Between Two Shapes, Using the Hausdorff Distance

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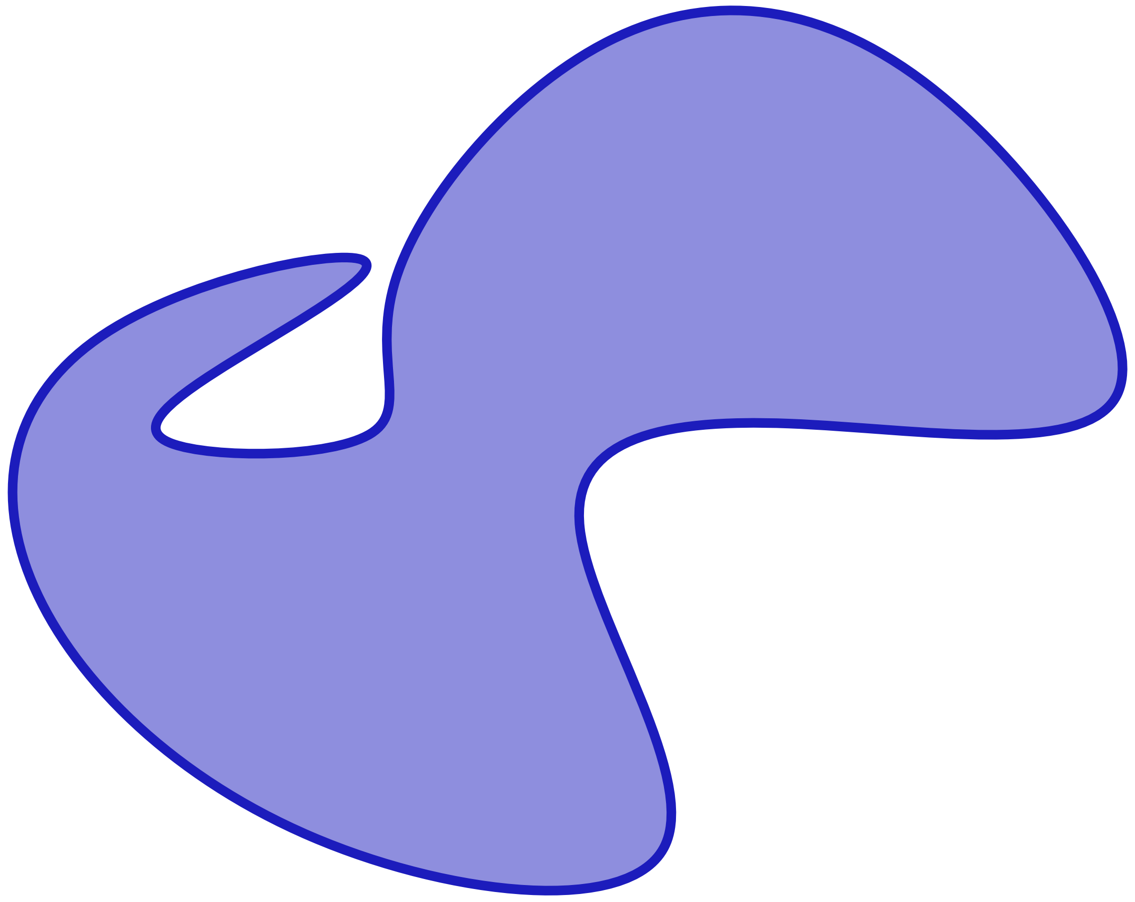
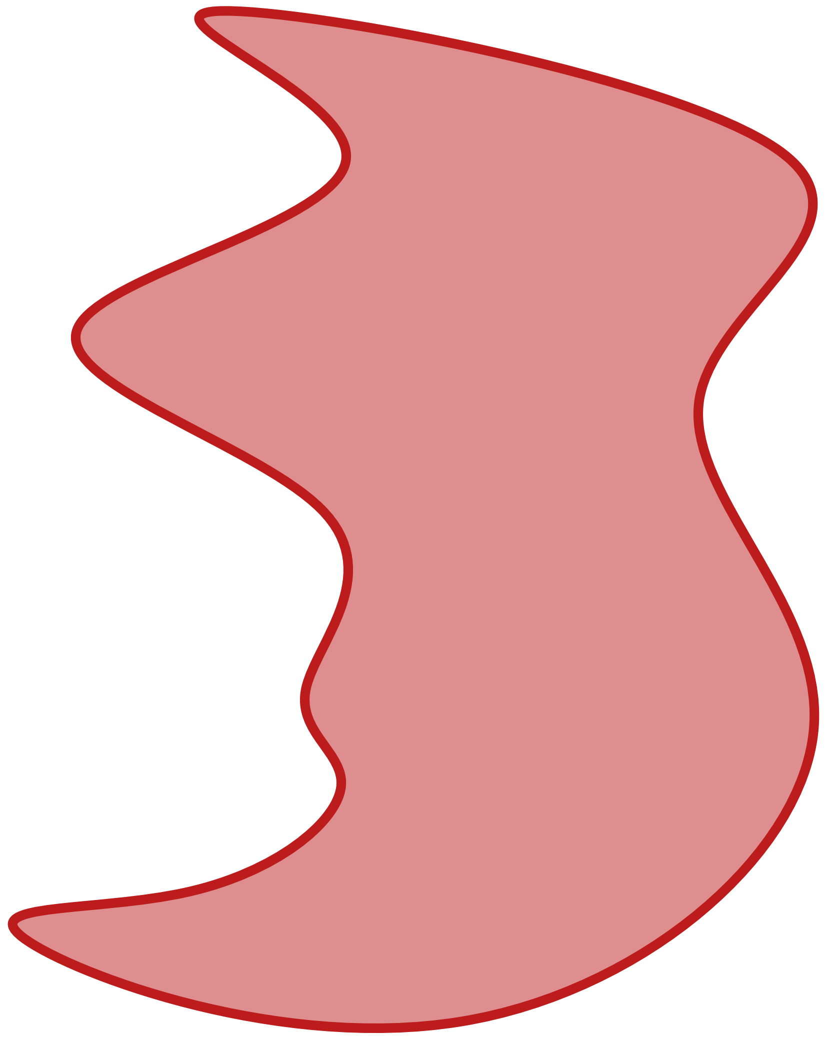
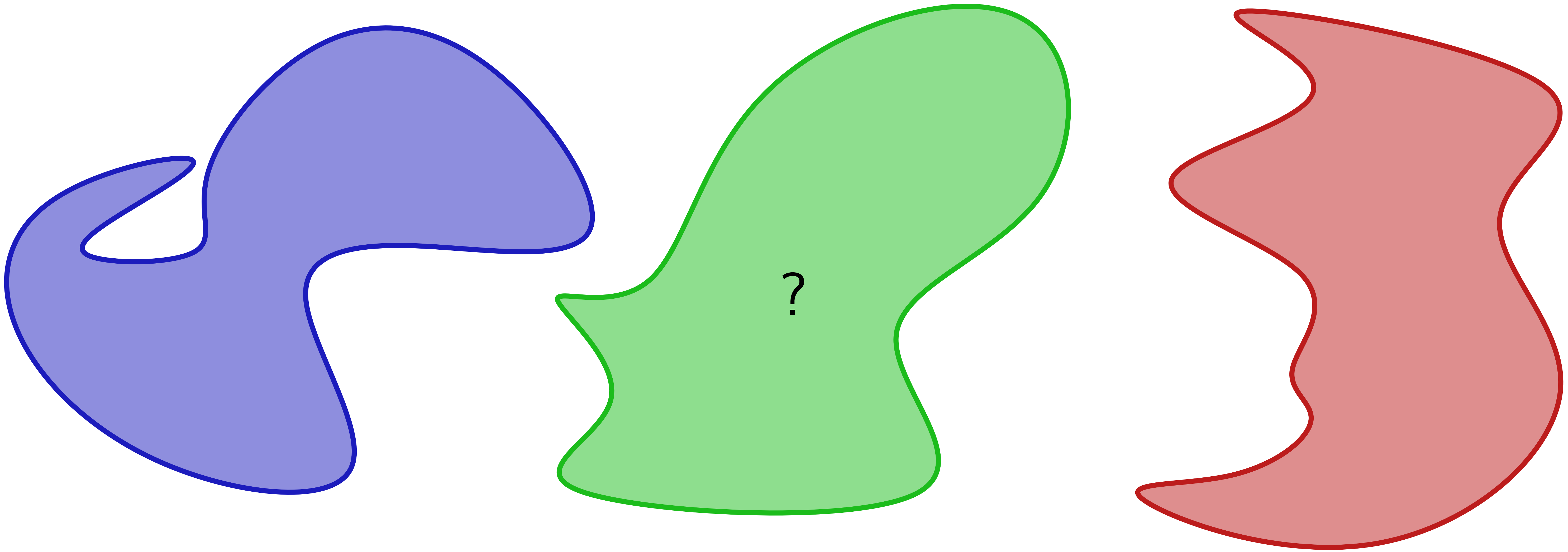
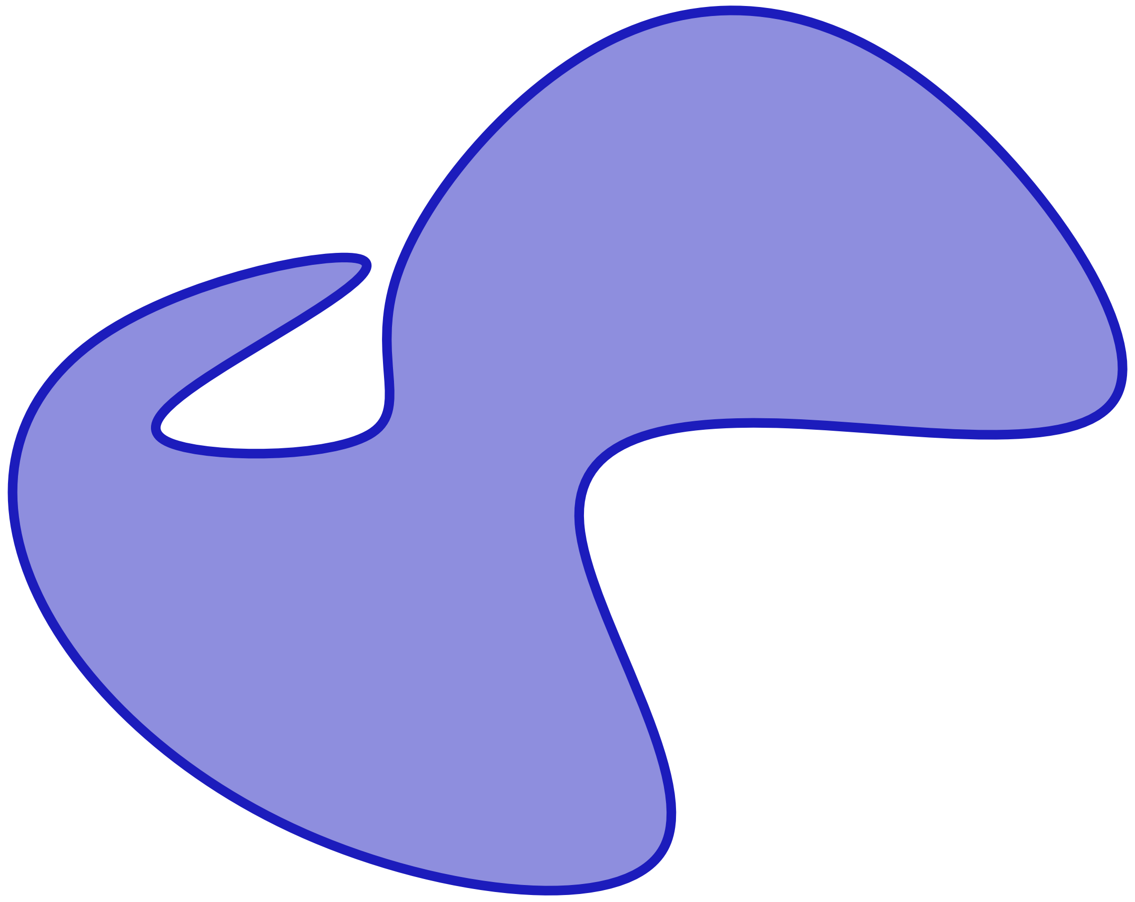
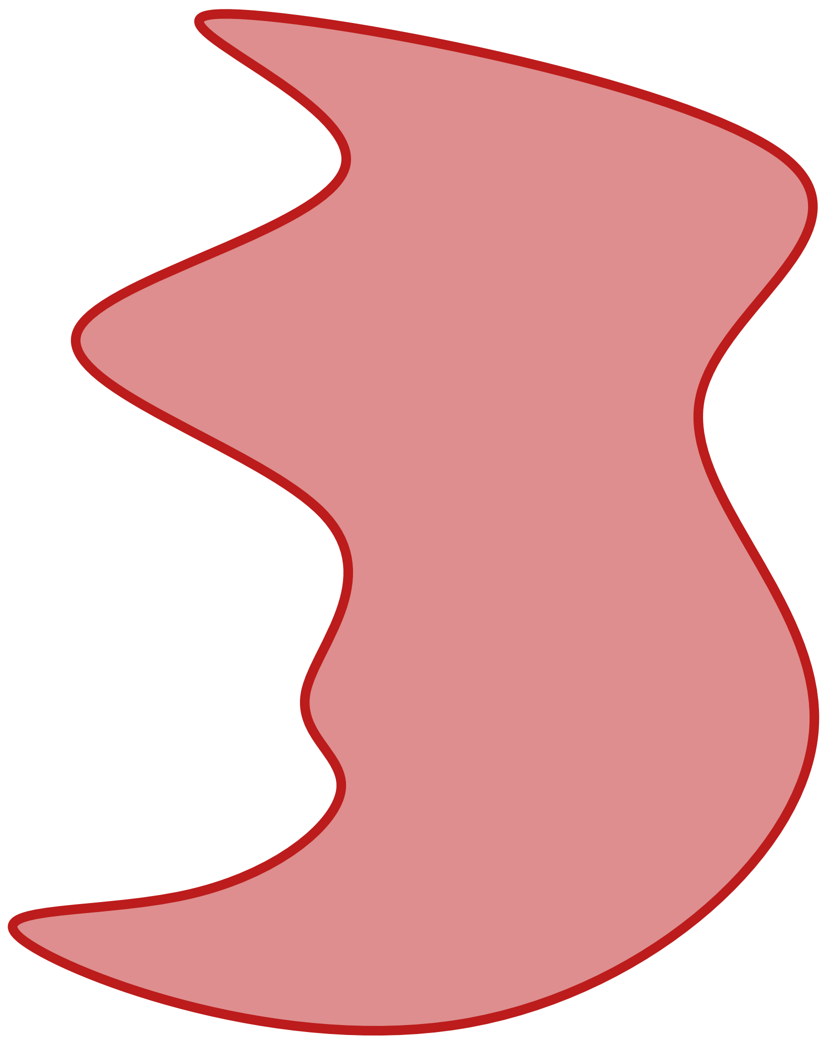
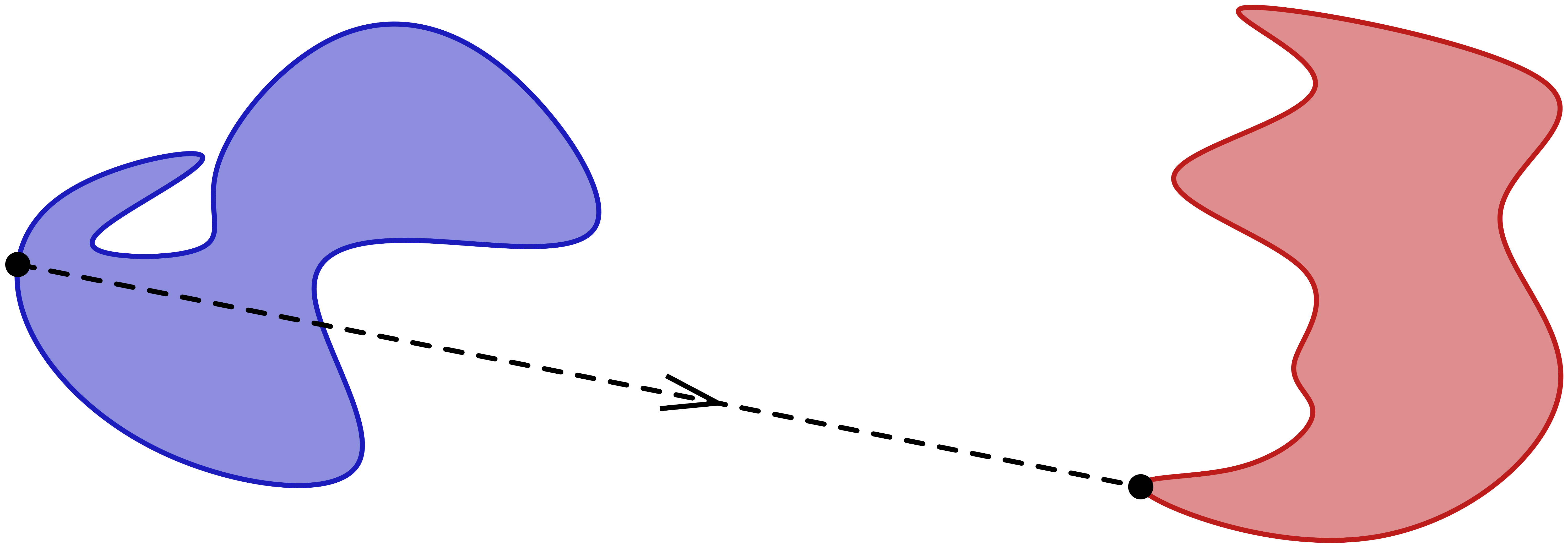





Directed Hausdorff distance $A \to B$.



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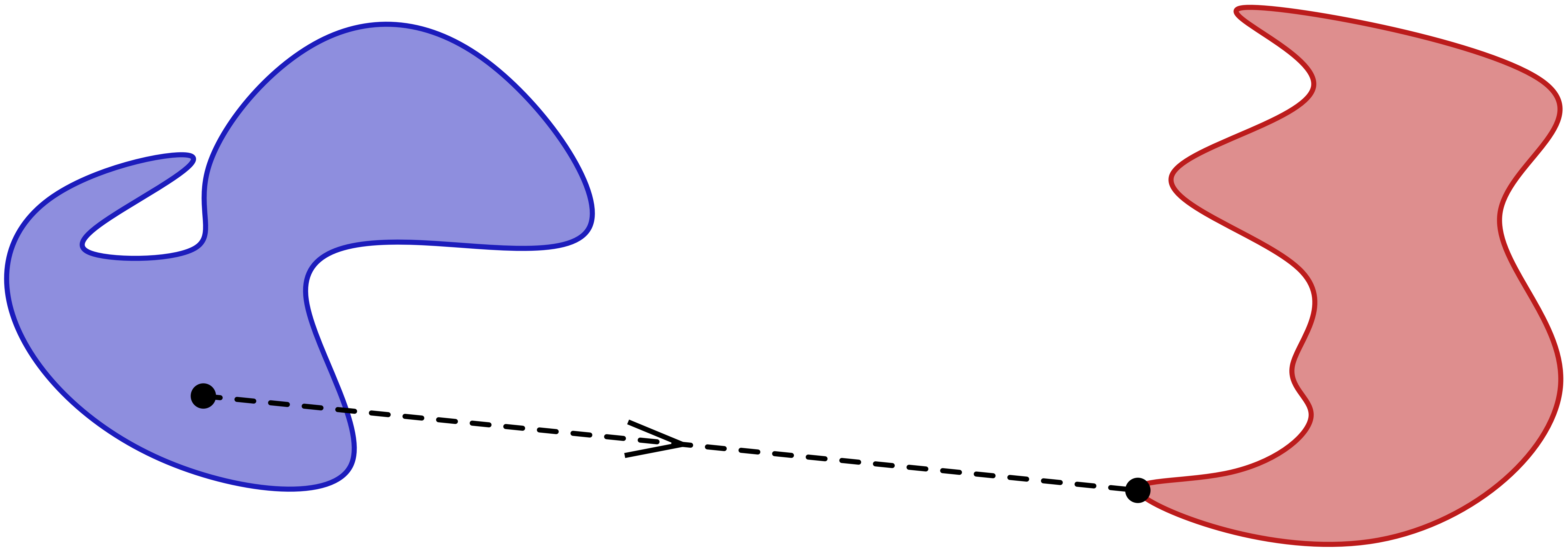

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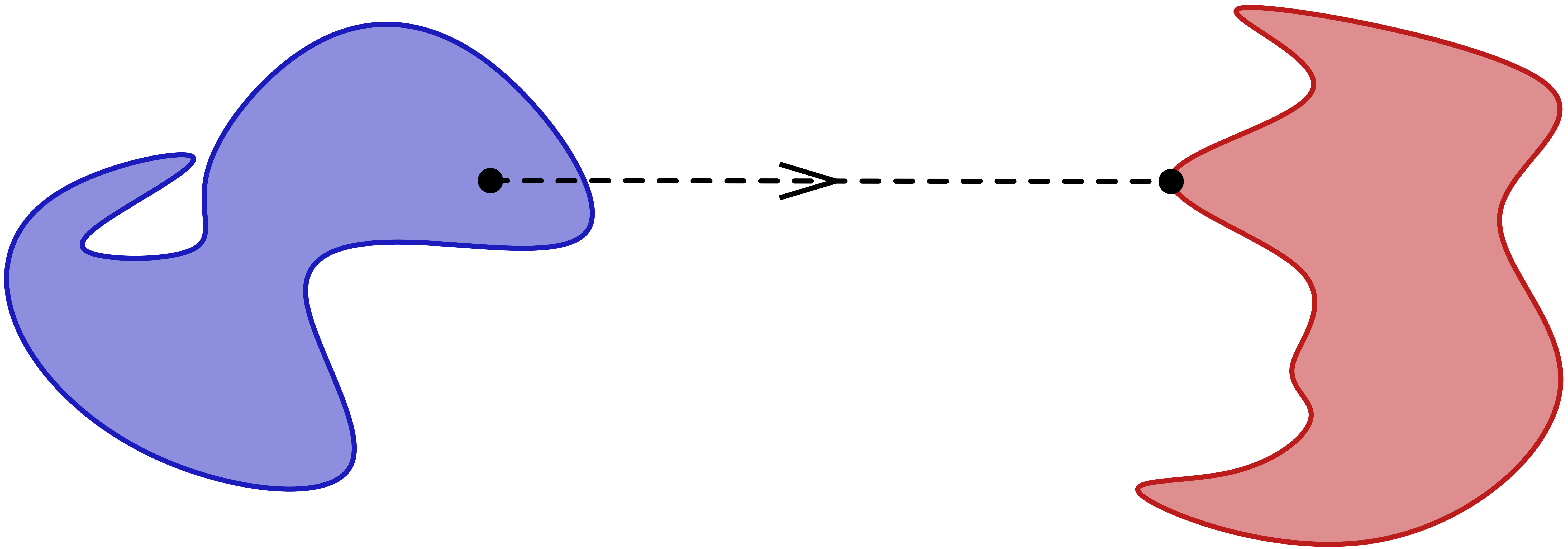
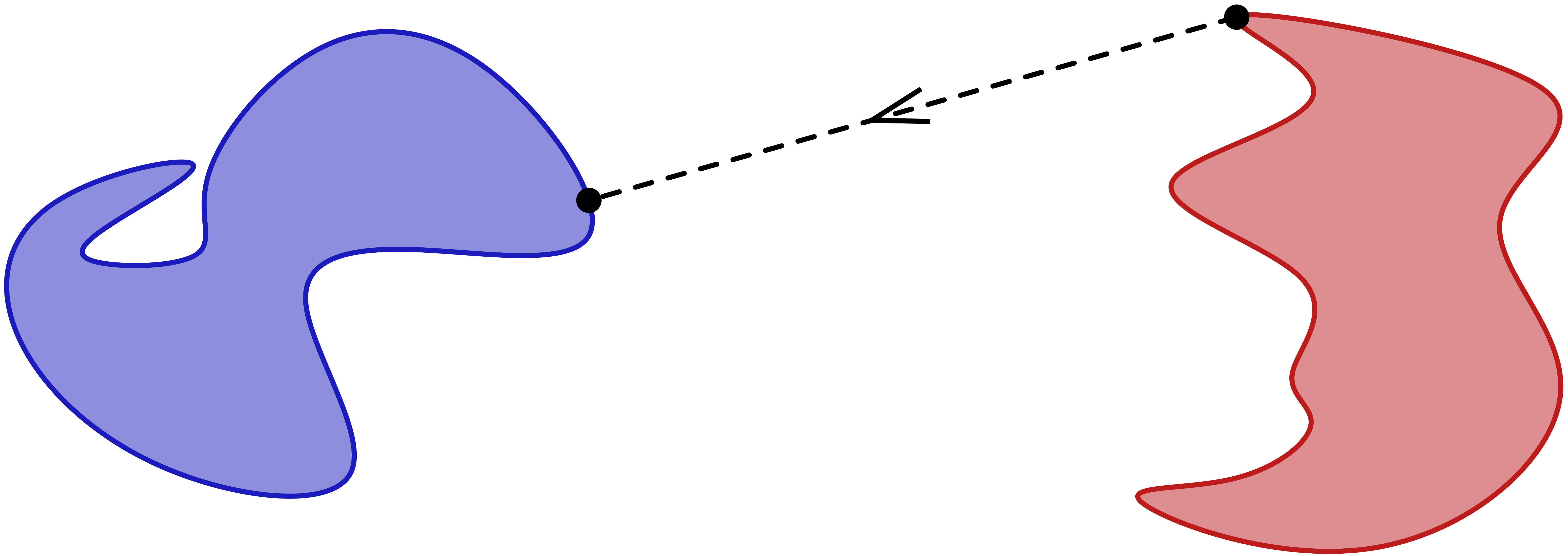


Directed Hausdorff distance $B \to A$.



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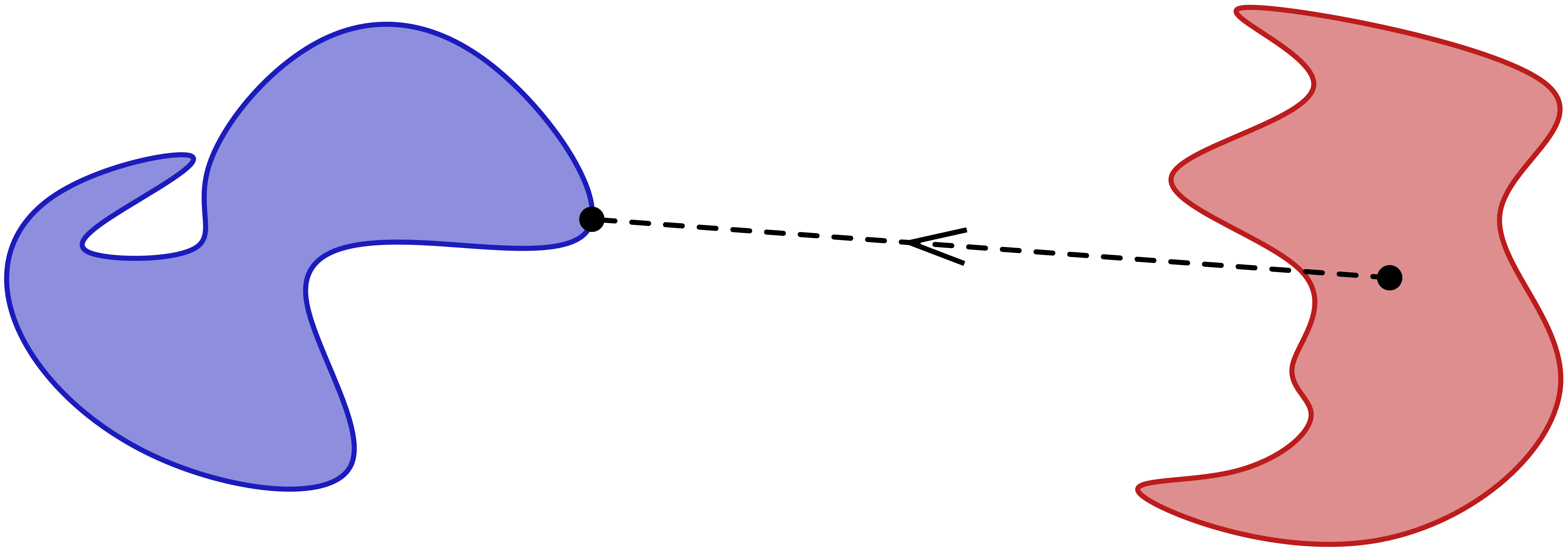

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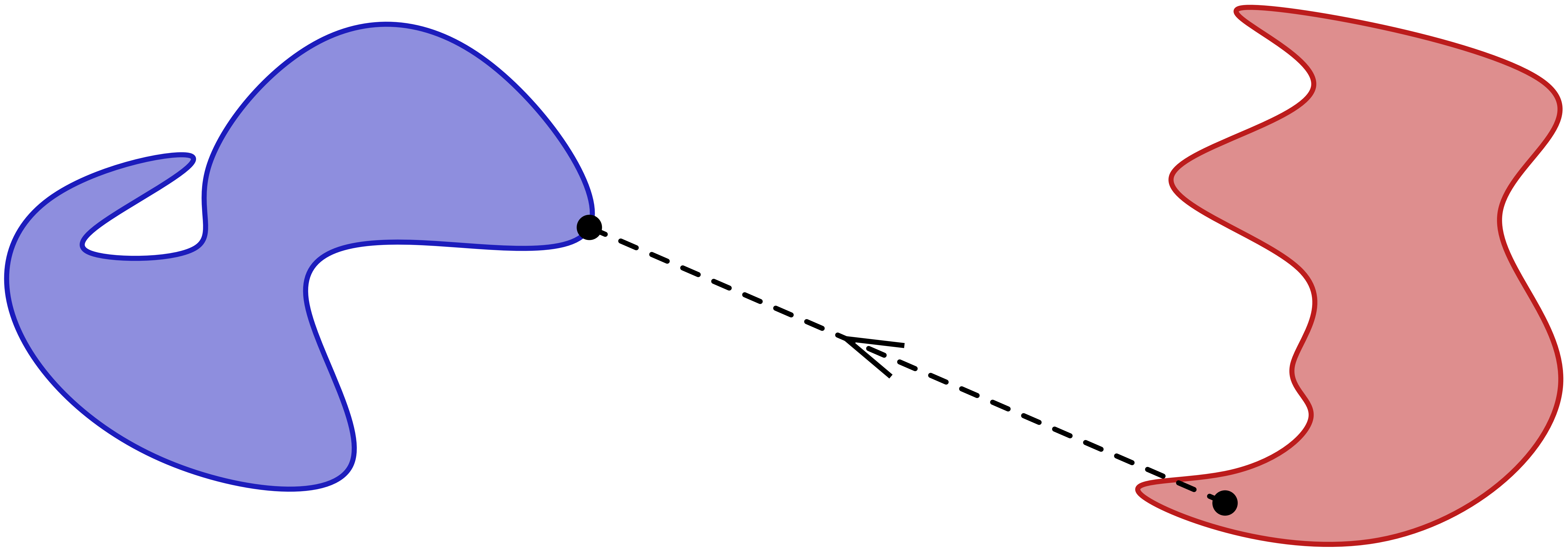
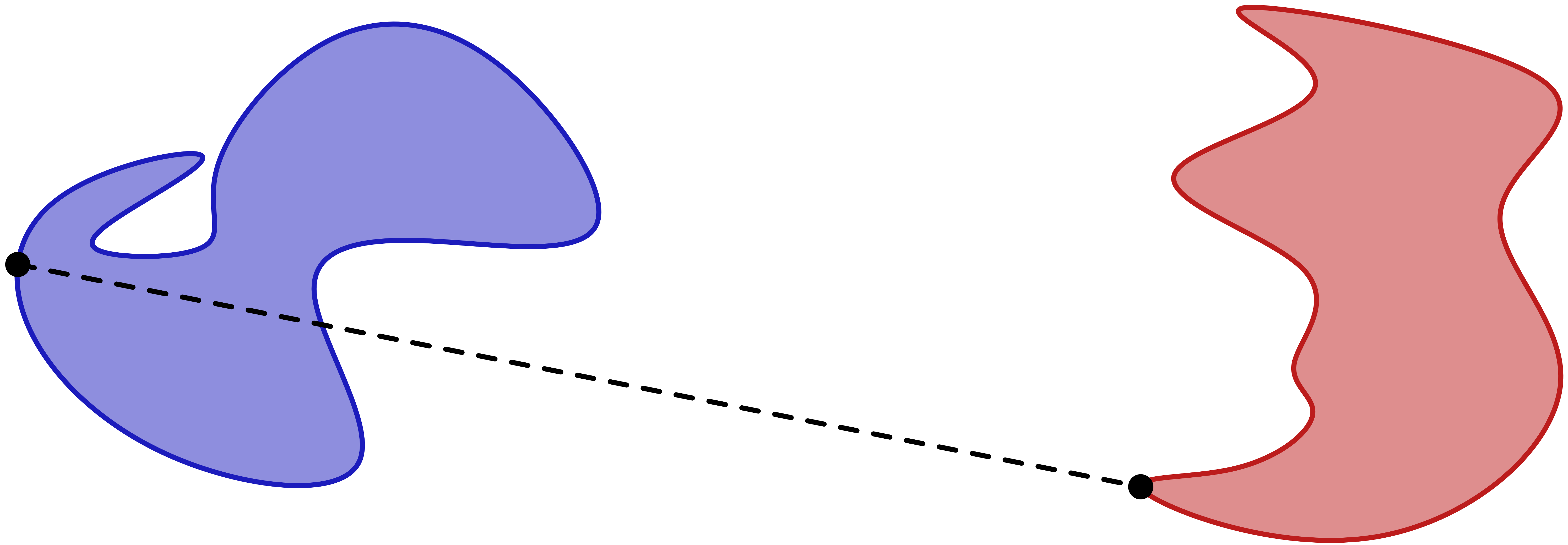
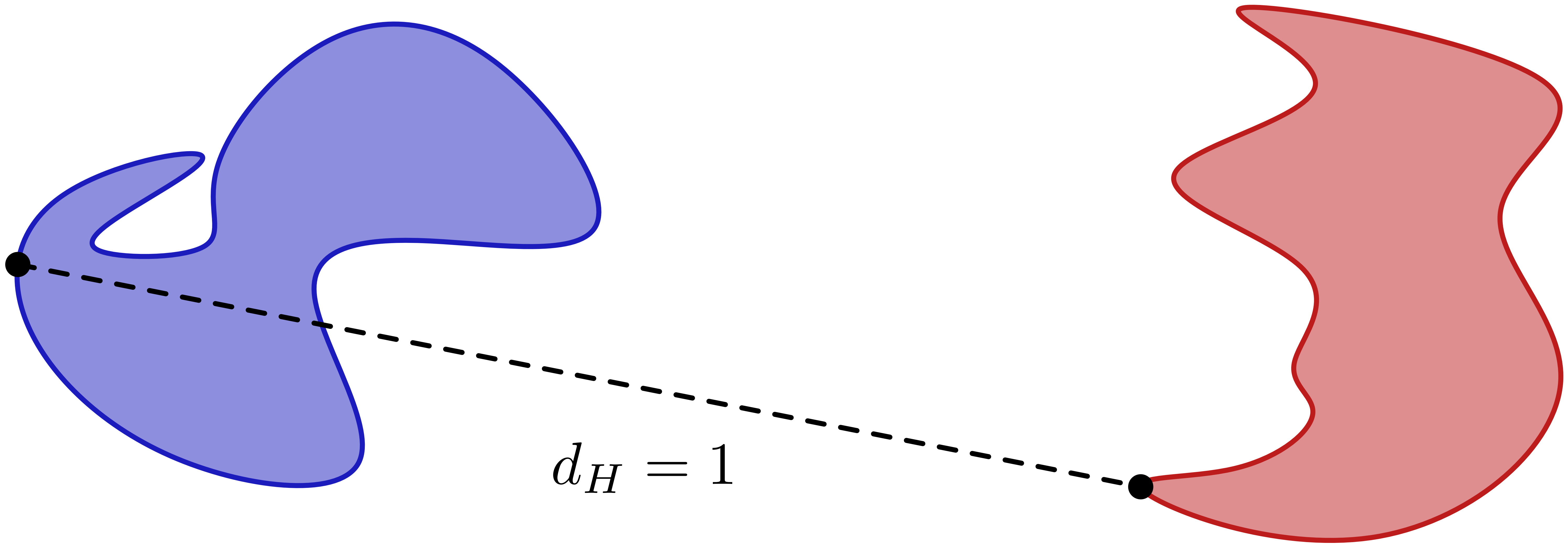
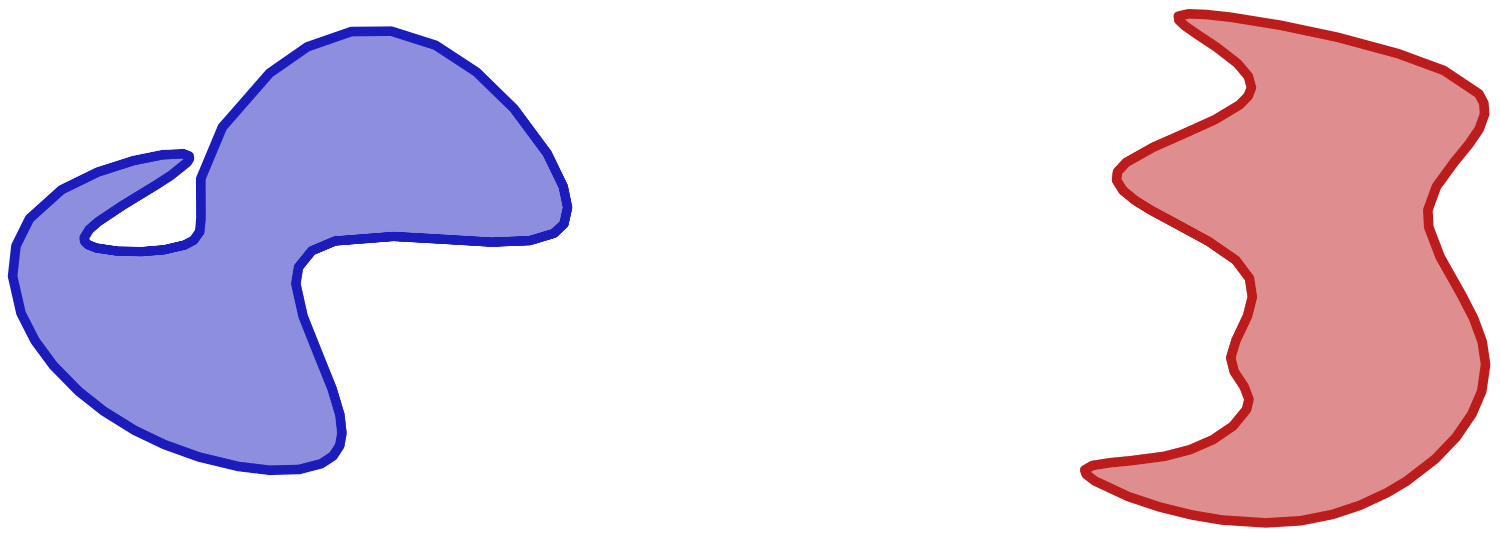


