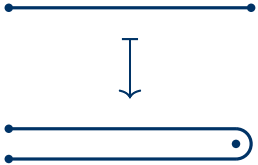

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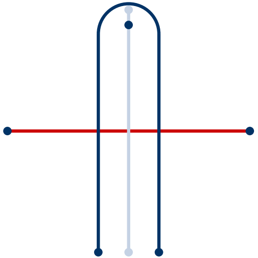

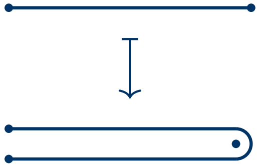

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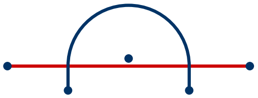

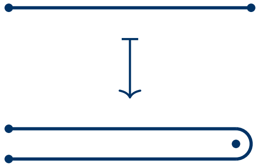

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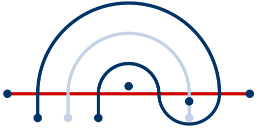

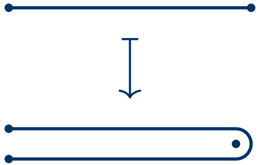

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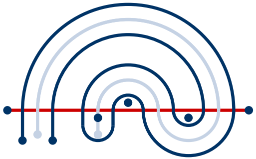
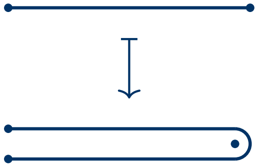
Lower Bound of maximum crossing number in K_n

	Simple Drawings $max\text{-}cr(K_n)$	Star-simple Drawings $max\text{-}cr^*_{neL}(K_n)$
Lower Bound	$\binom{n}{4}$ †	2^n ‡
Upper Bound	$\binom{n}{4}$ †	unbounded?





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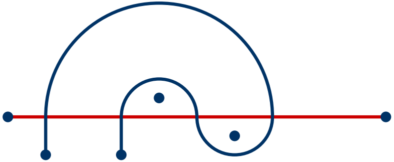
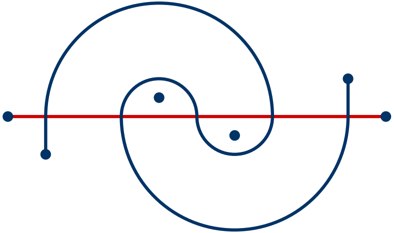
Slight Improvement of Lower Bound



2^{n-4}

Slight Improvement of Lower Bound



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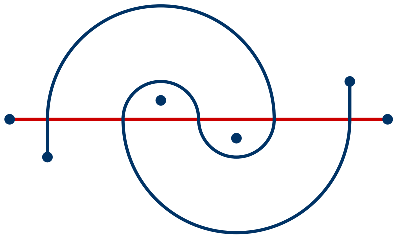
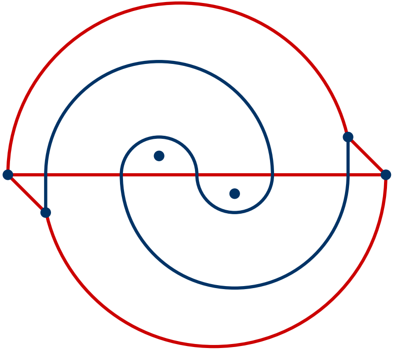


$$2^{n-4} \downarrow \frac{5}{4} \cdot 2^{n-4}$$

Slight Improvement of Lower Bound



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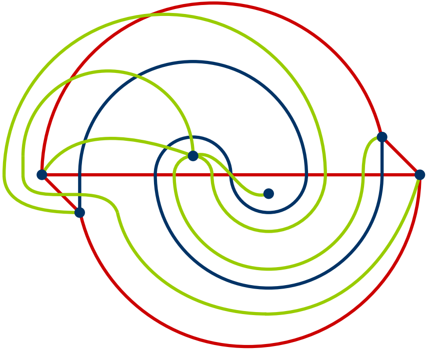
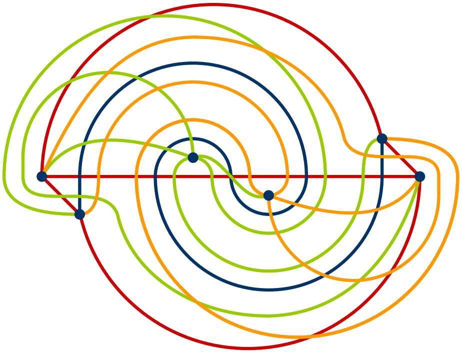


