Edge Guarding Plane Graphs

March 17, 2020
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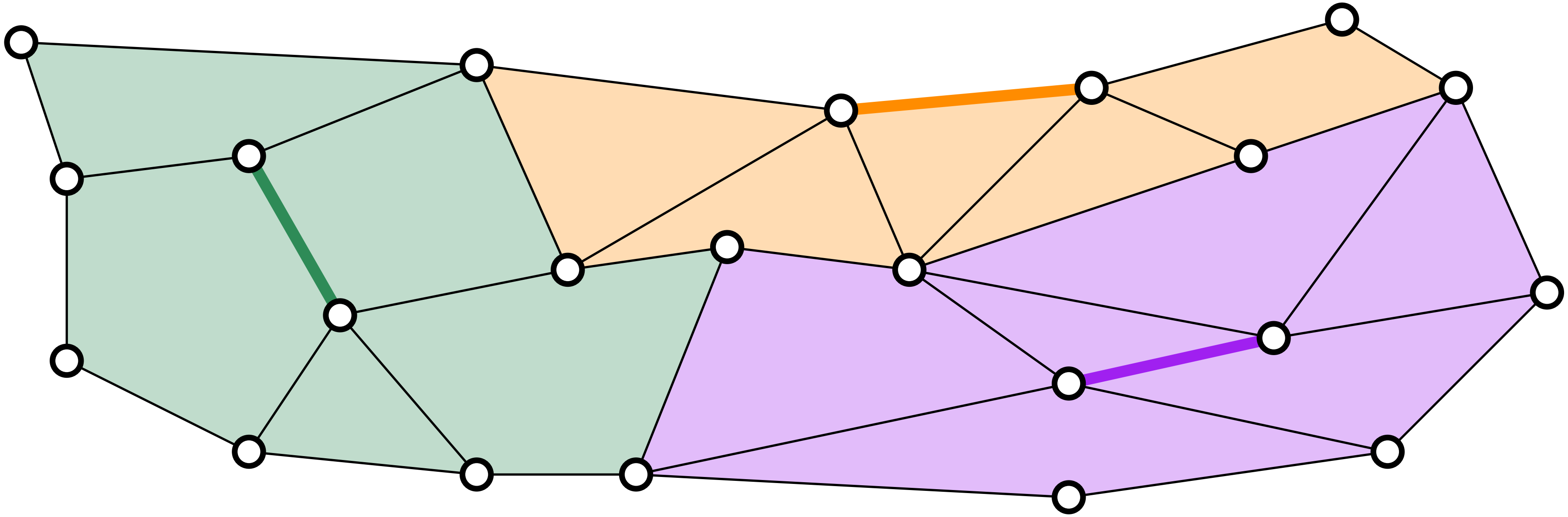
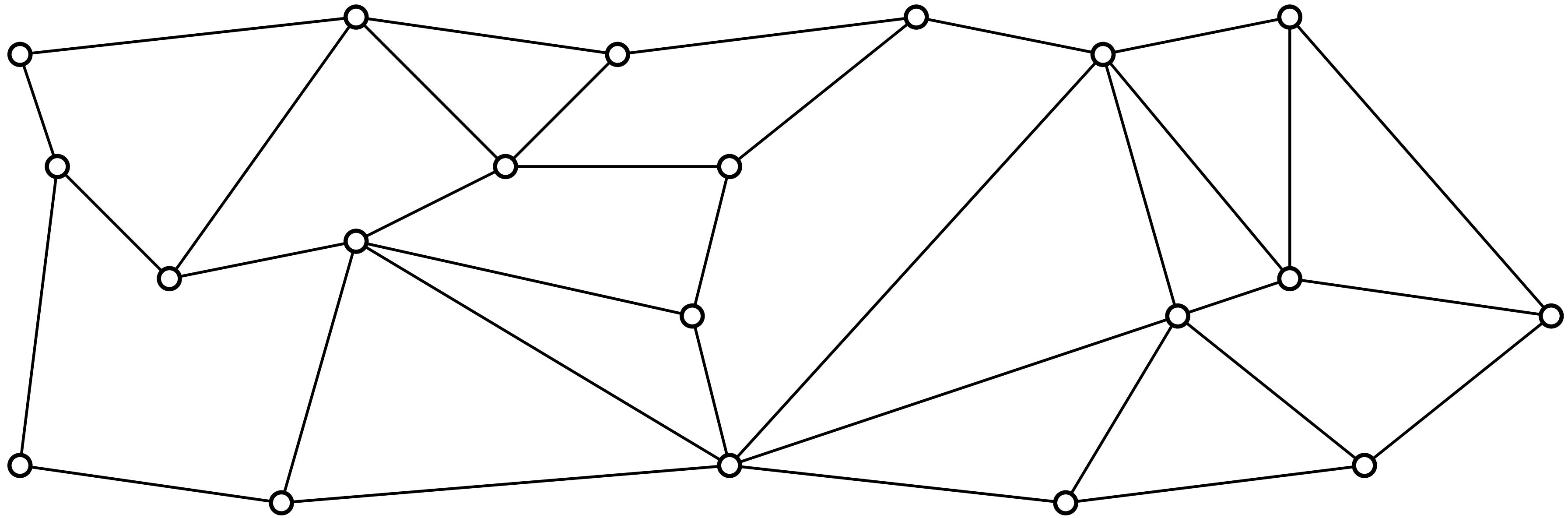
INSTITUTE OF THEORETICAL INFORMATICS · ALGORITHMICS GROUP



Edge Guarding

- $G = (V, E)$ plane graph.
- vw guards face f if at least one from $\{v, w\}$ is on the boundary of f.



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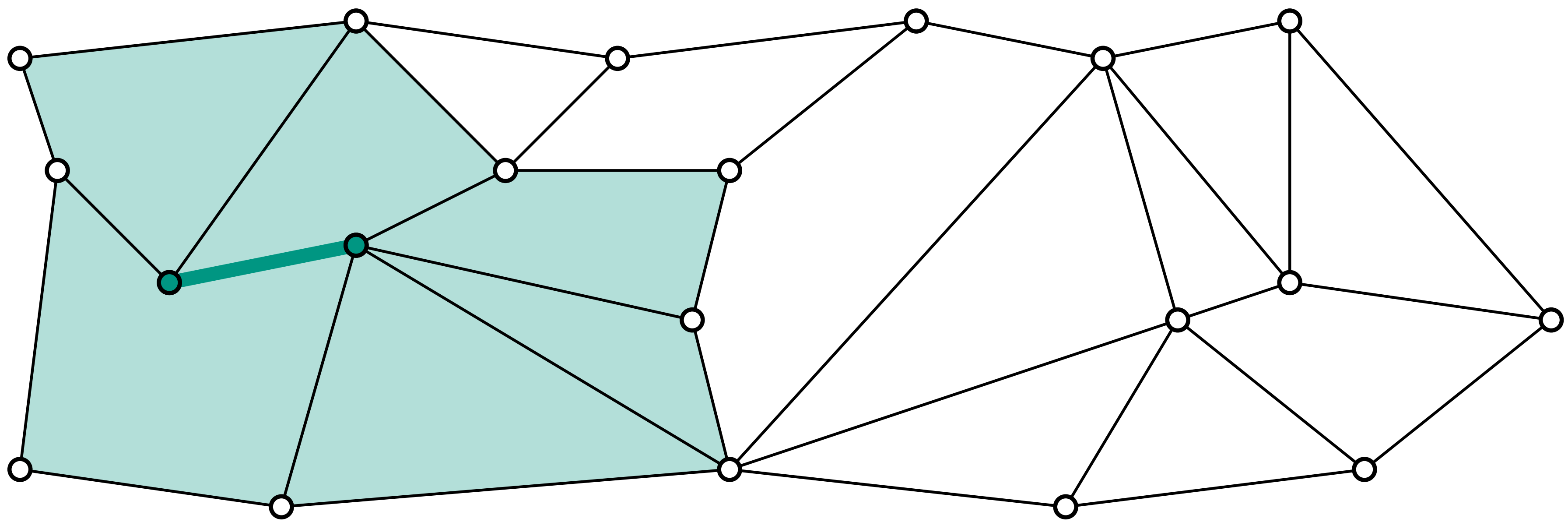

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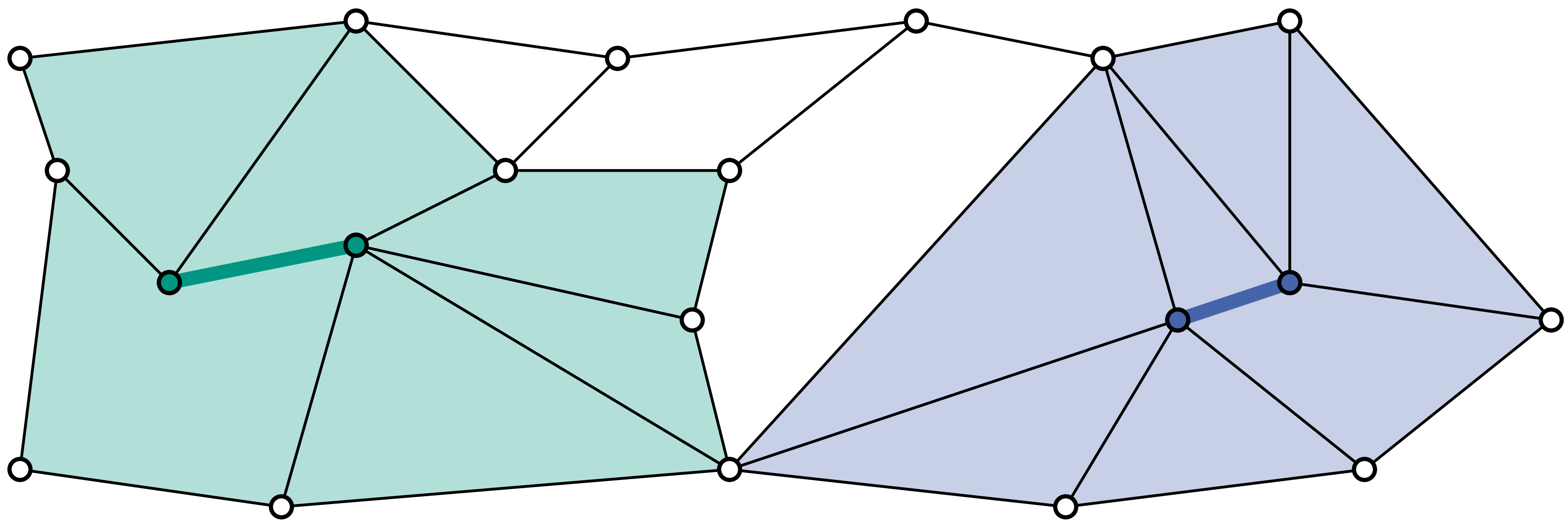

