Improved constant factor for the unit distance problem

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Current best known bounds: $5 \leq CNP \leq 7$. (de Grey (2018) and Isbell (1950))

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and upper bound:

$u(n) \leq \left\lfloor \frac{n}{n-2} \cdot u(n-1) \right\rfloor$ for $n \geq 1$
i.e. the maximal possible edge density of a UDG is monotonously decreasing.

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In 4 dimensions, the exact value is known: it is $\left\lfloor\frac{n^2}{4}\right\rfloor + n$ if n is divisible by 8 or 10 and $\left\lfloor\frac{n^2}{4}\right\rfloor + n - 1$ otherwise. (Brass (1997), van Wamelen (1999))

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The planar case for other norms: A unit distance graph with n vertices can have $\Omega\left(n^{4/3}\right)$ edges for an appropriately constructed norm. (Valtr (2005))

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The currently known best version of the crossing lemma: $cr(G) \geq \frac{e^3}{29n^2}$, if $e \geq 6.95n$. (Ackerman (2013))

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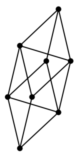

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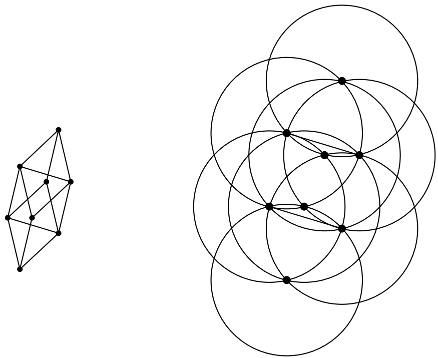

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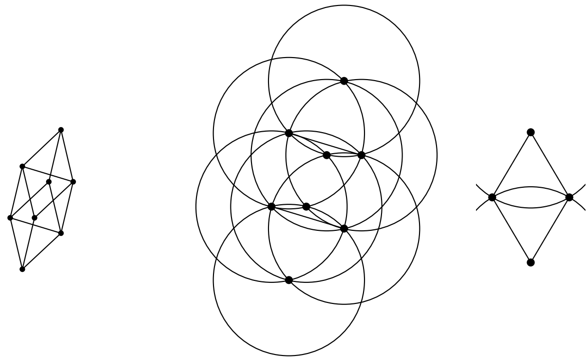

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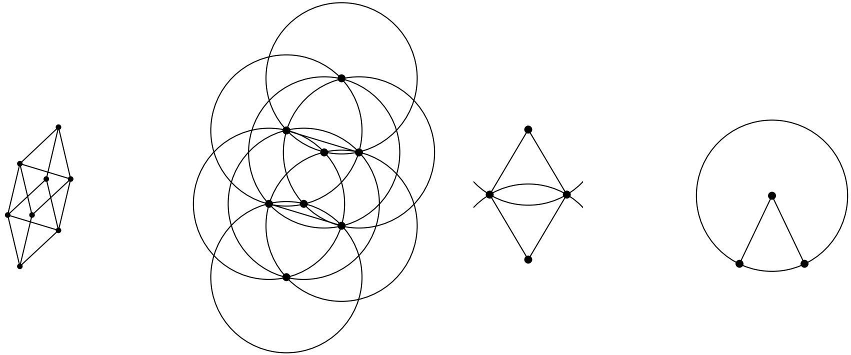

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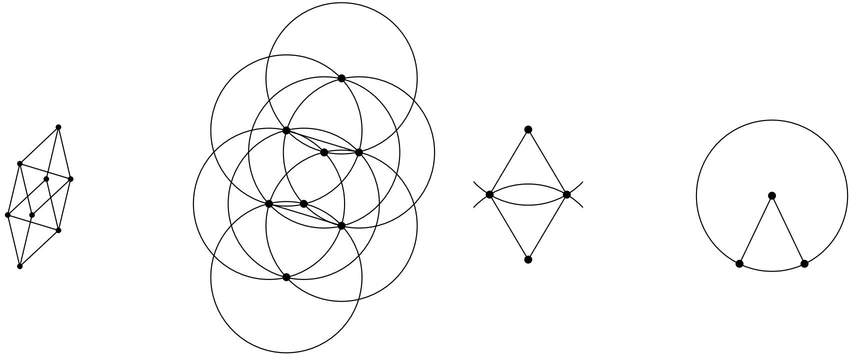

H has two times as many edges as G, and if we delete the duplicate edges, we have an upper bound for its crossings (it has at most $2 \cdot \binom{n}{2}$ of them), so we can use the crossing lemma to give an upper bound for the number of its edges.