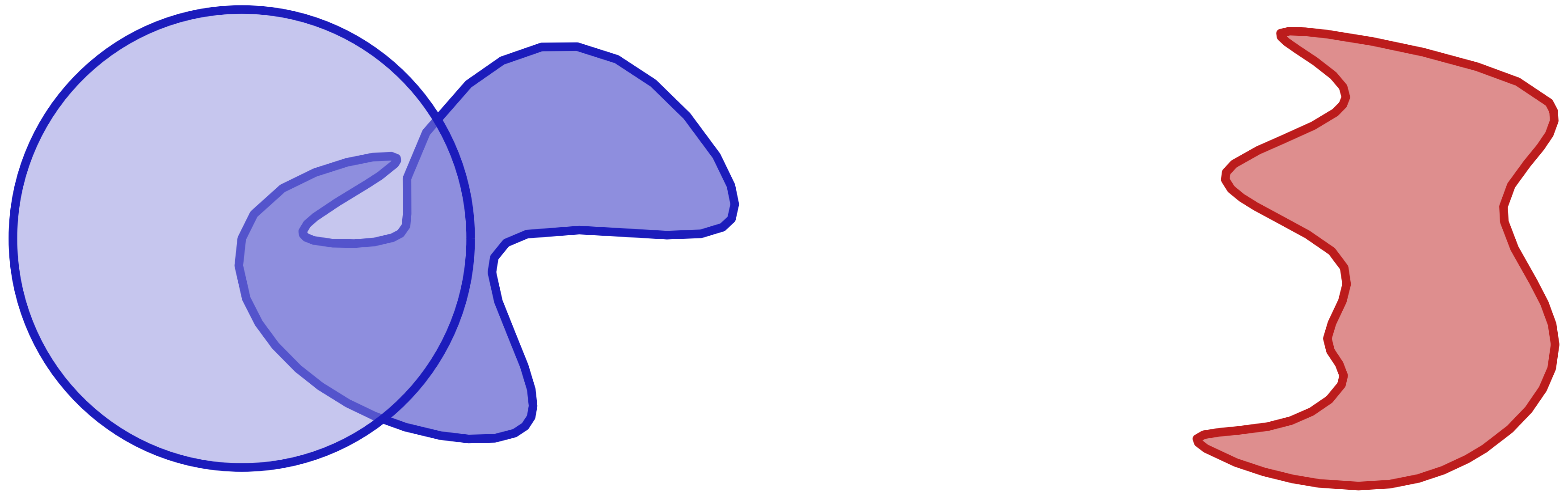
Undirected Hausdorff distance $A \leftrightarrow B$.

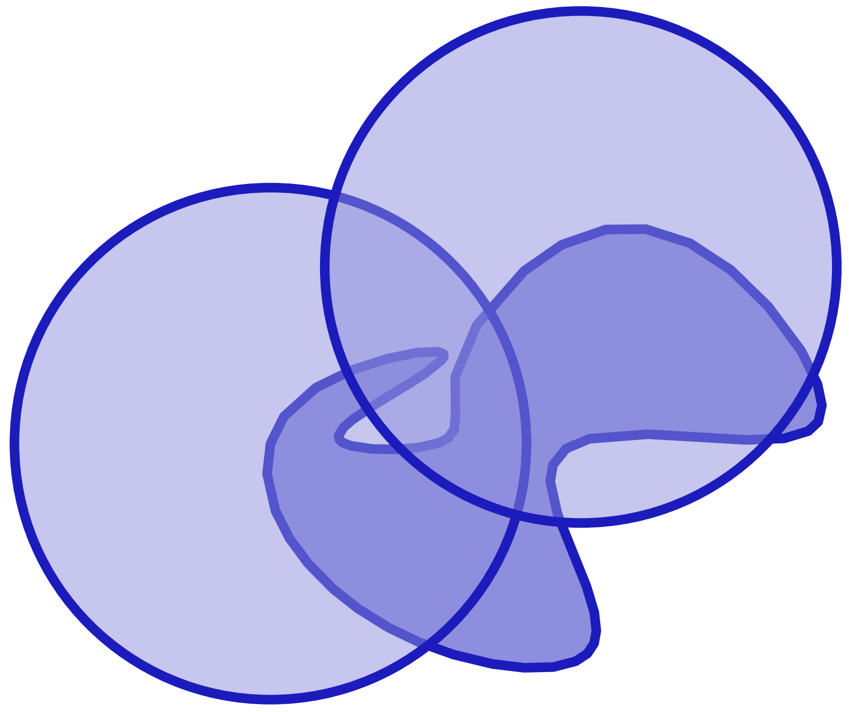


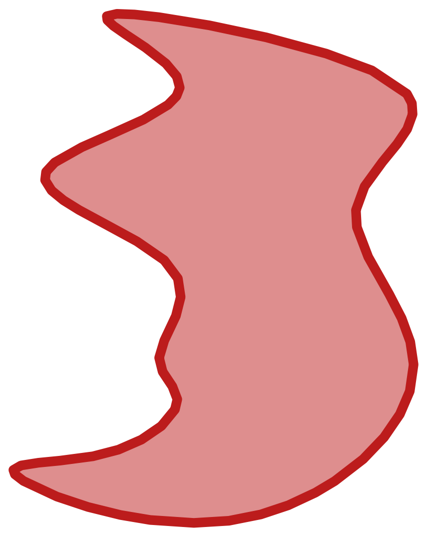
Undirected Hausdorff distance $A \leftrightarrow B$.