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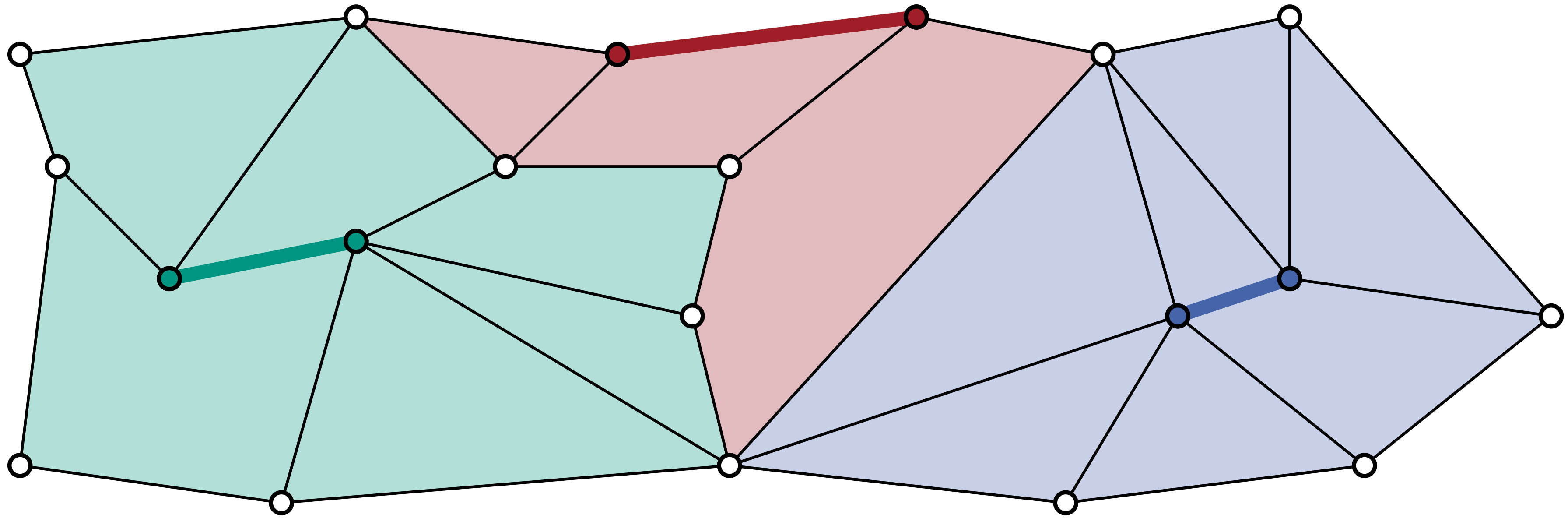

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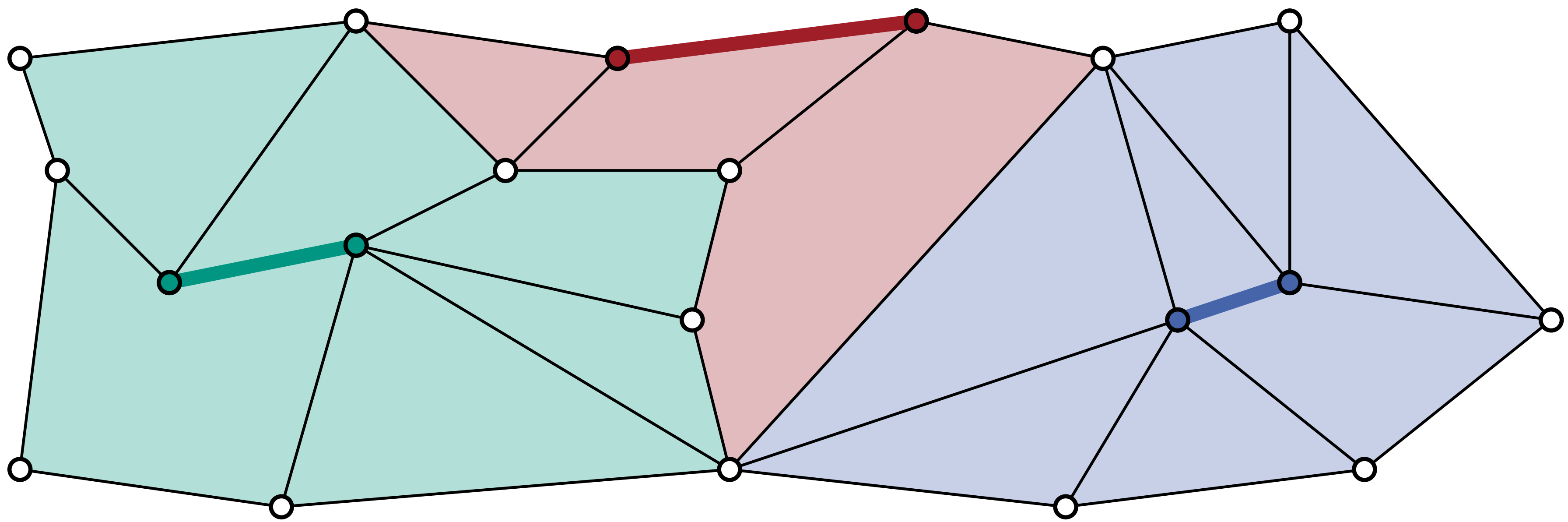

Question

For all n-vertex graphs of a planar graph class $\mathcal{C}$:
How many guards are sometimes necessary and always sufficient?

Previous Results

	Lower	Upper
Planar	$\left\lfloor \frac{n}{3} \right\rfloor$ [1]	$\min \left\{ \left\lfloor \frac{3n}{8} \right\rfloor, \left\lfloor \frac{n}{3} + \frac{\alpha}{9} \right\rfloor \right\}$ [2]
Triangulation	$\left\lfloor \frac{4n-8}{13} \right\rfloor$ [1]	$\left\lfloor \frac{n}{3} \right\rfloor$ [3]
Outerplanar	$\left\lfloor \frac{n}{3} \right\rfloor$ [1]	$\left\lfloor \frac{n}{3} \right\rfloor$ [4]
Max. Outerplanar	$\left\lfloor \frac{n}{4} \right\rfloor$ [5]	$\left\lfloor \frac{n}{4} \right\rfloor$ [5]

α: number of quadrilateral faces

[1] Bose, Shermer, Toussaint, Zhu 1997
[2] Biniaz, Bose, Ooms, Verdonschot 2019
[3] Everett, Rivera-Campo 1997
[4] Chvátal 1975
[5] O'Rourke 1983

Our Results

	Lower	Upper
Stacked Triangulations	$\left\lfloor \frac{2n-4}{7} \right\rfloor$	$\left\lfloor \frac{2n}{7} \right\rfloor$
Quadrangulations	$\left\lfloor \frac{n-2}{4} \right\rfloor$	$\left\lfloor \frac{n}{3} \right\rfloor$
2-Degenerate Quadrangulations	$\left\lfloor \frac{n-2}{4} \right\rfloor$	$\left\lfloor \frac{n}{4} \right\rfloor$

Our Results

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	Lower	Upper	
Stacked Triangulations	$\left\lfloor \frac{2n-4}{7} \right\rfloor$	$\left\lfloor \frac{2n}{7} \right\rfloor$	**Today!**
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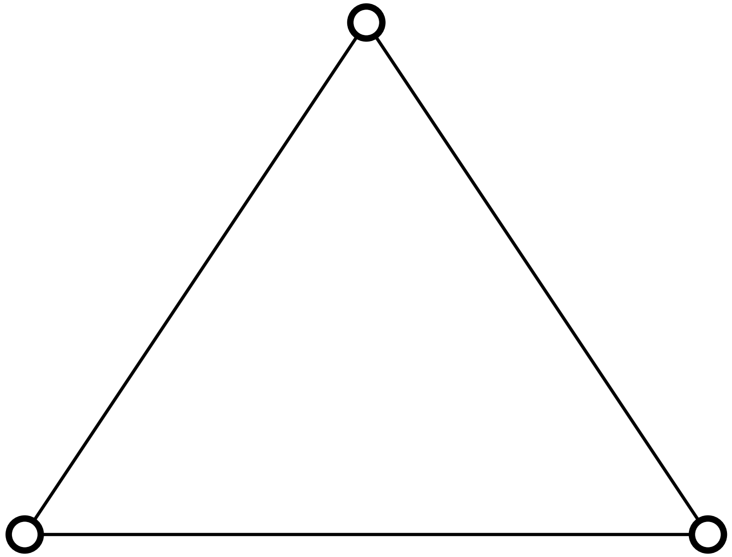
Definition: Stacked Triangulations



- A triangle is a stacked triangulation.
- Let f be an inner face of a stacked triangulation:
  Adding a new vertex into f and subdividing it into three new faces gives a stacked triangulation.

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Definition: Stacked Triangulations

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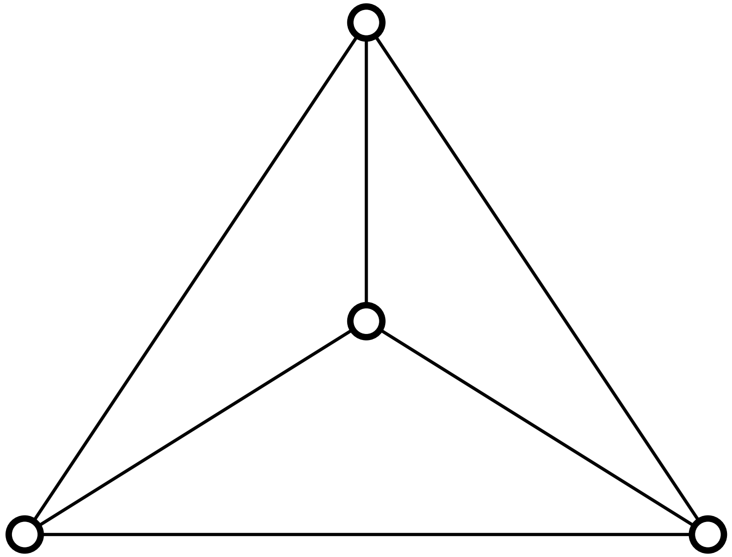

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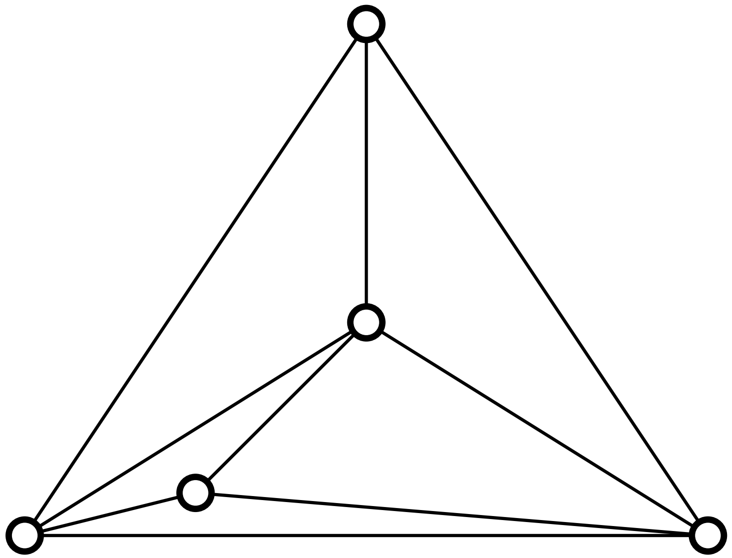