Our improvement to Székely's proof

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Take a graph G with n vertices and $u(n)$ edges and transform it to a graph H the same way Székely did.

Suppose that G has no vertex with degree ≤ 2. Then H has $2u(n)$ edges counted with multiplicity and all of the edges of H have multiplicity at most 2.

Again, there are at most $n^2 - n$ crossings among the edges of H. Let us first forget about the crossings of the circles that occur in vertices and thus they are not crossings in H.

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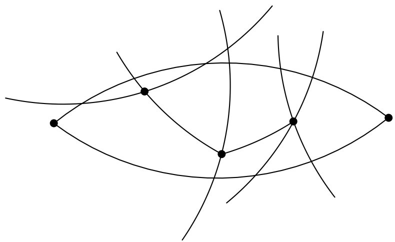

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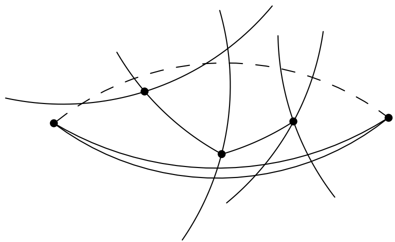

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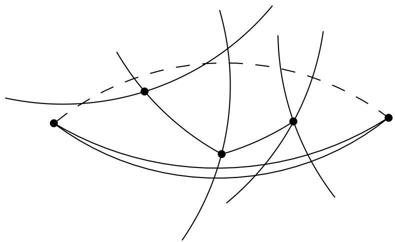

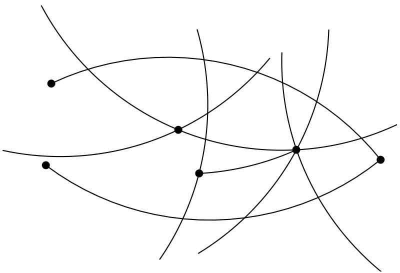

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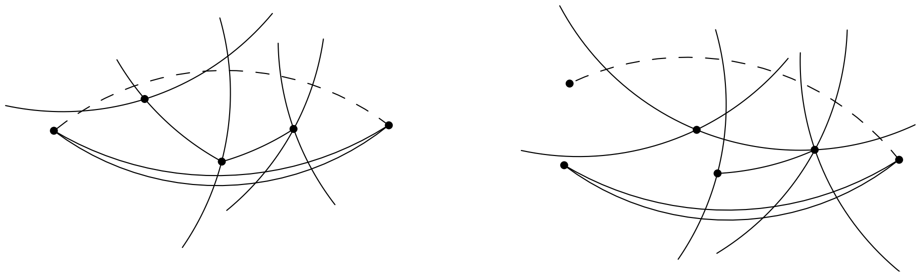

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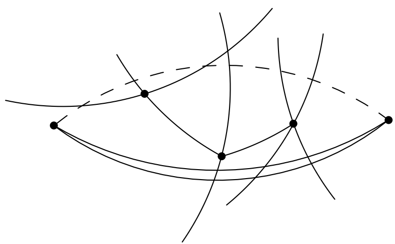

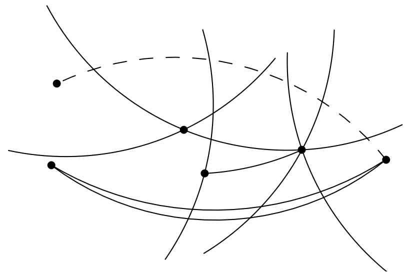

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If the condition of the crossing lemma applies, the double edges have at least $\frac{(u(n))^3}{29n^2}$ crossings counted without multiplicity, thus at least $\frac{4u(n)^3}{29n^2}$ crossings counted with multiplicity. From this we get the inequality $u(n) \leq \sqrt[3]{\frac{29}{4} \cdot (n^4 - n^3)}$.

