Balanced Independent Sets on Colored Interval Graphs

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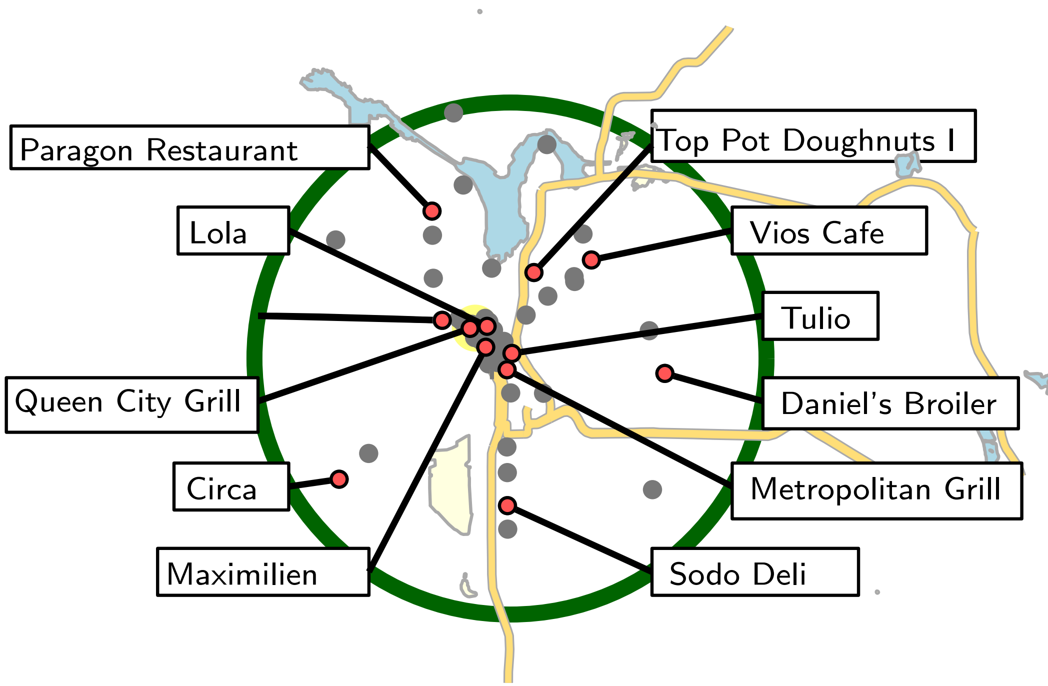
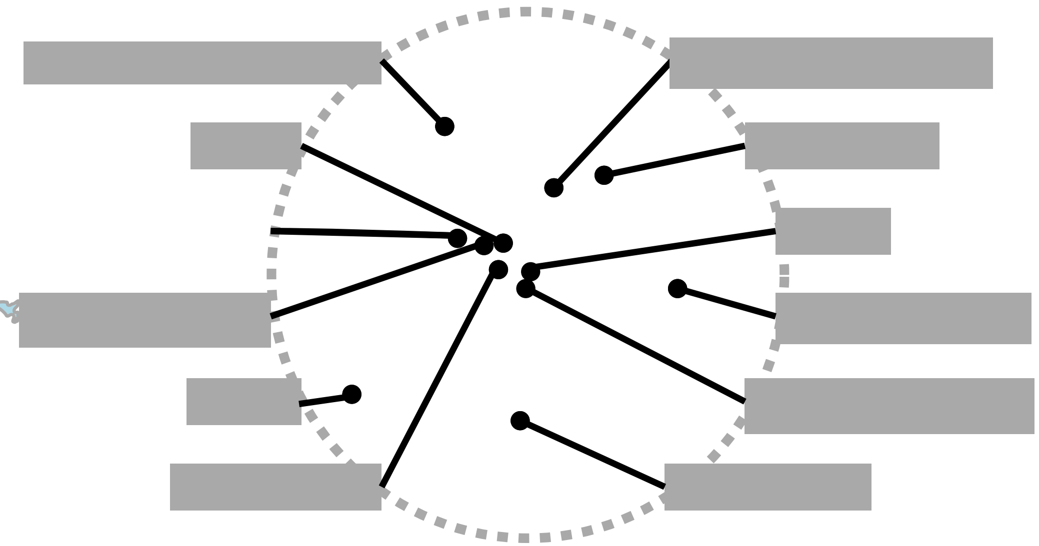
Boundary labeling