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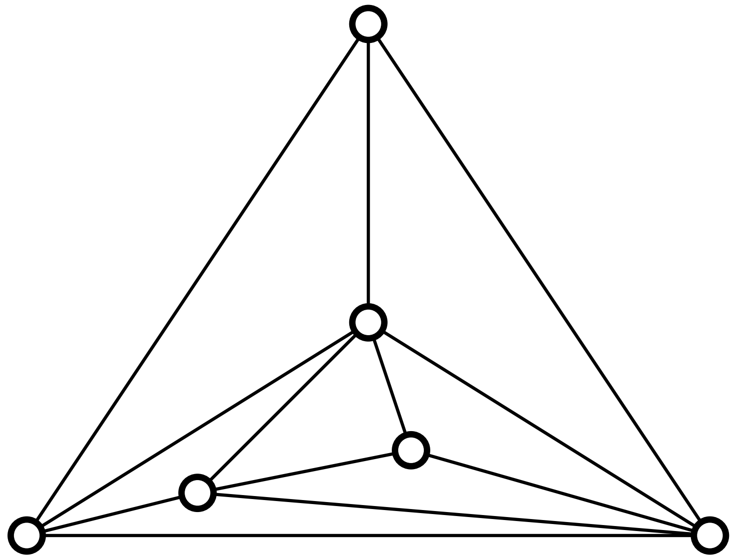

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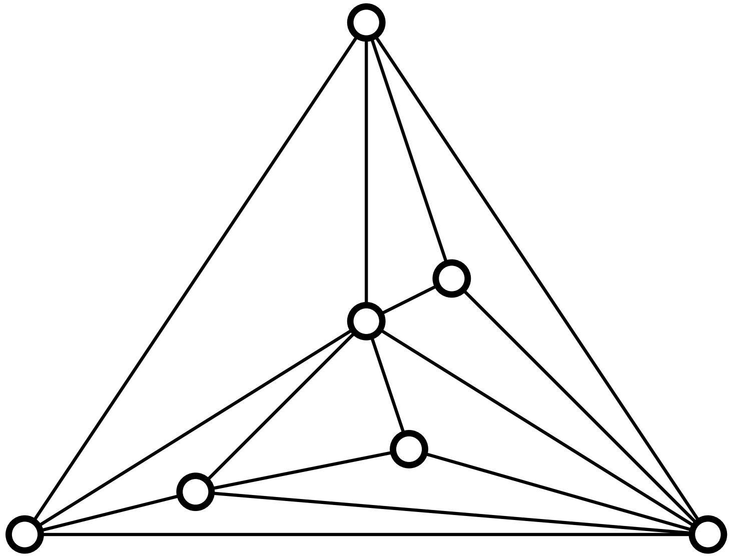

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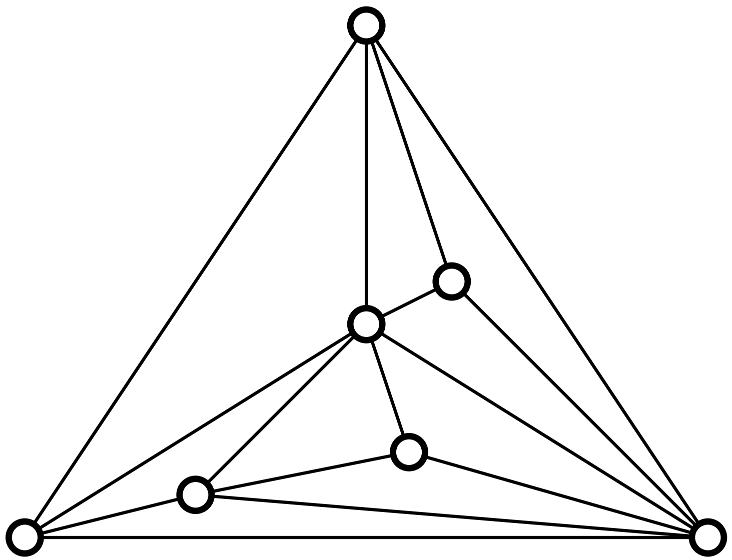

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Theorem [J. 2019]

For n-vertex stacked triangulations $\left\lfloor \frac{2n}{7} \right\rfloor$ edge guards are always sufficient.

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Induction via Vertex Deletion

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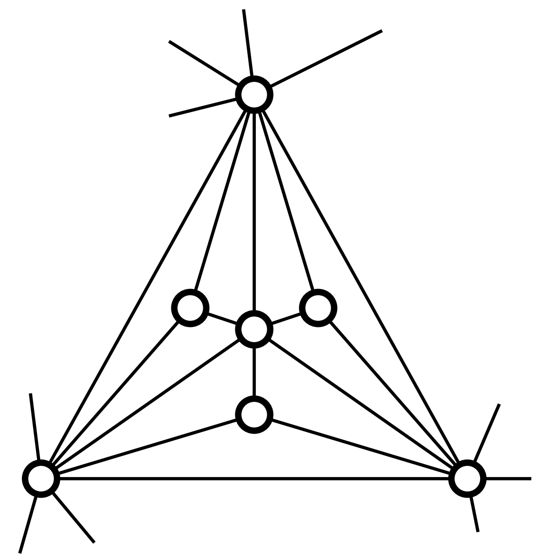
- Use induction on the number n of vertices:
  1. Create smaller graph G' of size $|G'| = |G| - k$.
  2. Apply induction hypothesis on G' to get edge guard set Γ'.
  3. Reinsert old vertices.
  4. Use ℓ additional edges to augment Γ' into Γ for G.



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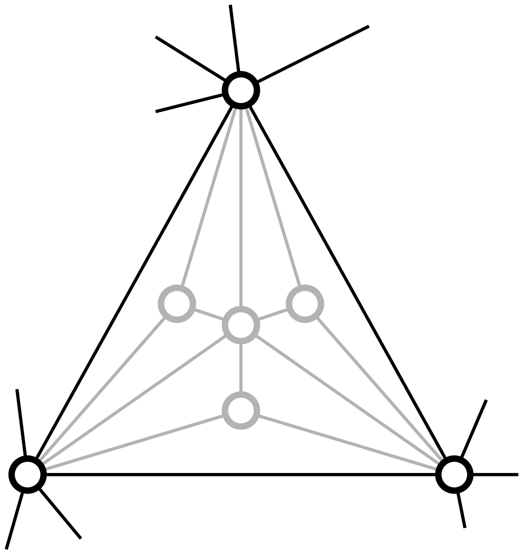

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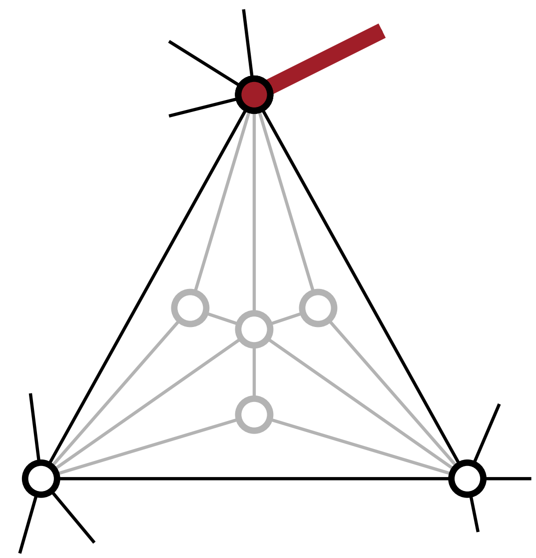

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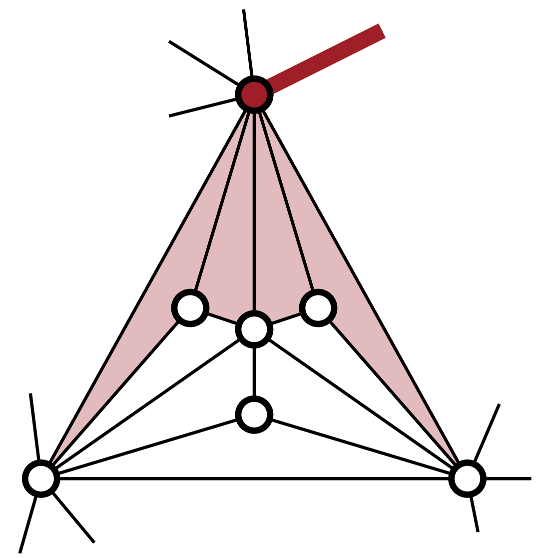

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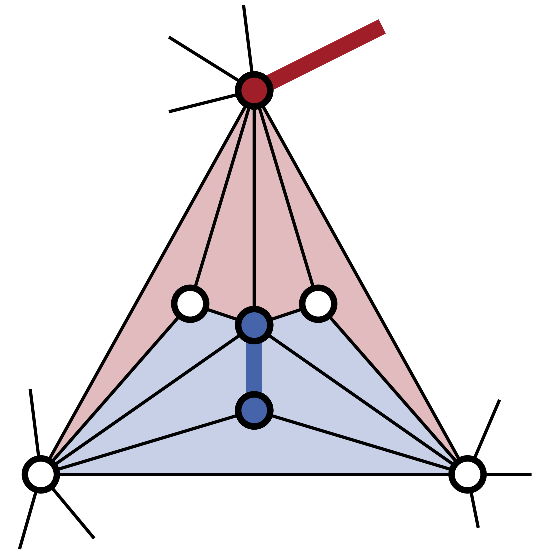

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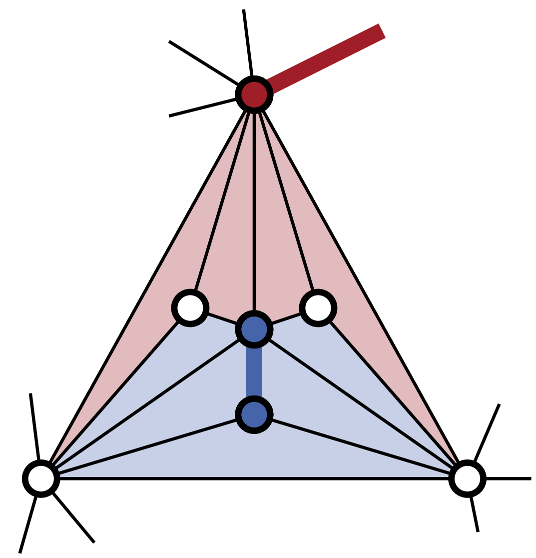

- $\frac{\ell}{k} \leq \frac{2}{7}$ in all cases $\Rightarrow$ edge guard set of size $\left\lfloor \frac{2n}{7} \right\rfloor$

