Weighted ε-Nets

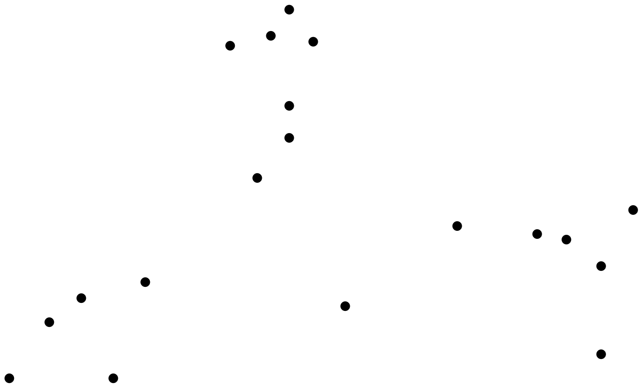
EuroCG 2020

Daniel Bertschinger and Patrick Schnider

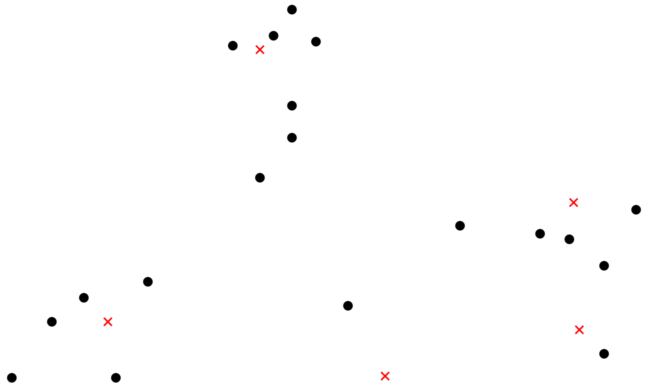
Department of Computer Science, ETH Zürich