$$2^{n-4}$$
$$\downarrow$$
$$\frac{5}{4} \cdot 2^{n-4}$$

Slight Improvement of Lower Bound



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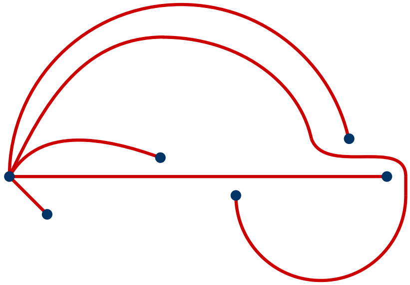

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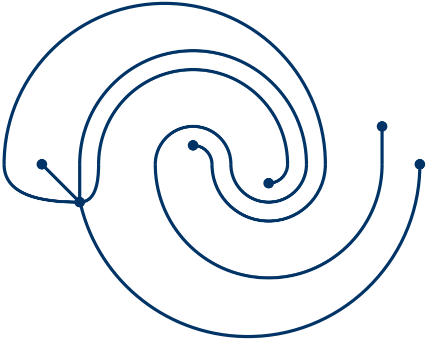

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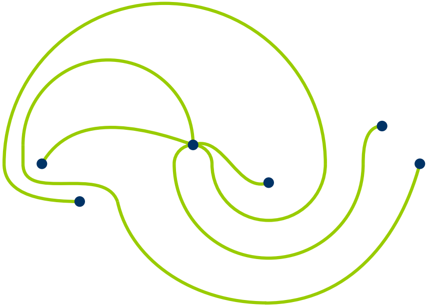

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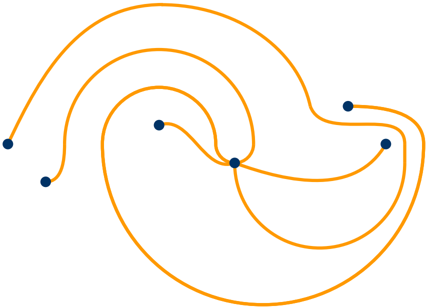
$\downarrow$

$\frac{5}{4} \cdot 2^{n-4}$

Slight Improvement of Lower Bound



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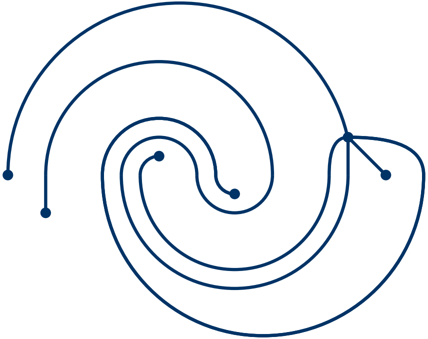

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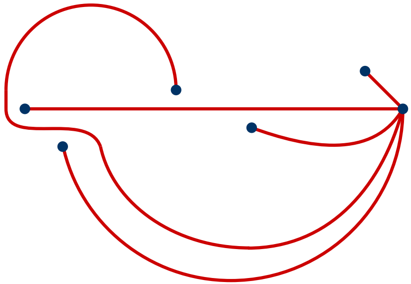

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Upper Bound for maximum crossing number

Theorem

Consider $D^*_{neL}(K_n)$*:*

Upper Bound for maximum crossing number

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If $C(k)$ is the maximum number of crossings of a pair of edges that

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Upper Bound for maximum crossing number

Theorem

Consider $D^*_{neL}(K_n)$:
If $C(k)$ is the maximum number of crossings of a pair of edges that
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then $C(k) \leq e \cdot k!$.