- labels are at the boundary of the focus region
- a leader connects a label with its corresponding POI
- task: select a large conflict-free labeling

M. Fink, J.-H. Haunert, A. Schulz, J. Spoerhase und A. Wolff.
Algorithms for labeling focus regions.
IEEE Transactions on Visualization and Computer Graphics (Proc. InfoVis'12),
18(12):2583–2592, 2012.

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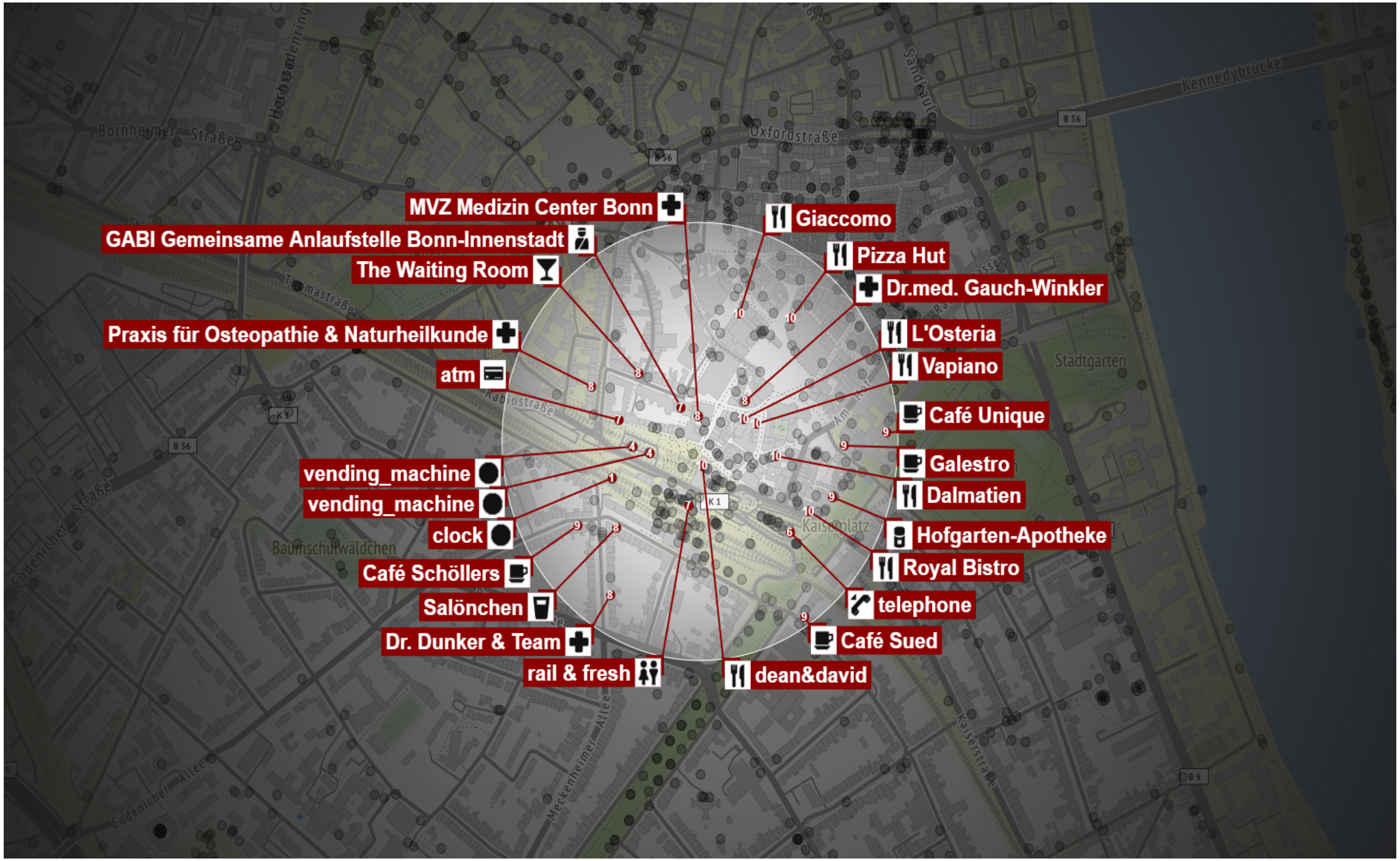
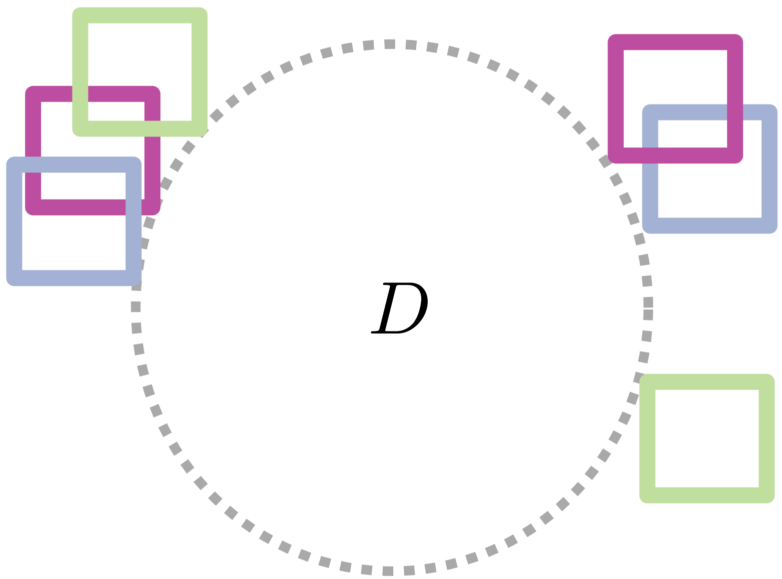