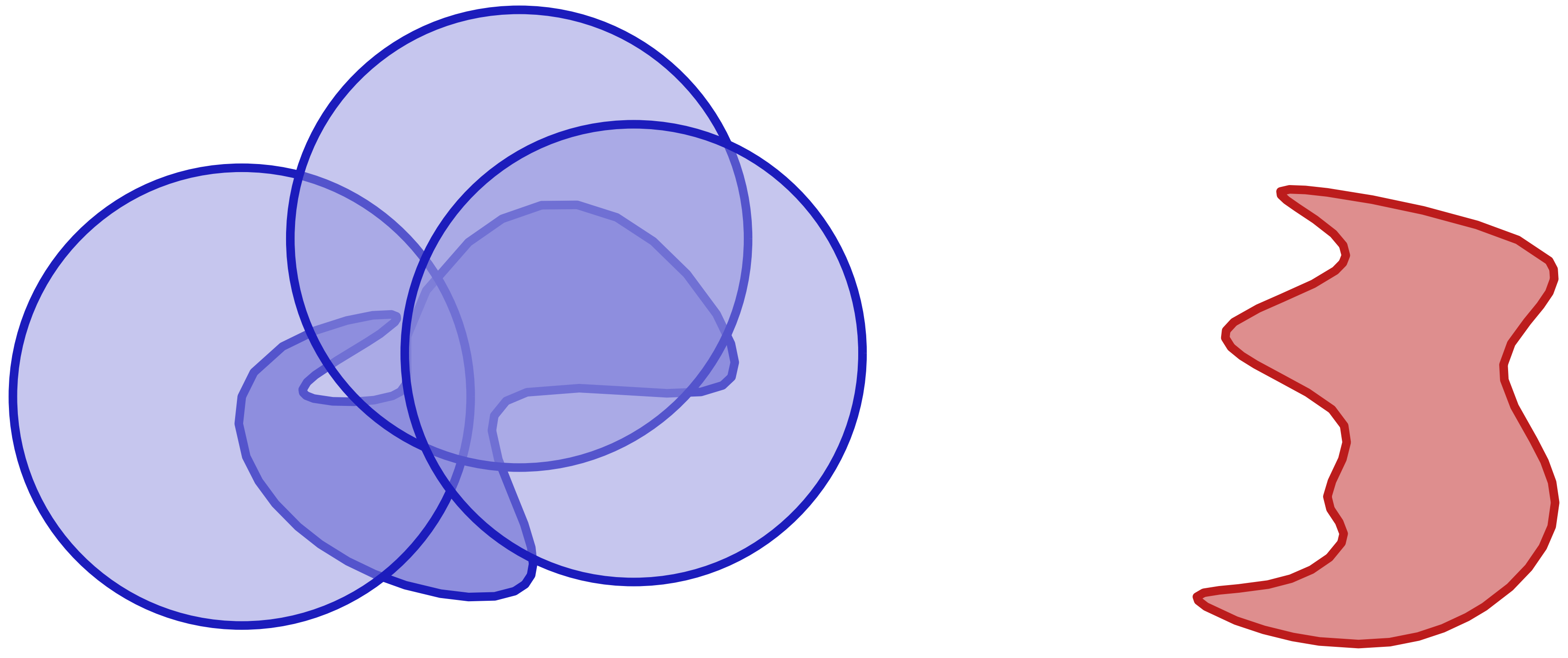


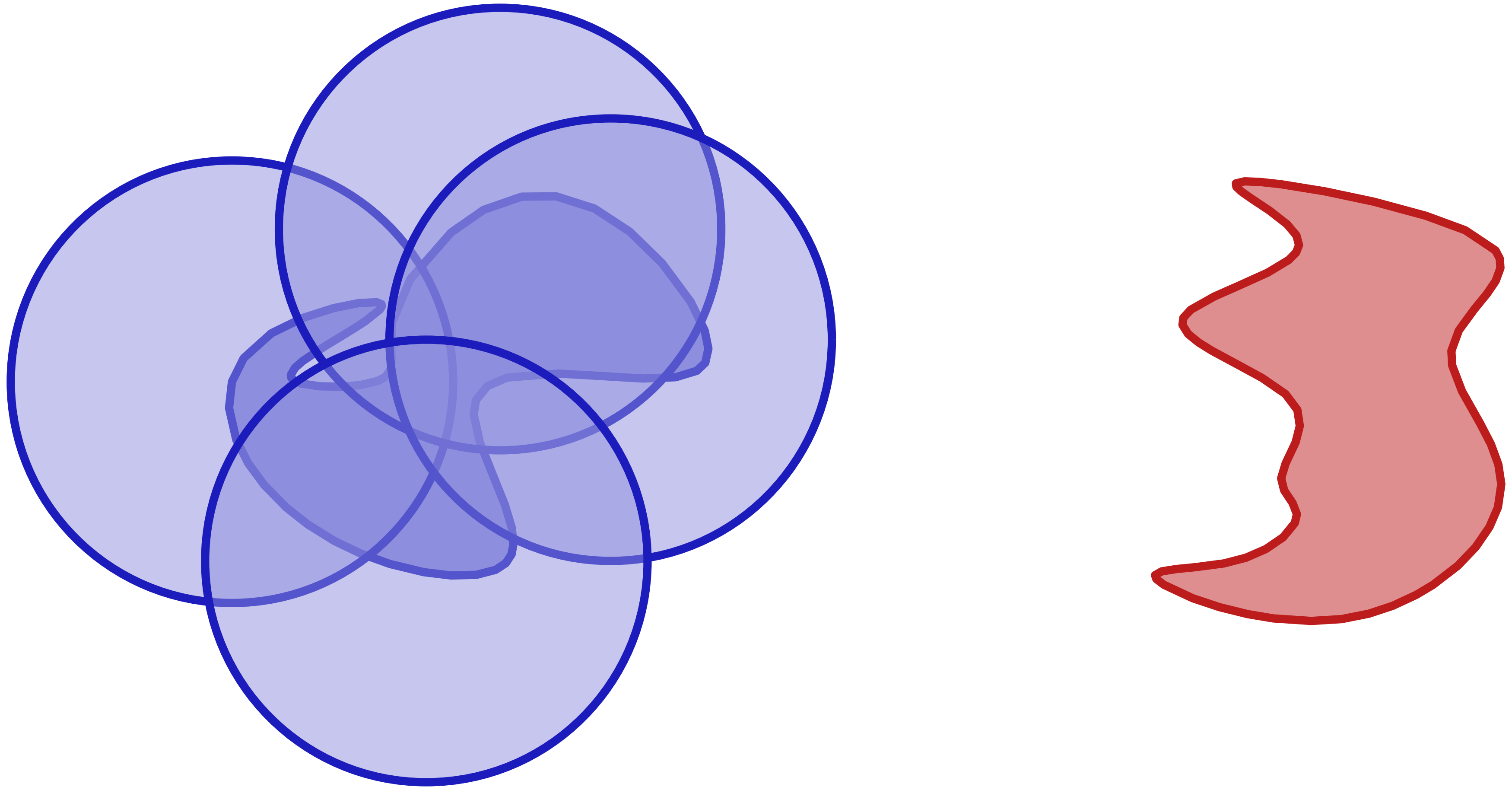


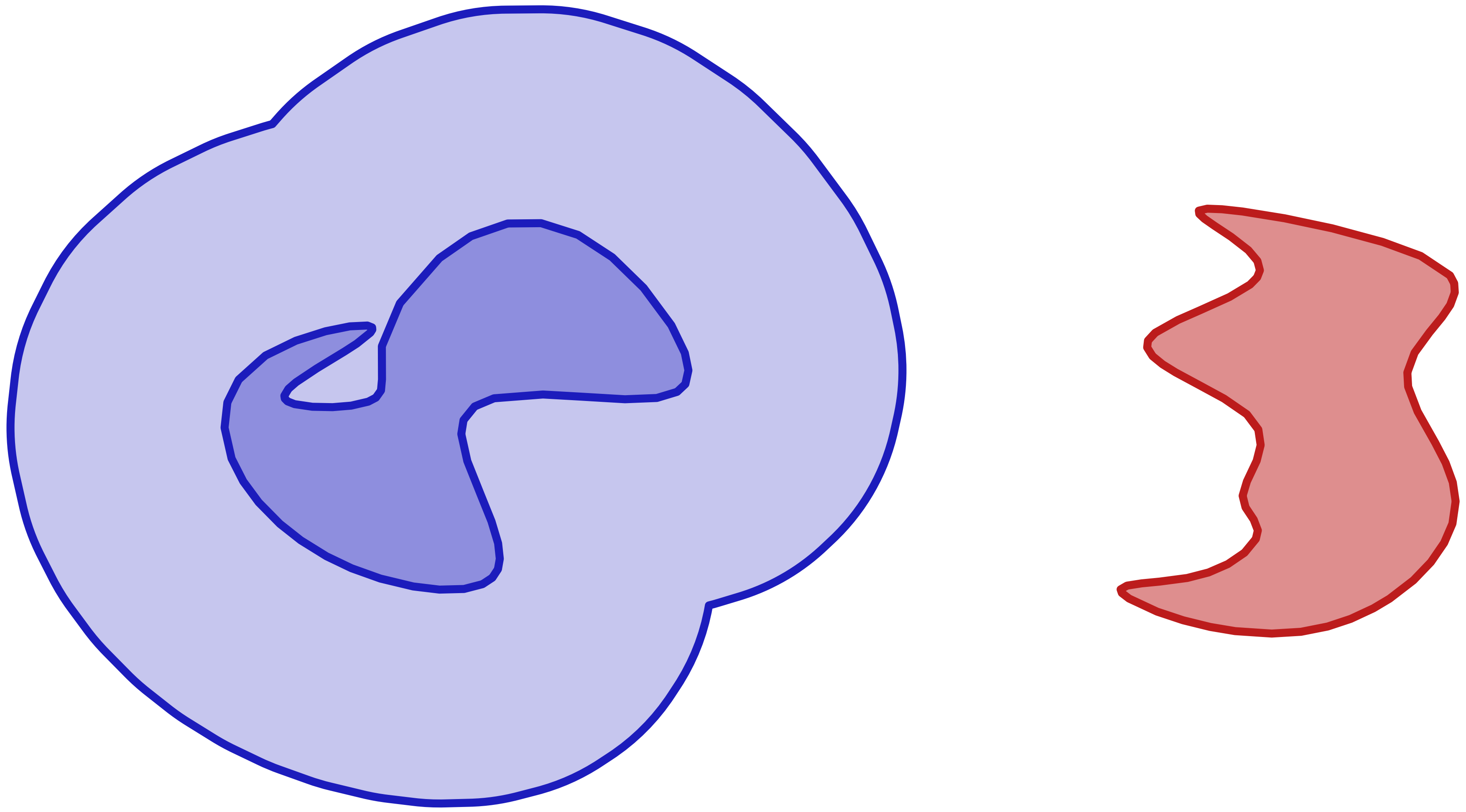


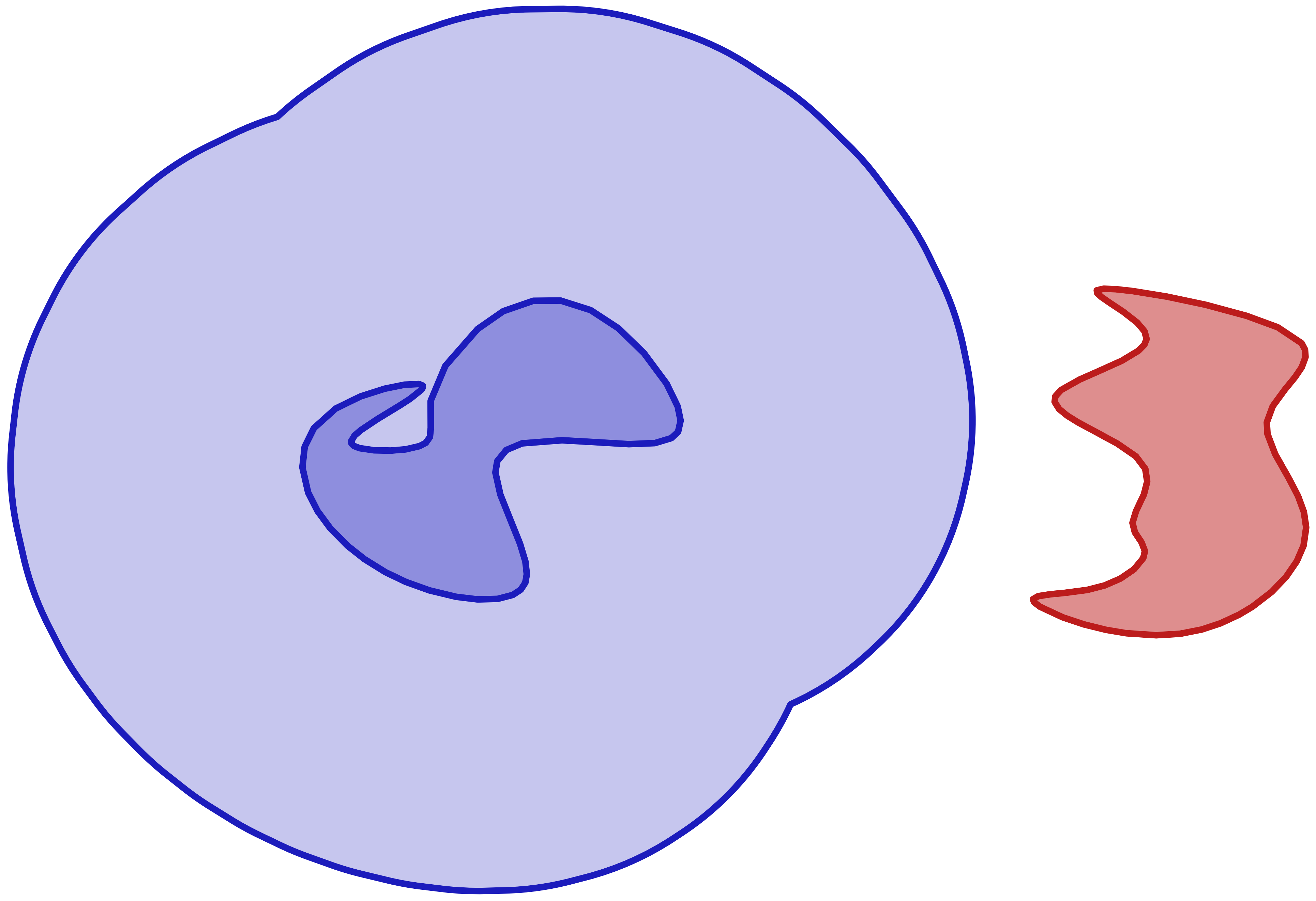


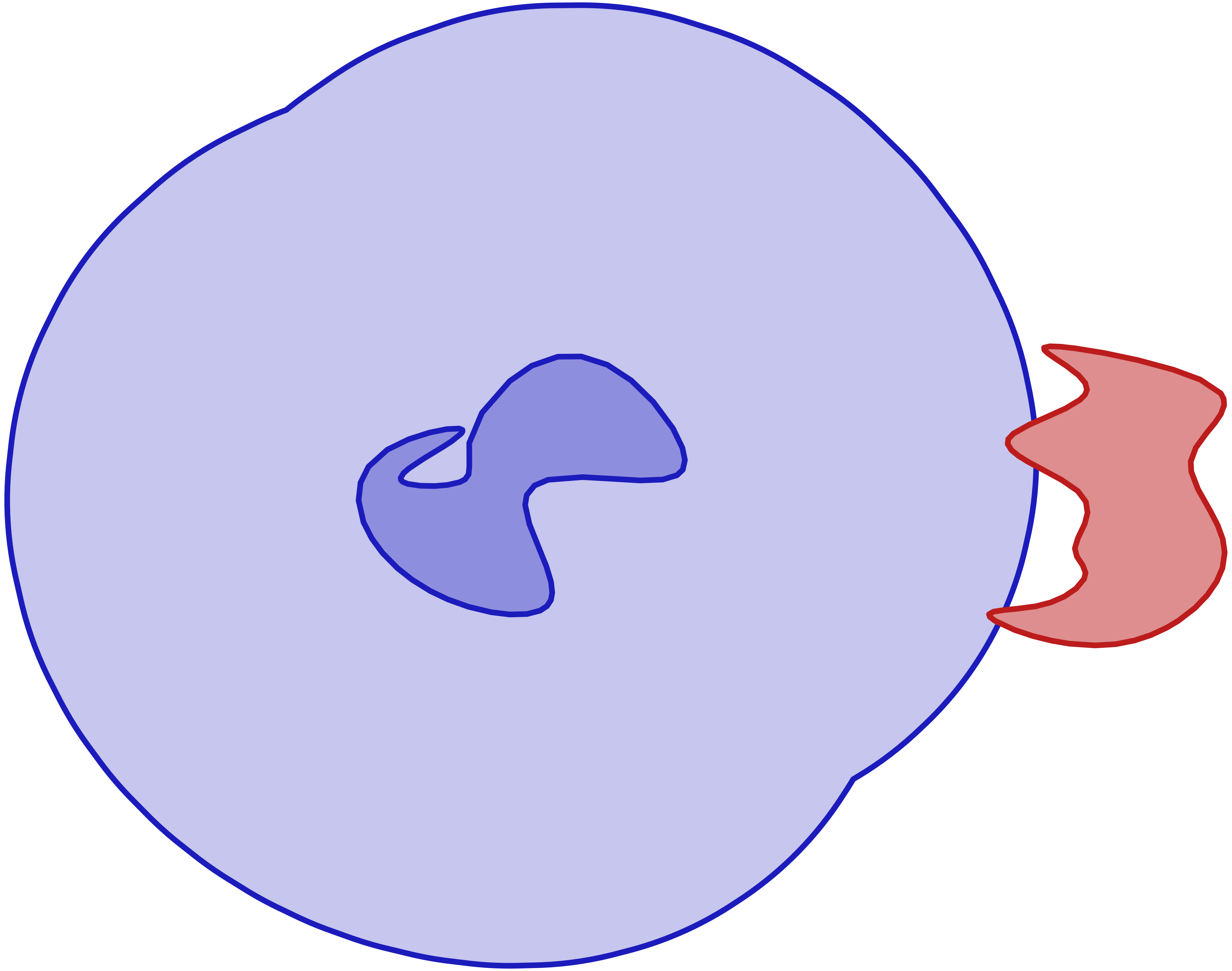


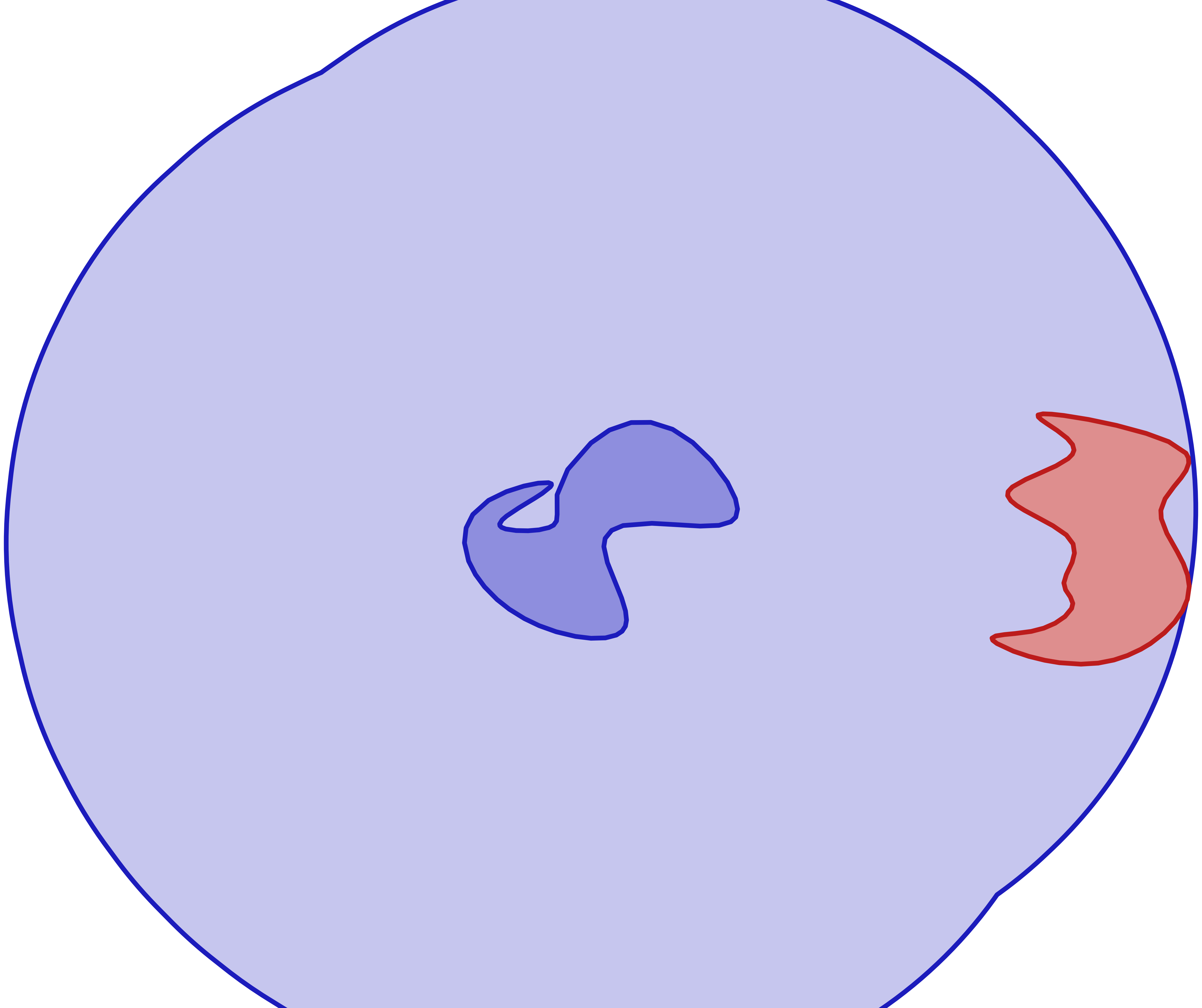




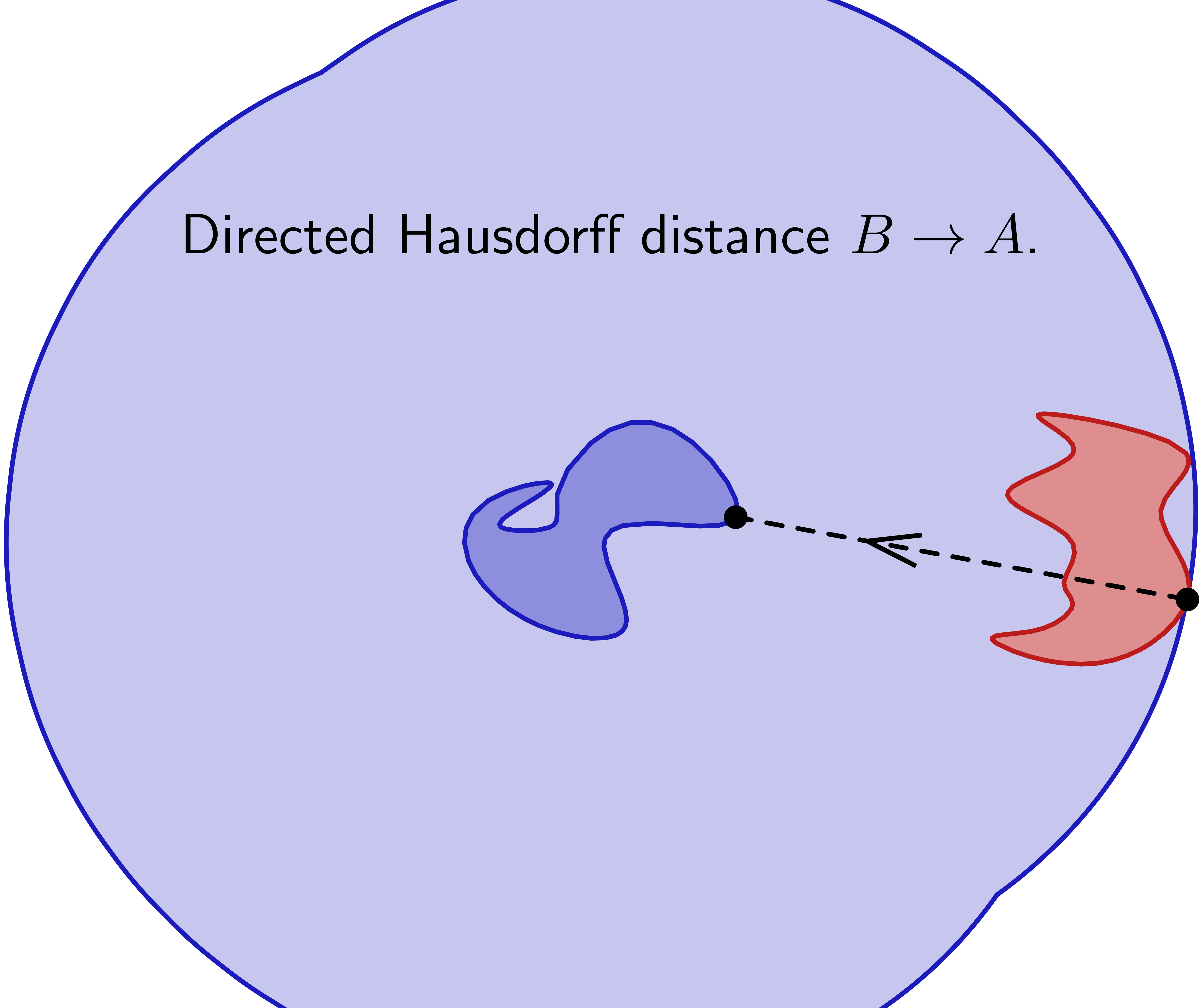


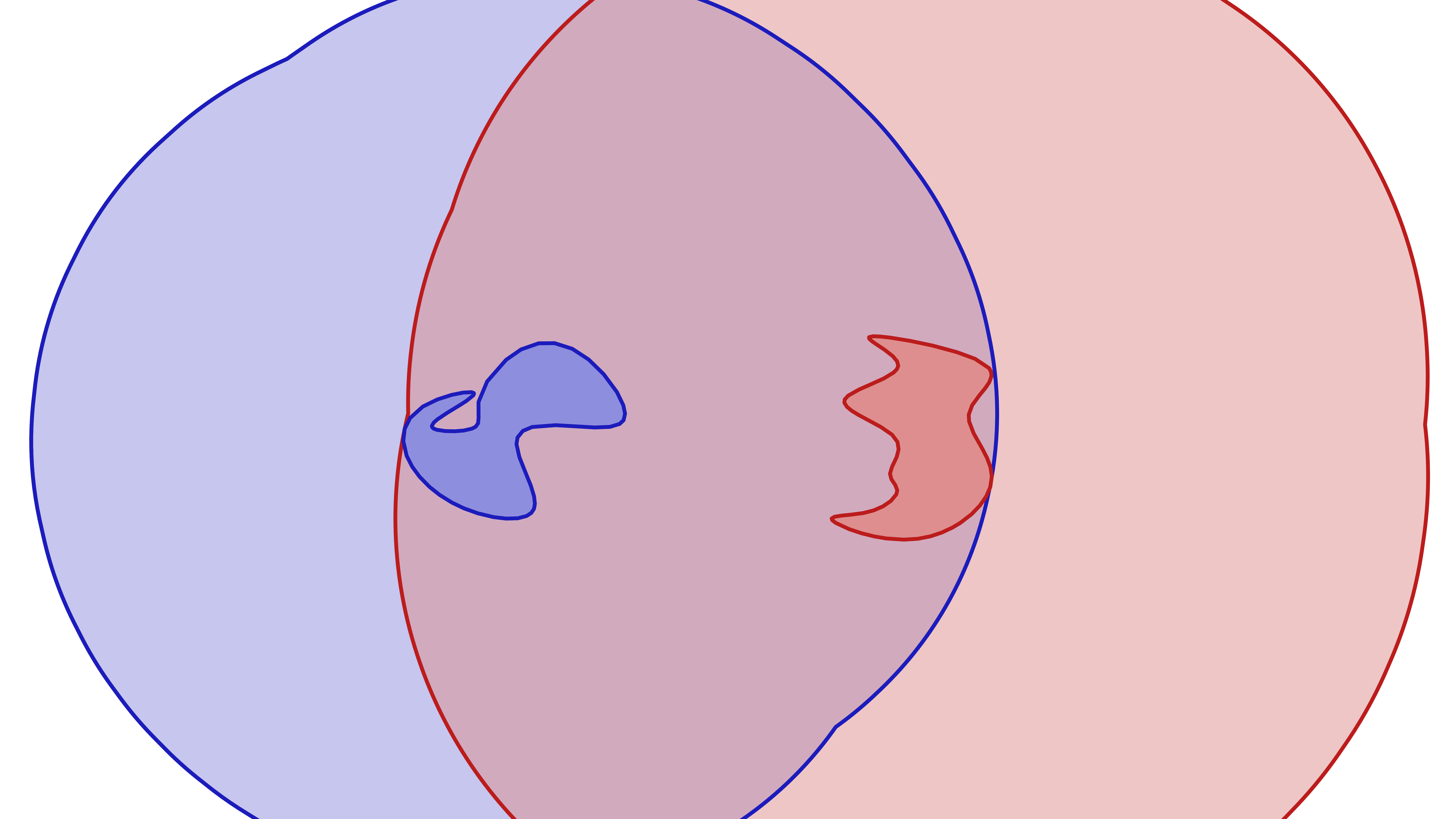


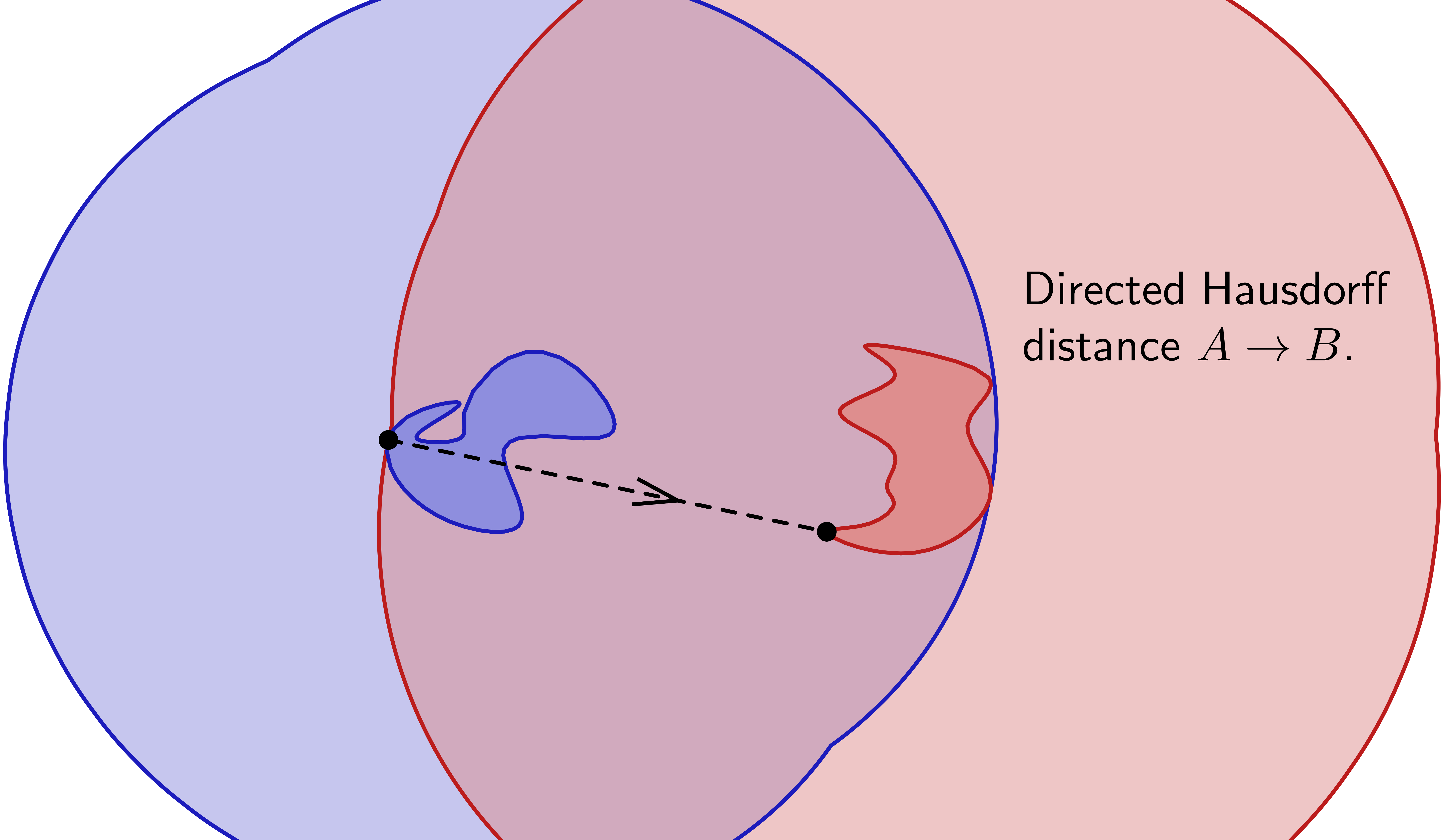




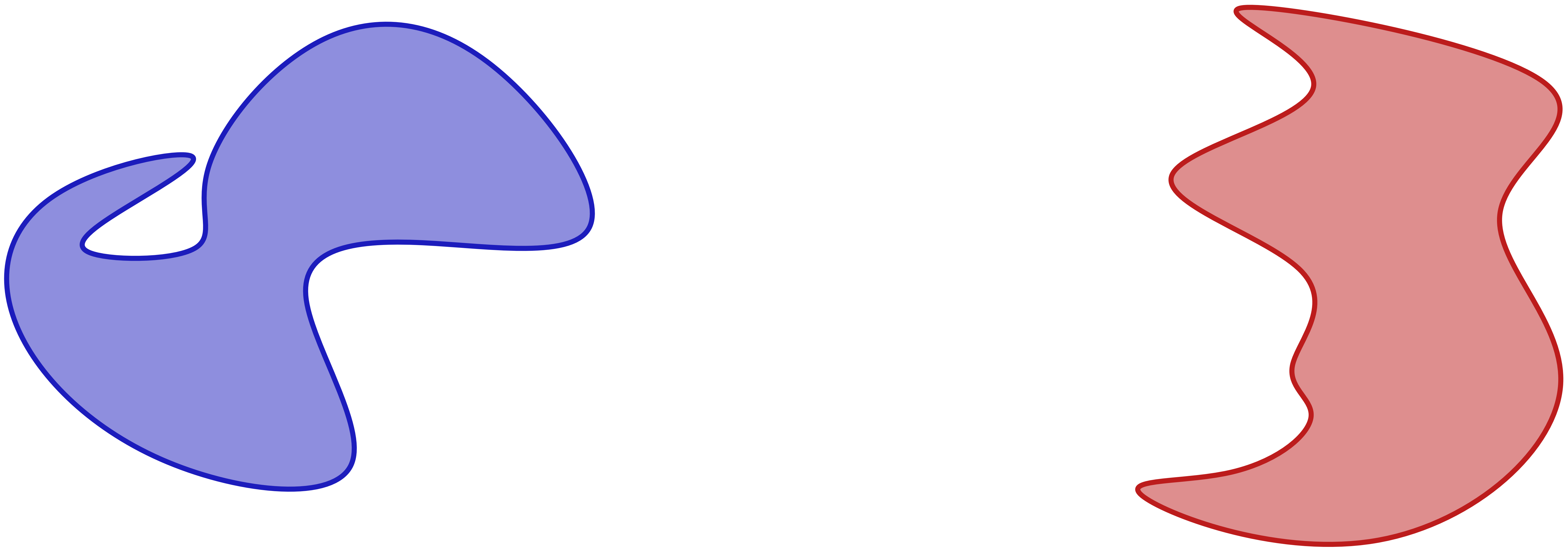
Directed Hausdorff distance $B \to A$.

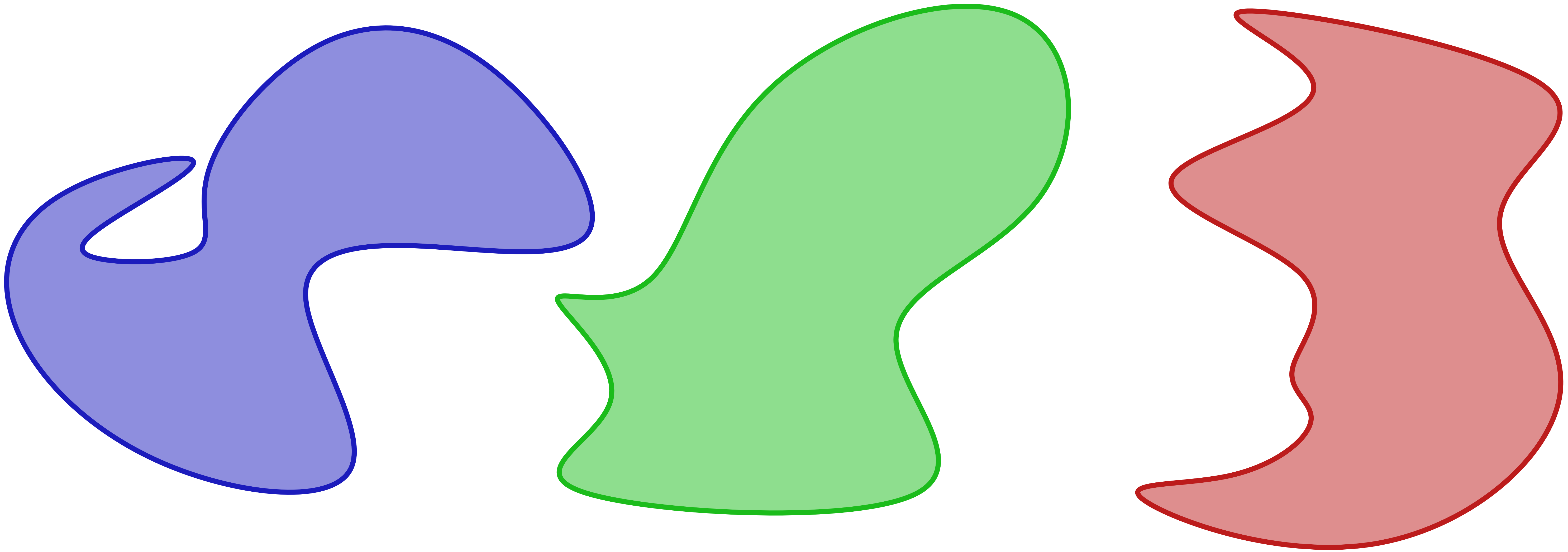


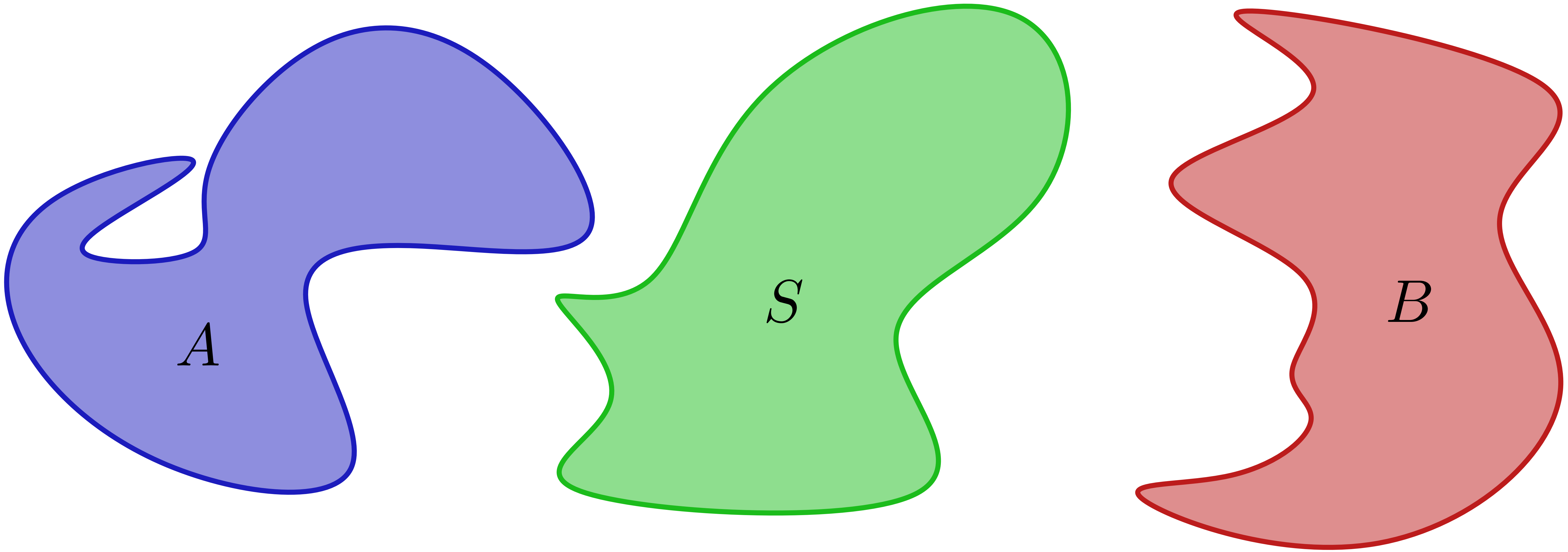




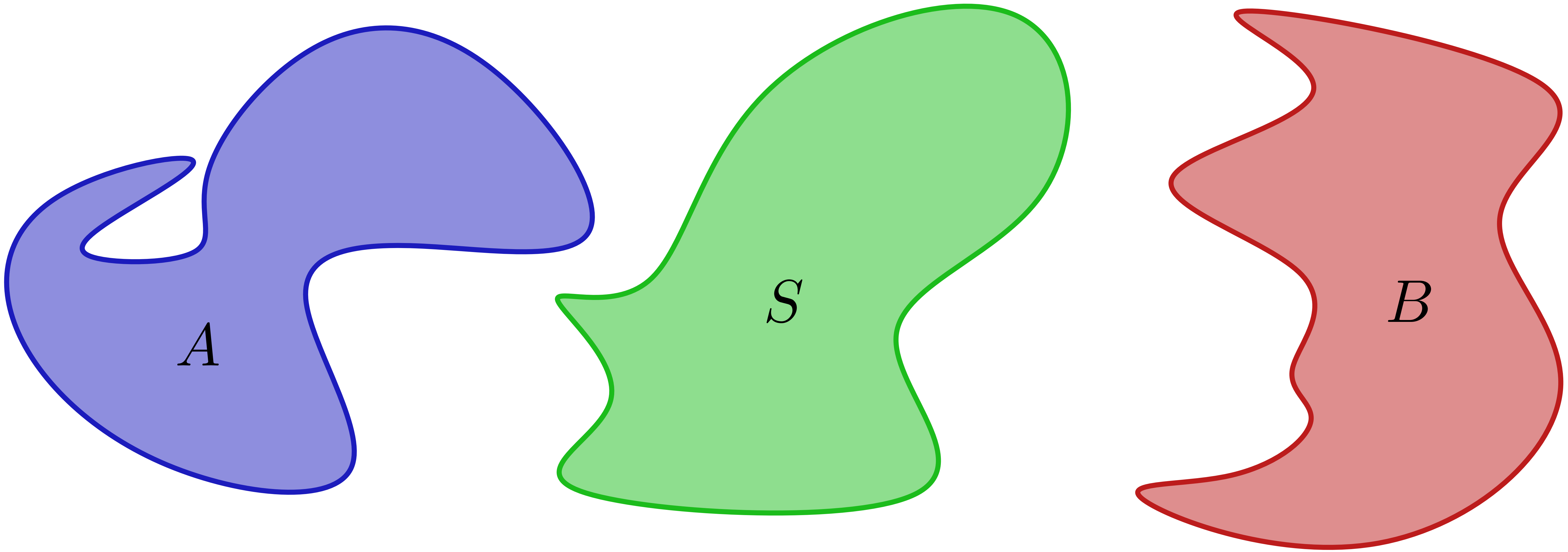
Directed Hausdorff distance $A \to B$.






A
S
B

Find S with minimal Hausdorff distance to A and B.



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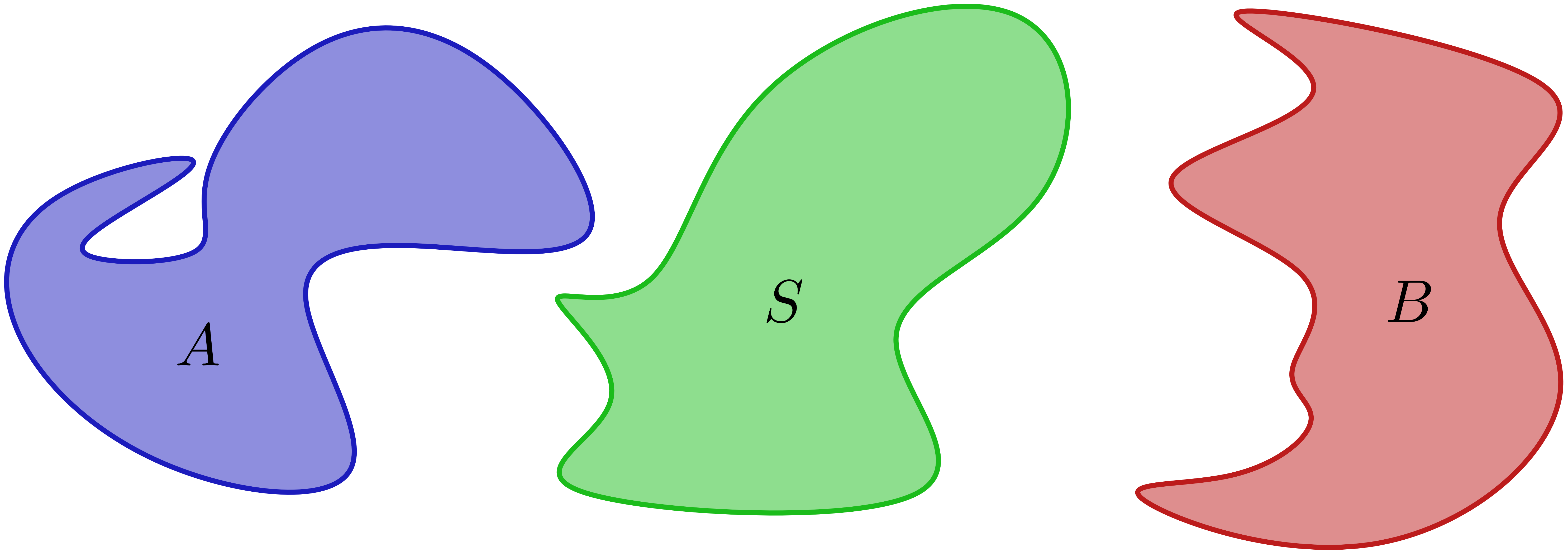
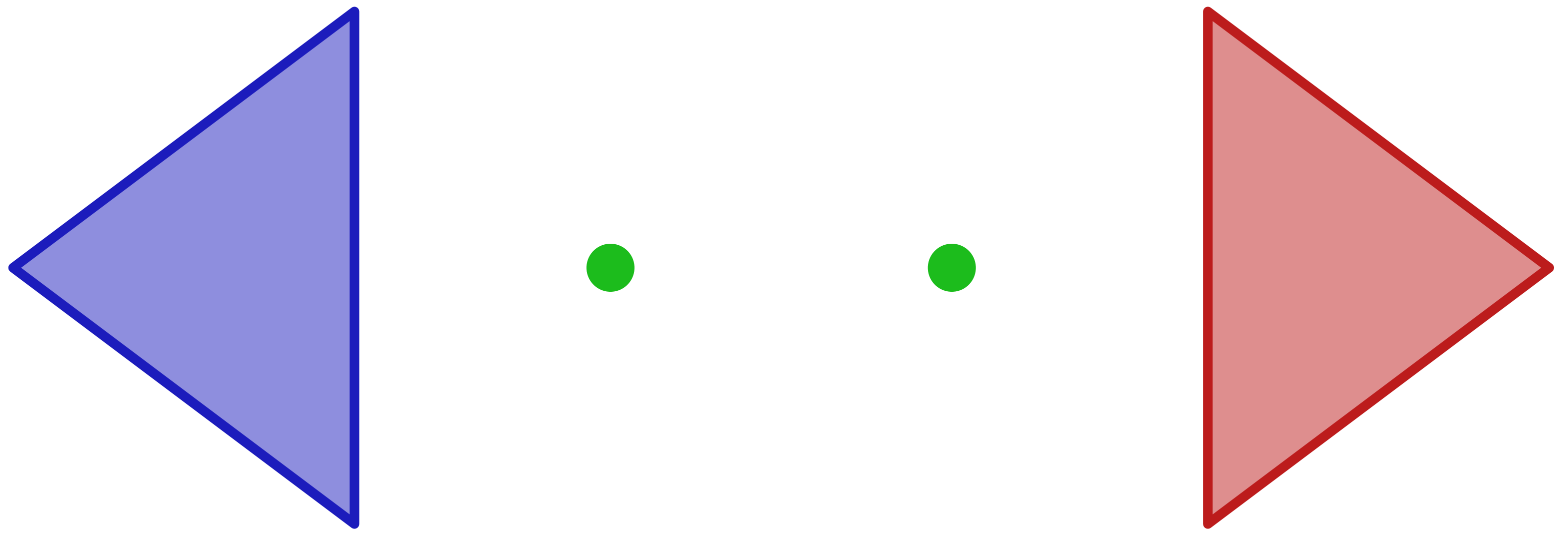
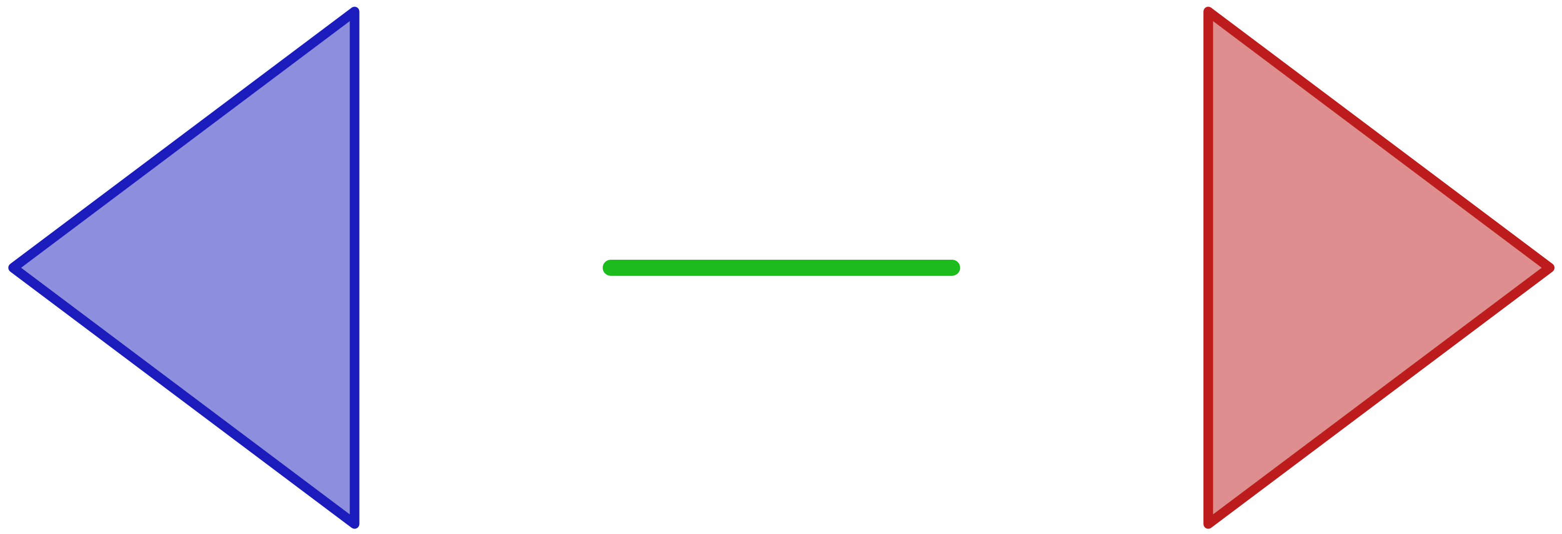
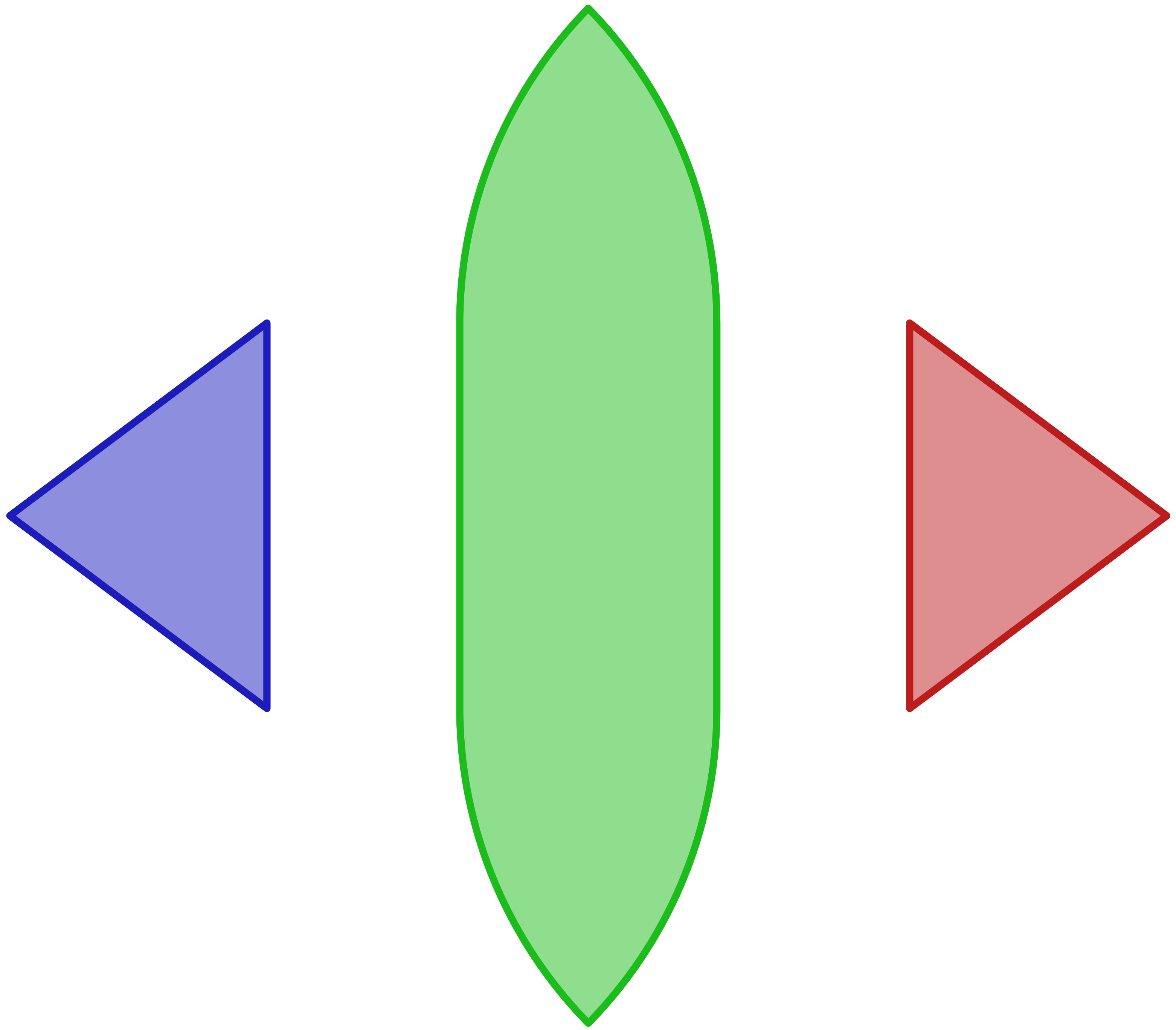
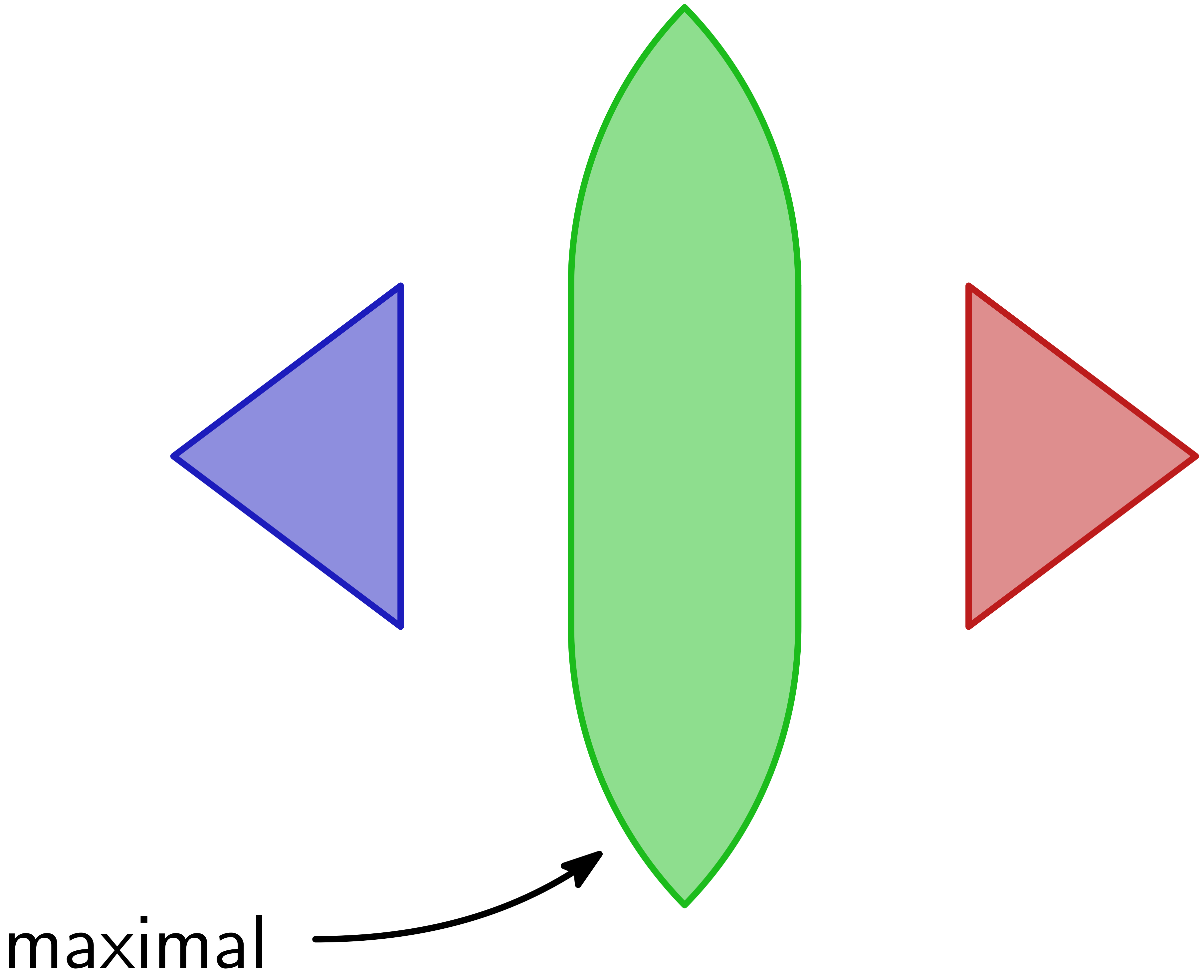


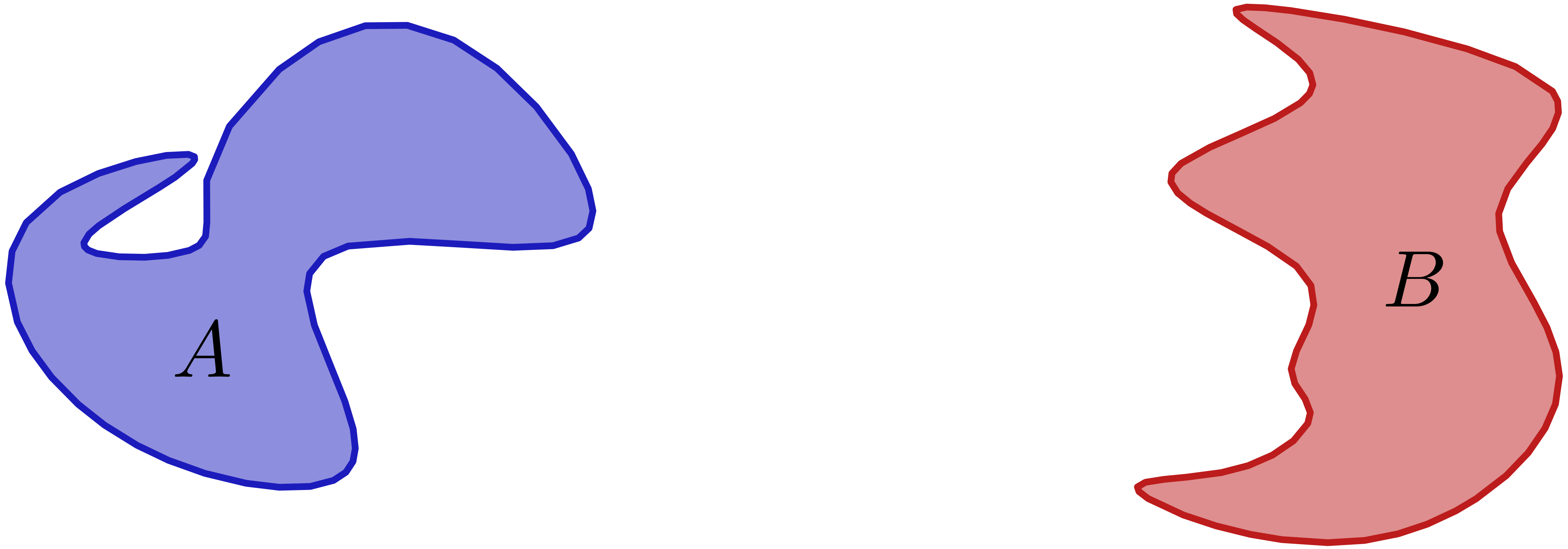
Result: distance $1/2$ is always possible.