March 17, 2020

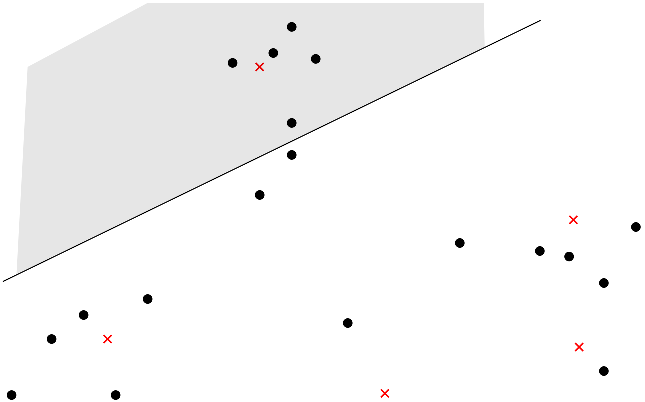
Motivation



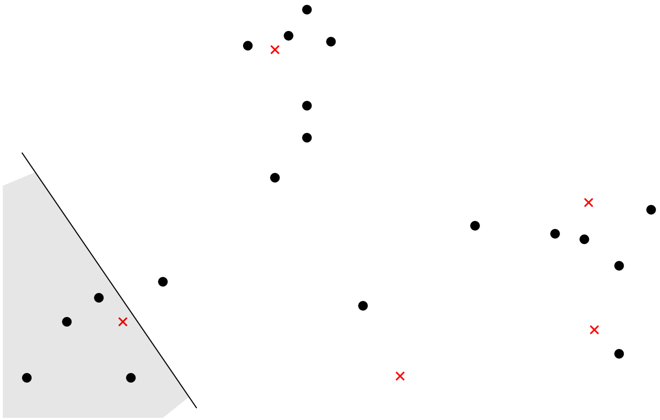
Motivation



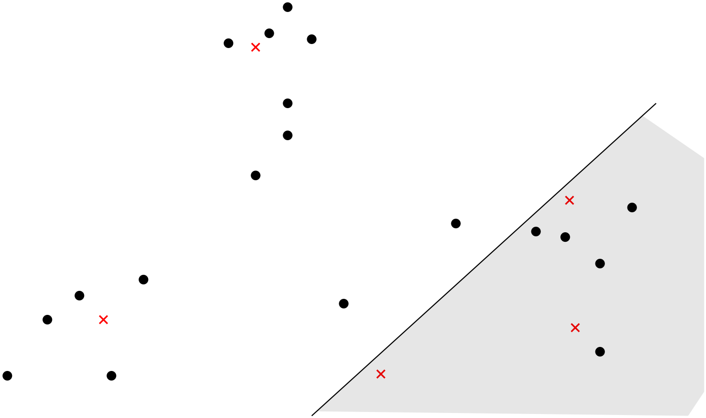
Motivation



Motivation



Motivation



Motivation

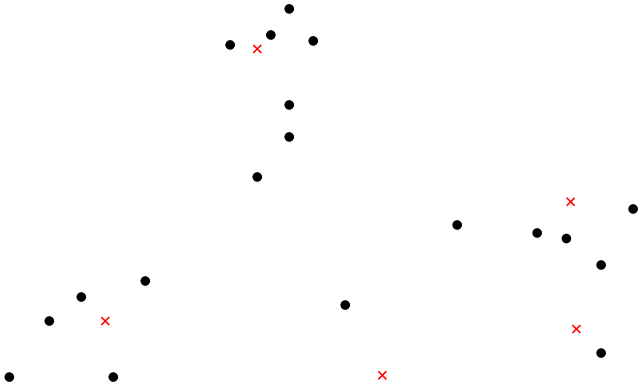
Definition

Definition

An ε-*net on a point set* $P \subseteq \mathbb{R}^d$ is a set $N \subseteq \mathbb{R}^d$ such that every halfspace H with $|H \cap P| \geq \varepsilon|P|$ has nonempty intersection with N.