- labels represent objects of multiple categories
- task: select a good mixture of different object types

Model

input:

- a set of n colored axis-parallel unit squres touching a disk D
  - rectangle: icon



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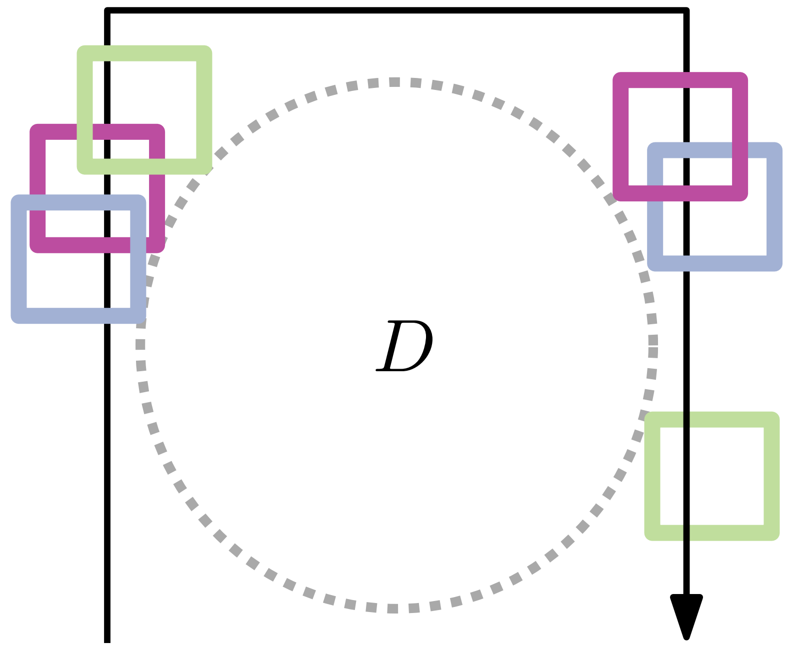