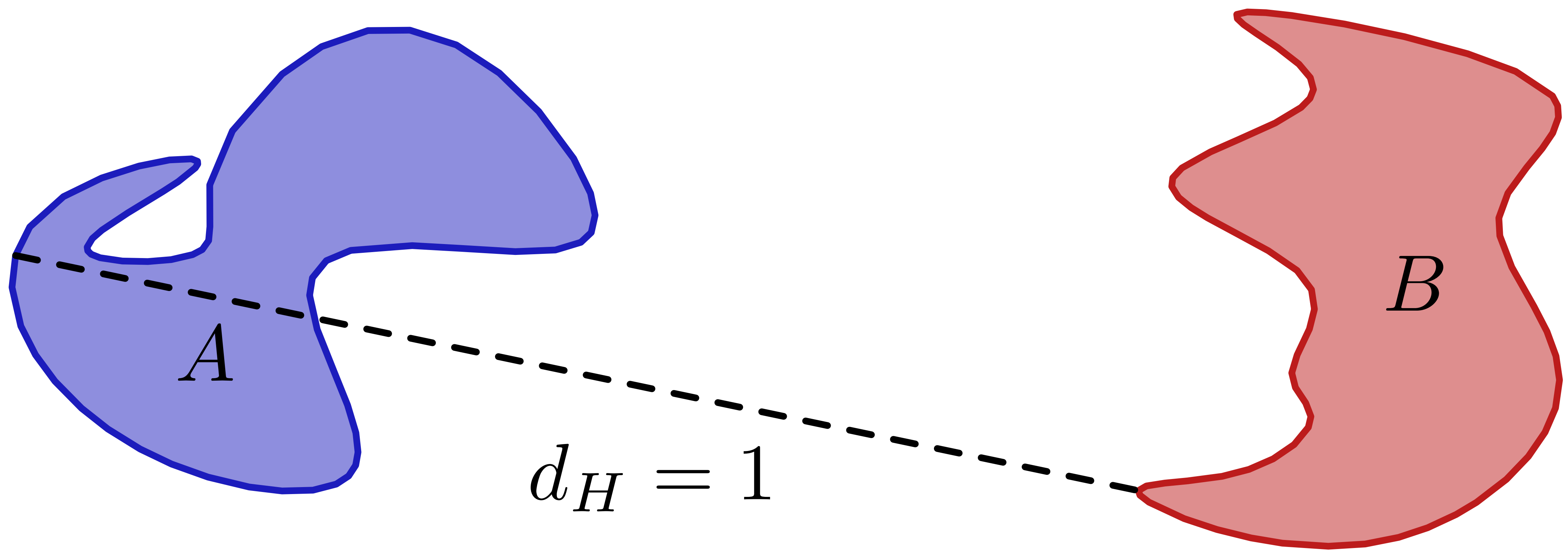


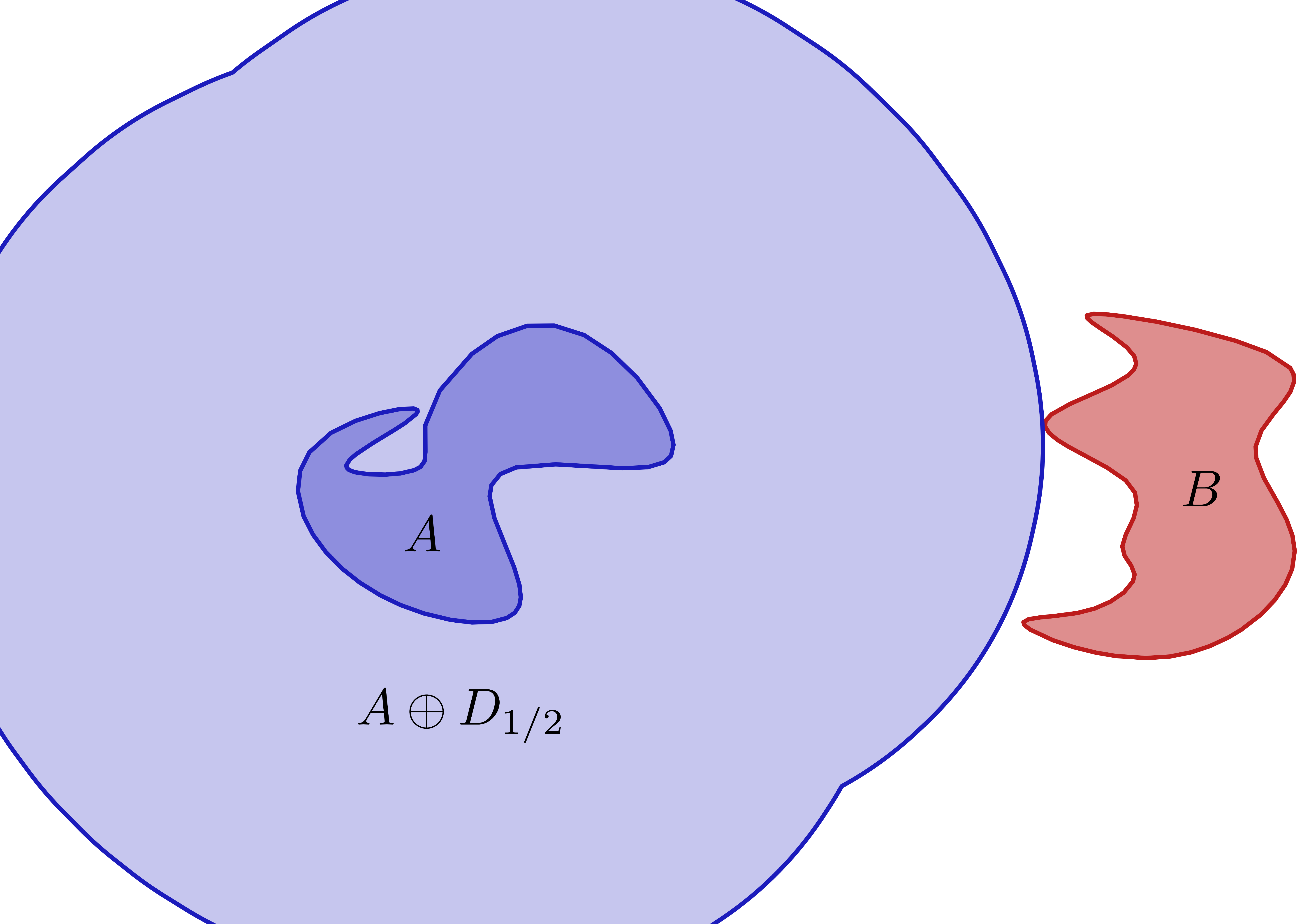



maximal

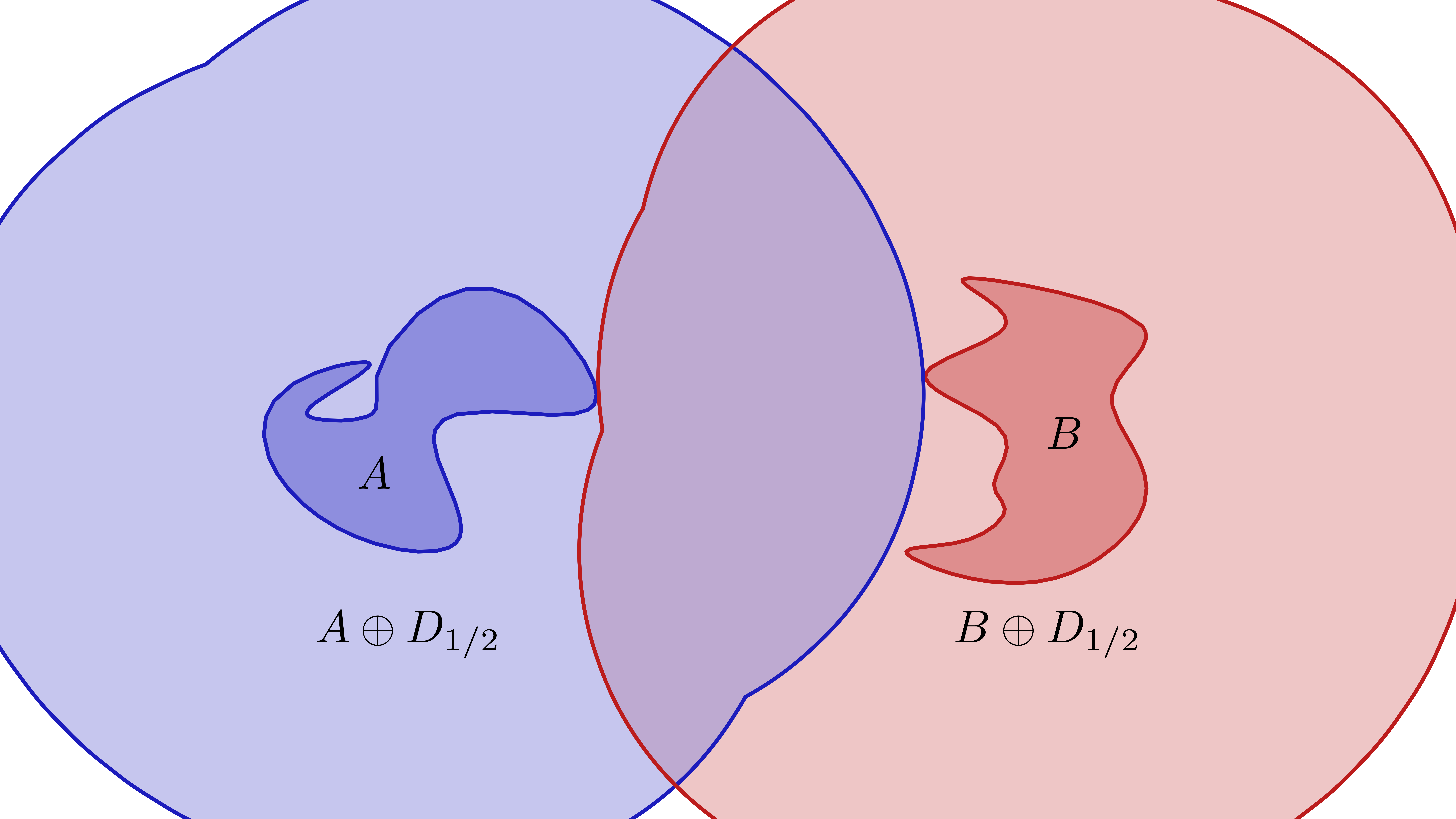


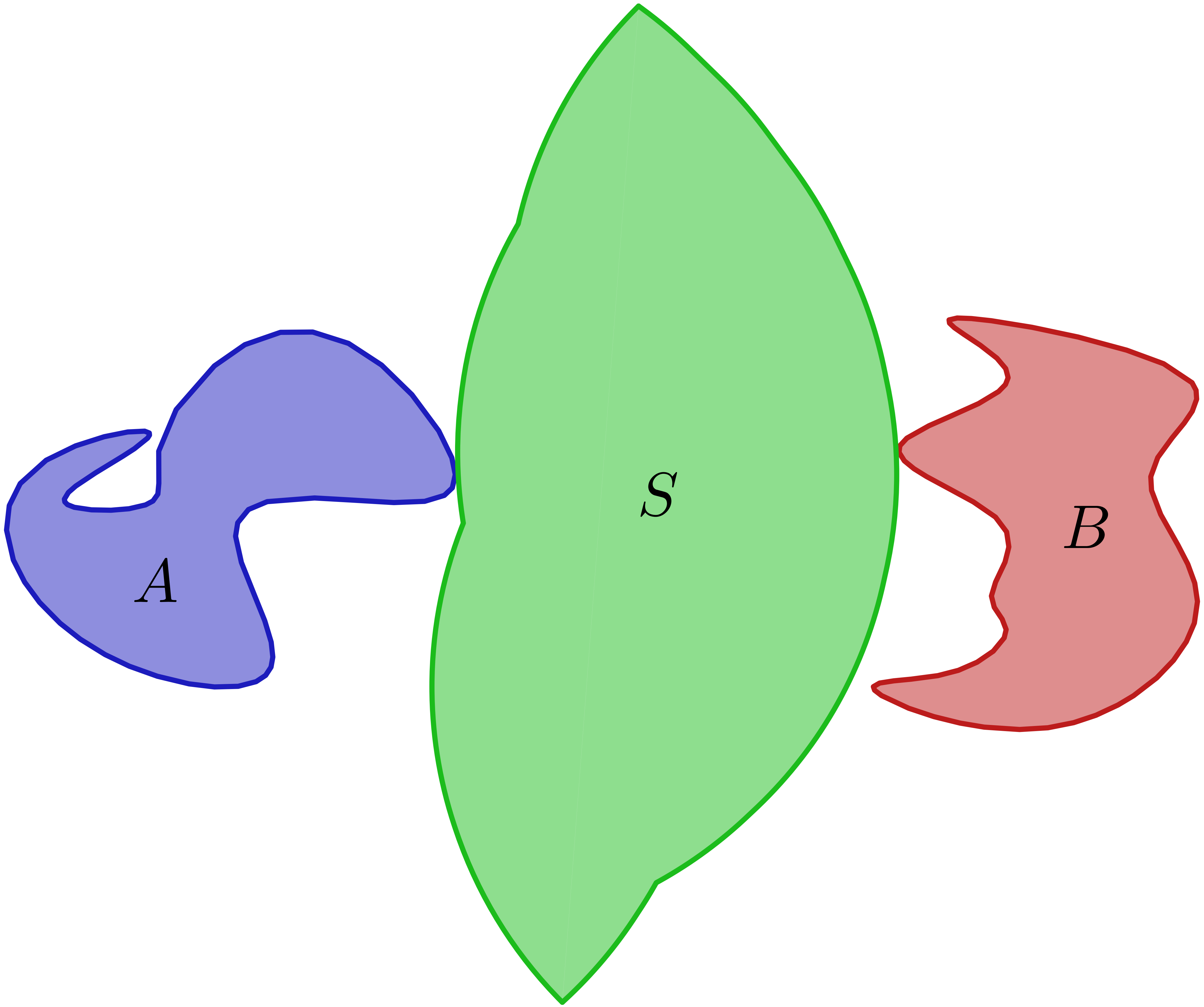
A
B


A
B
$d_H = 1$

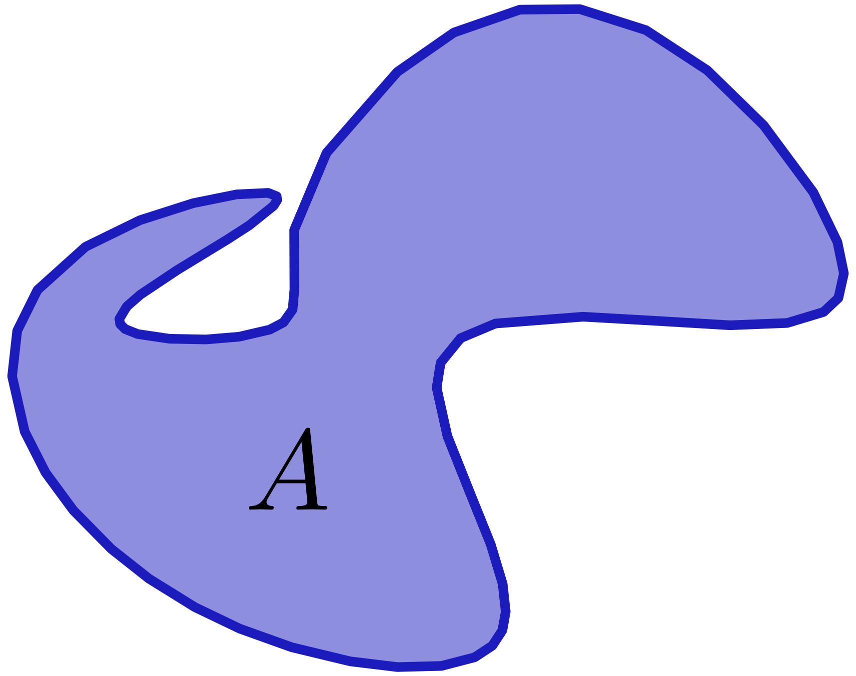
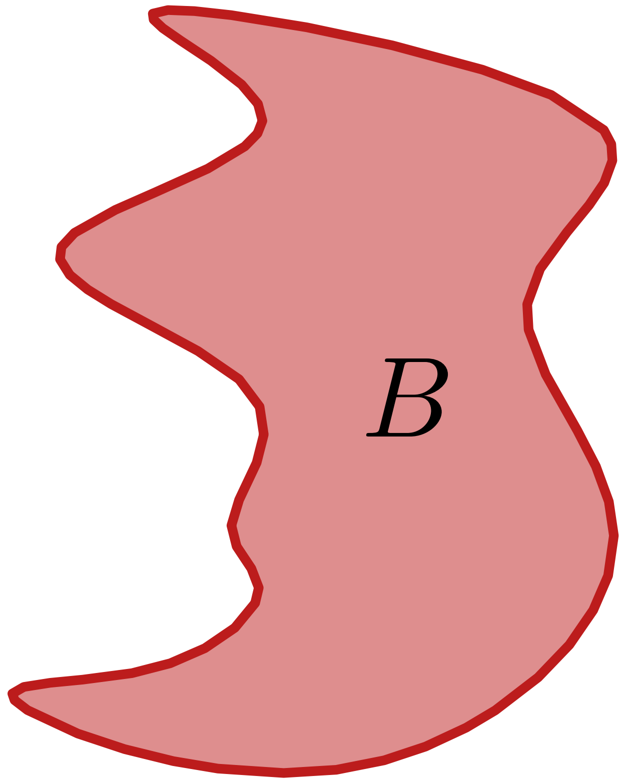