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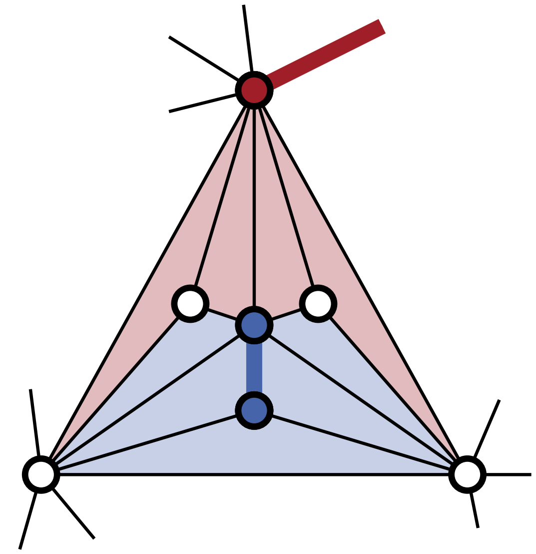
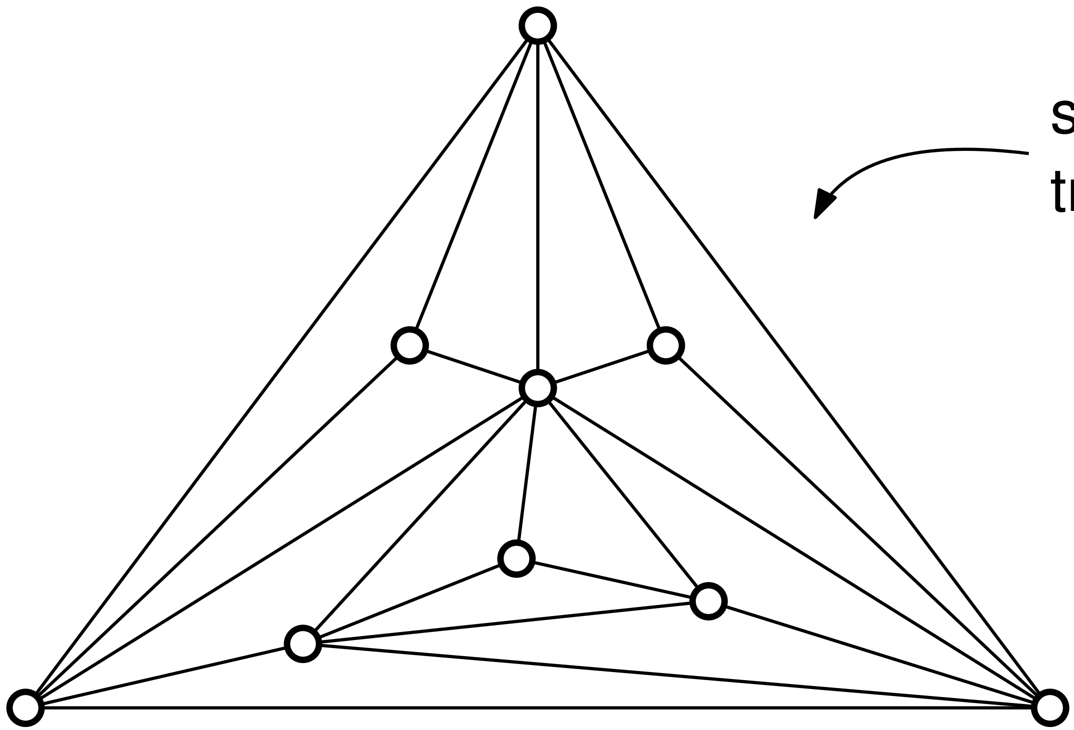


- $\frac{\ell}{k} \leq \frac{2}{7}$ in all cases $\Rightarrow$ edge guard set of size $\left\lfloor \frac{2n}{7} \right\rfloor$
- Also applied successfully for 2-Degenerate Quadrangulations $\left(\frac{\ell}{k} \leq \frac{1}{4}\right)$.

Induction: Examples



stacked triangulation

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Induction: Examples

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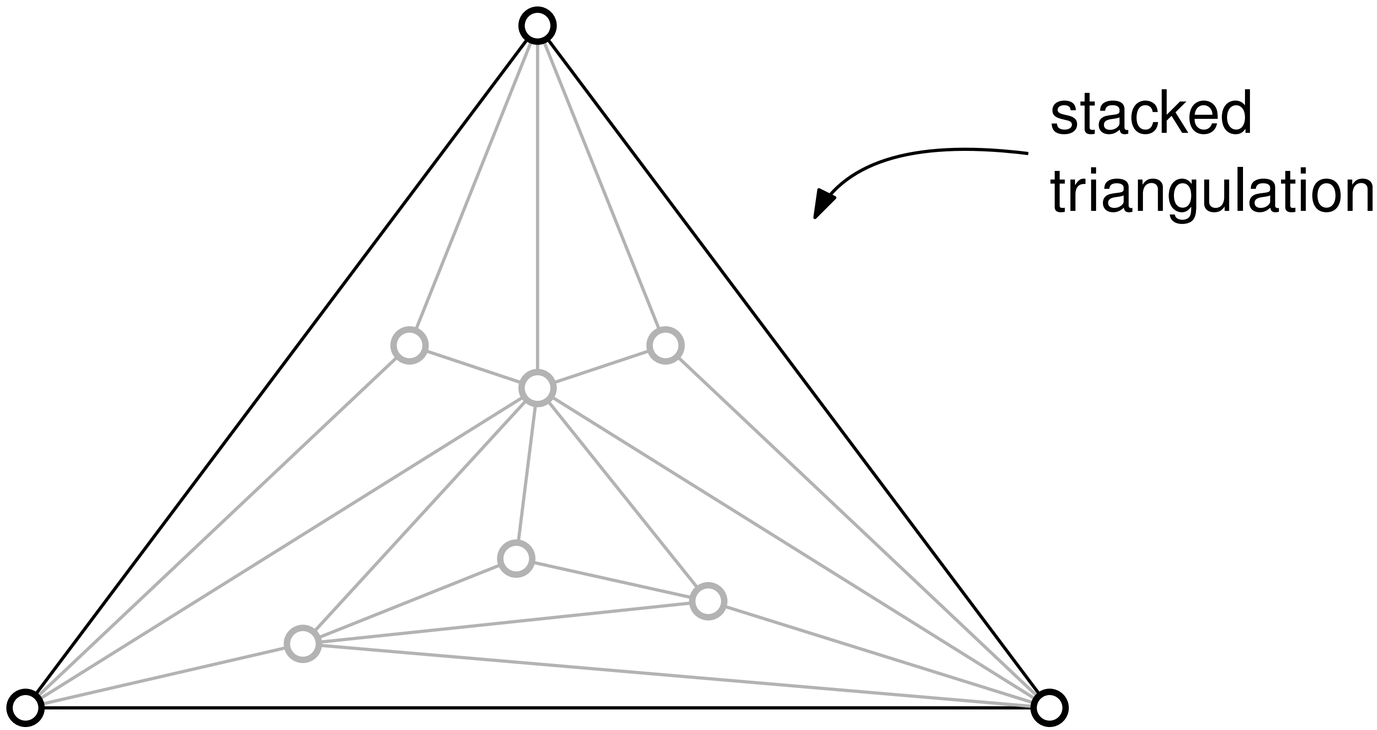

- Remove inner vertices (k = 6).

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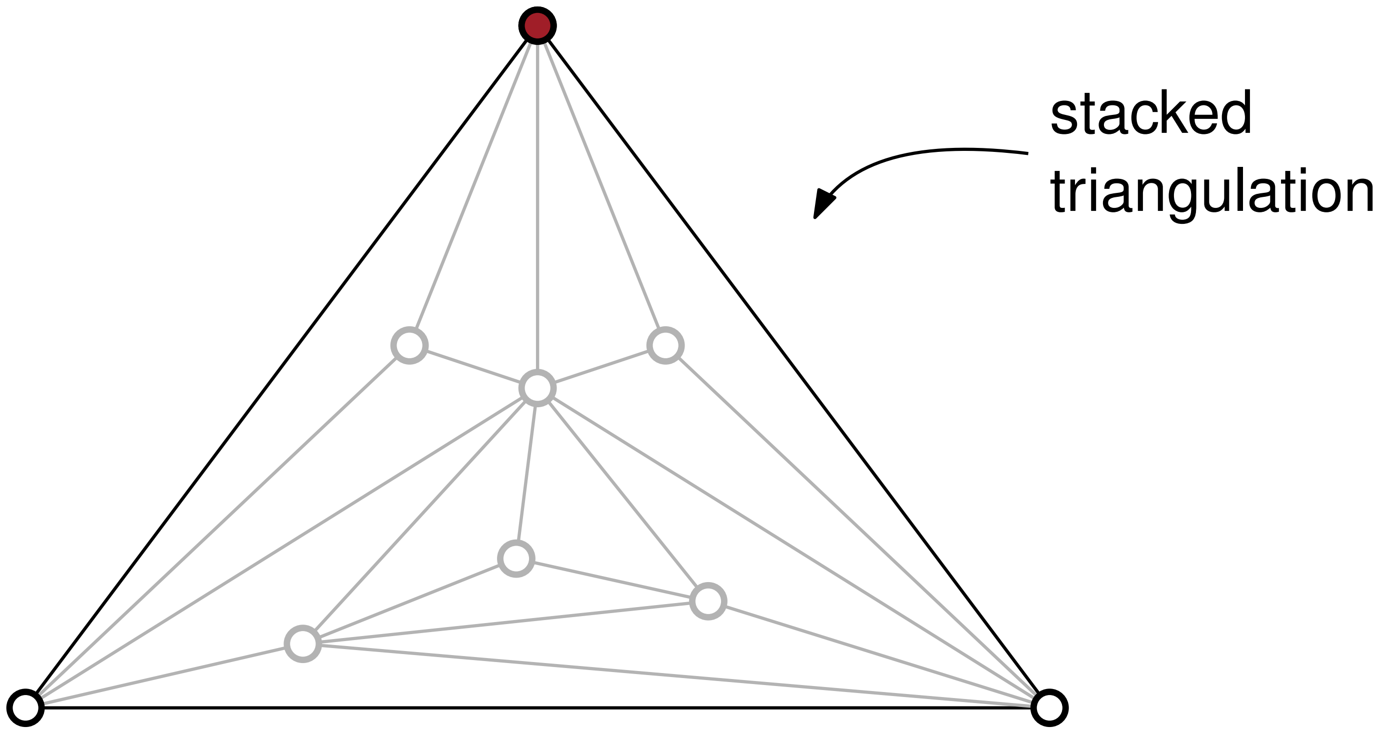
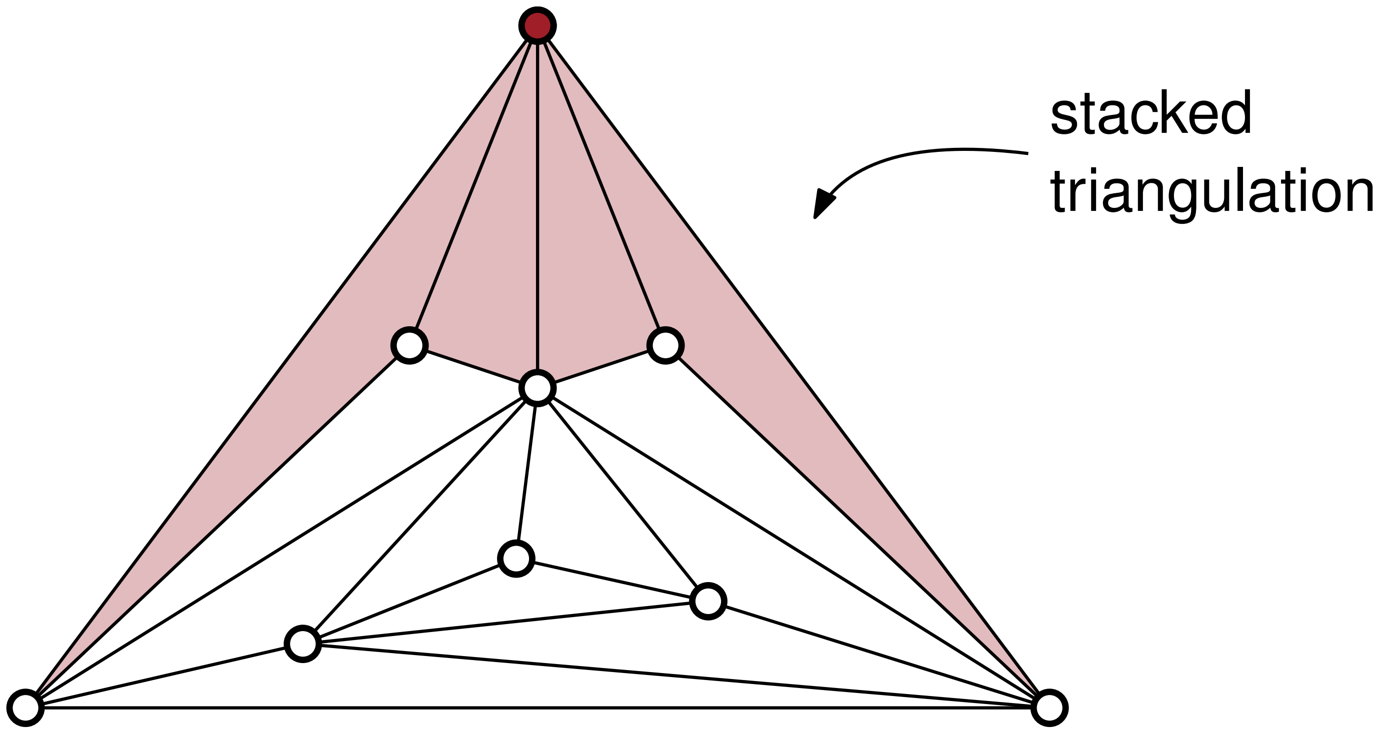