- interval representation of its intersection model

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- M contains exactly f elements from each of k color classes

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1-BIS

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reduction from 3-bounded 3SAT

- each variable x_i appears in ≤ 3 clauses
  each clause C_j has 2 or 3 literals

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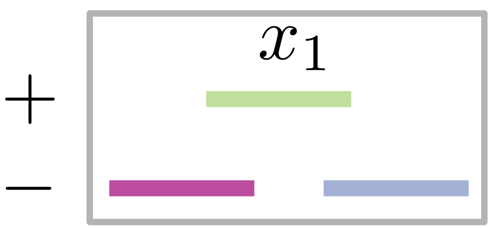
  - intersection: each pair of opposite literals



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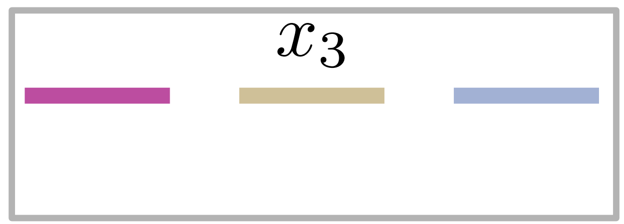

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1-BIS Problem: NP hardness

- Correctness

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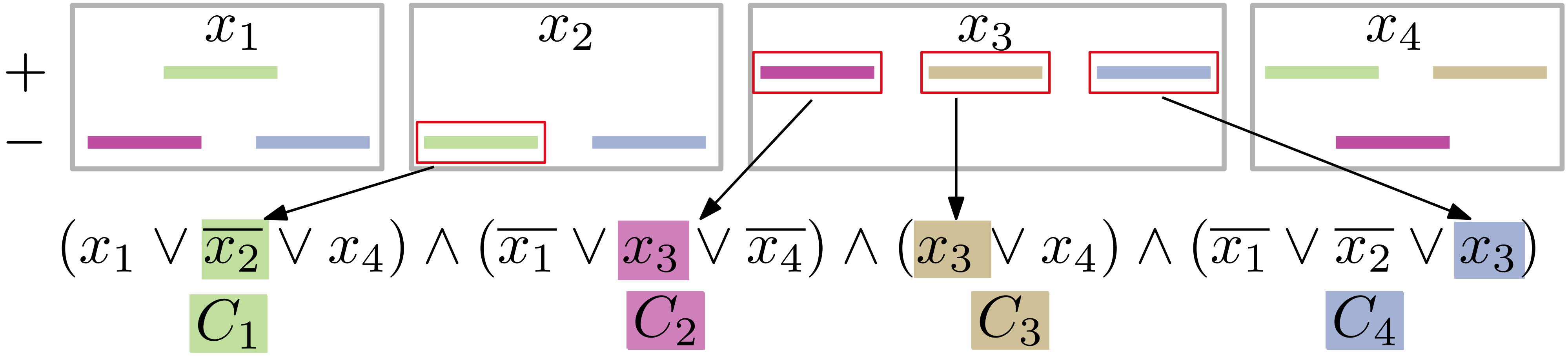