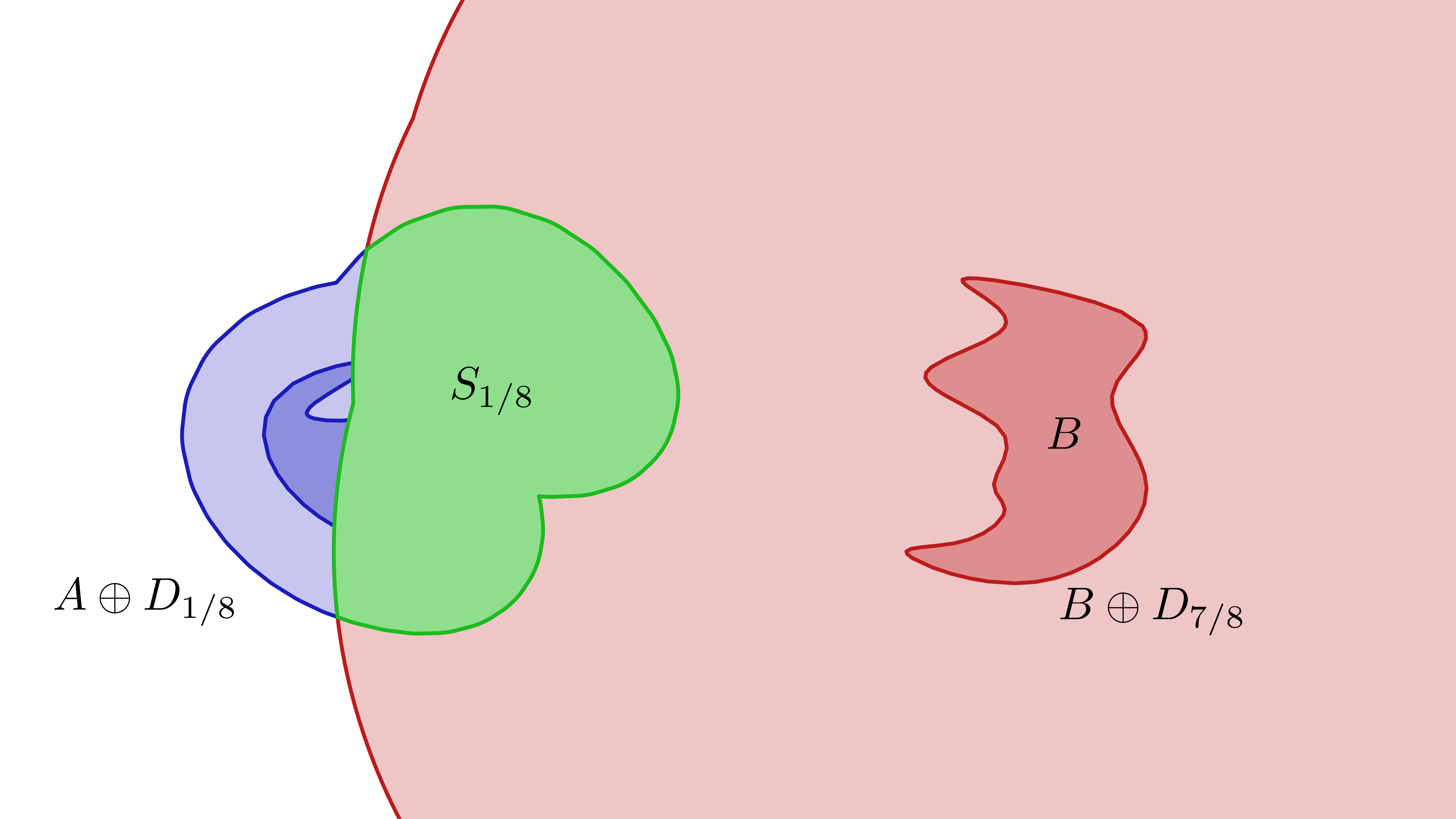
A
B
$A \oplus D_{1/2}$

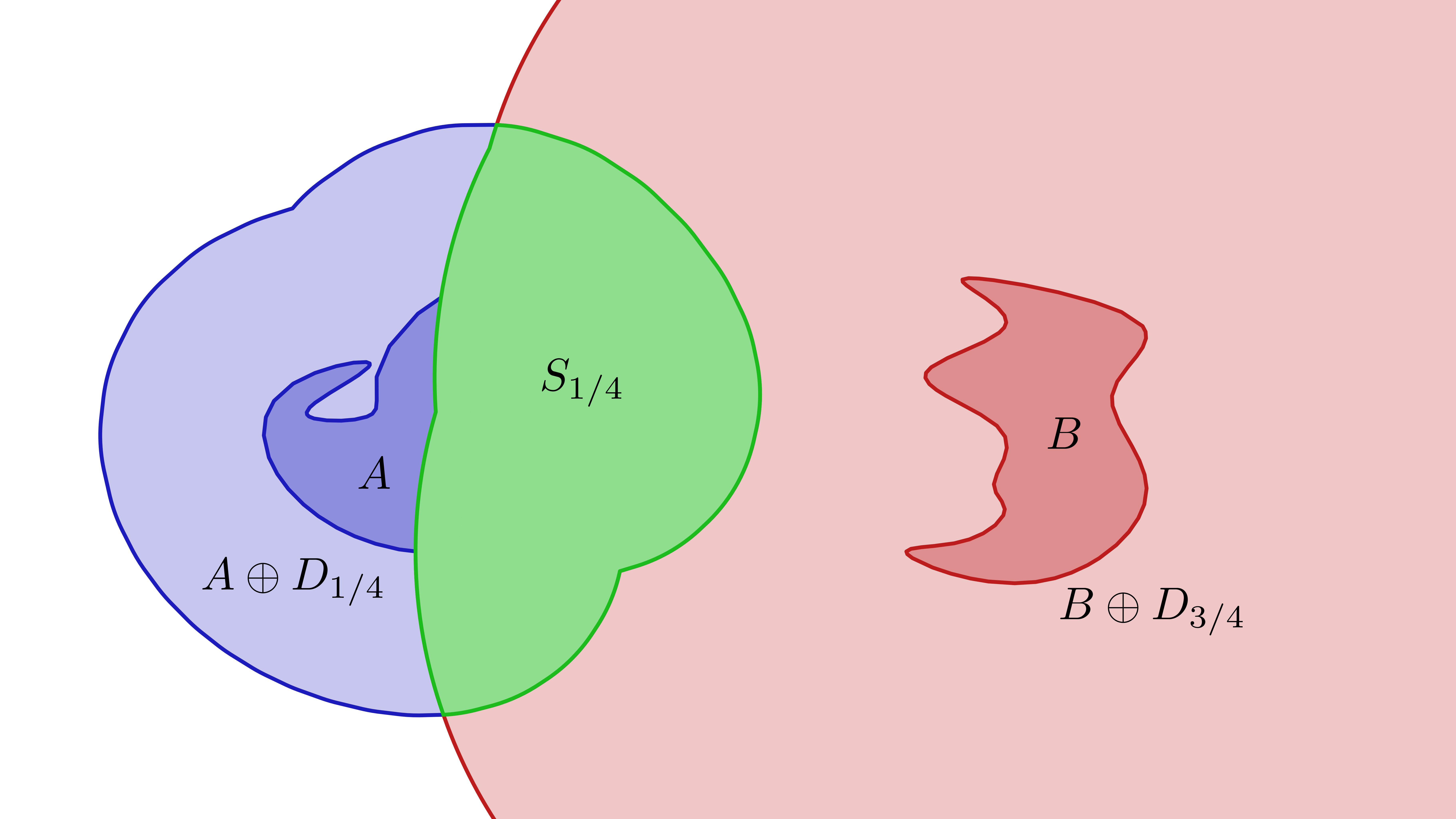

A
B
$A \oplus D_{1/2}$
$B \oplus D_{1/2}$



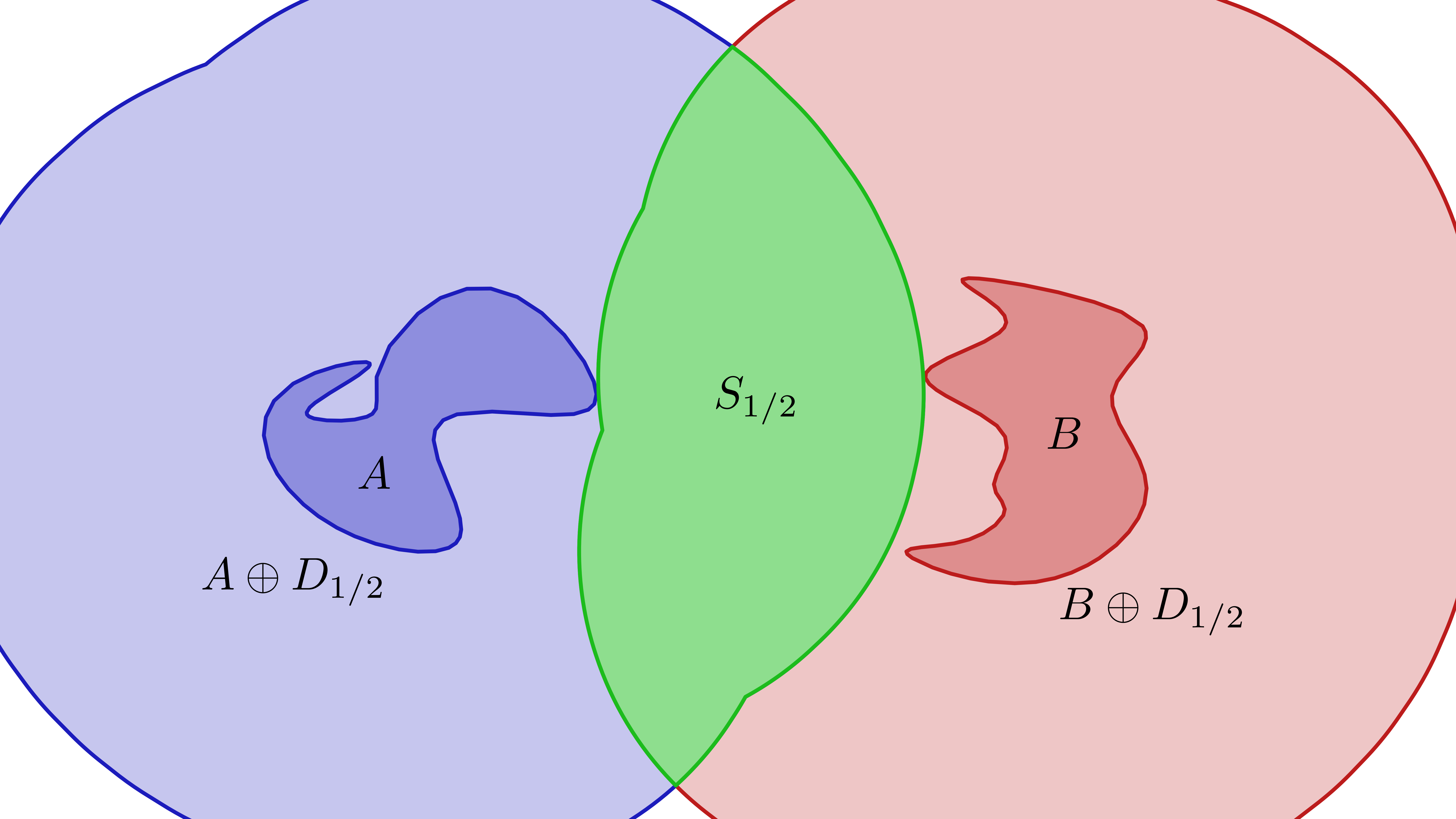
A
S
B


A

B

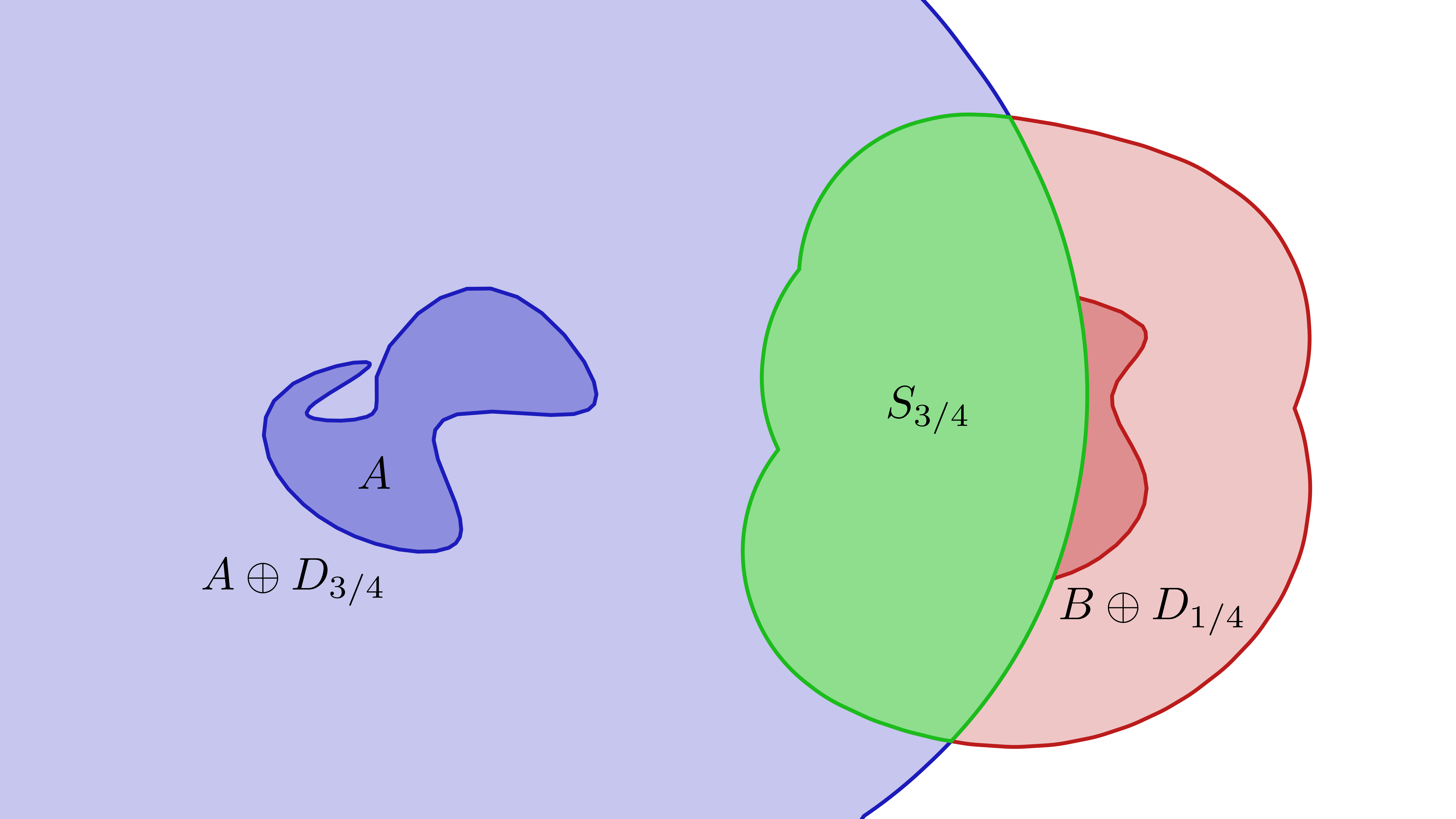

$S_{1/8}$
B
$A \oplus D_{1/8}$
$B \oplus D_{7/8}$

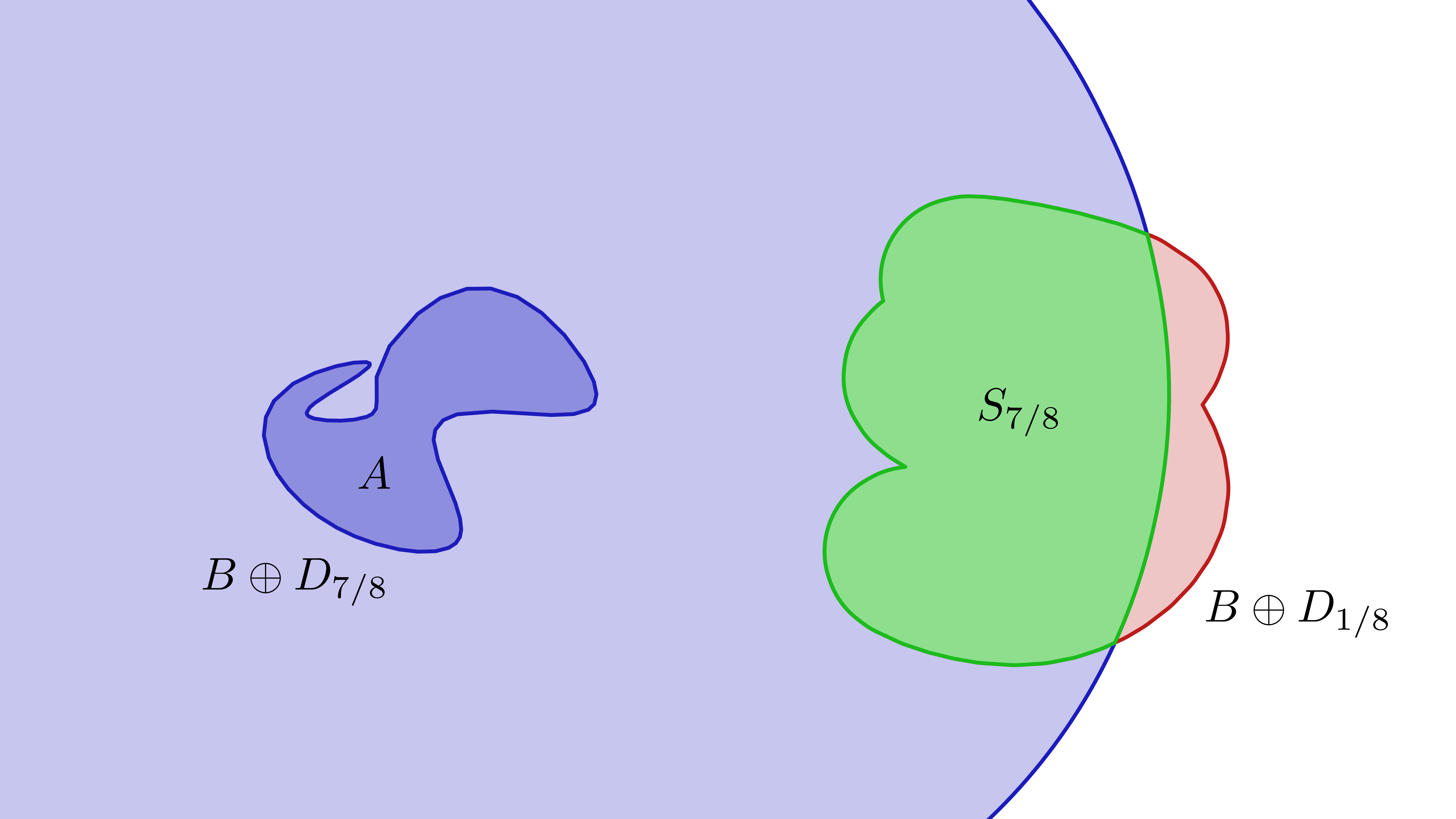


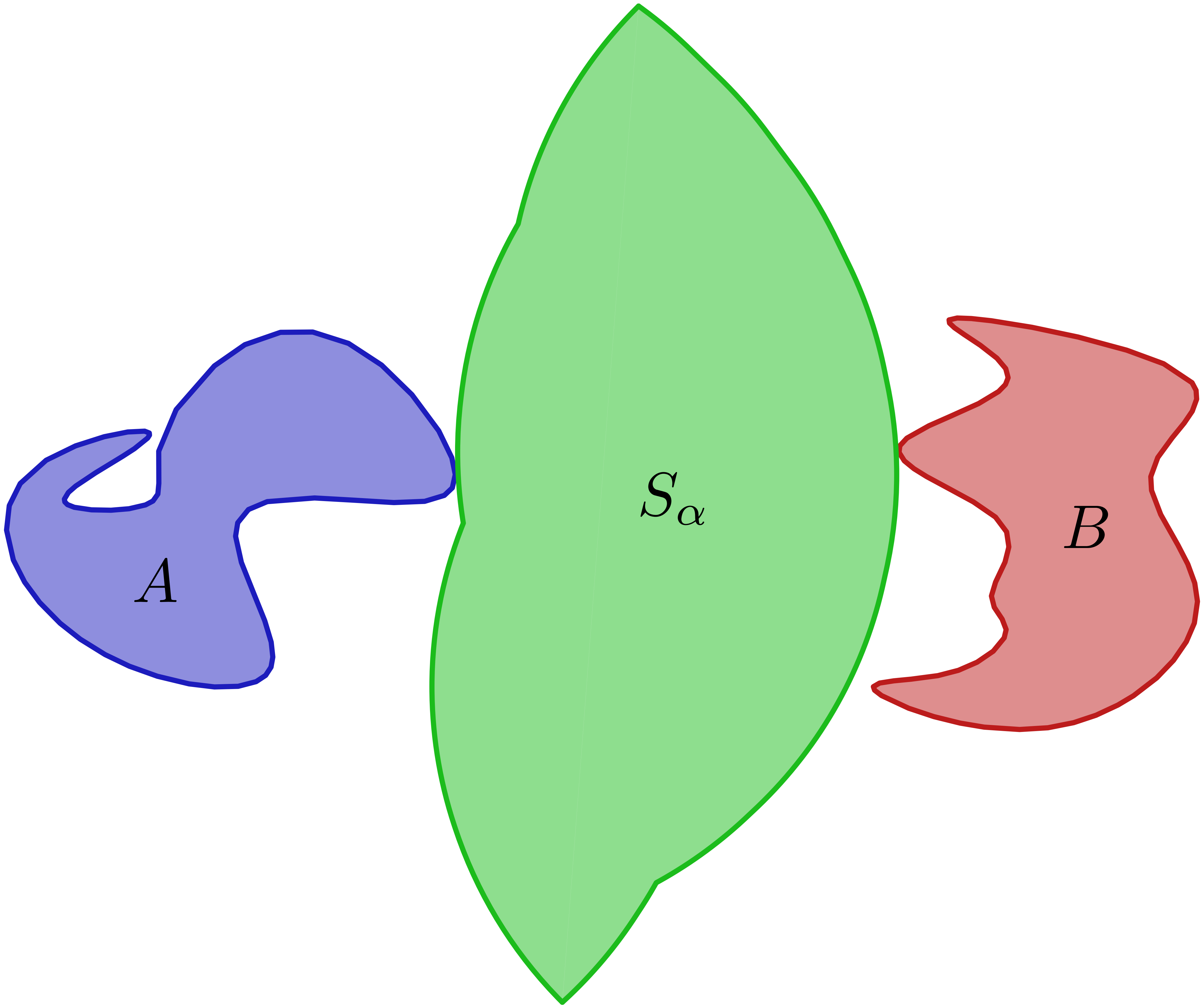
$S_{1/4}$
B
A
$A \oplus D_{1/4}$
$B \oplus D_{3/4}$



$S_{1/2}$
A
B
$A \oplus D_{1/2}$
$B \oplus D_{1/2}$


A
$A \oplus D_{3/4}$
$S_{3/4}$
$B \oplus D_{1/4}$

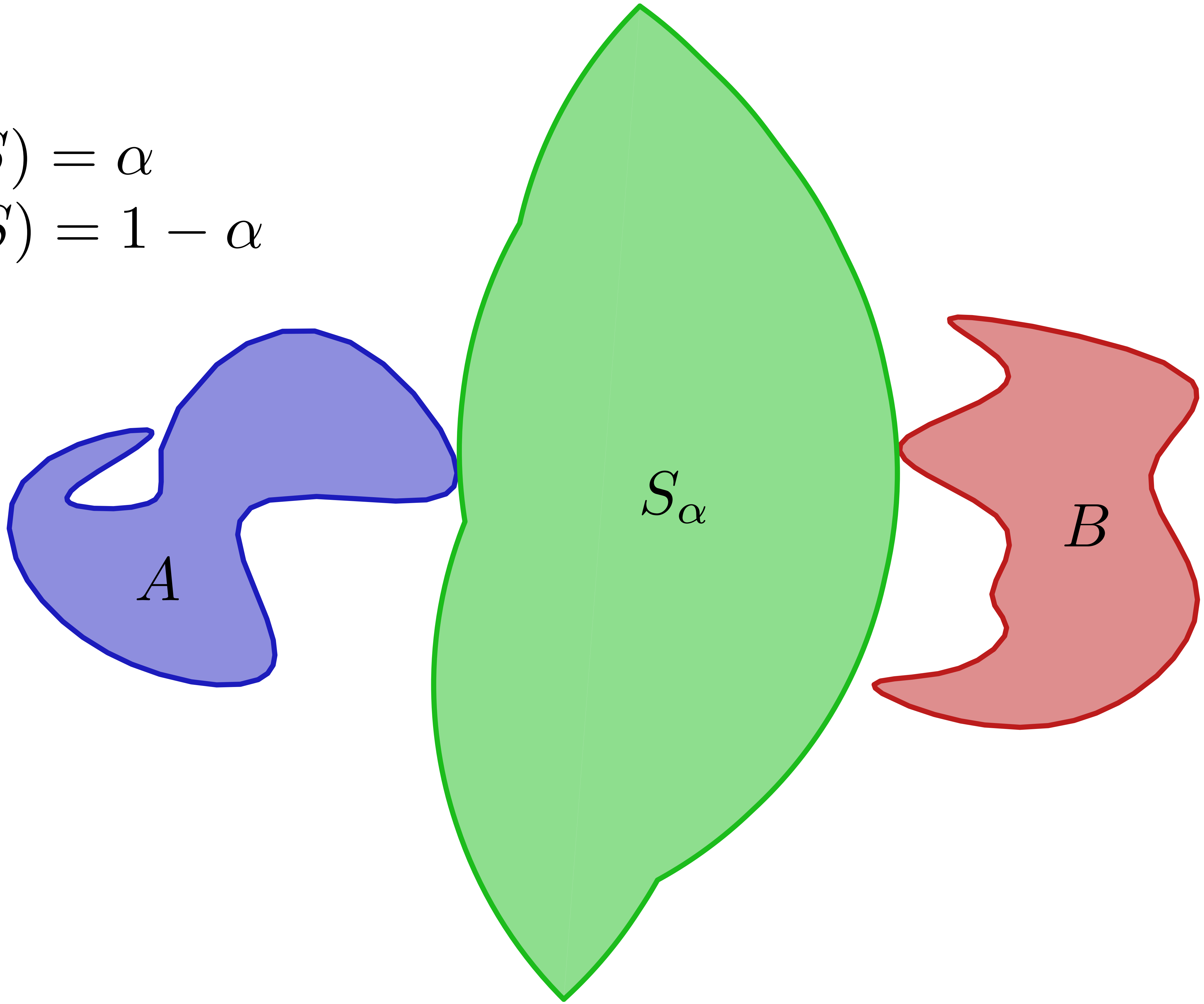

A
$B \oplus D_{7/8}$
$S_{7/8}$
$B \oplus D_{1/8}$



A
S_α
B

Claim:

- $d_H(A, S) = \alpha$
- $d_H(B, S) = 1 - \alpha$



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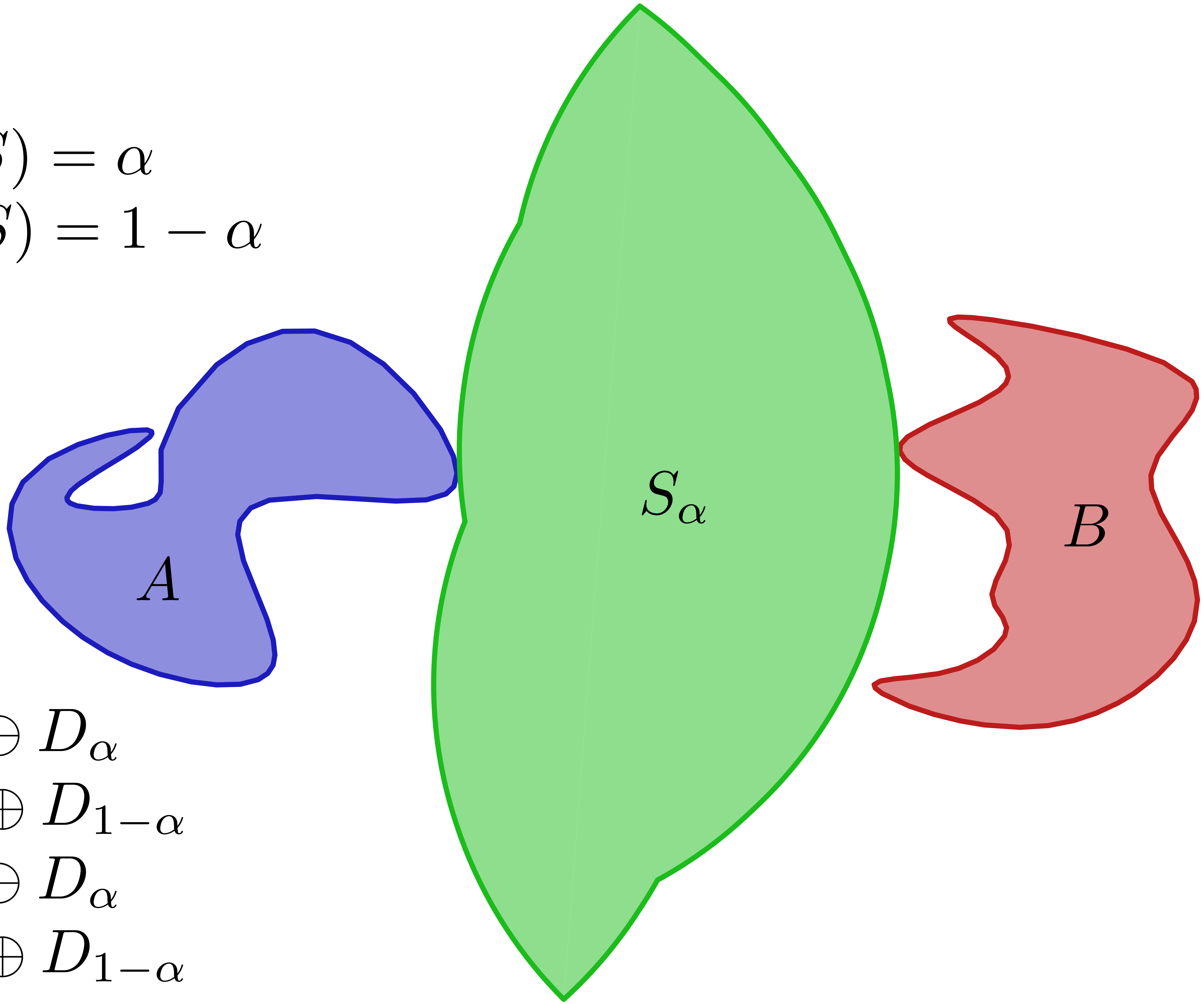

To show:

- $S \subseteq A \oplus D_\alpha$
- $S \subseteq B \oplus D_{1-\alpha}$
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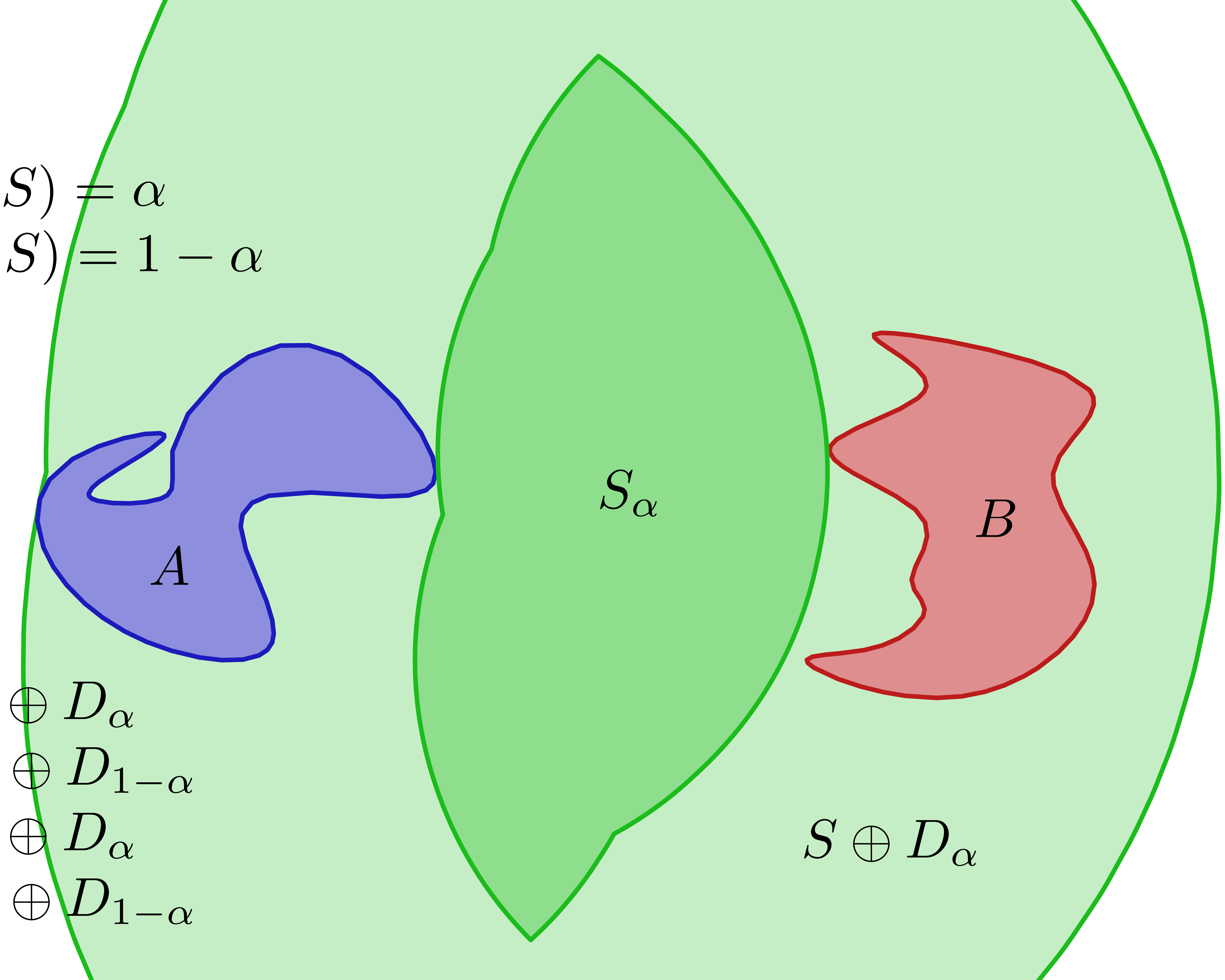

Lemma: $S_\alpha = (A \oplus D_\alpha) \cap (B \oplus D_{1-\alpha})$ has $d_H(A, S_\alpha) = \alpha$ and $d_H(B, S_\alpha) = 1 - \alpha$.

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- $A \subseteq S_\alpha \oplus D_\alpha$

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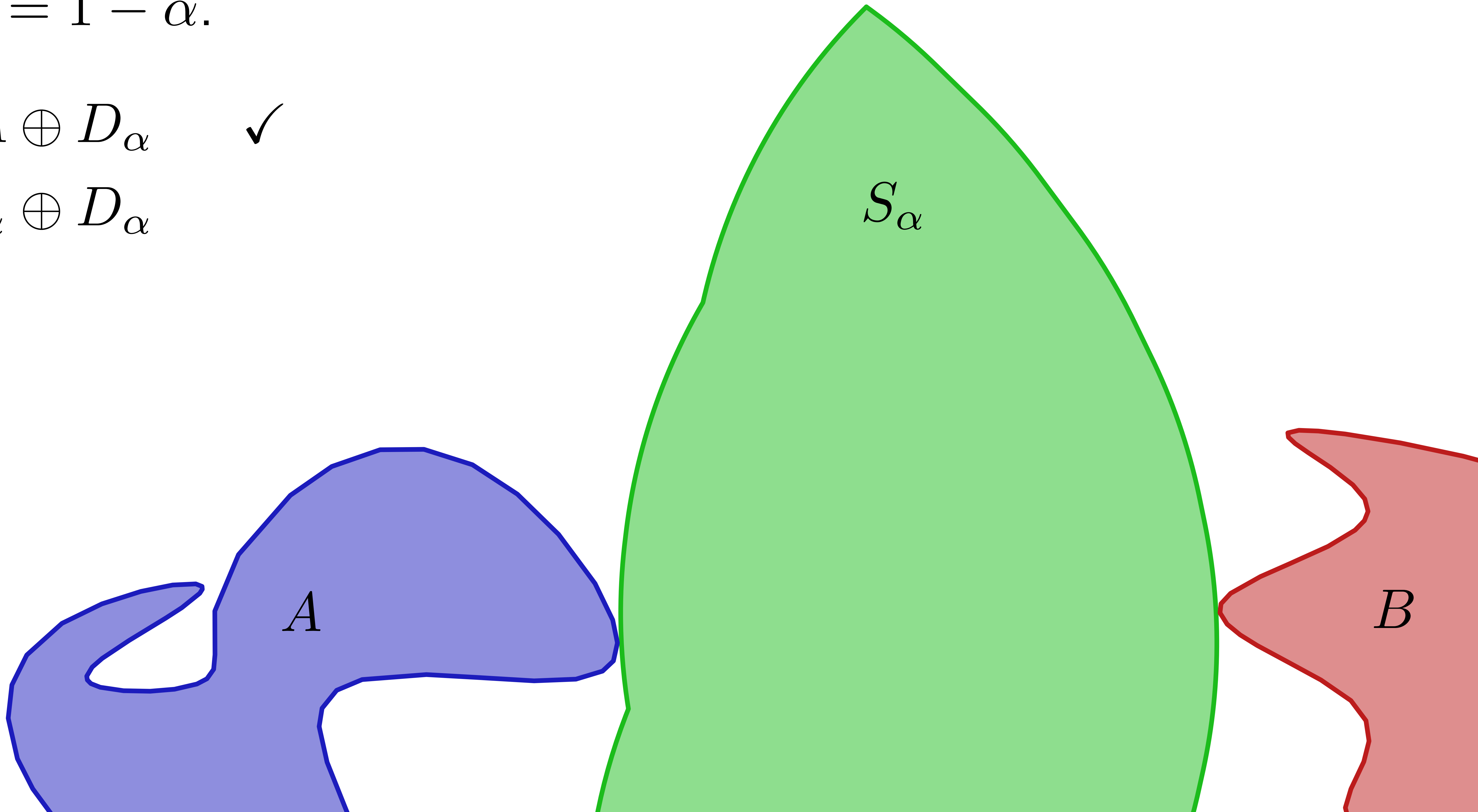

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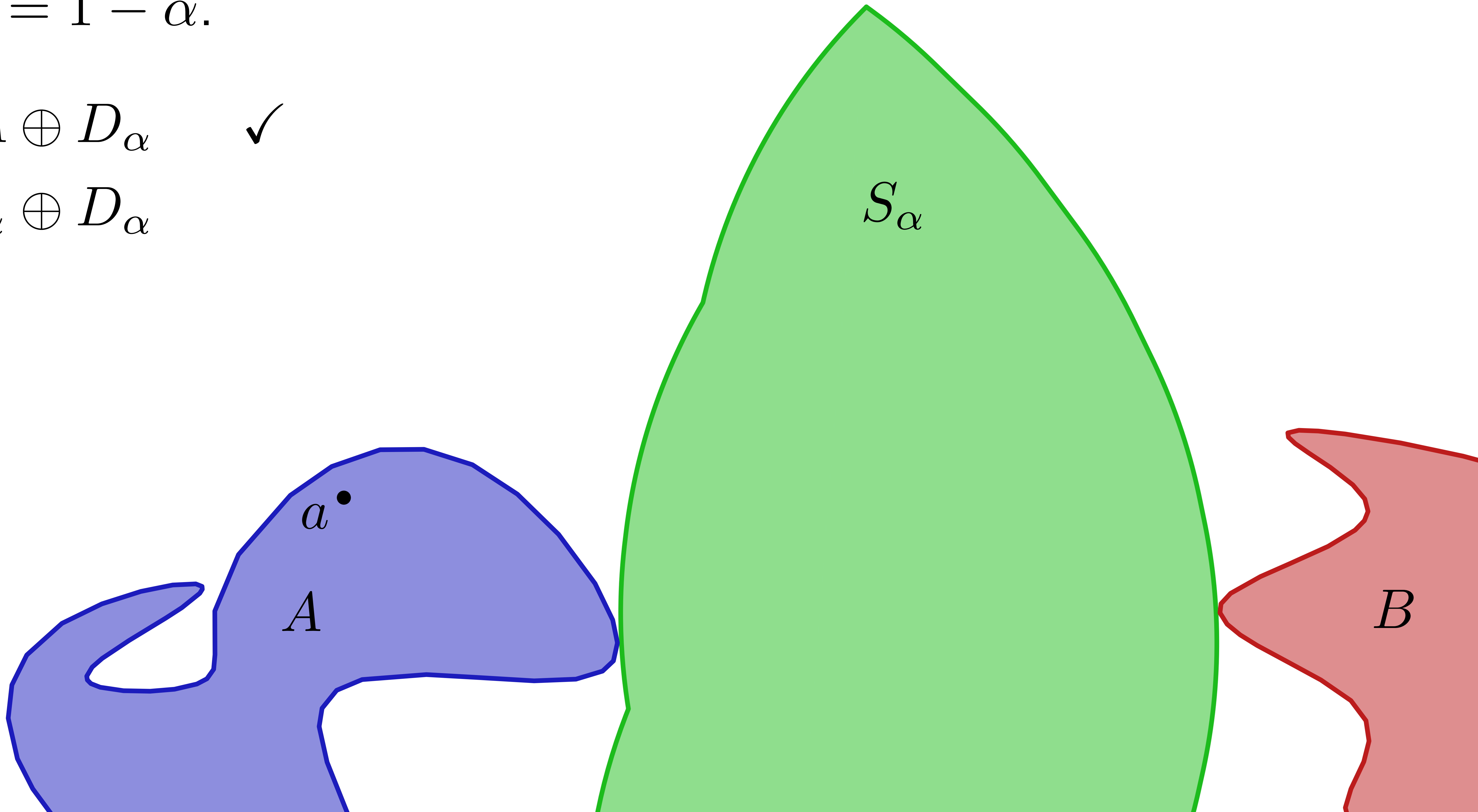

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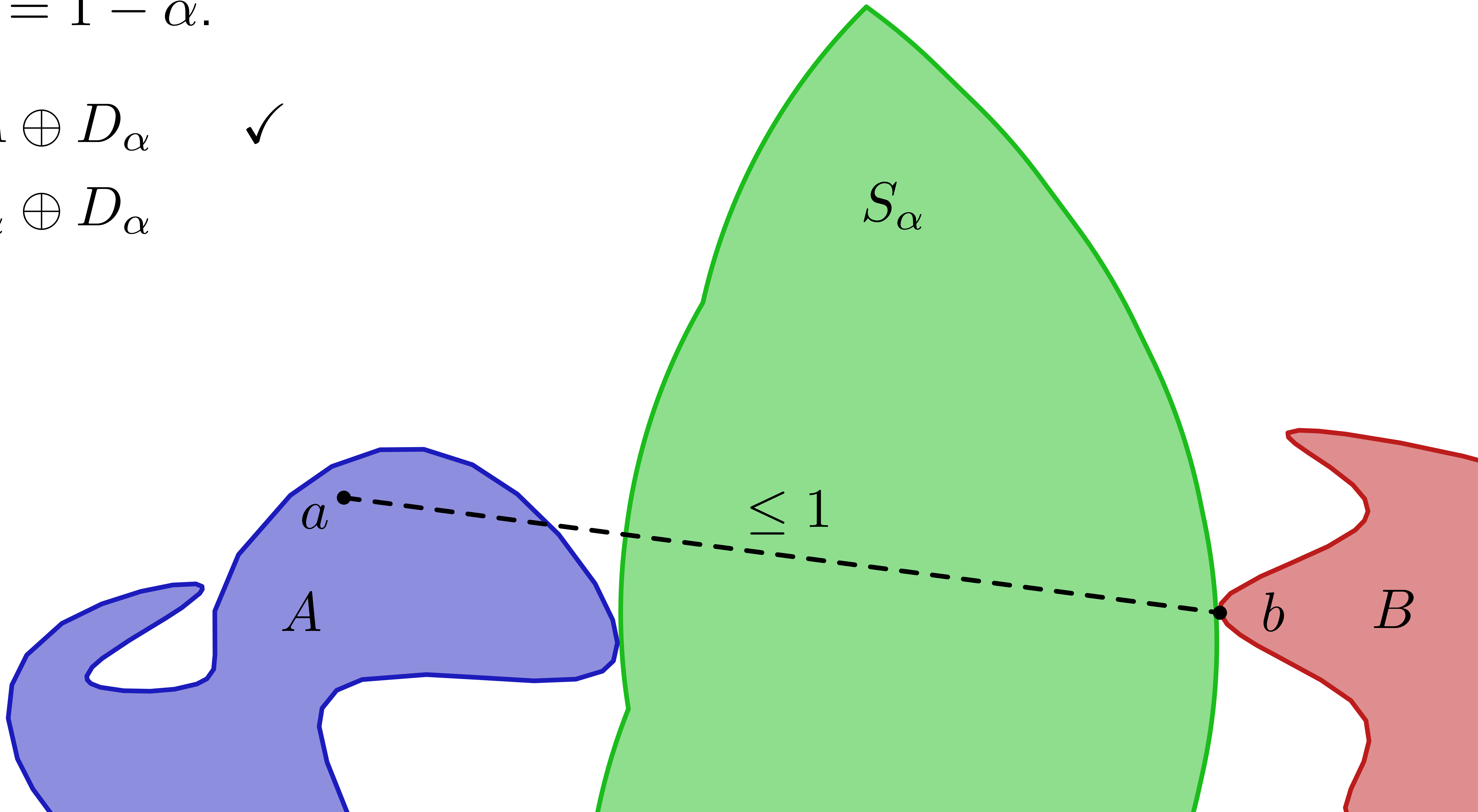

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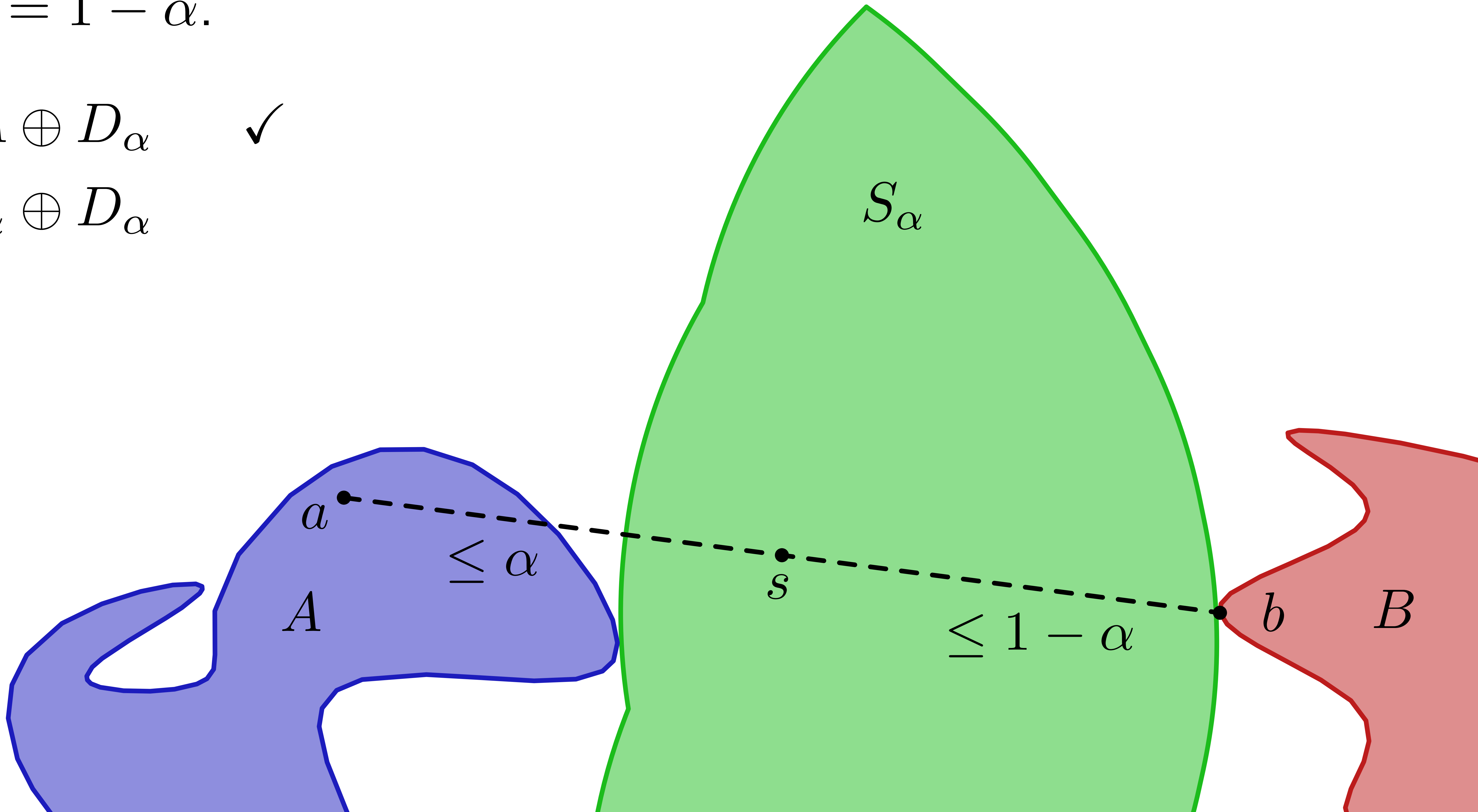

Lemma: $S_\alpha = (A \oplus D_\alpha) \cap (B \oplus D_{1-\alpha})$ has $d_H(A, S_\alpha) = \alpha$ and $d_H(B, S_\alpha) = 1 - \alpha$.