- Remove inner vertices ($k = 6$).
- Apply induction.

Induction: Examples



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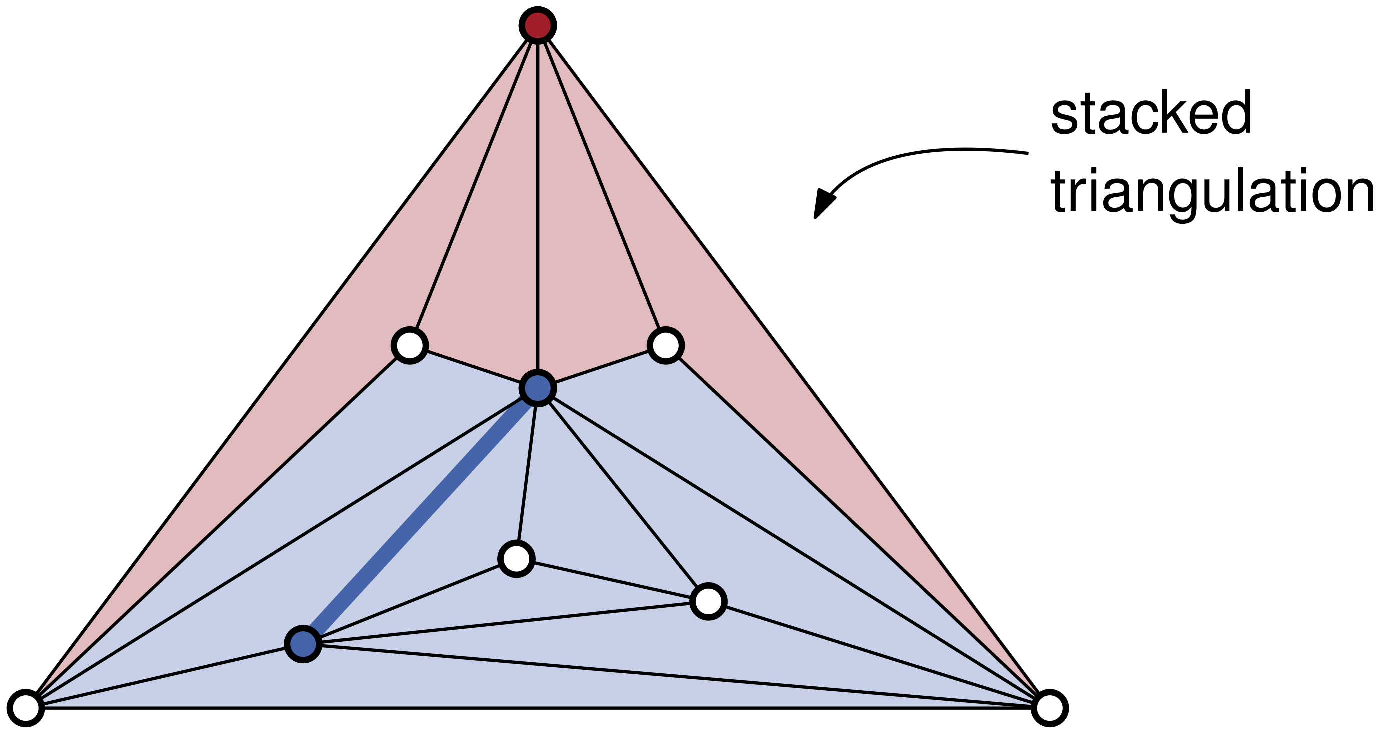

- Remove inner vertices ($k = 6$).
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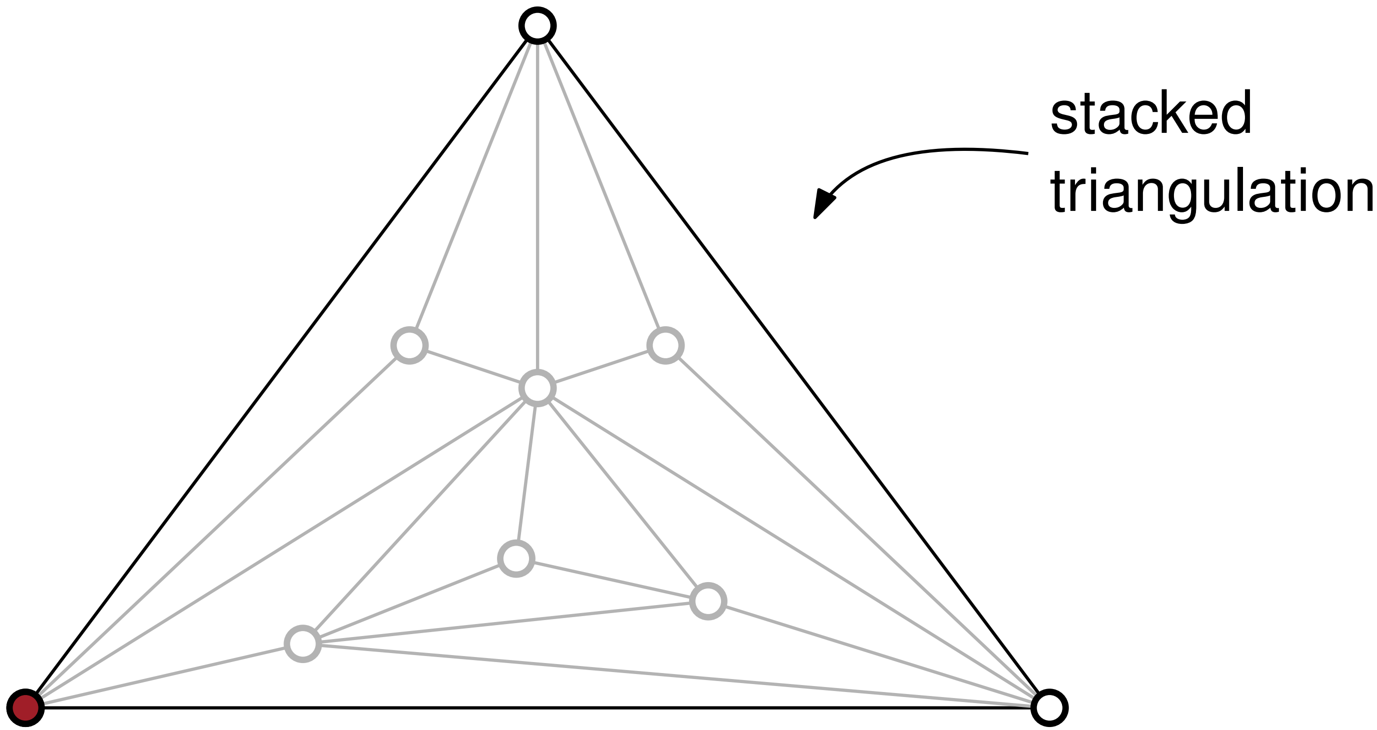
- Add addtional edge ($\ell = 1$), so $\frac{\ell}{k} = \frac{1}{6} \leq \frac{2}{7}$.