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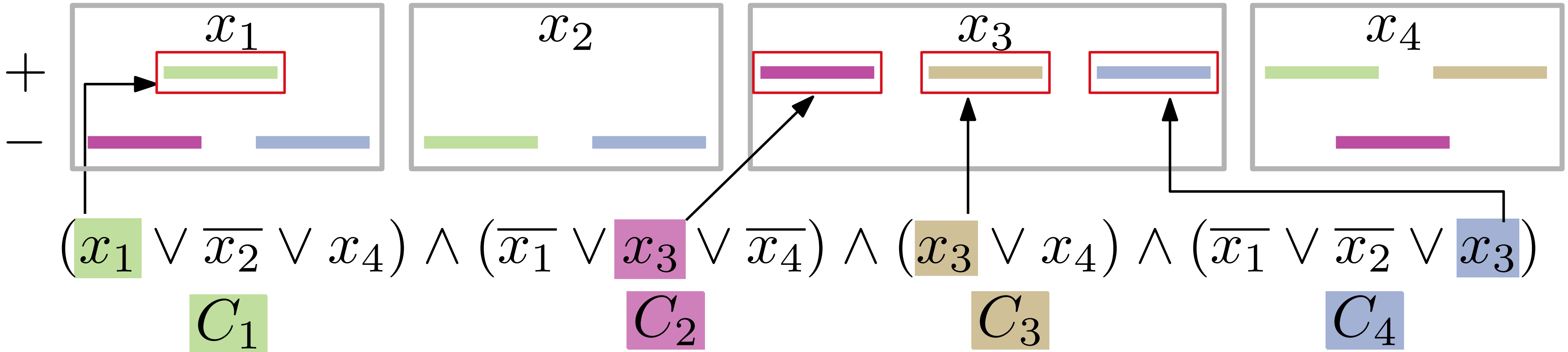

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f-BIS: An *FPT* Algorithm by (f, k)

- sorted set of intervals $\mathcal{I} = \{I_1, \ldots, I_n\}$ sorted by right-endpoints

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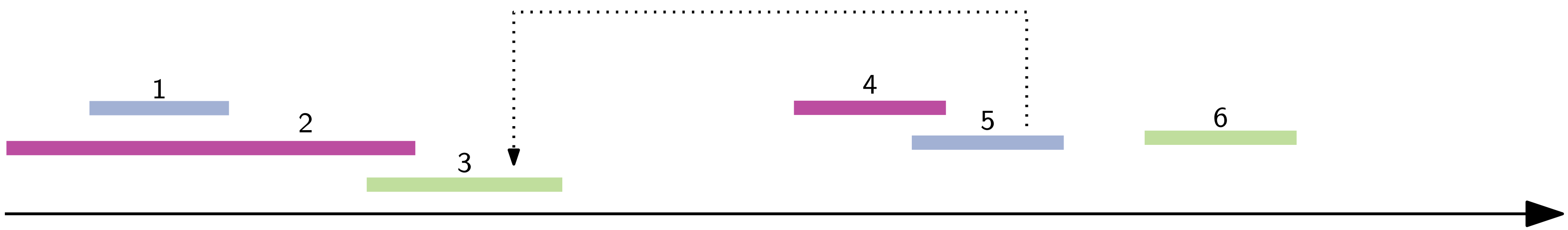

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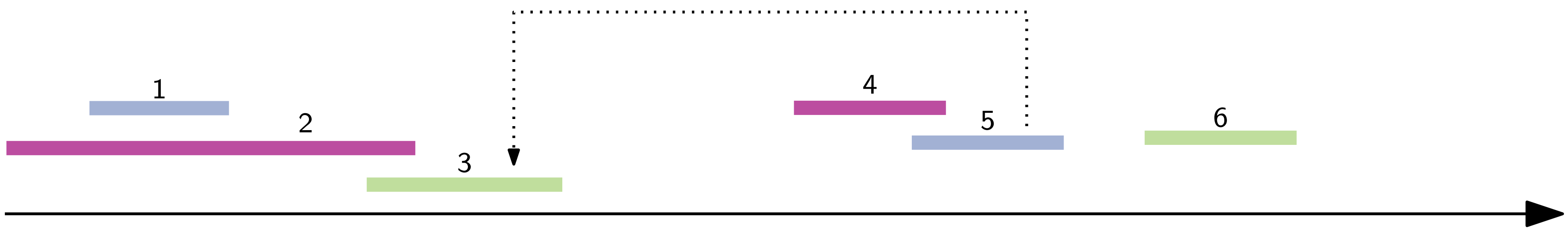