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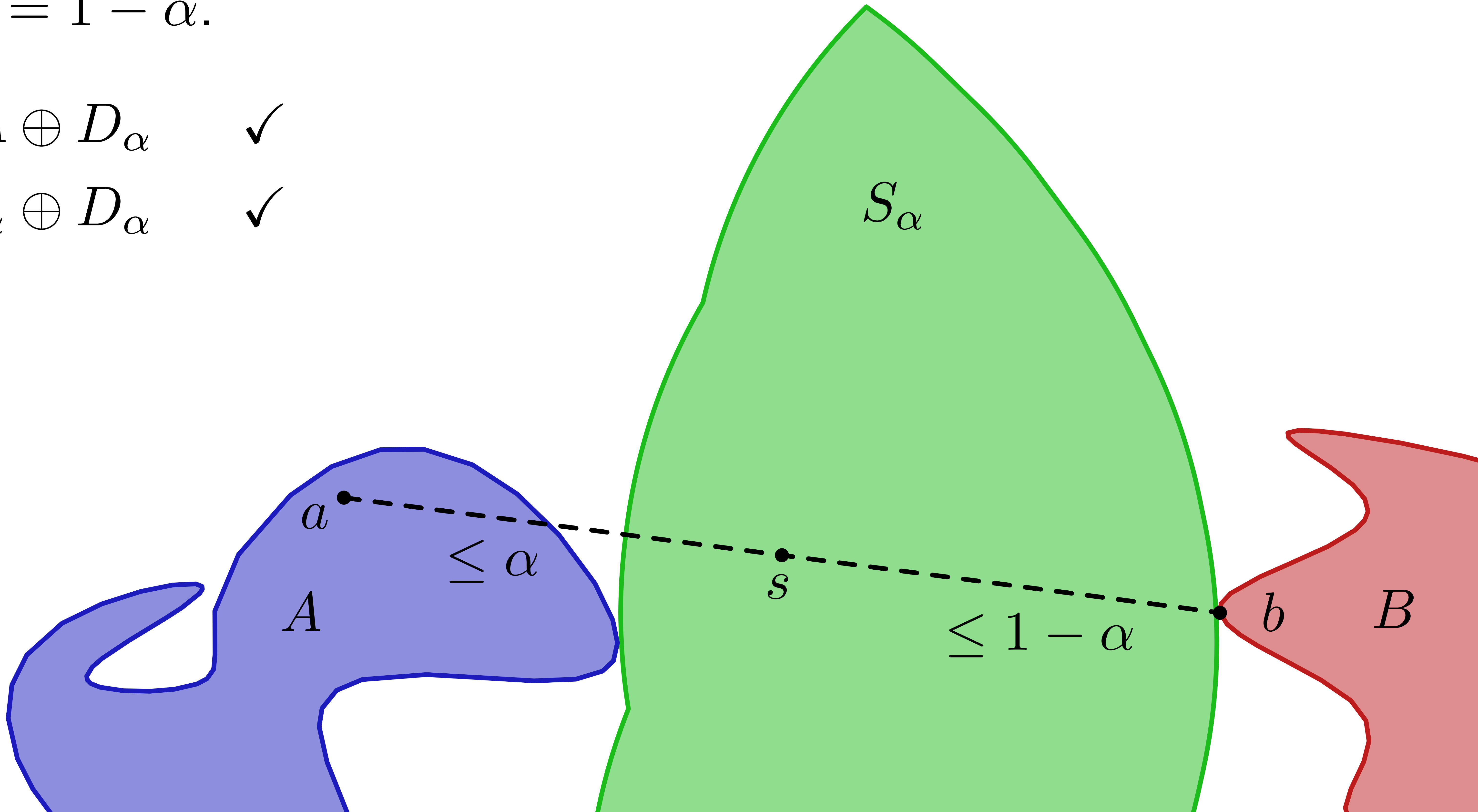
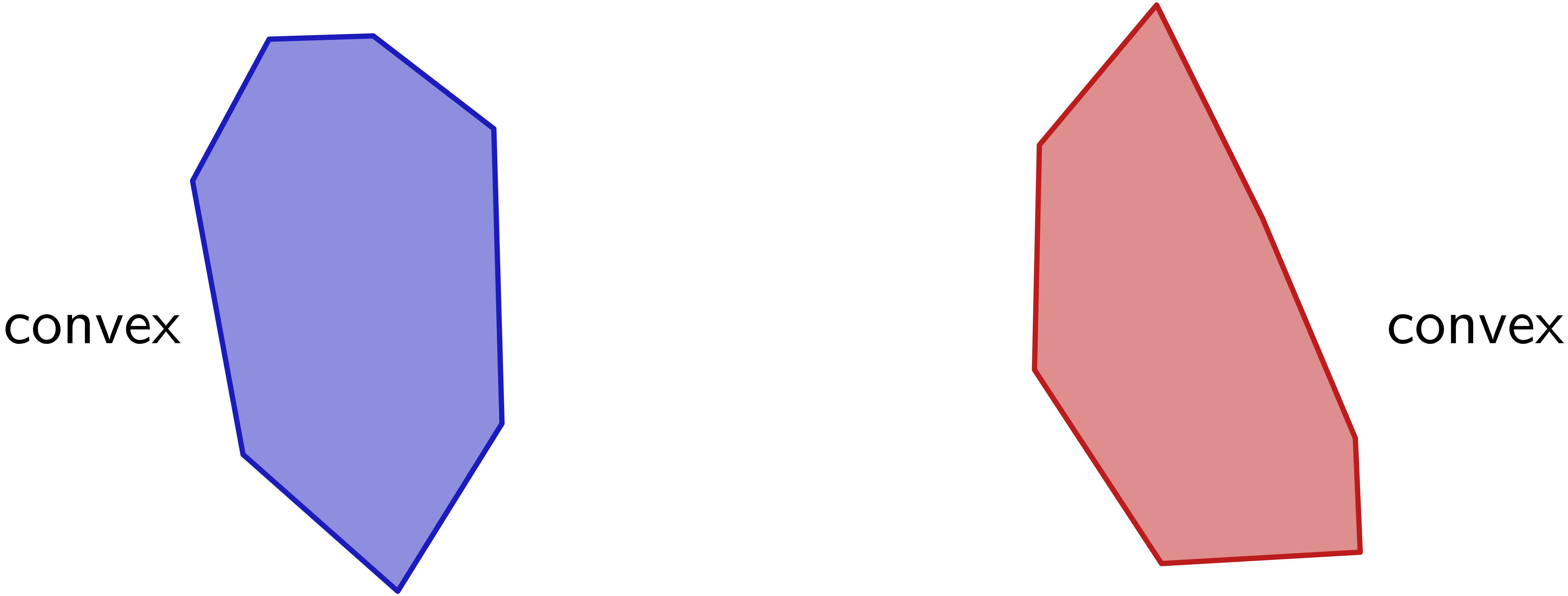
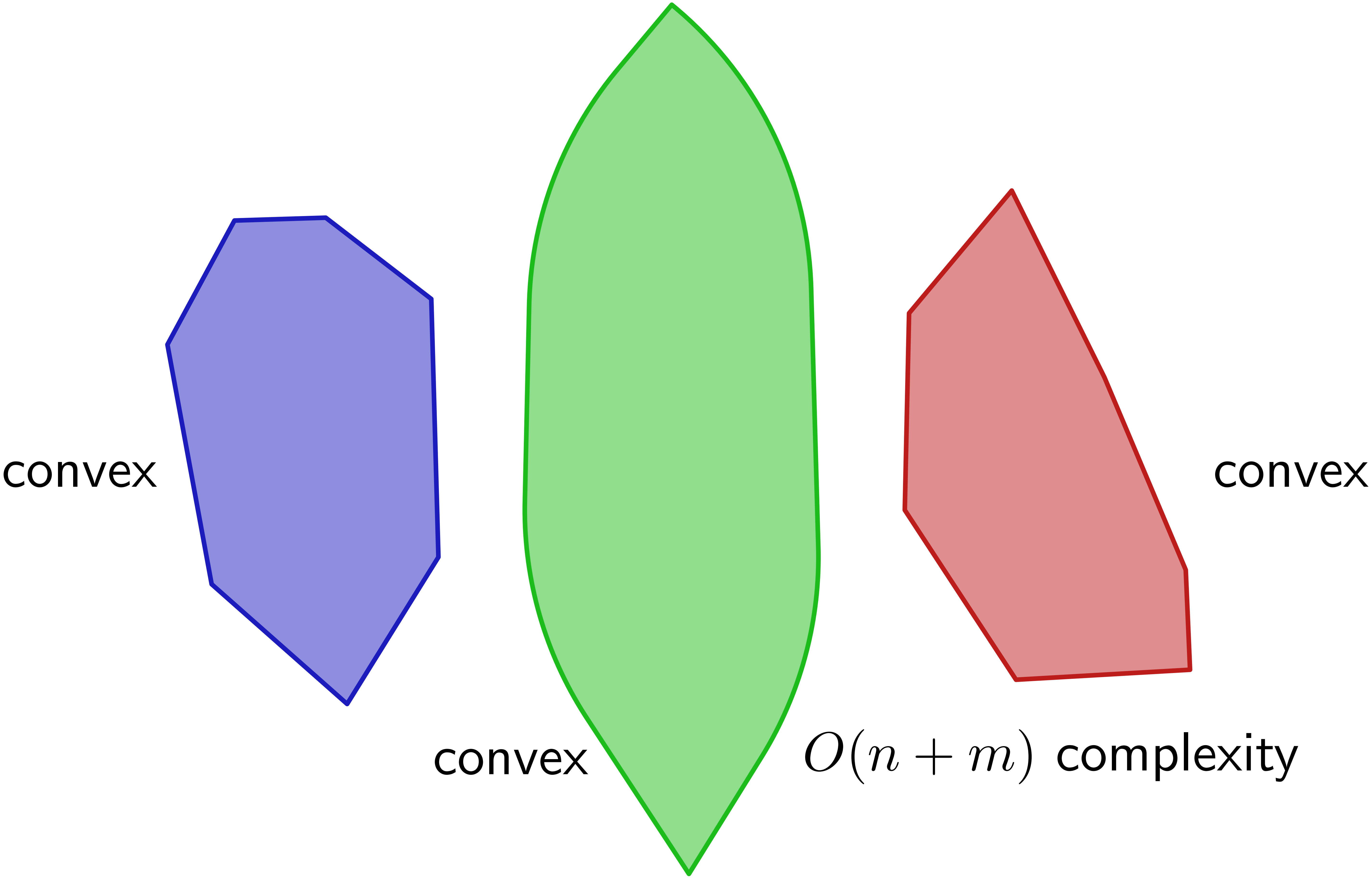
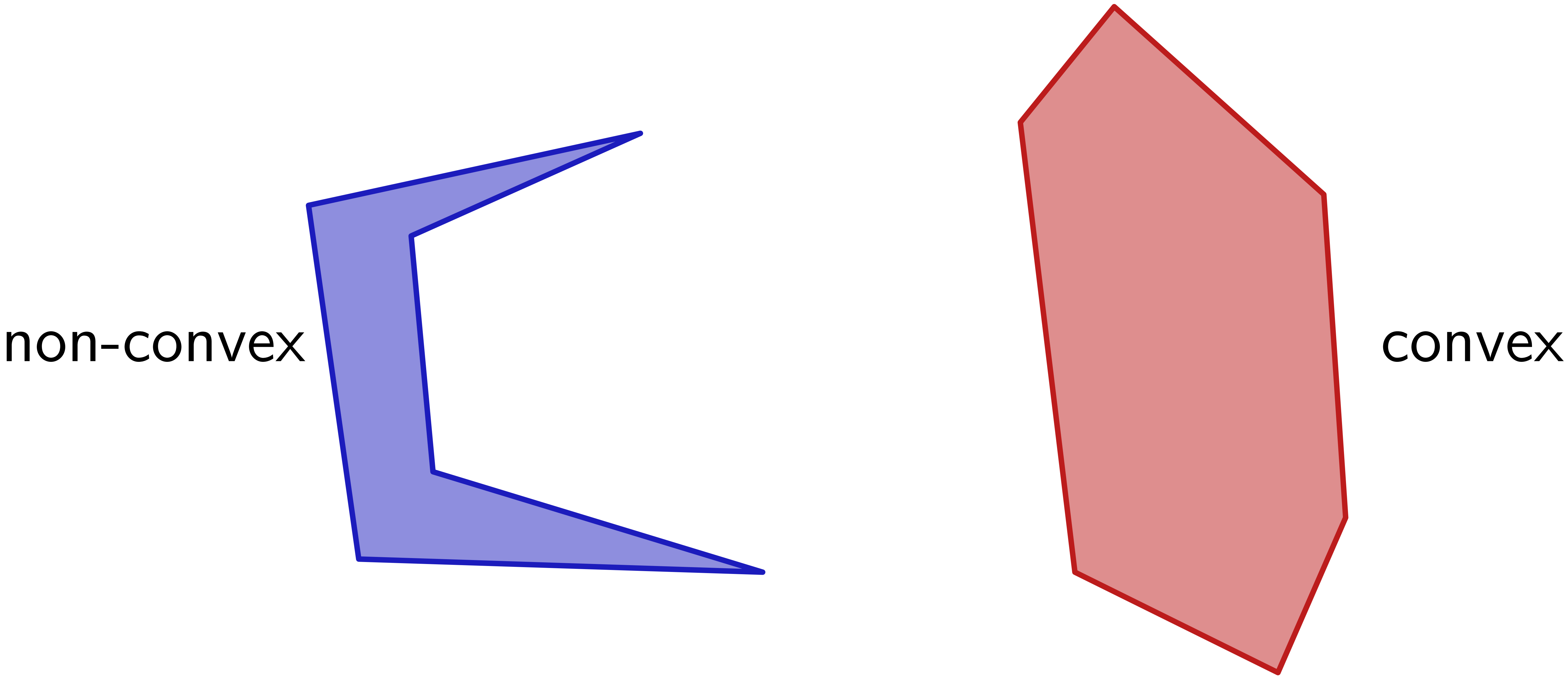
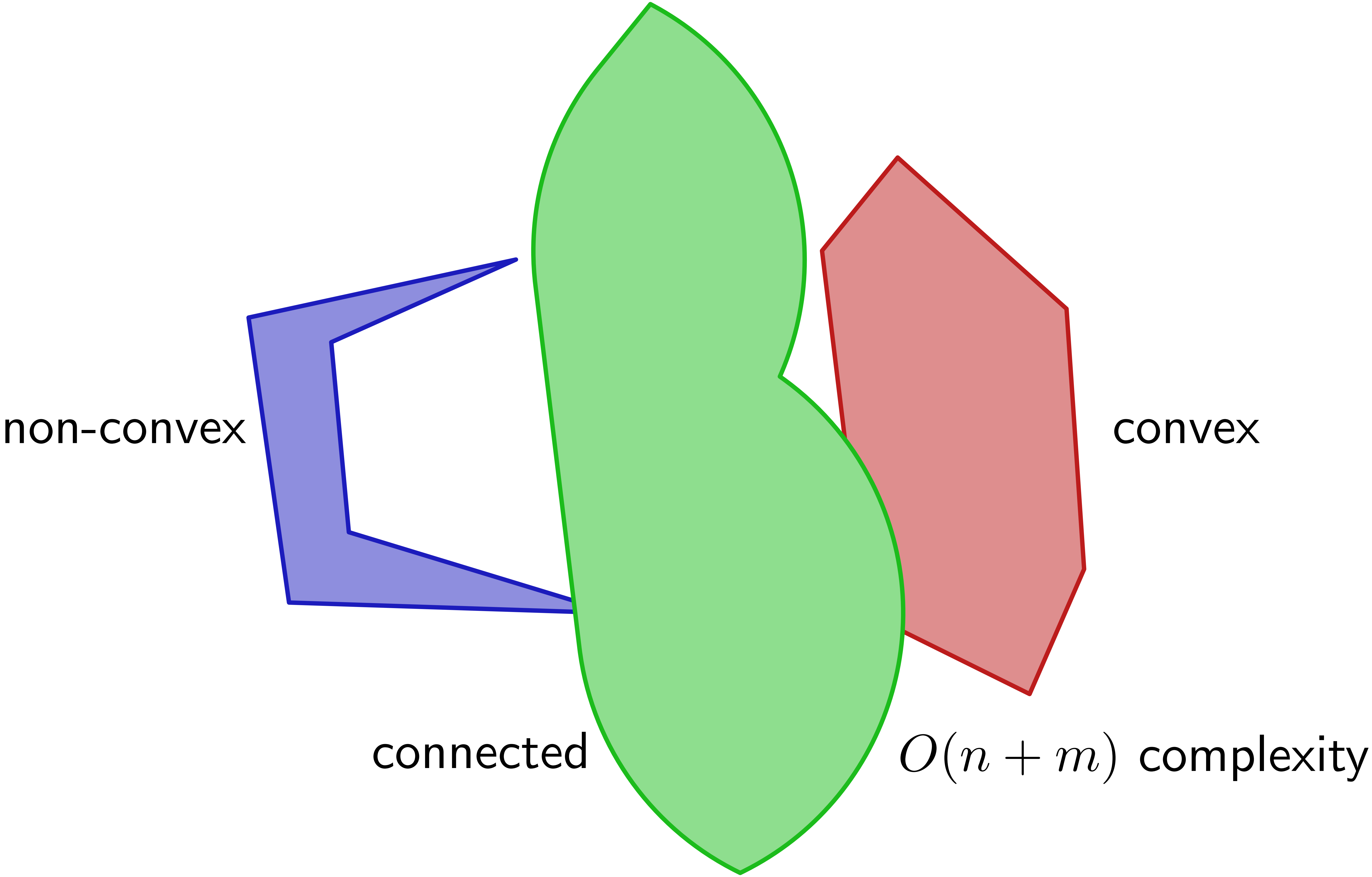