Upper Bound for maximum crossing number

Setting

Upper Bound for maximum crossing number

Setting

- crossing edges: e, e'
- vertices of e, e' on outer face

Upper Bound for maximum crossing number

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Upper Bound for maximum crossing number

Setting

- crossing edges: e, e'
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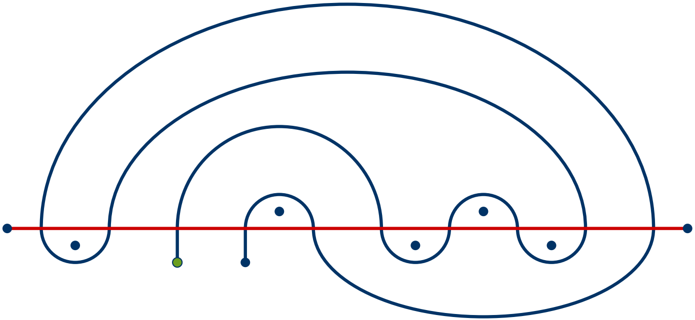
- $e = \{u, v\}$ meander edge



Upper Bound for maximum crossing number

Setting

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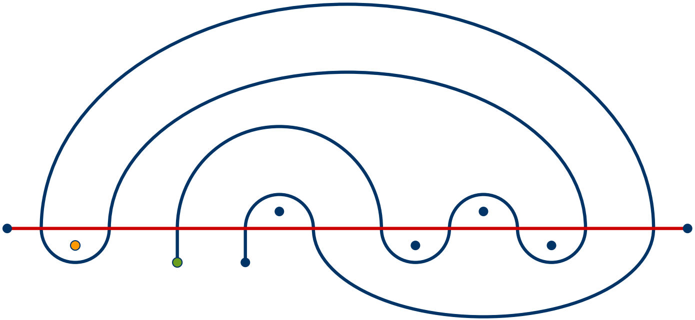
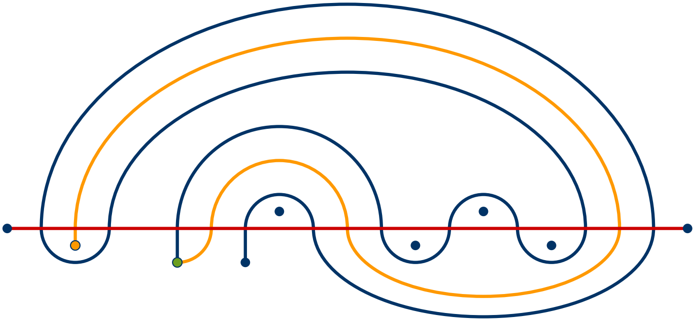
- p_i lens node



Upper Bound for maximum crossing number

Setting

- crossing edges: e, e'
- vertices of e, e' on outer face
- e' vertical straight line
- $e = \{u, v\}$ meander edge
- p_i lens node
- $e_i = \{p_i, u\}$



Upper Bound for maximum crossing number

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(a) Between any two crossings of e and e' from left to right, ≤ one crossing of e' with one of the e_i

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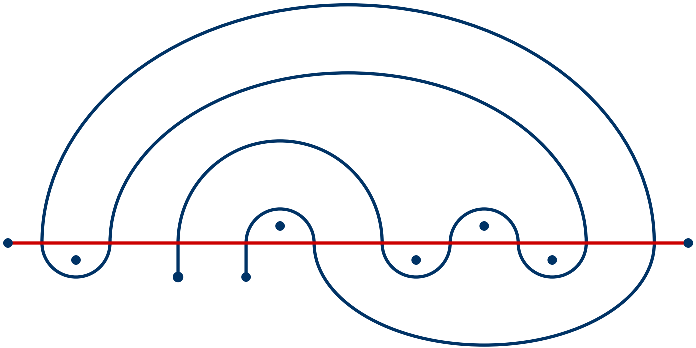

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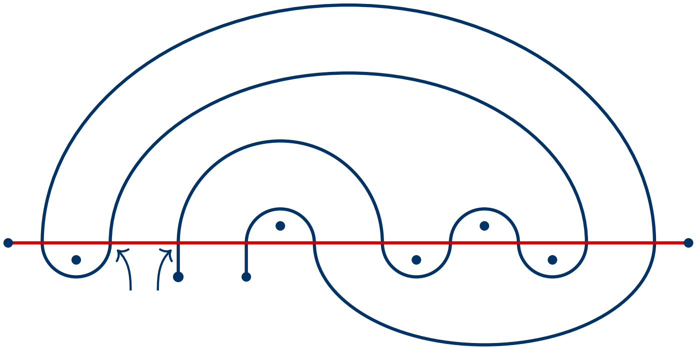