Induction: Examples



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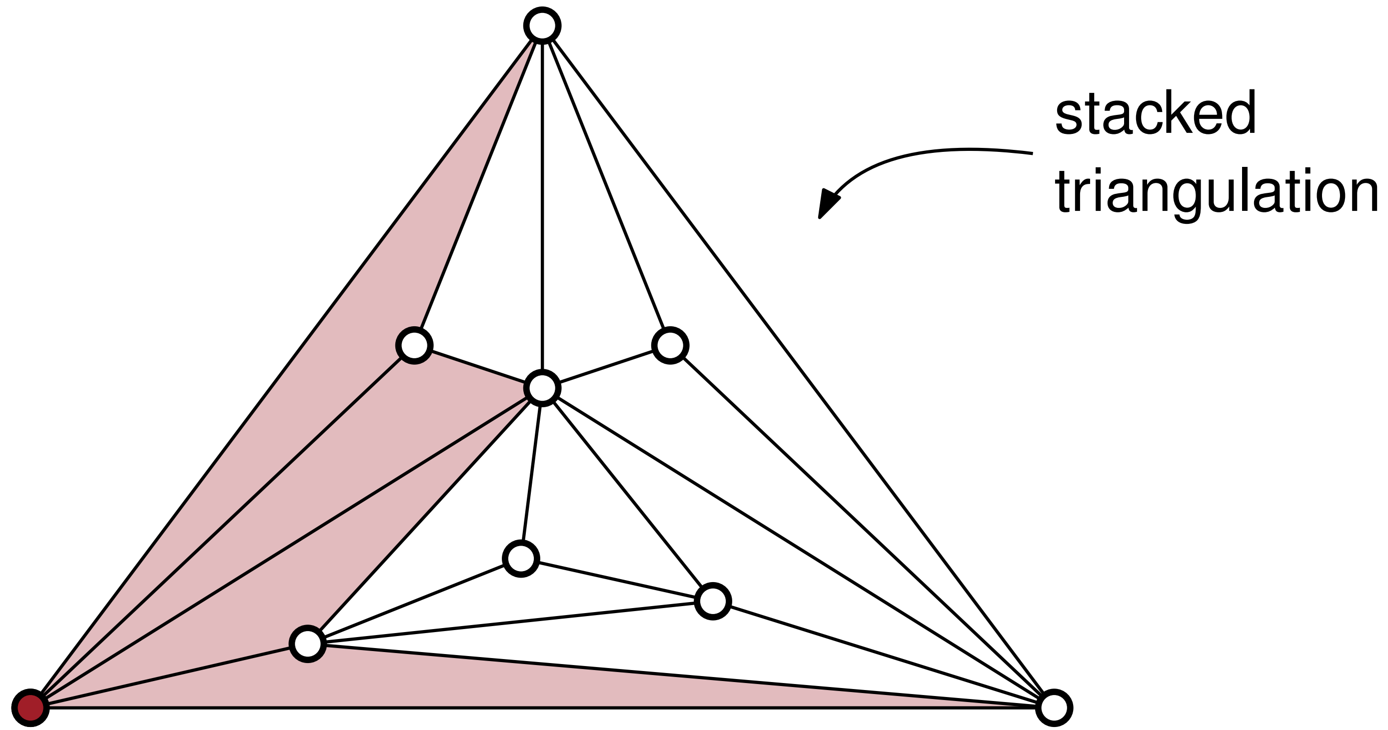

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Induction: Examples

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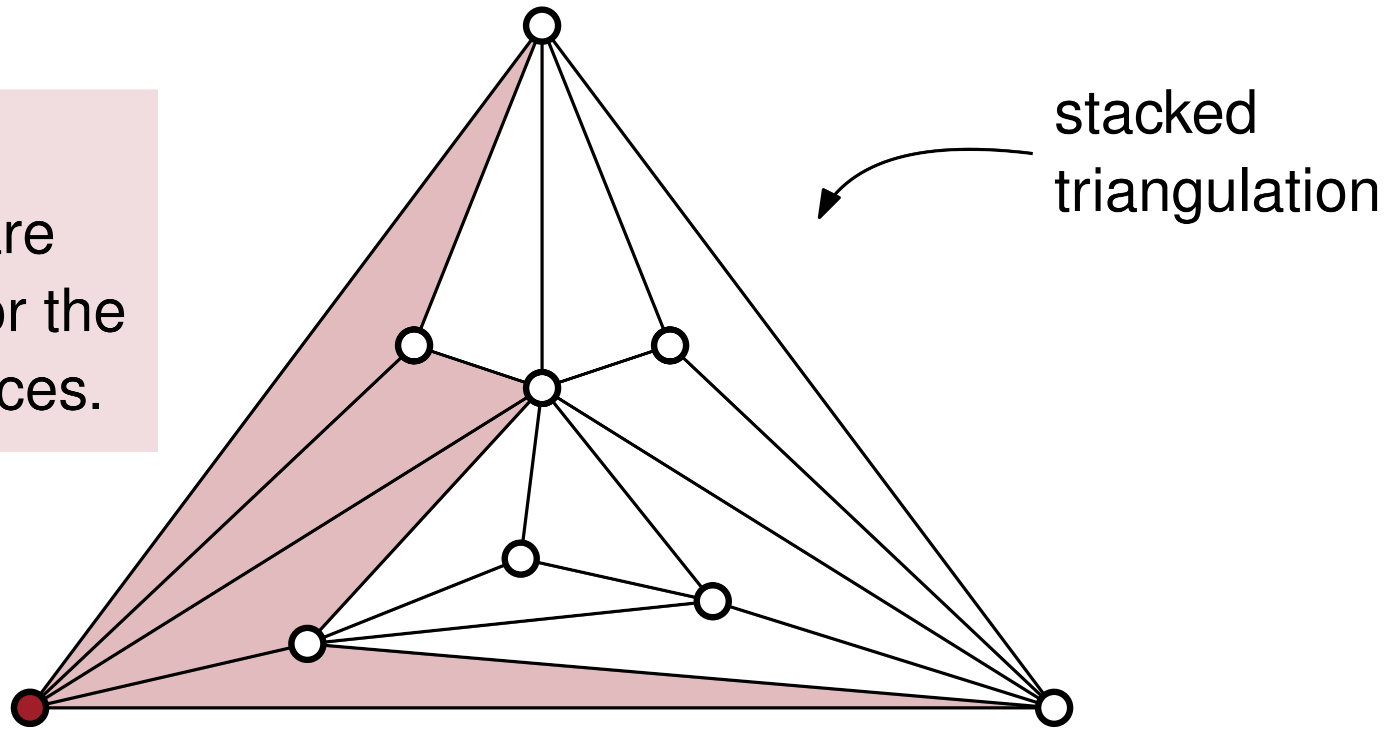
Two edges are necessary for the remaining faces.



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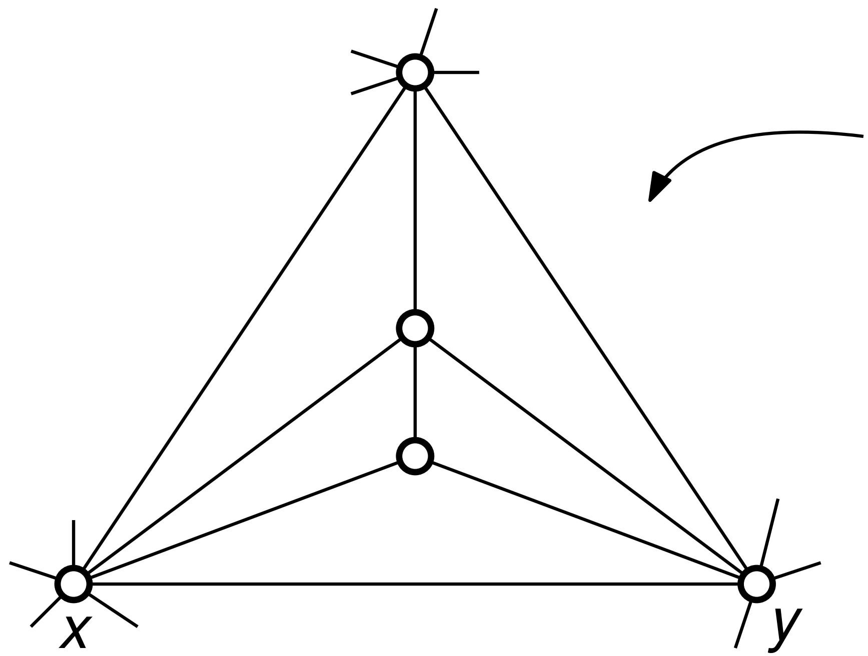
Induction: Trick



somewhere in a stacked triangulation

Lemma

There is a minimum size edge guard set Γ with $x, y \in V(\Gamma)$.

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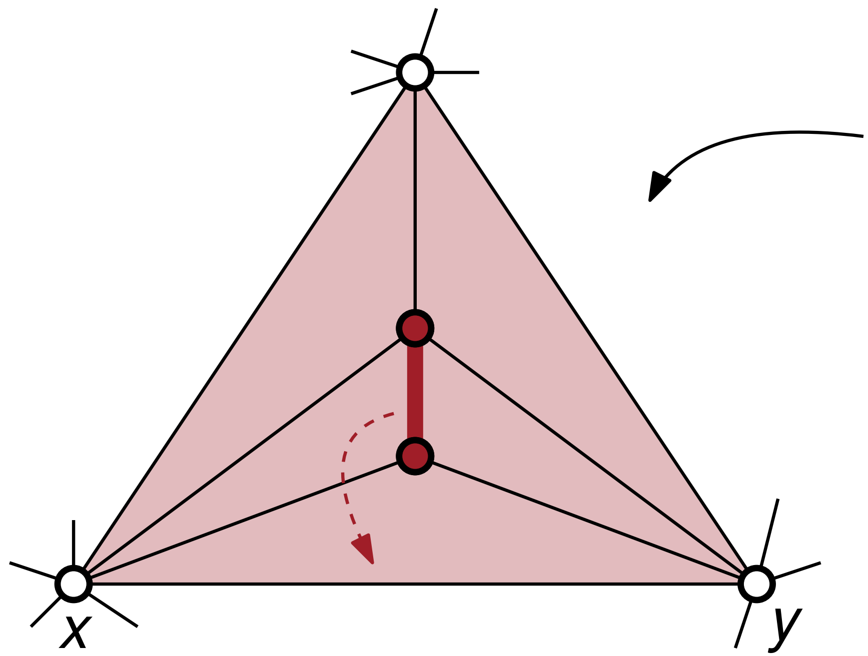

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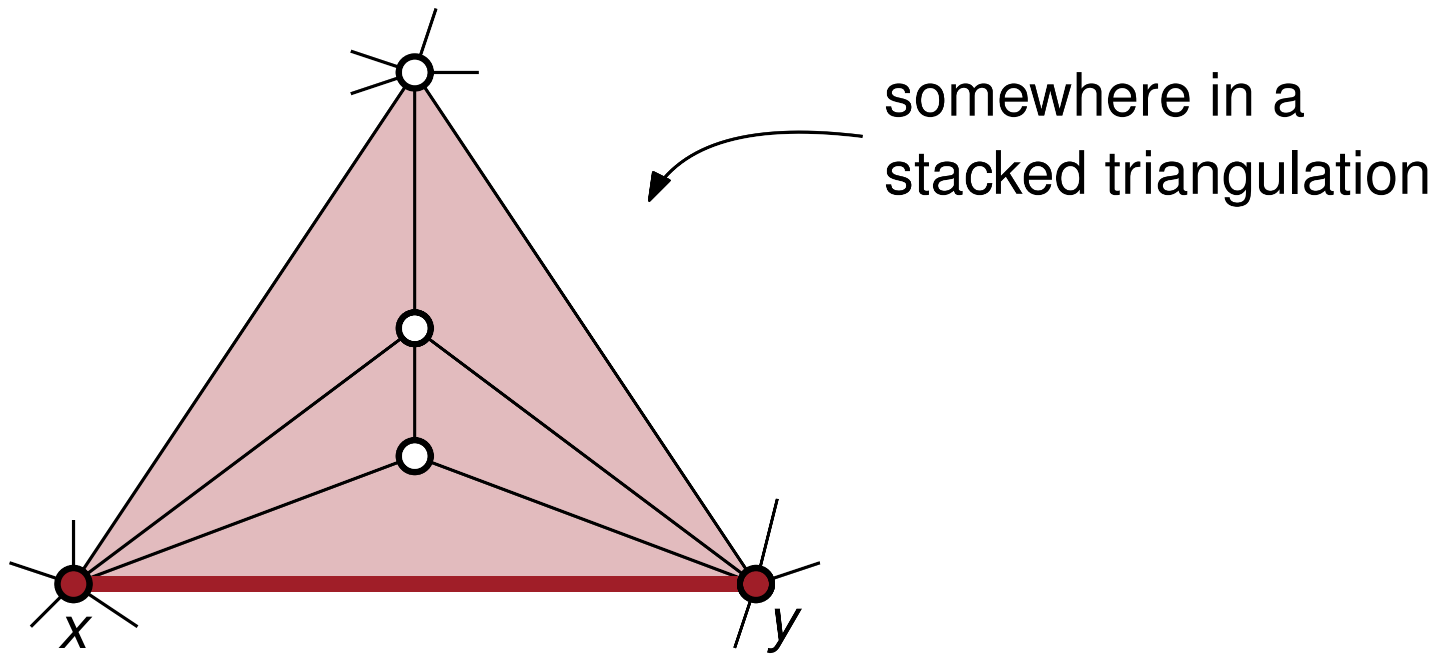