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- *cardinality vector* $C_{\mathcal{I}'}$: k-dimensional vector $(c_1, \ldots, c_i, \ldots, c_k)$

  cardinality of intervals of color i in $\mathcal{I}'$

  - $C_{\mathcal{I}'}$ is *valid*: $\mathcal{I}'$ is independent and $c_i \leq f$

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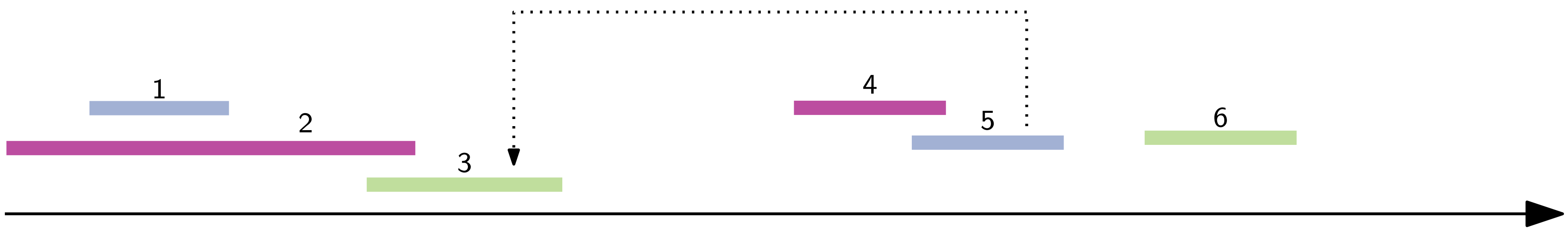

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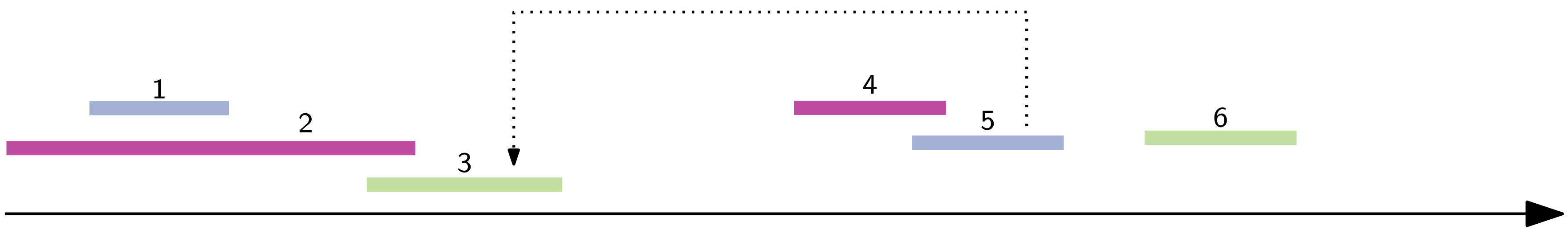

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    $\hat{e}_{c(I_j)} = (0 \dots, 1, \dots, 0)$, where the 1 is at position $c(I_j)$

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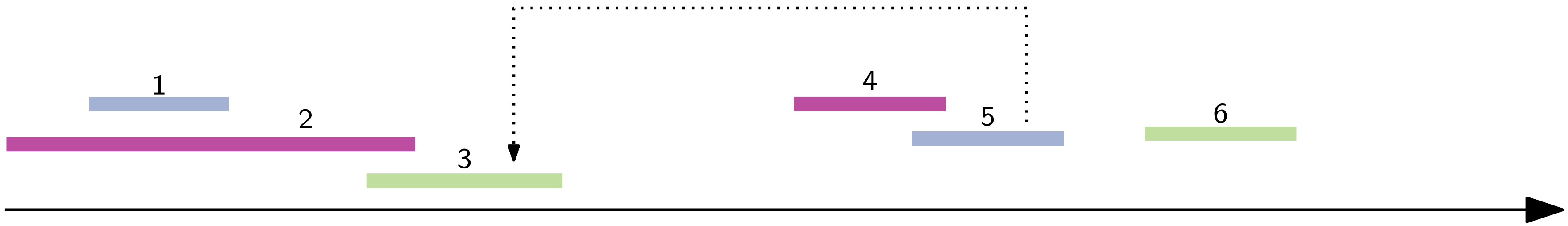