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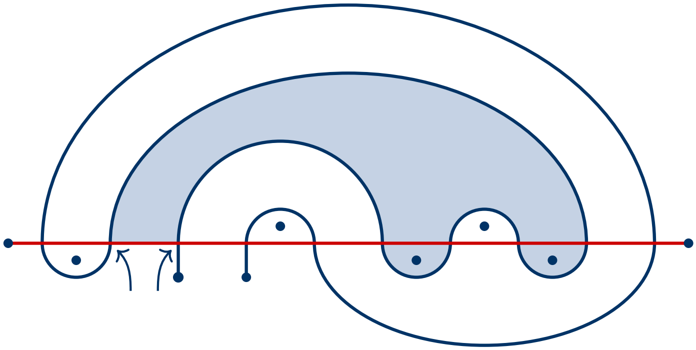

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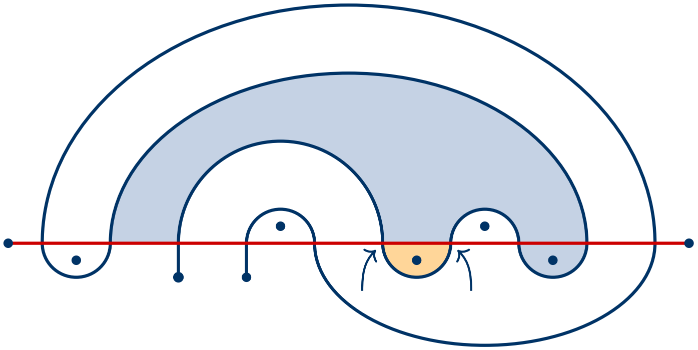

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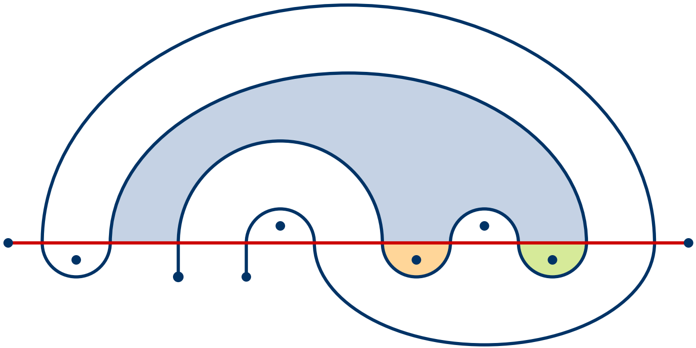

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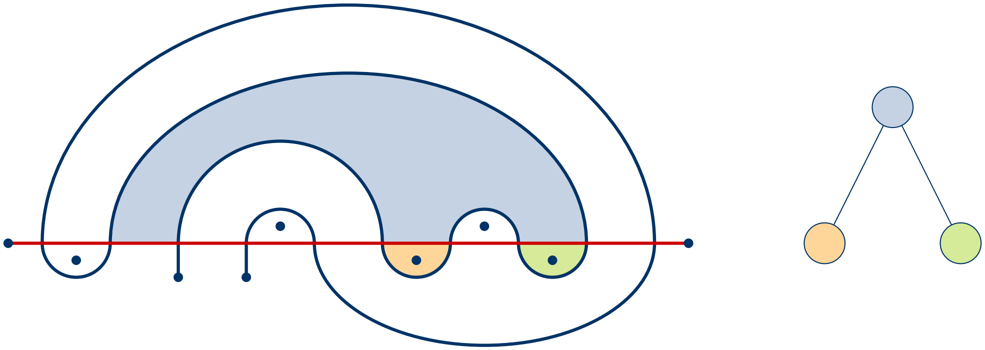

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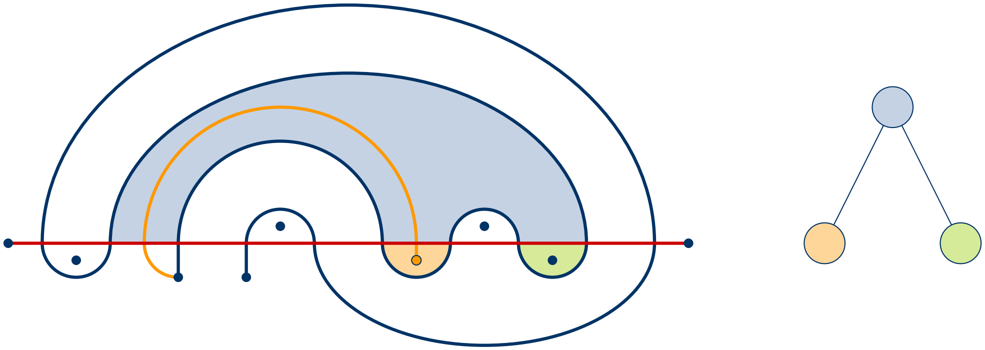