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Induction Example: Revisited

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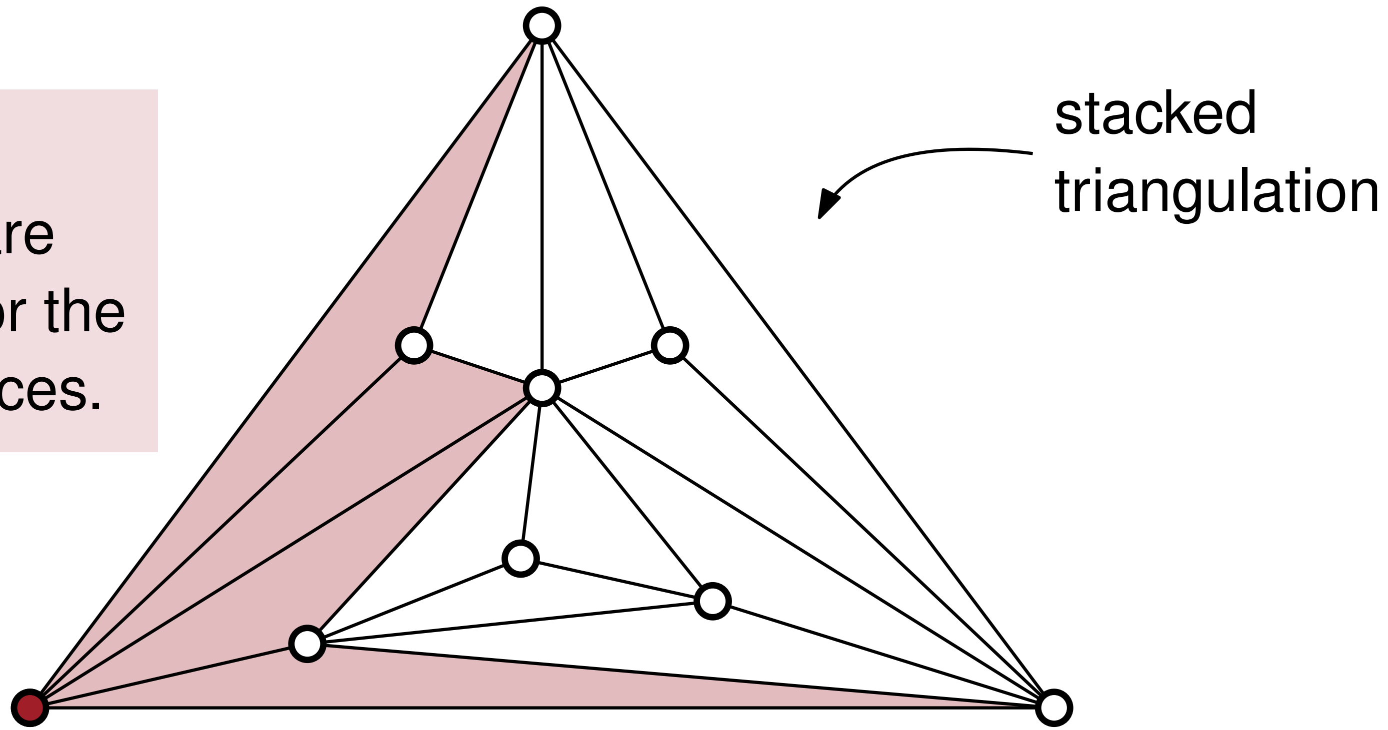

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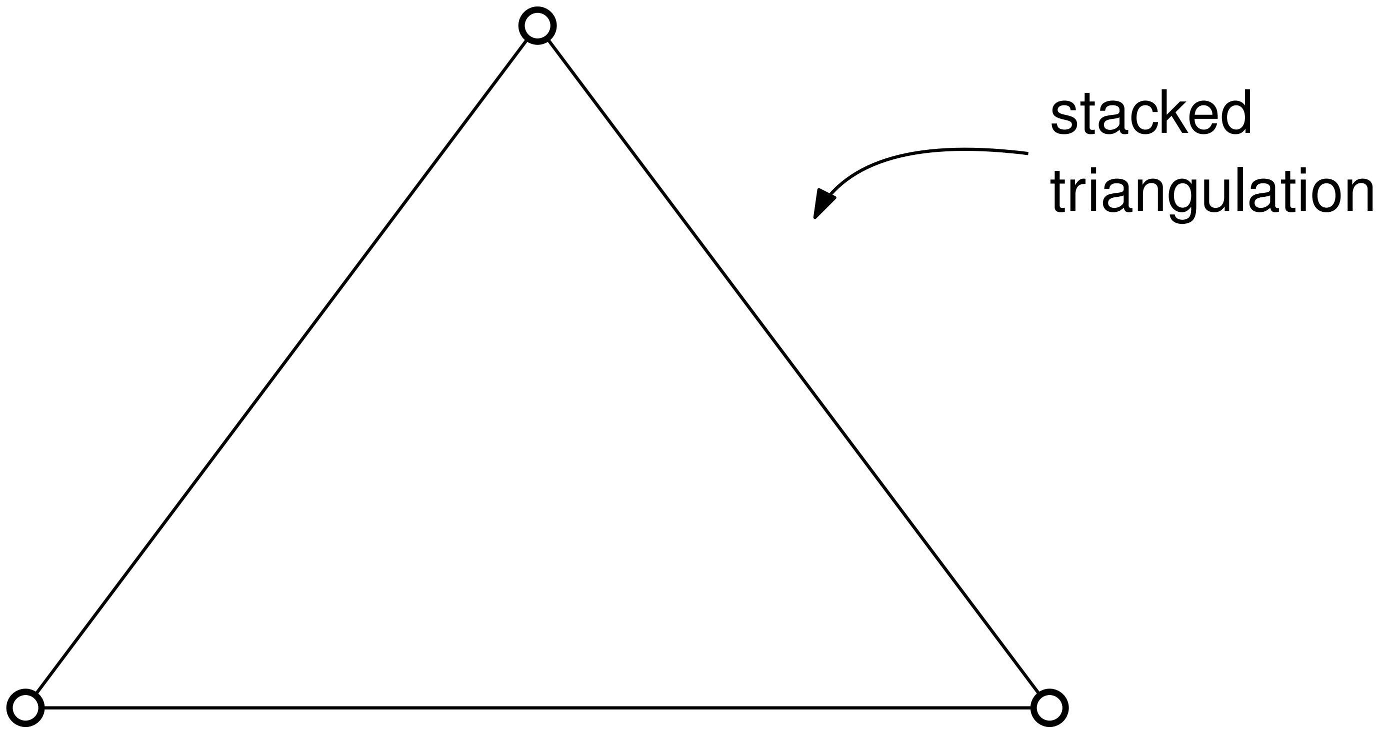

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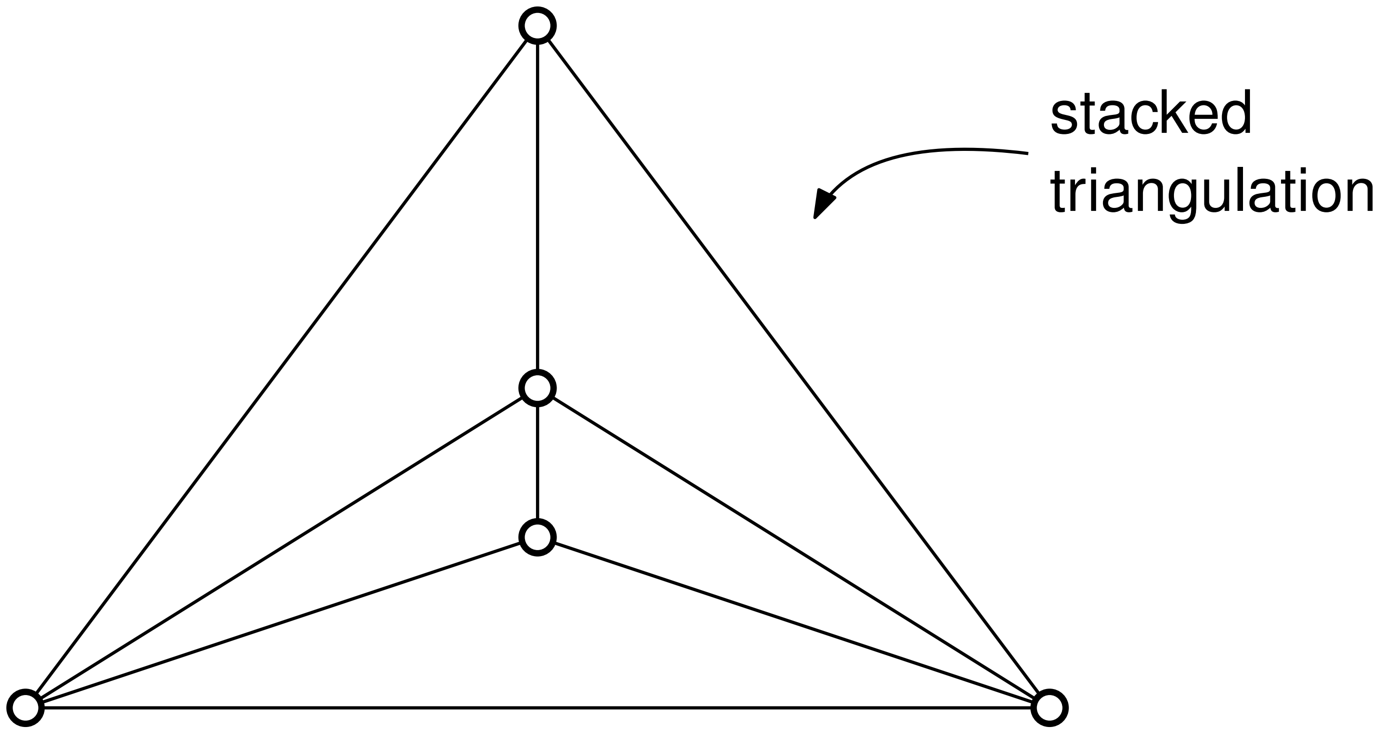