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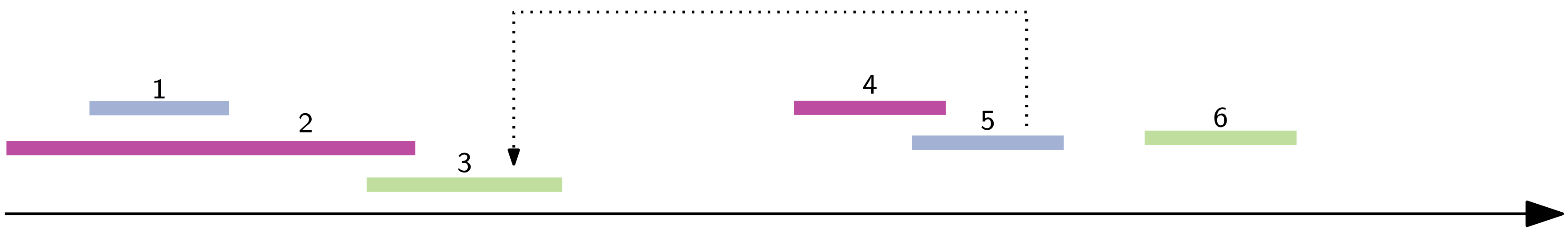

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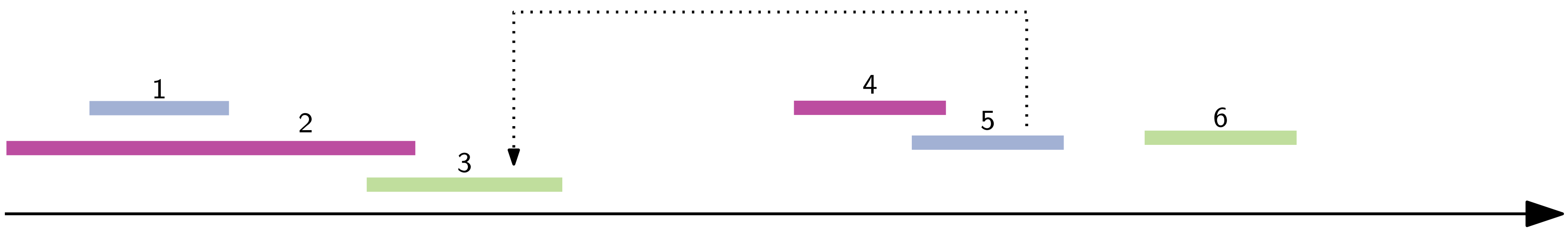

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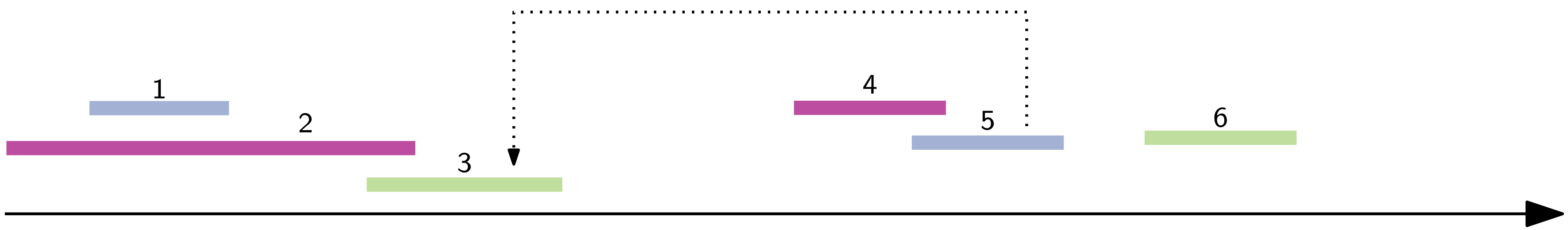

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- runtime: $O(n \log n + n f^k \alpha(f^k))$

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