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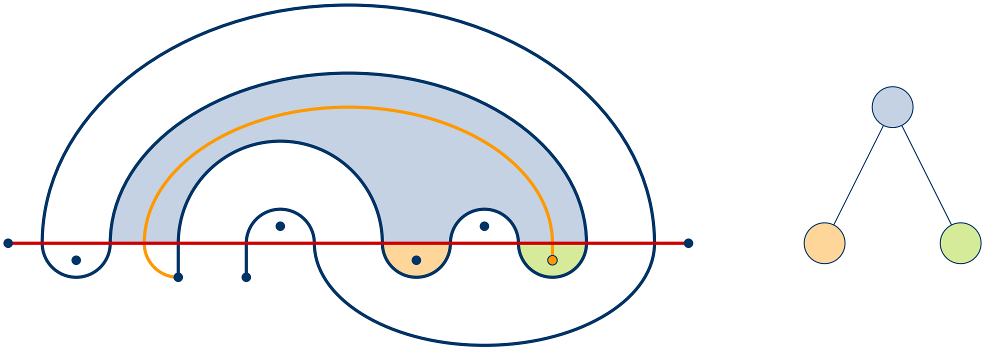

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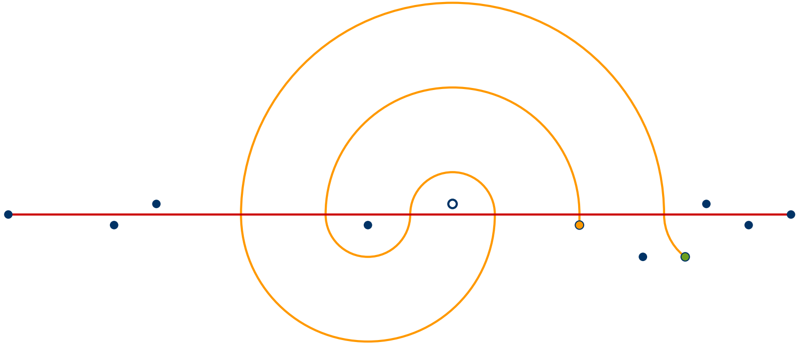

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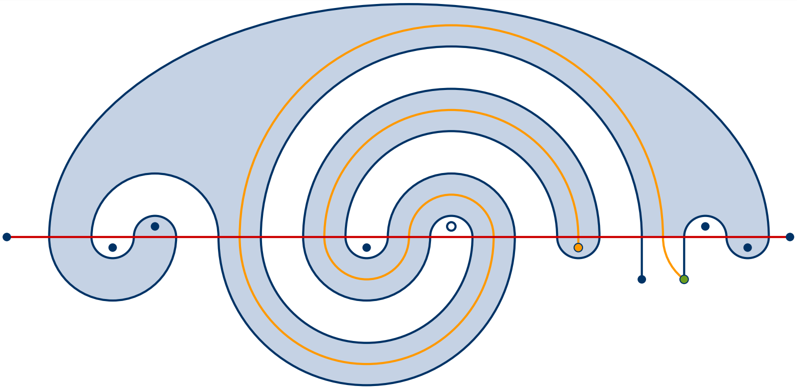

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Result

$C(k) \leq k \cdot C(k-1) + 1$

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$$k \leq (n-4) \Rightarrow \mathrm{e} \cdot (n-4)! \Rightarrow \textit{max-cr}^*_{neL}(K_n) \leq n!$$

Summary

	Simple Drawings $max\text{-}cr(K_n)$	Star-simple Drawings $max\text{-}cr^*_{neL}(K_n)$
Lower Bound	$\binom{n}{4}$ †	2^n ‡
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Open Problem

- Big gap between upper bound and lower bound
- Further graph classes

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Thank you!

Questions?