- Remove inner vertices ($k = 6$).
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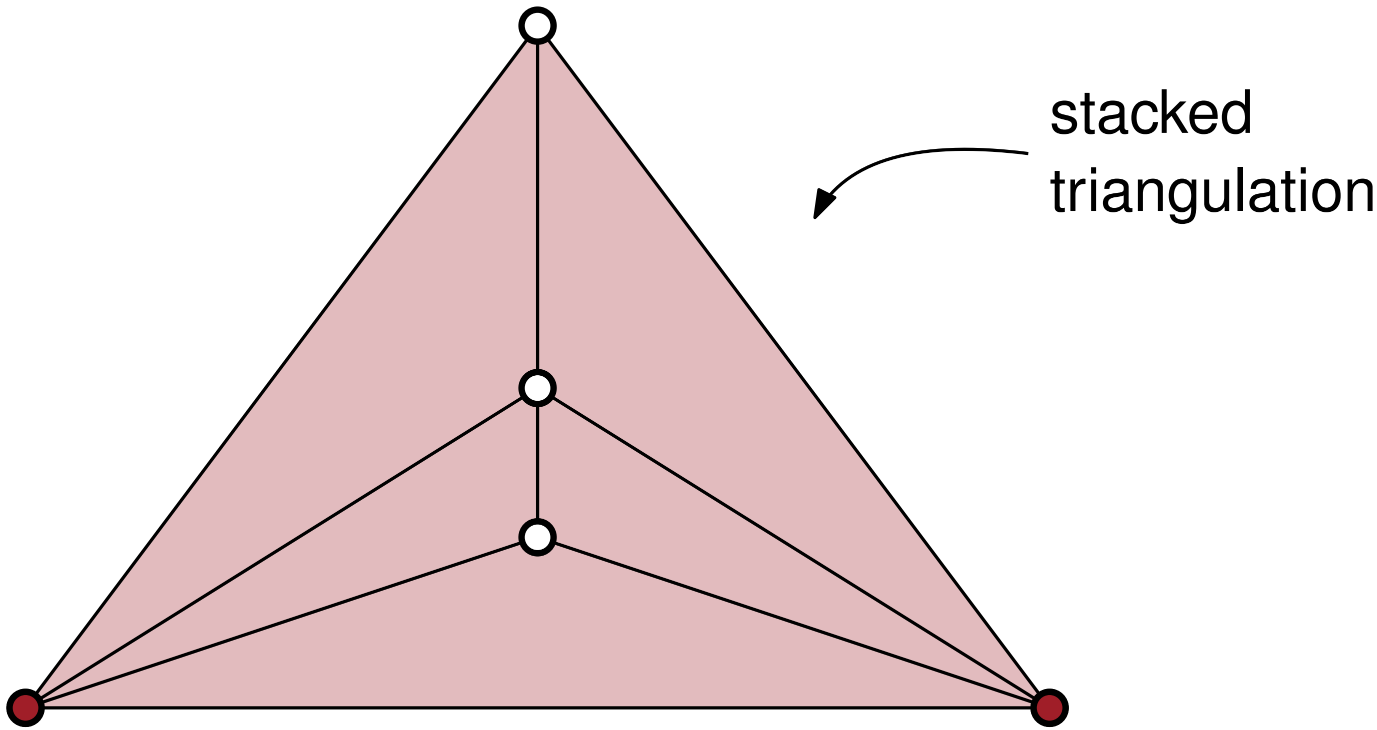

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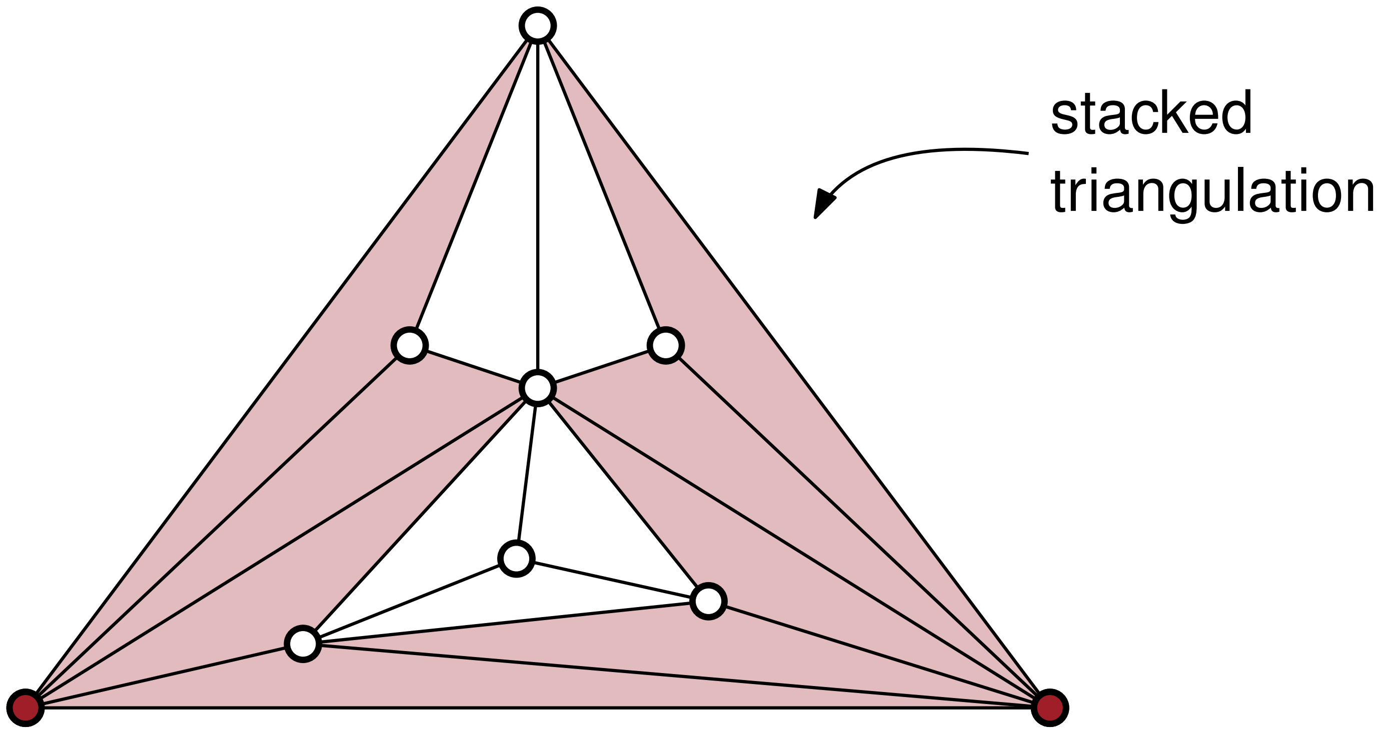

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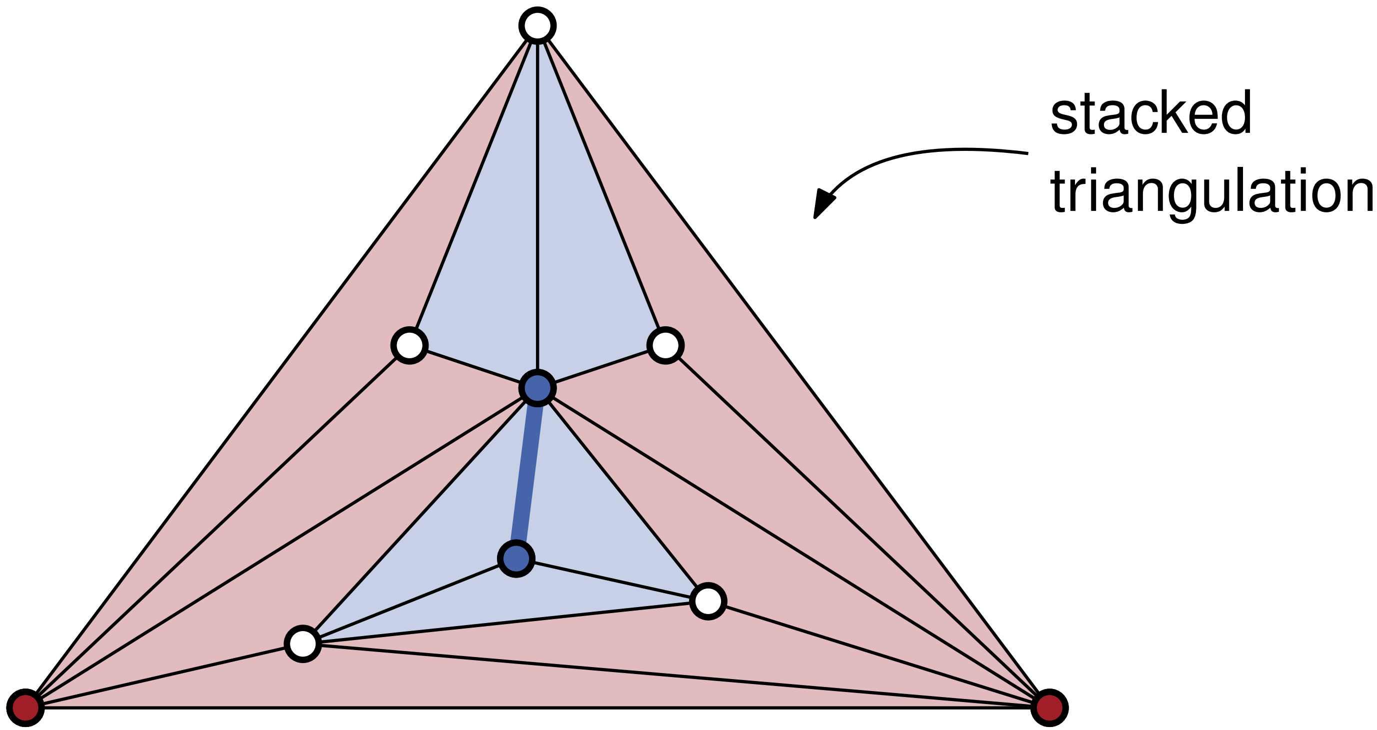

- Remove inner vertices ($k = 6$).
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- Apply lemma from last slide.
- Reinsert old vertices. One more edge suffices ($\ell = 1$), so $\frac{\ell}{k} = \frac{1}{4} \leq \frac{2}{7}$.

Open Problems

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How many edge guards are always sufficient for

- general plane graphs?
- (4-connected) triangulations?
- quadrangulations?

Thank your for your attention.

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