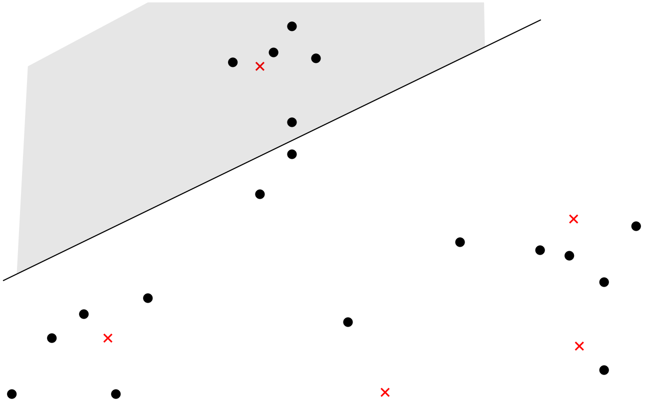
Motivation

Examples



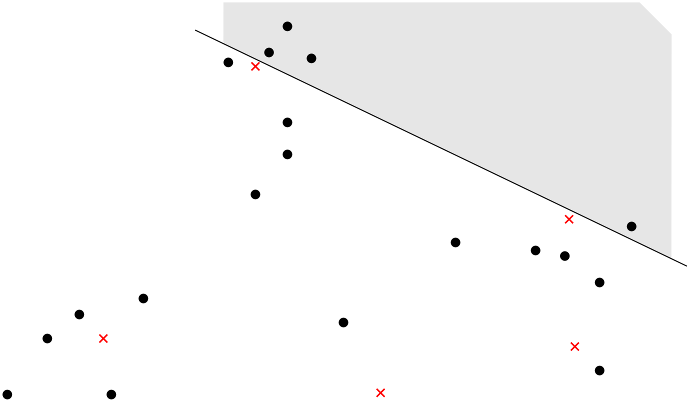
Motivation

Examples



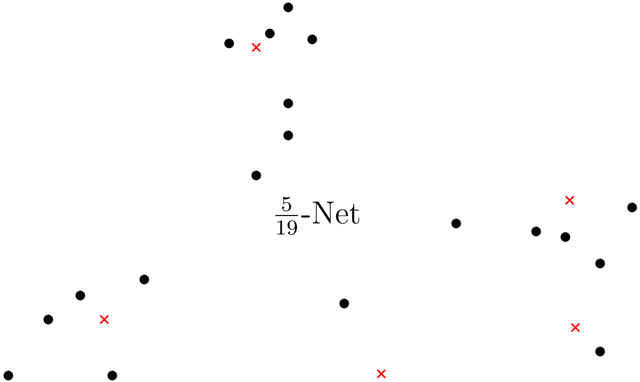
Motivation

Examples



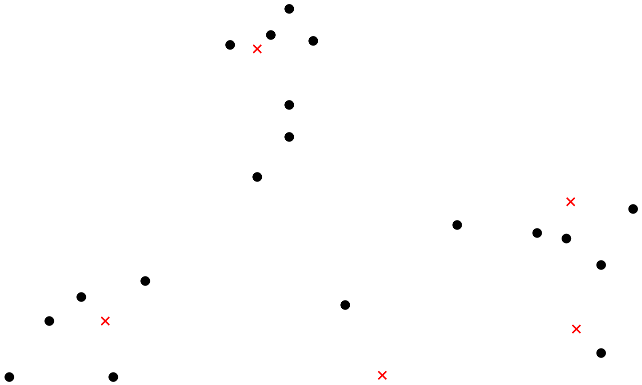
Motivation

Examples



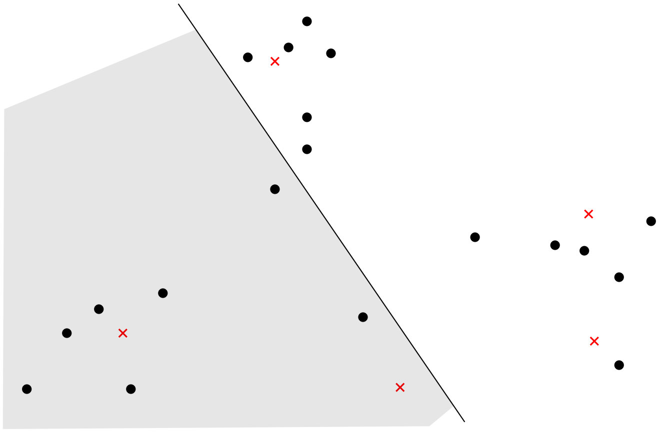
Motivation

Examples



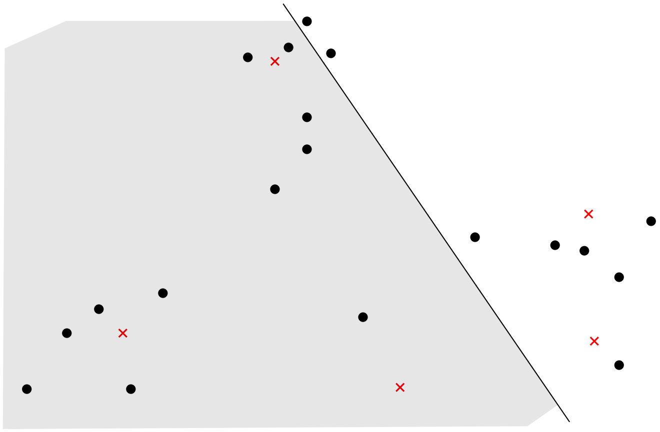
Motivation

Examples



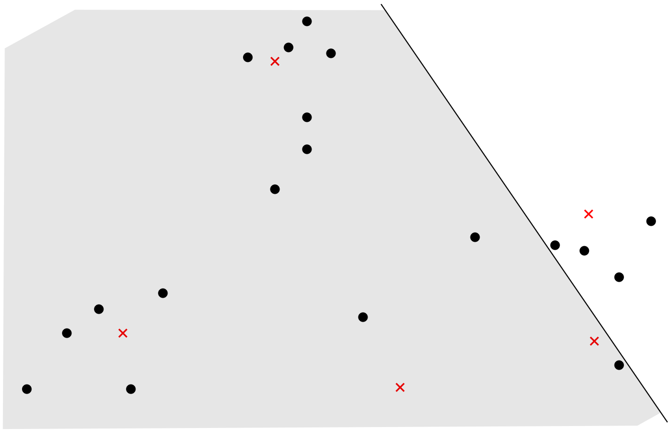
Motivation

Examples



Motivation

Examples



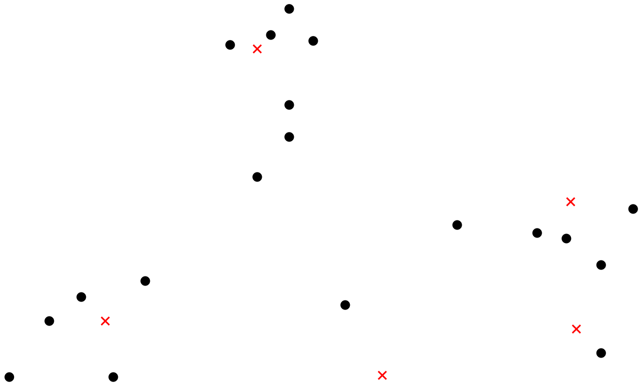
Definition

Let $P \subseteq \mathbb{R}^d$ be a finite point set and $0 \leq \varepsilon \leq 1$. A set $A \subseteq X$ is called an *ε-approximation* of P if, for each halfspace H, we have

$$\left| \frac{|H \cap P|}{|P|} - \frac{|H \cap A|}{|A|} \right| \leq \varepsilon.$$