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So now for all n's we have $u(n) \leq \sqrt[3]{\frac{29}{4}\cdot(n^4-n^3)} < \sqrt[3]{\frac{29}{4}}n^{4/3} < 1.936\cdot n^{4/3}$.

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Combining this with the lower bound for the number of crossings among the circles forming the edges of H that occur in vertices, we get

$n^2 - n \geq 4 \cdot |E(G)| - 12n + 24 + \sum_{v \in V(G)} \binom{deg(v)}{2} \geq$

$4 \cdot u(n) - 12n + 24 + n \cdot \left(1 - \left\{\frac{2u(n)}{n}\right\}\right) \cdot \binom{\left\lfloor \frac{2u(n)}{n} \right\rfloor}{2} + n \cdot \left\{\frac{2u(n)}{n}\right\} \cdot \binom{\left\lceil \frac{2u(n)}{n} \right\rceil}{2}$ (where $\{x\}$ denotes the fractional part of x) or $u(n) \leq u(n-1) + 2$.

n	$u(n)$	Lower bounding graph(s)
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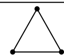
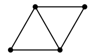
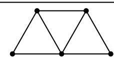
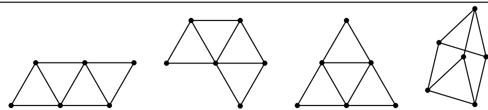
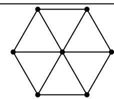
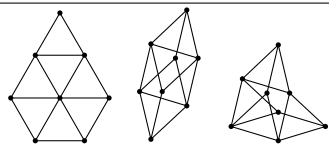
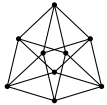
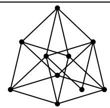
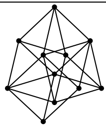
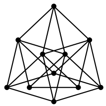
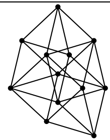
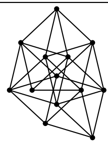
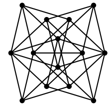
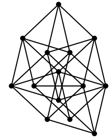
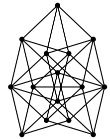
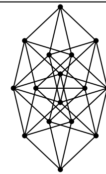
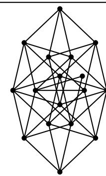
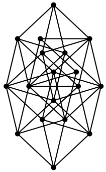
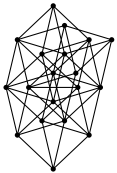
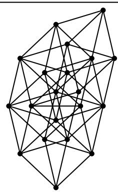
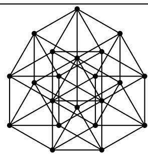
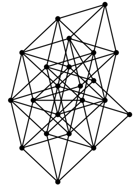
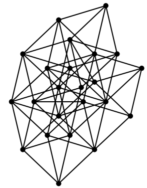
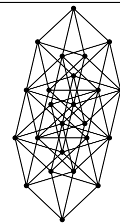
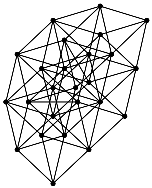
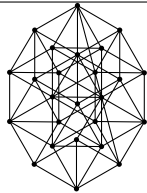
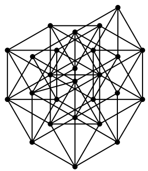
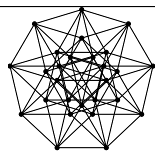
2	1	
3	3	
4	5^*	
5	7^*	
6	9^*	
7	12	
8	14^*	

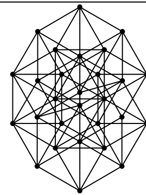
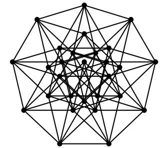
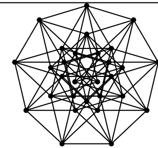
9	18	
10	20*	
11	23*	 
12	27	
13	30*	

14	33*	 
15	37 or 38	
16	41 or 42*	
17	43–47	

18	46–52	
19	50–57*	
20	54–63	
21	57–68*	

22	60–**72**	
23	64–**77**	
24	68–**82**	

25	72–**87**	 
26	76–**92**	
27	81–**97**	

28	85-**102**	 
29	**89–108**	
30	**93–113**