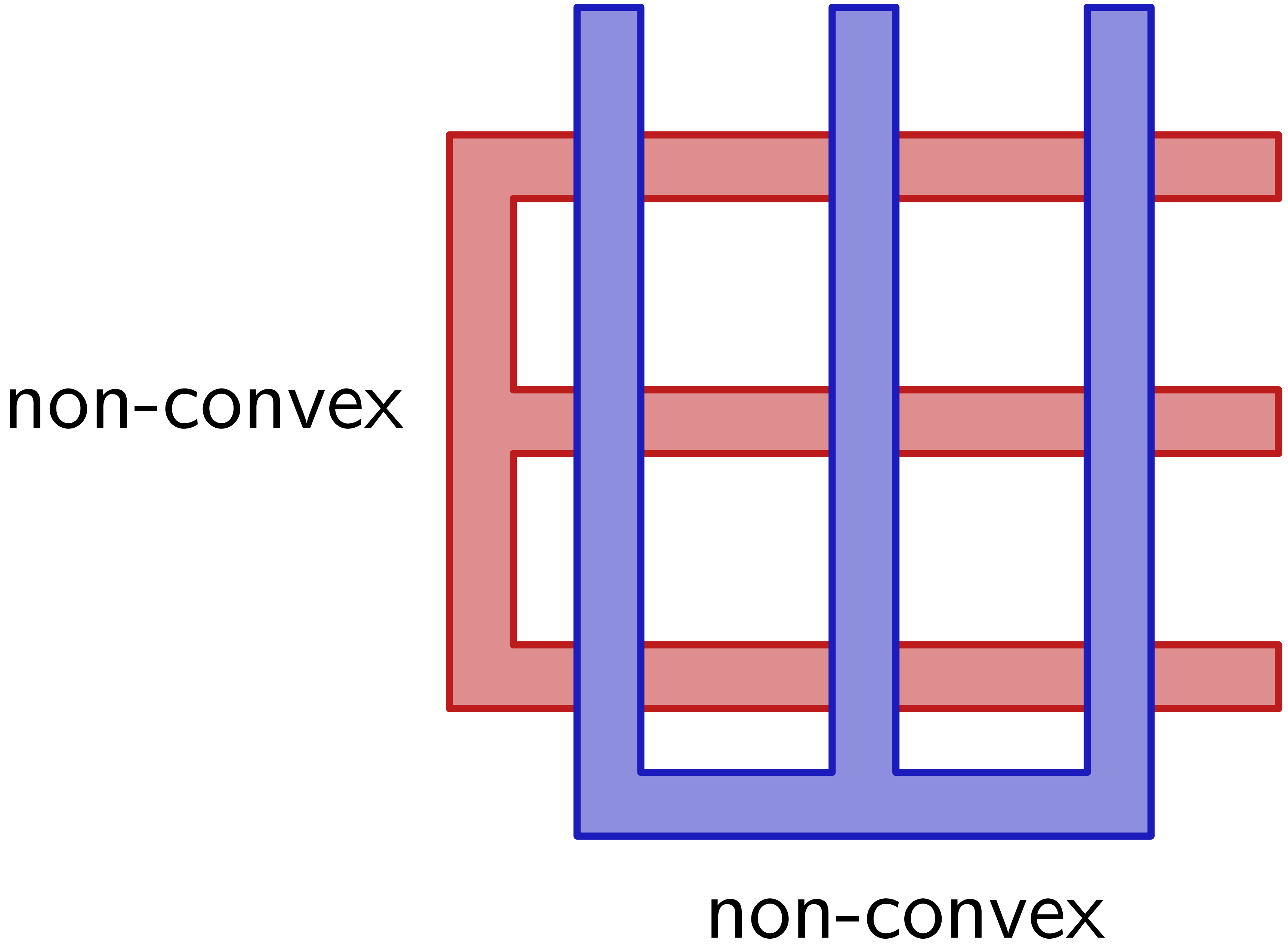

convex
convex

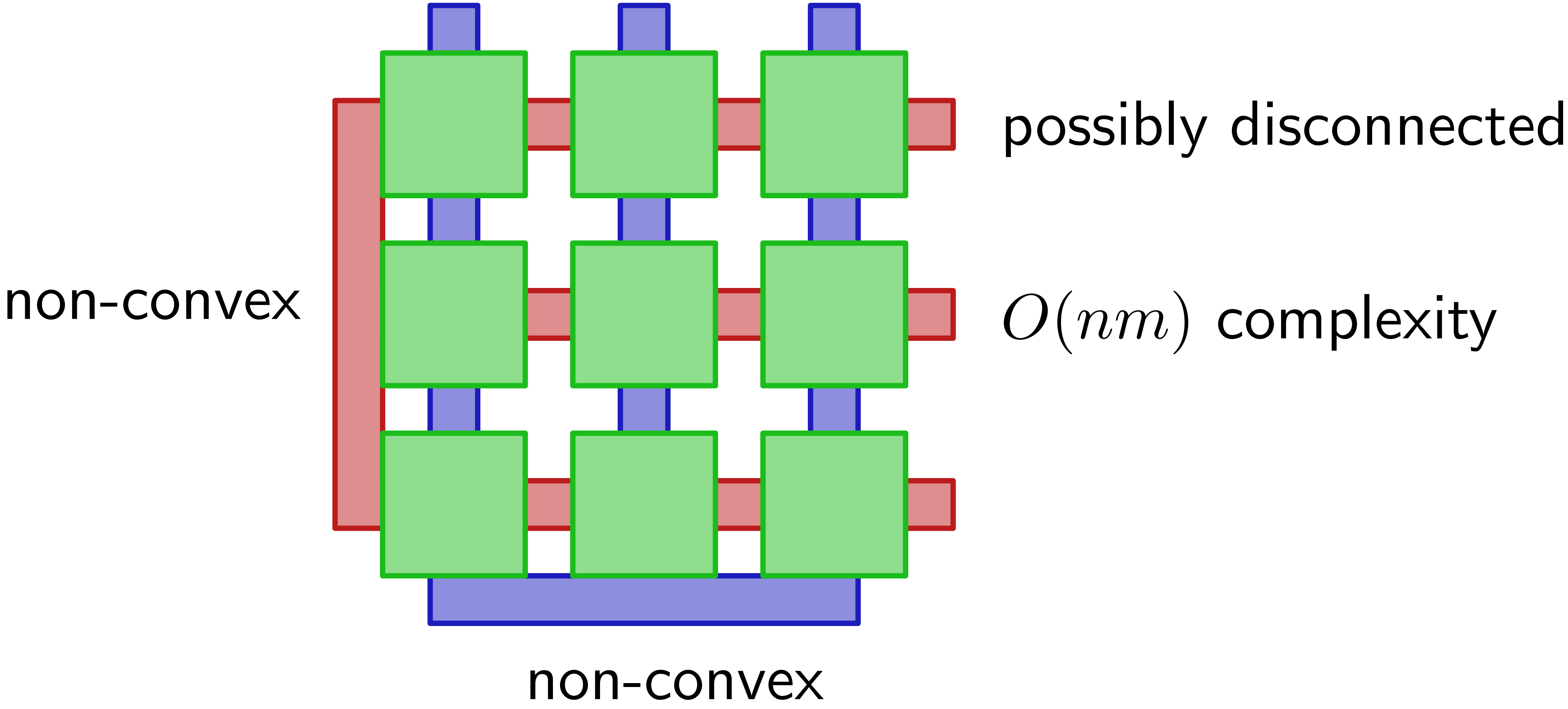

convex
convex
convex
$O(n+m)$ complexity

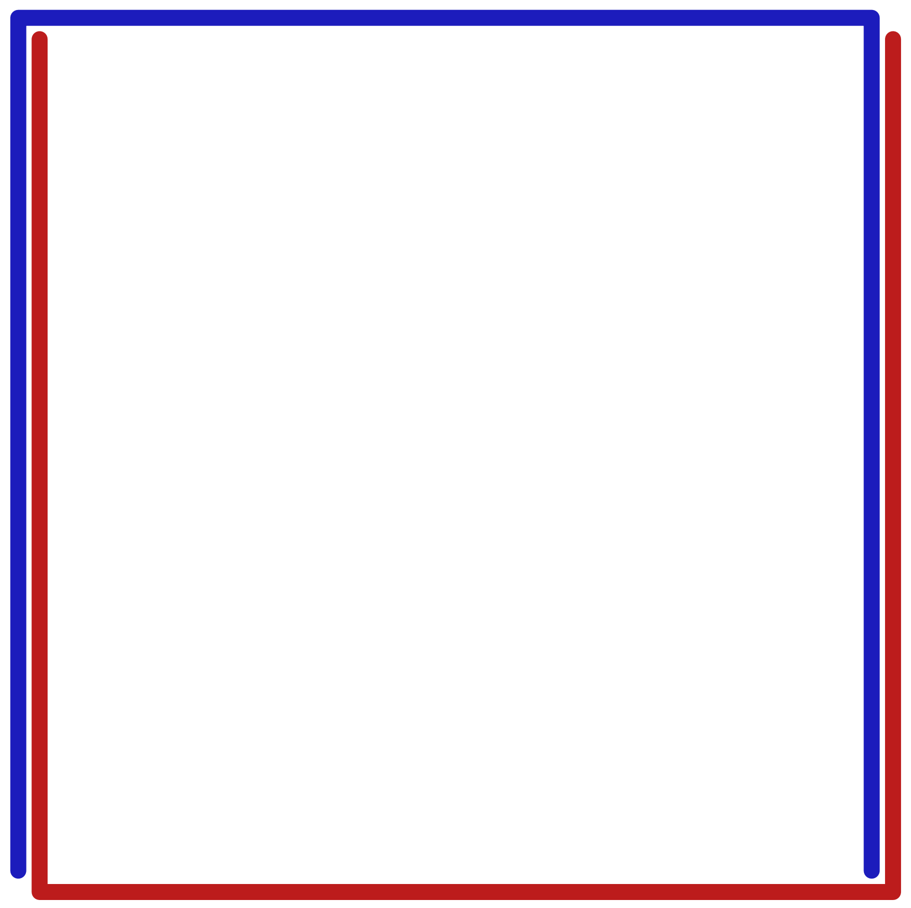

non-convex
convex

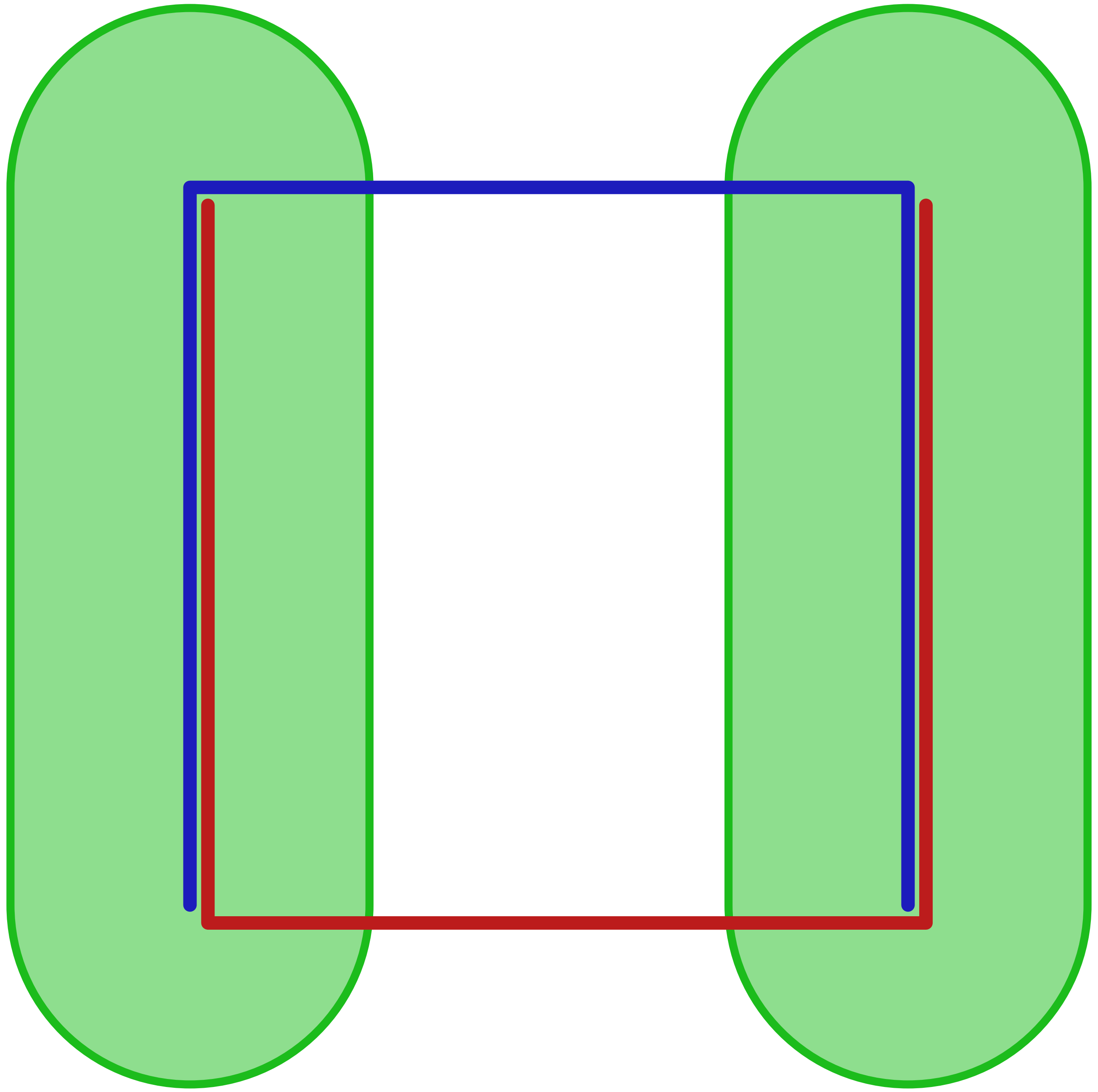


non-convex
convex
connected
$O(n+m)$ complexity

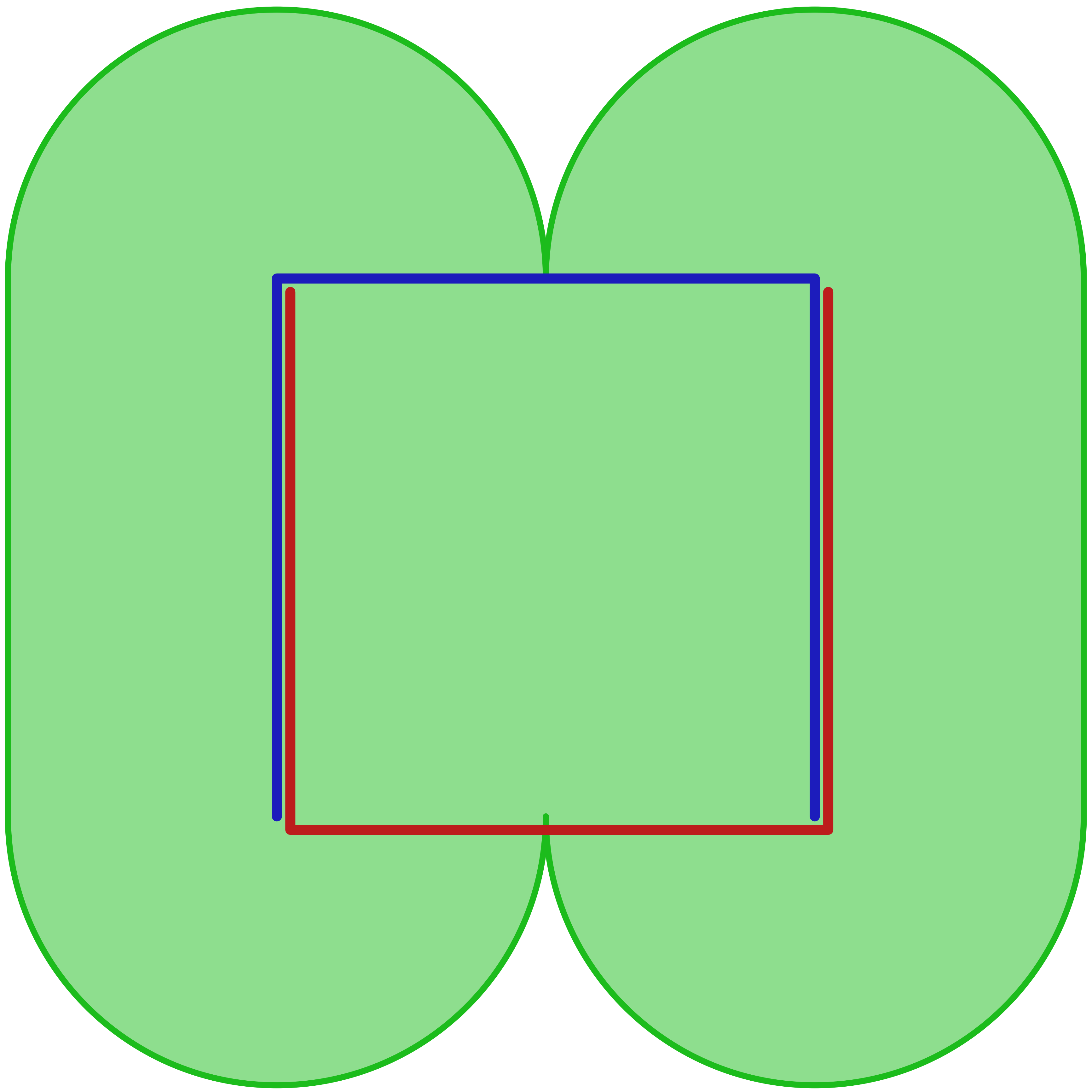

non-convex
non-convex


possibly disconnected
$O(nm)$ complexity
non-convex
non-convex

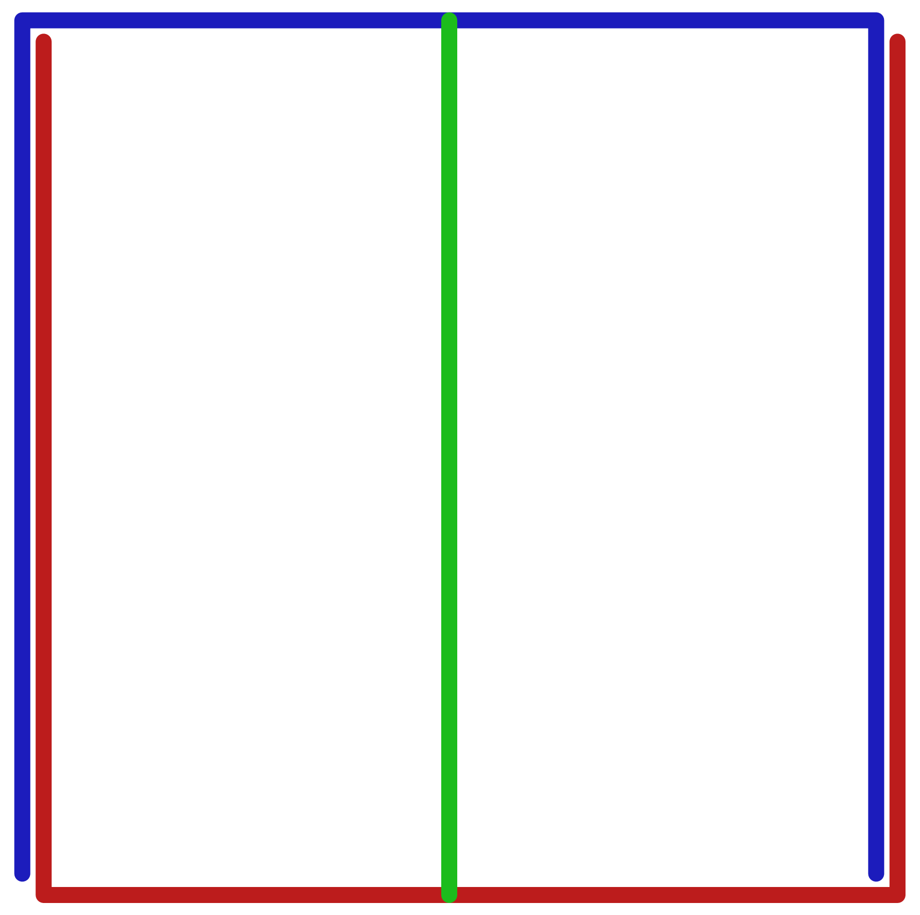


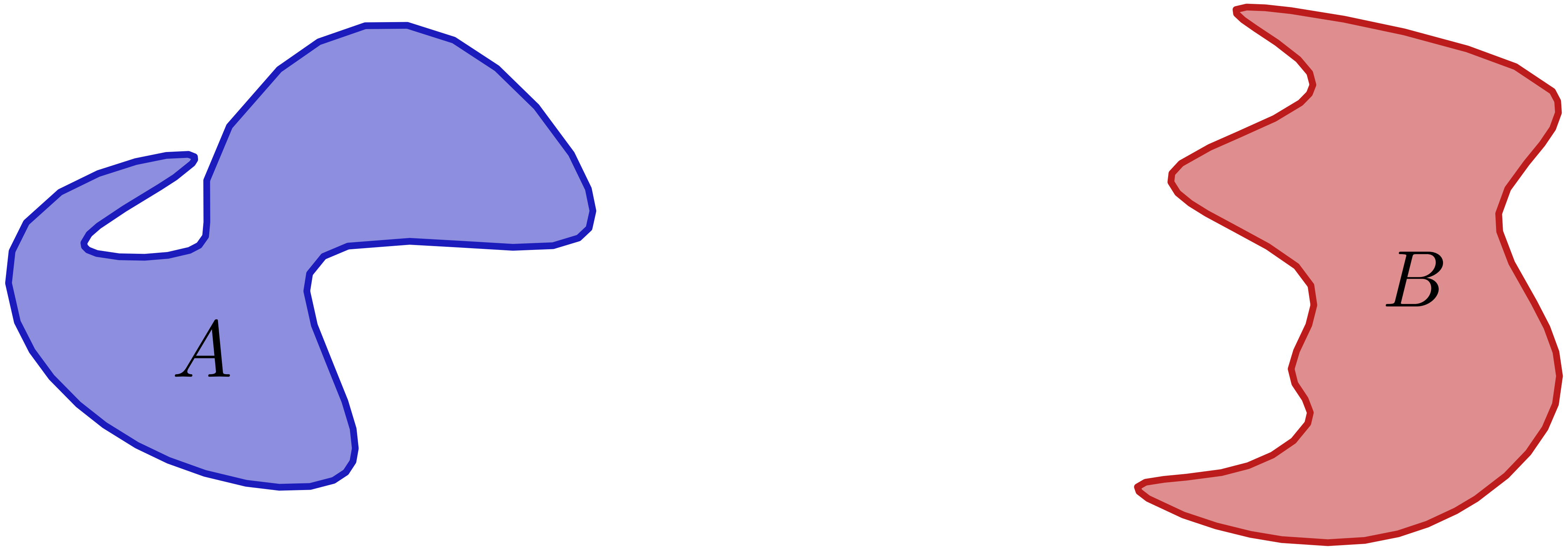


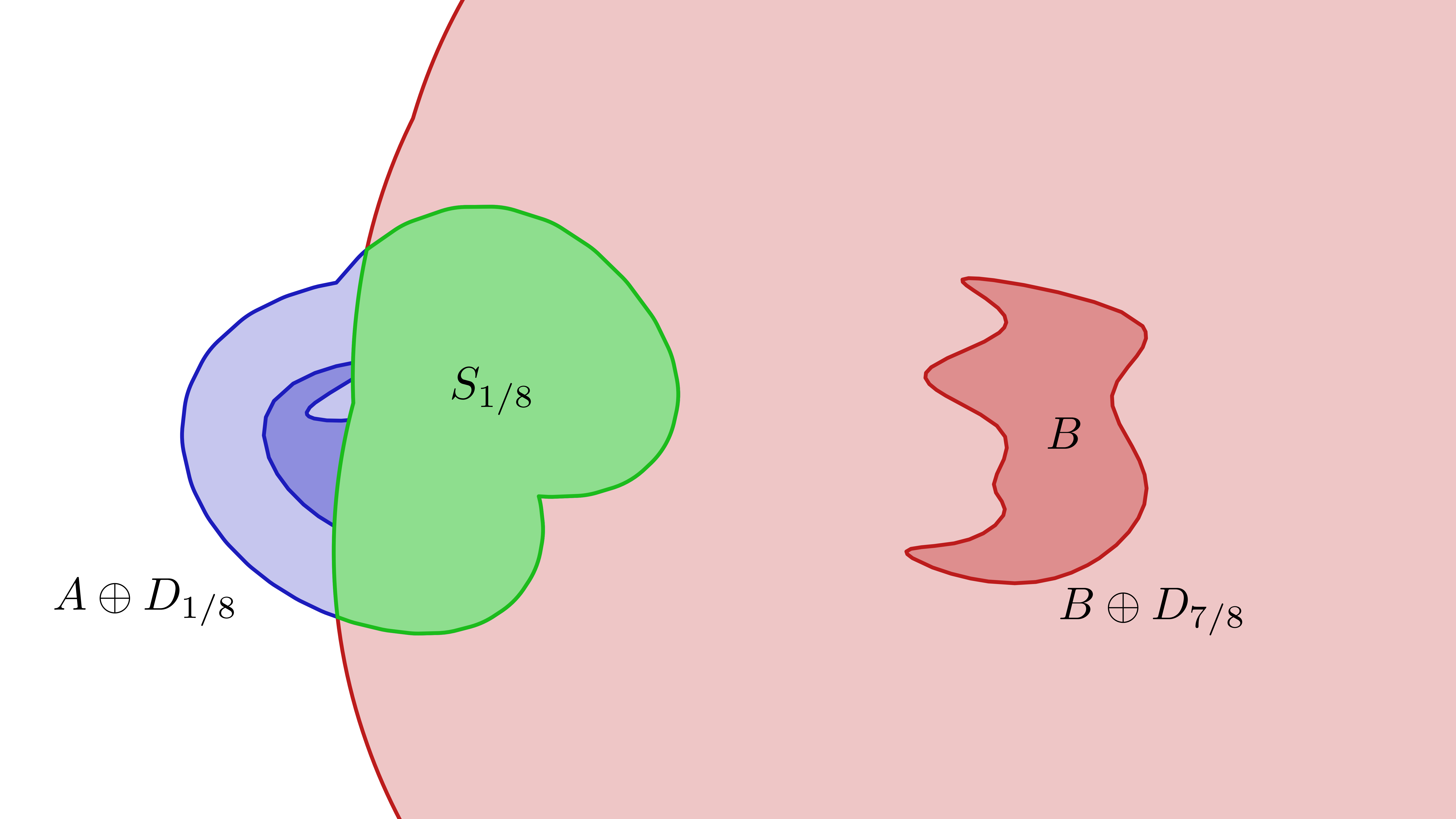
maximal S

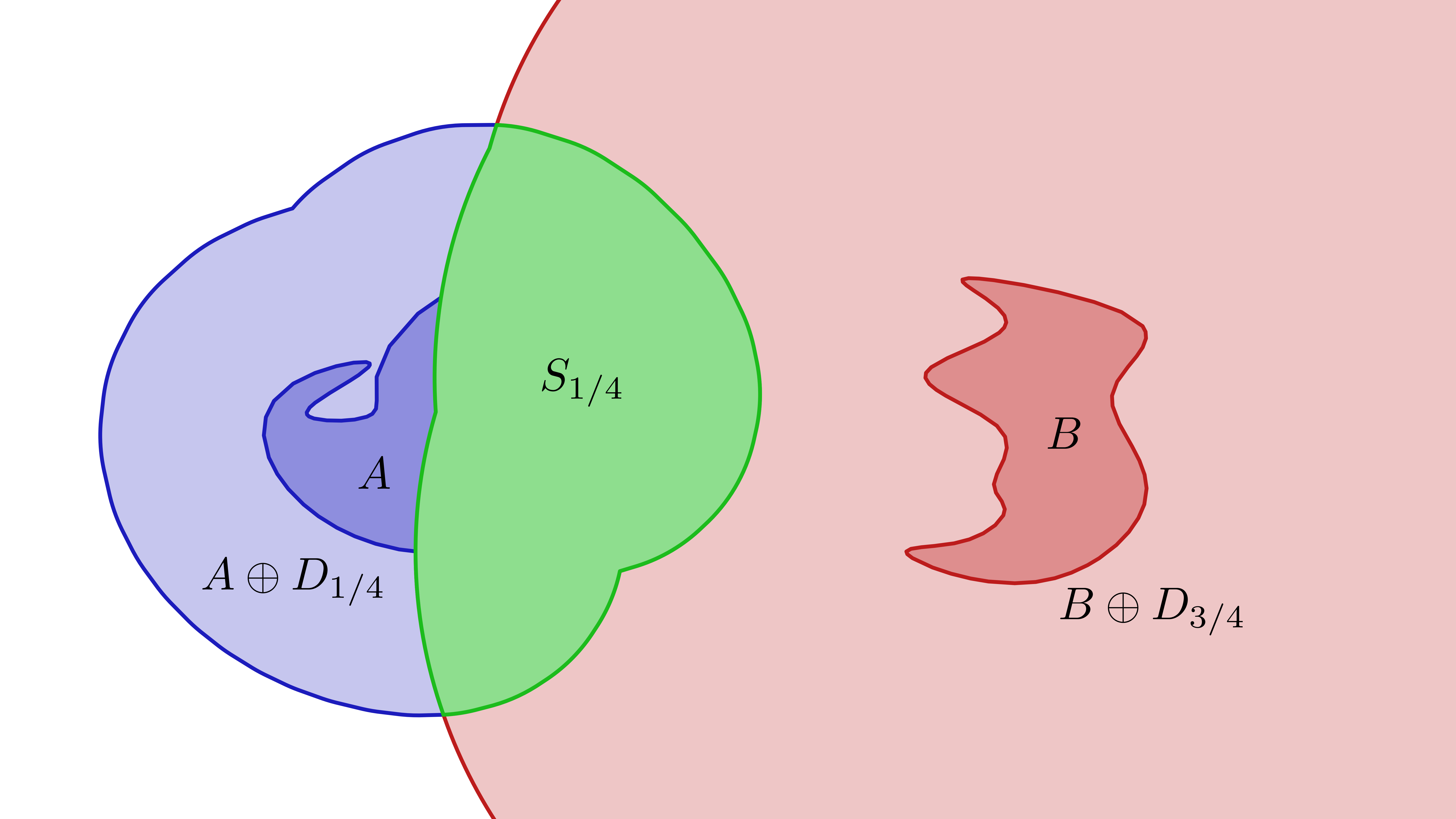


minimal S

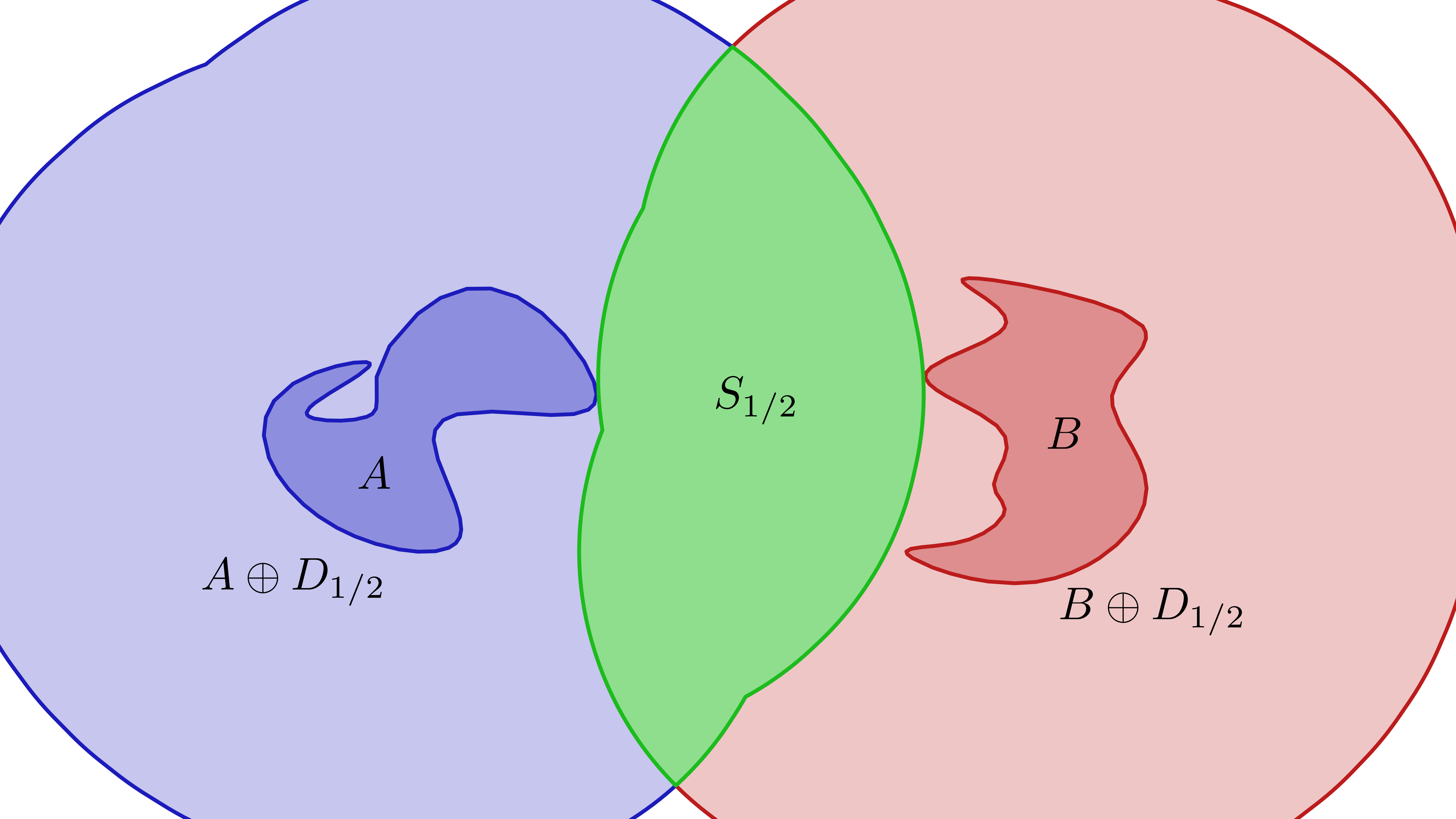



A
B

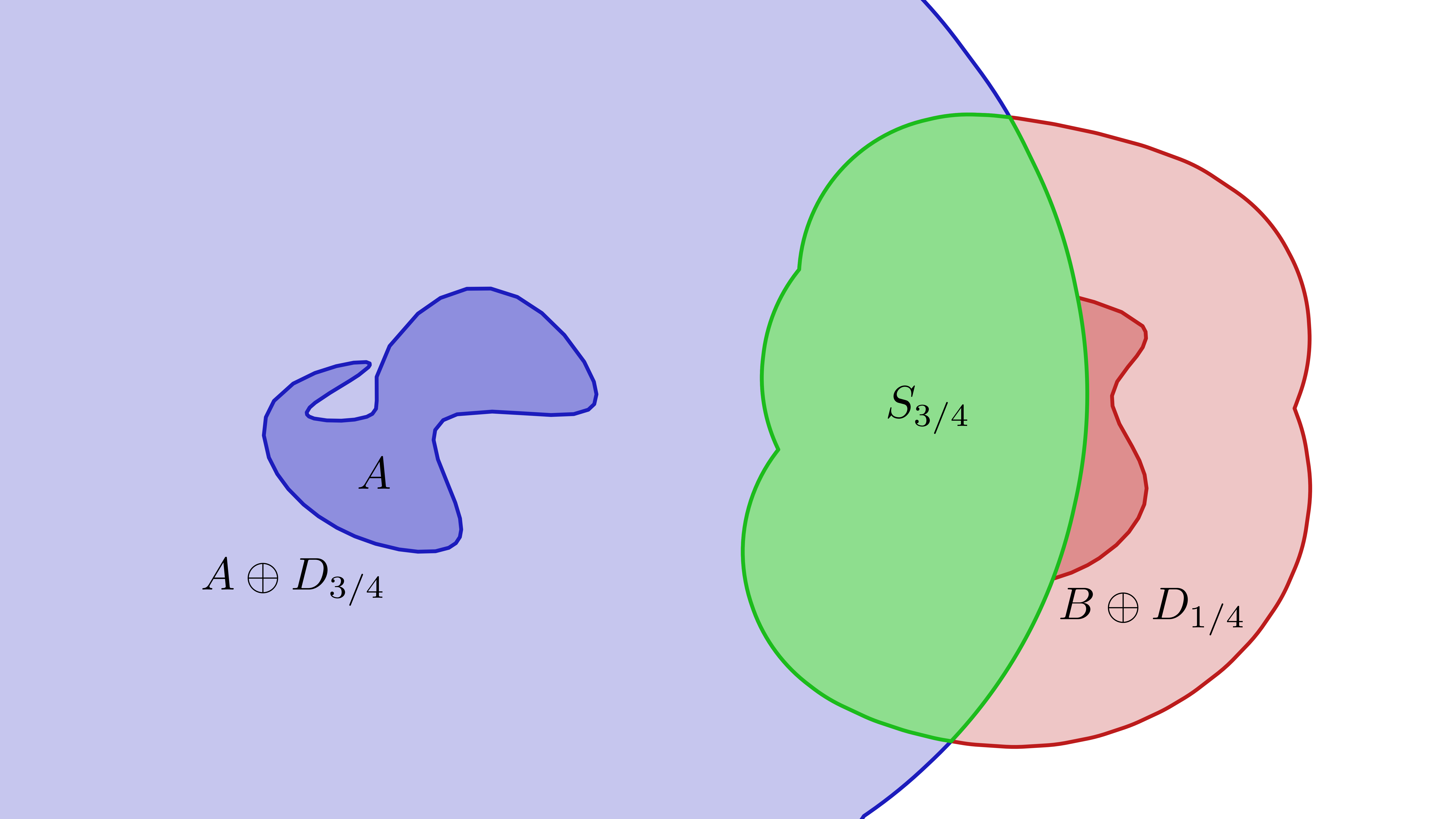

$S_{1/8}$
B
$A \oplus D_{1/8}$
$B \oplus D_{7/8}$



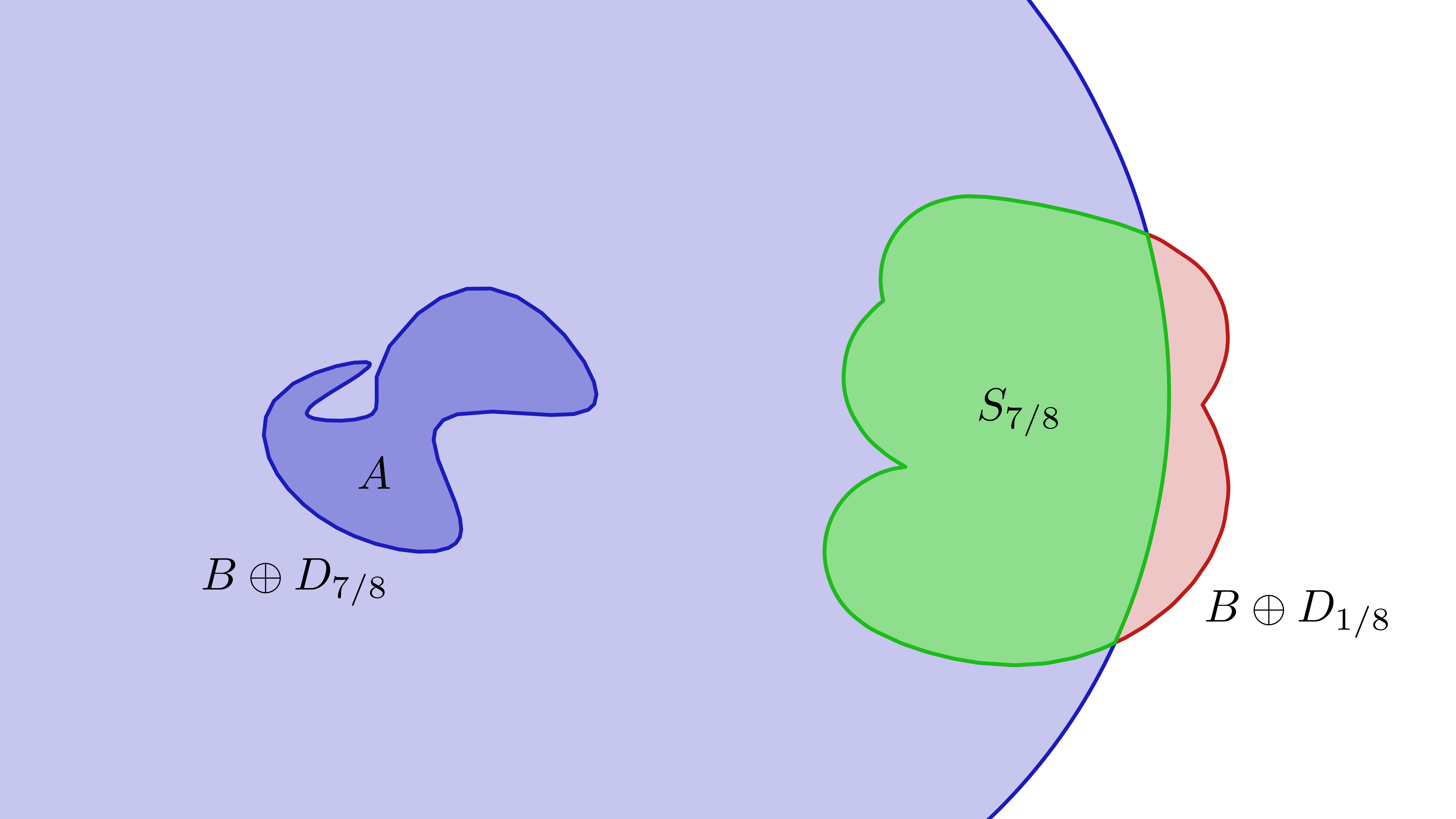
$S_{1/4}$
A
$A \oplus D_{1/4}$
B
$B \oplus D_{3/4}$



$S_{1/2}$
A
B
$A \oplus D_{1/2}$
$B \oplus D_{1/2}$



A
$A \oplus D_{3/4}$
$S_{3/4}$
$B \oplus D_{1/4}$



A
$B \oplus D_{7/8}$
$S_{7/8}$
$B \oplus D_{1/8}$

Conclusion:

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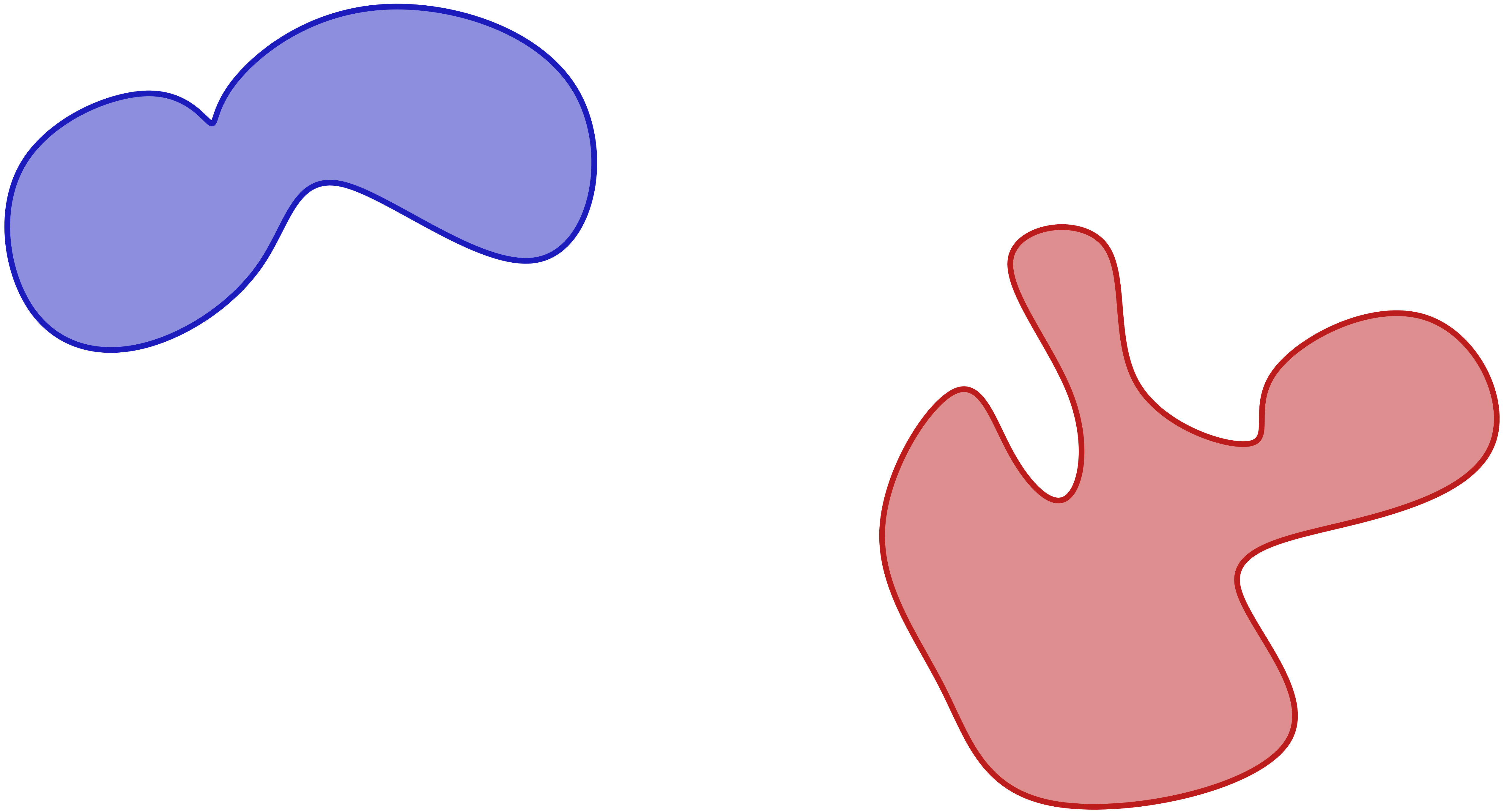
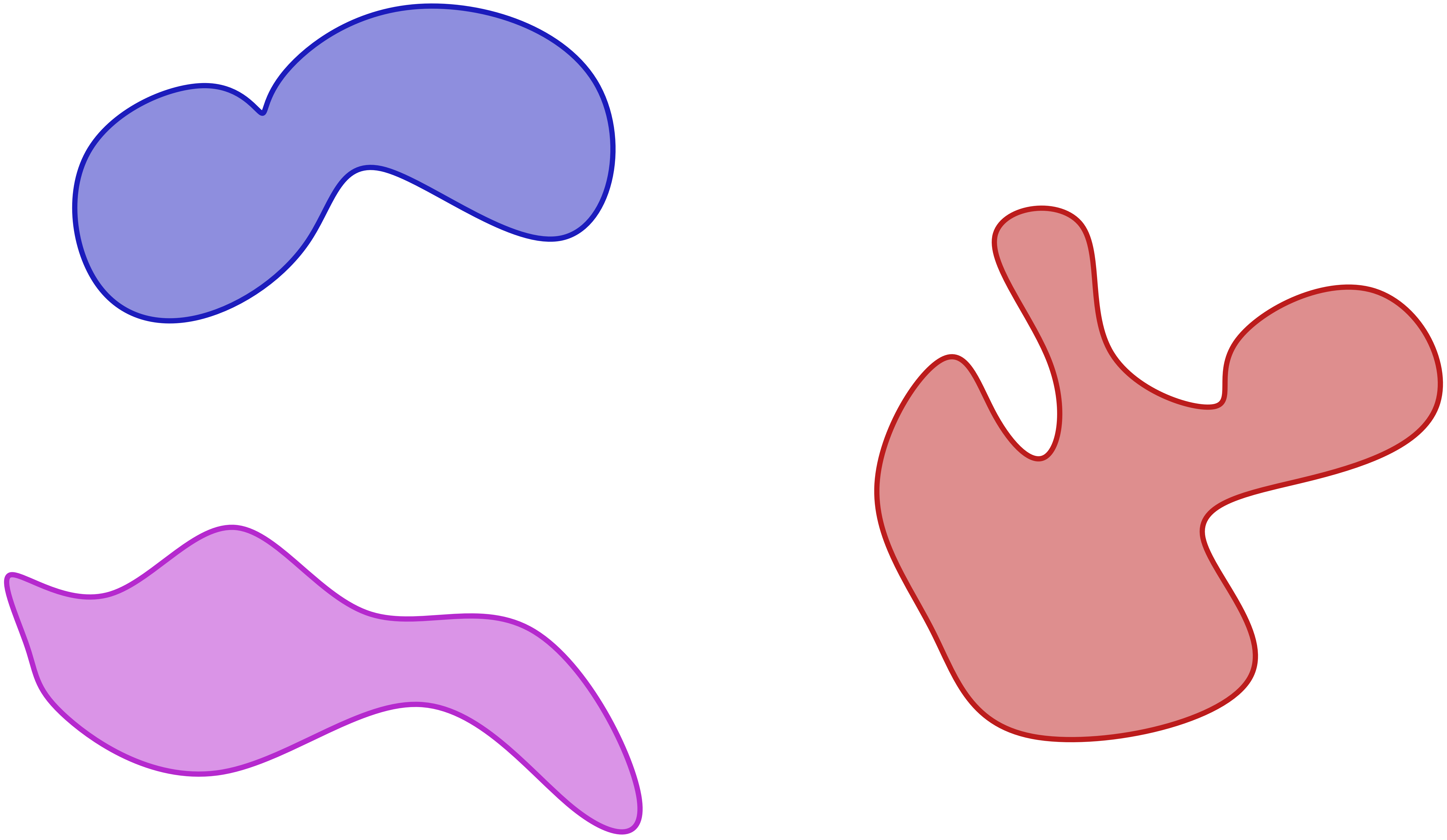
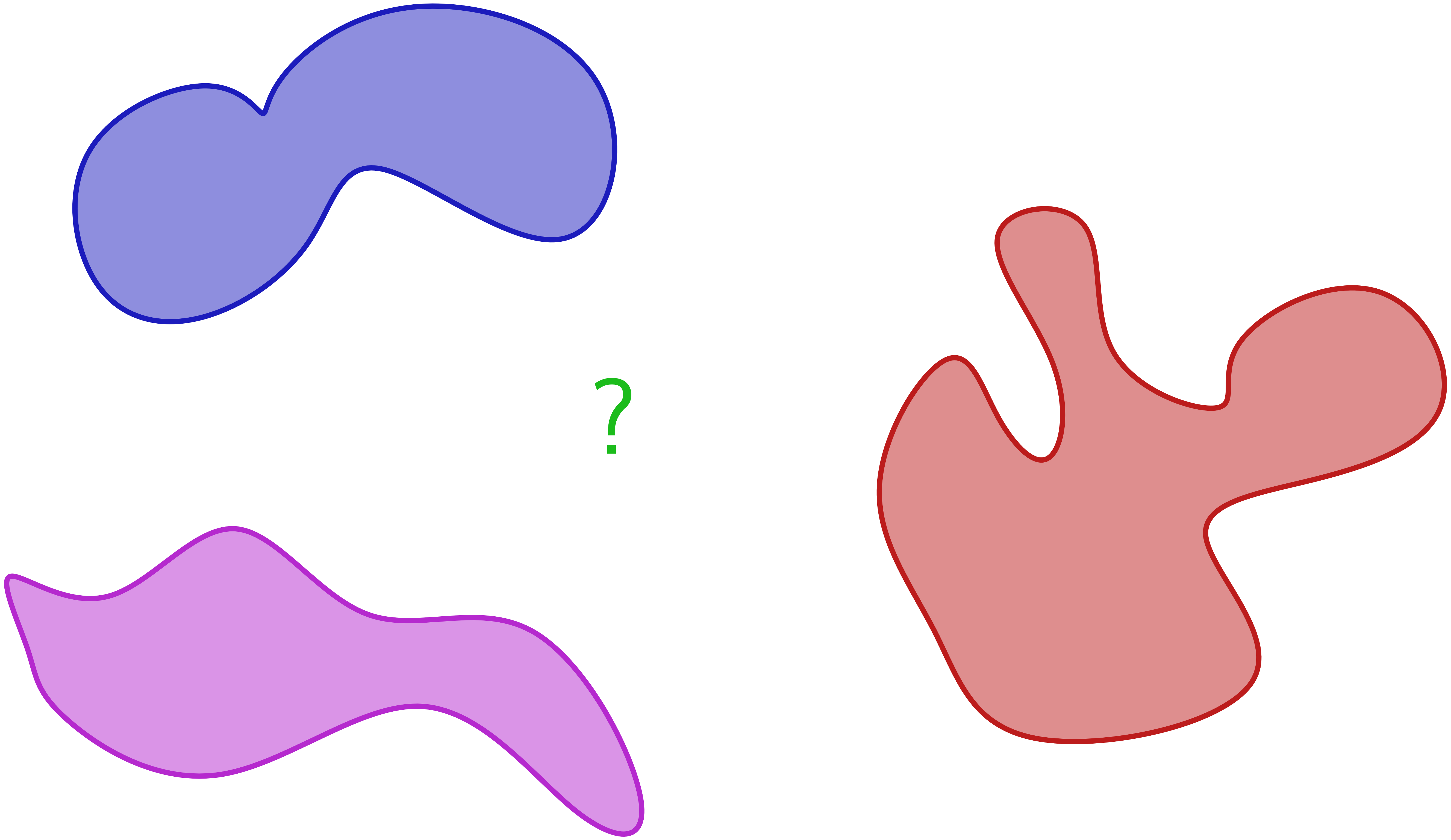
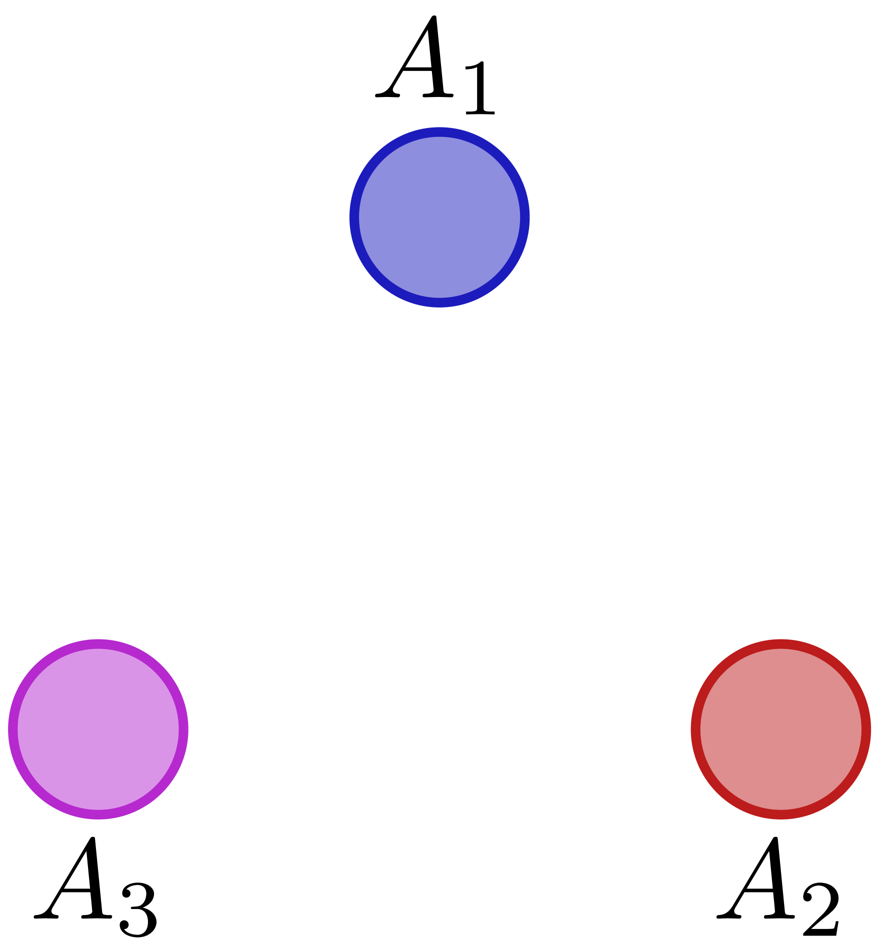
  - Not yet clear how to do morphing between three shapes







Input sets $\{A_1, \ldots, A_k\}$



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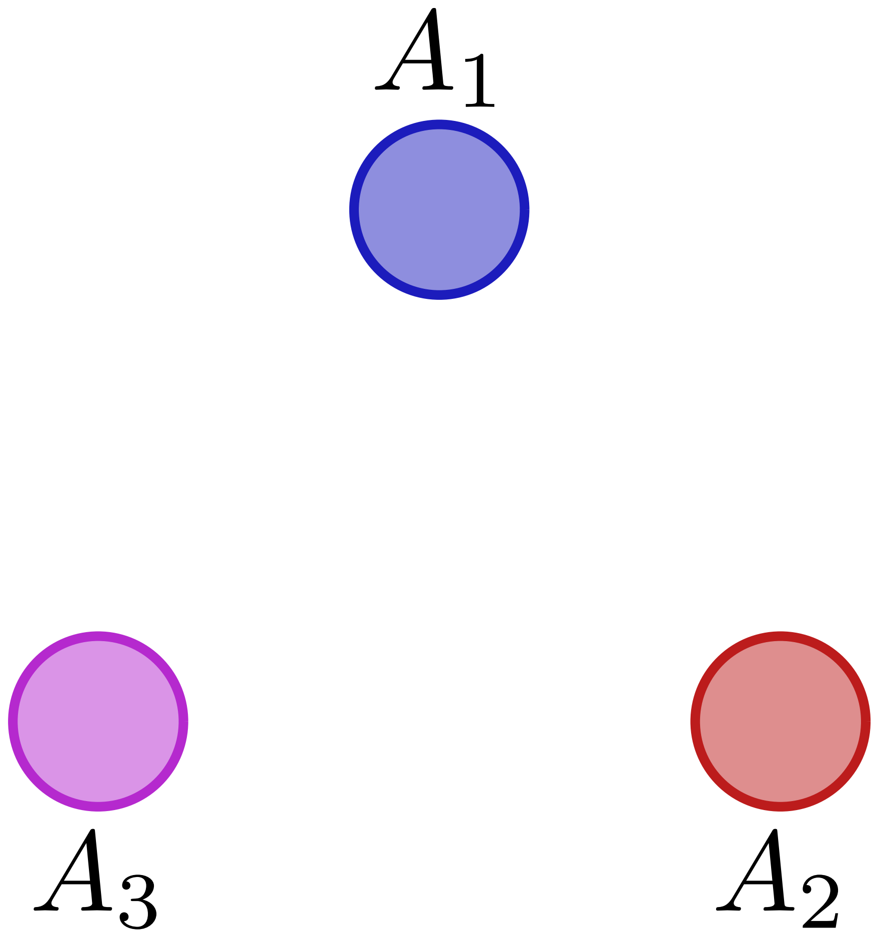

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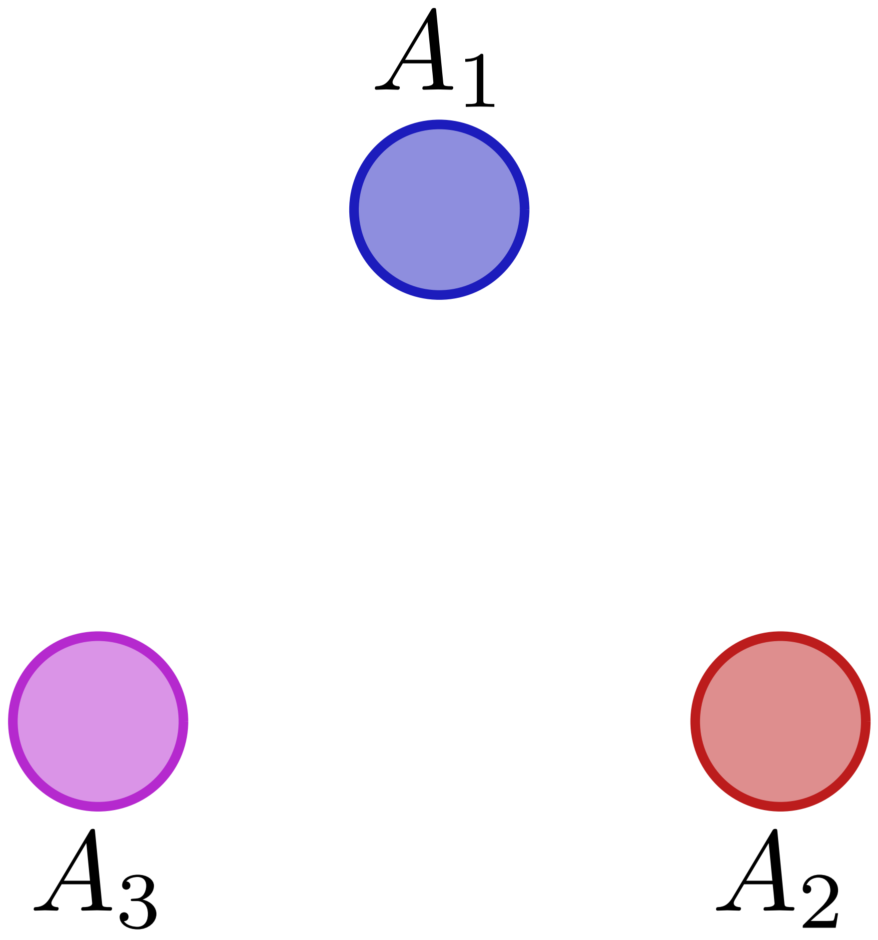
Find smallest α s.t. $A_i \subseteq S_\alpha \oplus D_\alpha$



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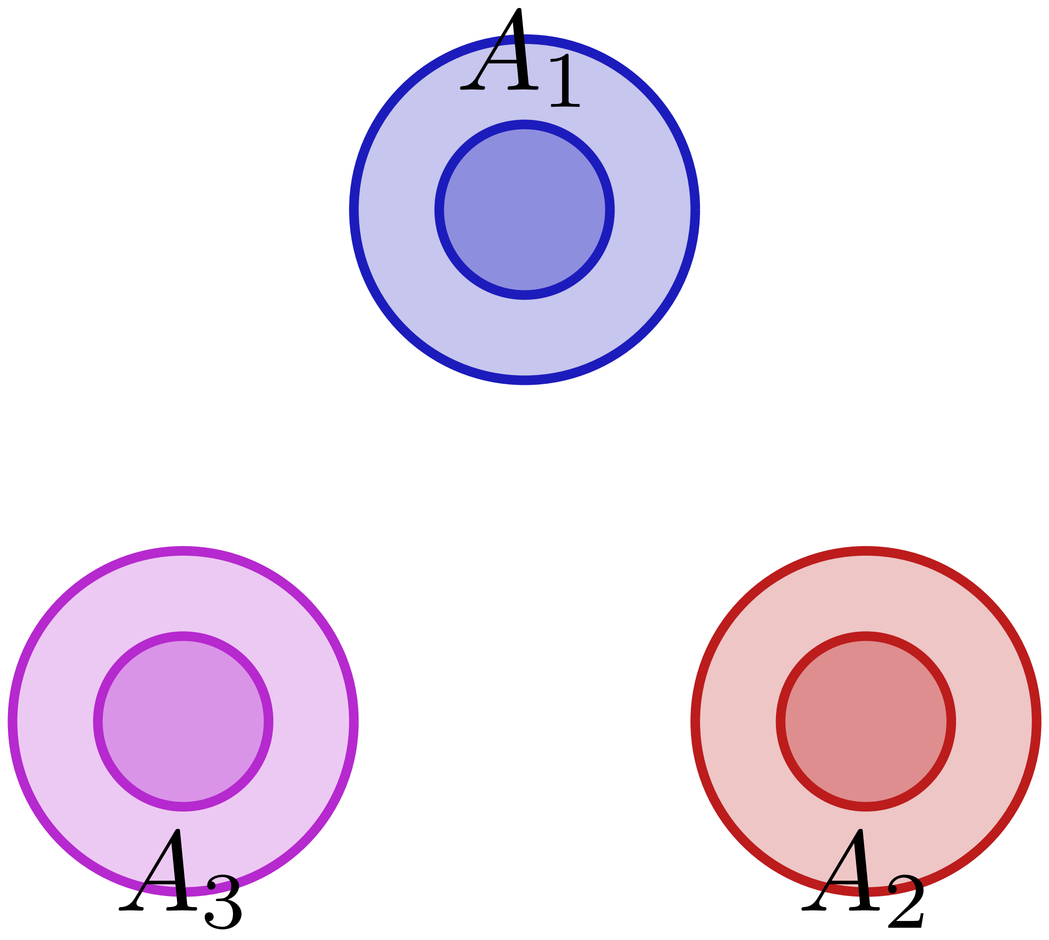

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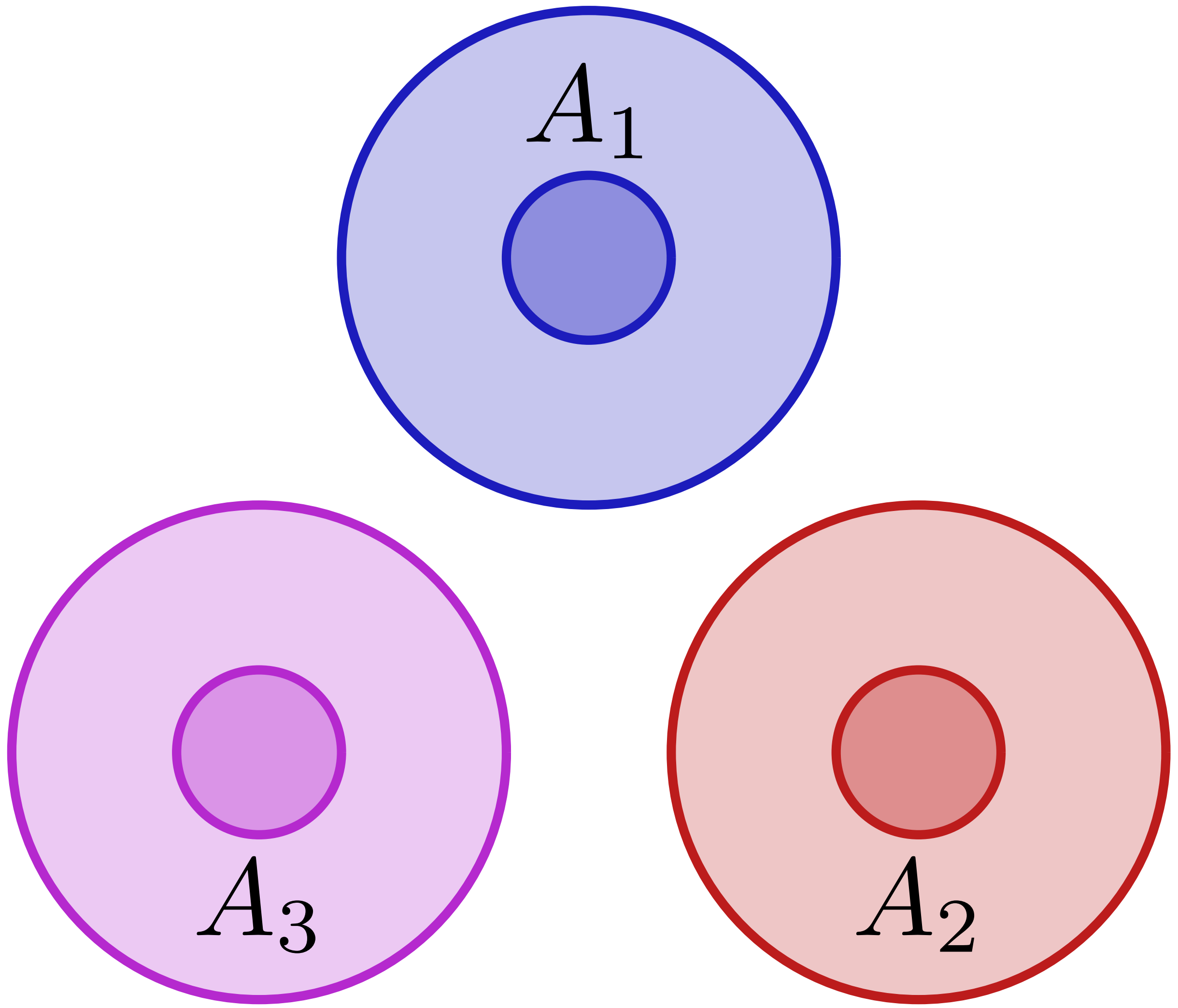

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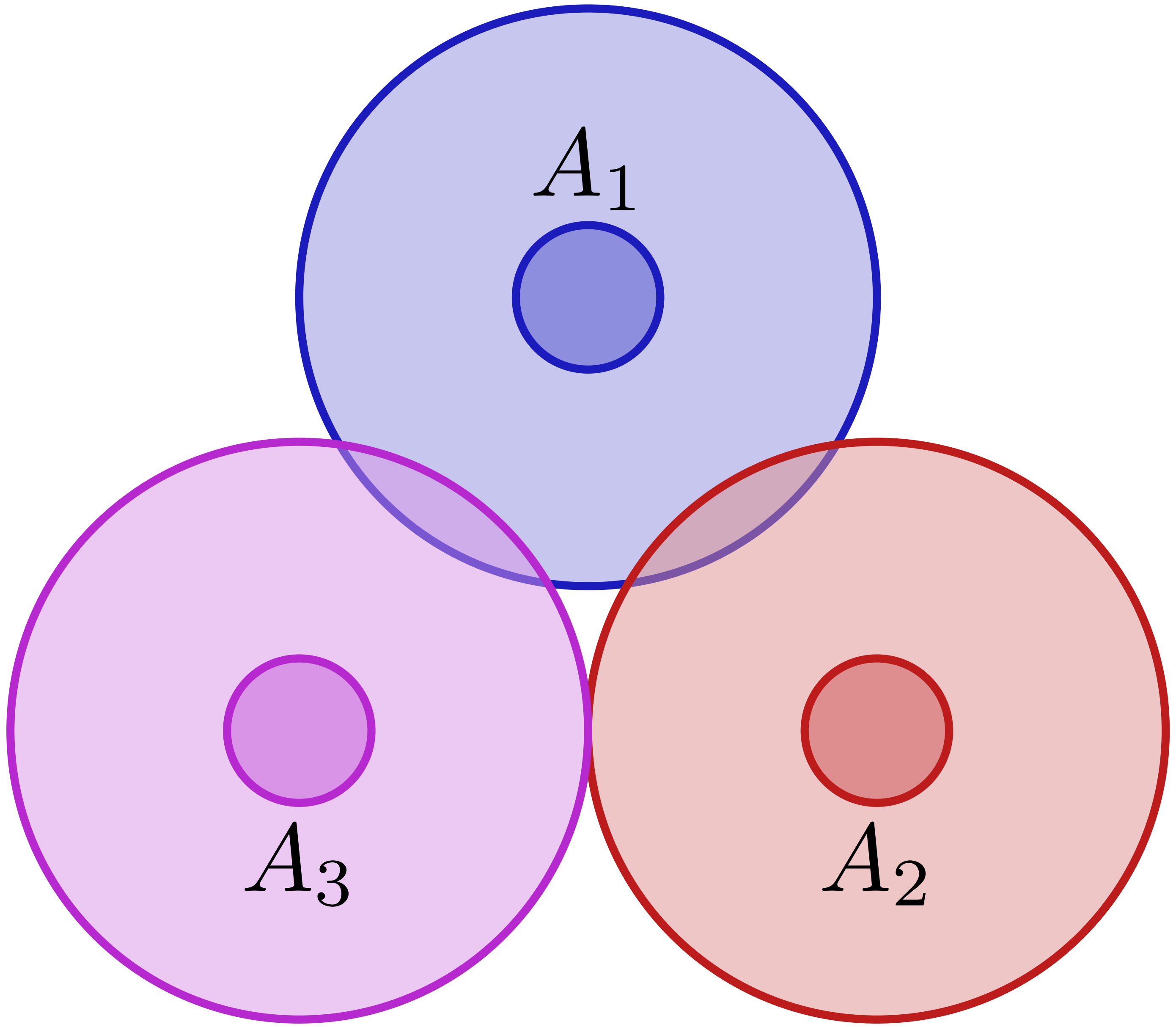

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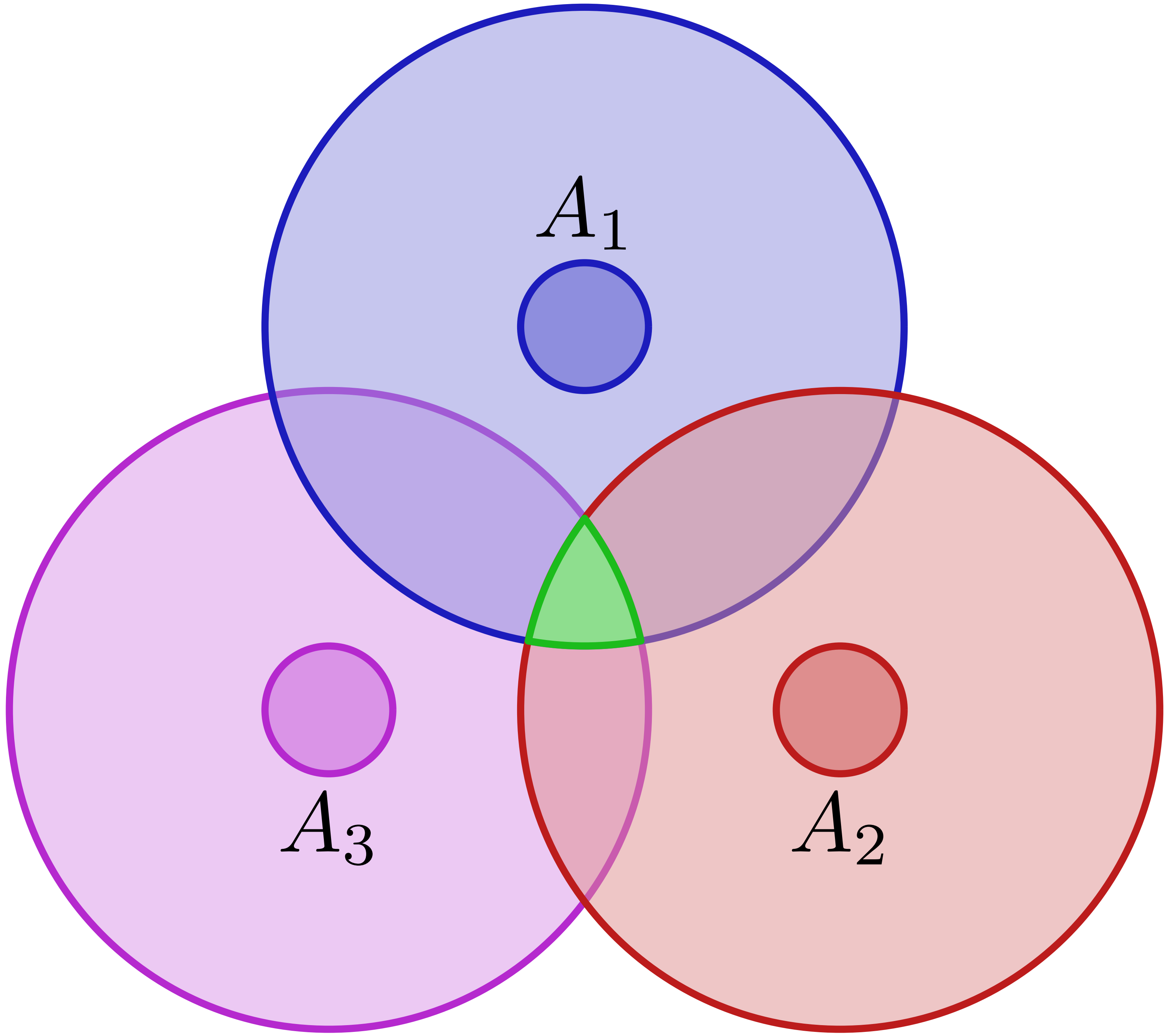

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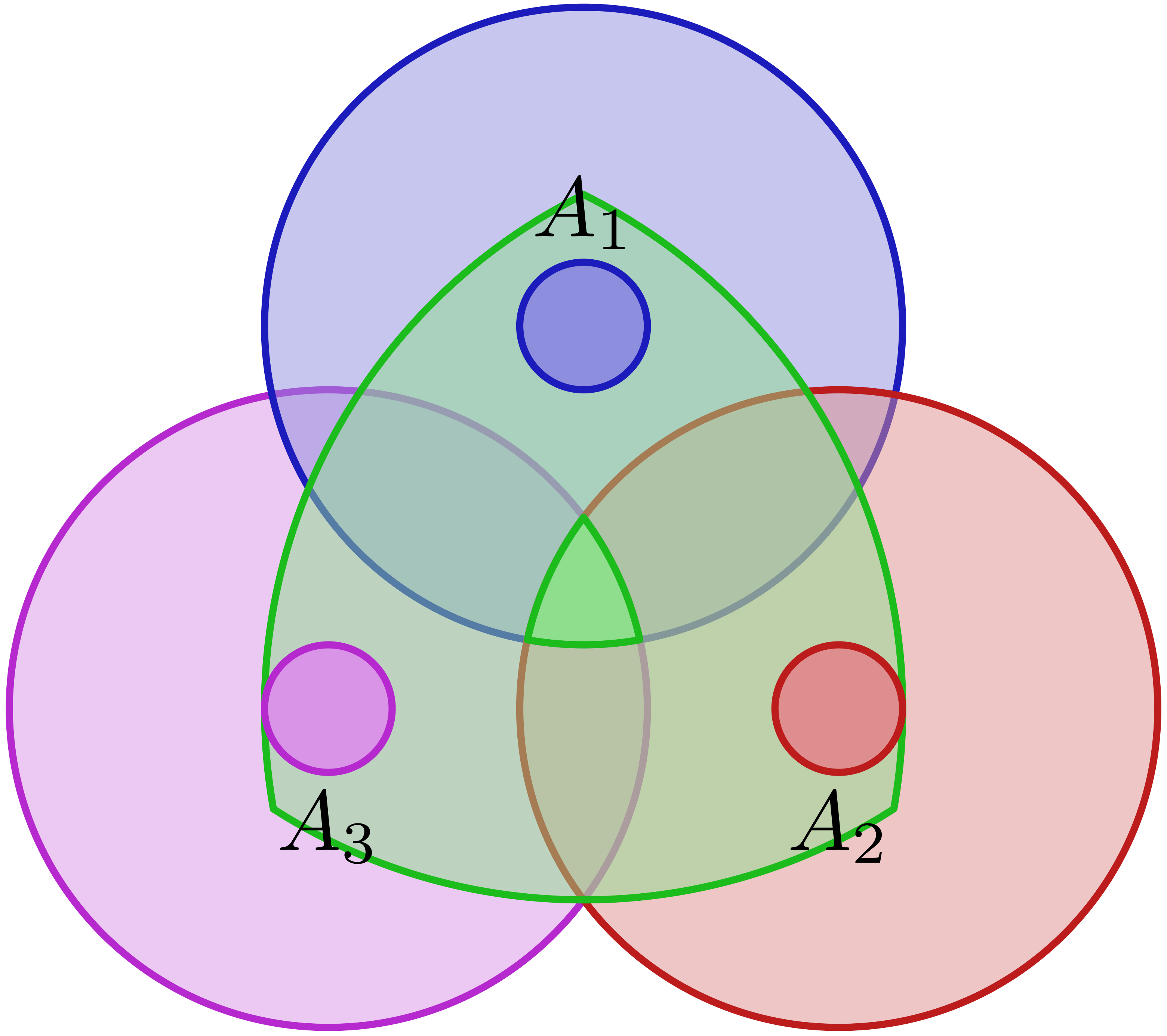

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