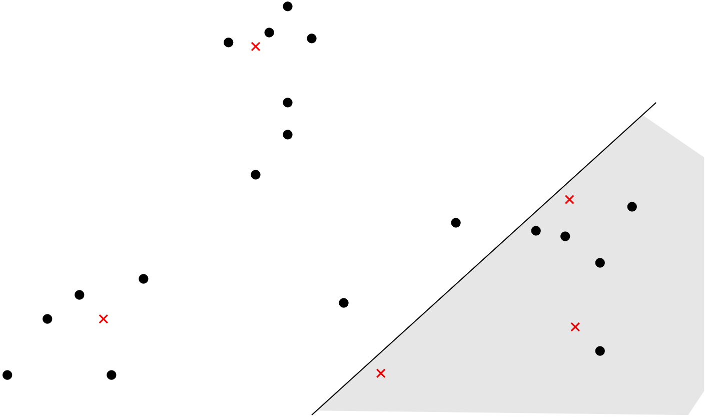
Motivation

Examples



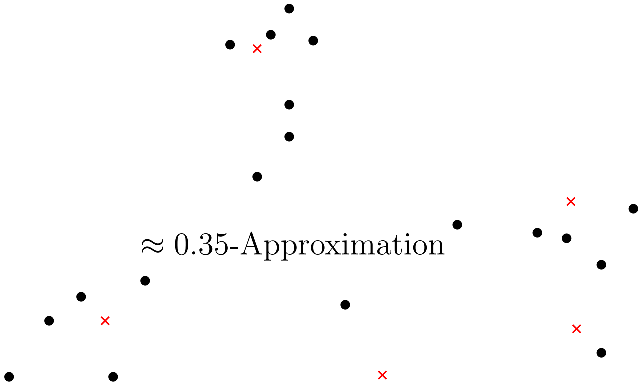
Motivation

Examples



Motivation

Examples



Remark

Instead of halfspaces, we can work with different *ranges*, e.g. convex sets, boxes, polyhedrons, simplices.

Remark

Instead of halfspaces, we can work with different *ranges*, e.g. convex sets, boxes, polyhedrons, simplices.

Theorem [Haussler, Welzl, '87]

Every range space of VC-dimension d has an ε-net of size at most $\mathcal{O}(\frac{d}{\varepsilon}\log\frac{1}{\varepsilon})$.

Remark

Instead of halfspaces, we can work with different *ranges*, e.g. convex sets, boxes, polyhedrons, simplices.

Theorem [Haussler, Welzl, '87]

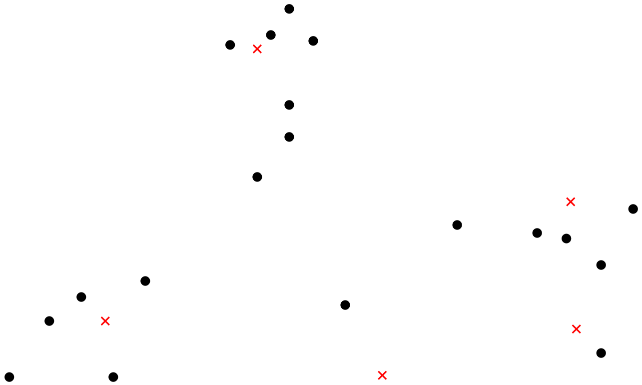
Every range space of VC-dimension d has an ε-net of size at most $\mathcal{O}(\frac{d}{\varepsilon}\log\frac{1}{\varepsilon})$.

Theorem [Matoušek et al. '93]

Every range space of VC-dimension d attains an ε-approximation of size $\mathcal{O}(\frac{d}{\varepsilon^2}\log\frac{1}{\varepsilon})$.

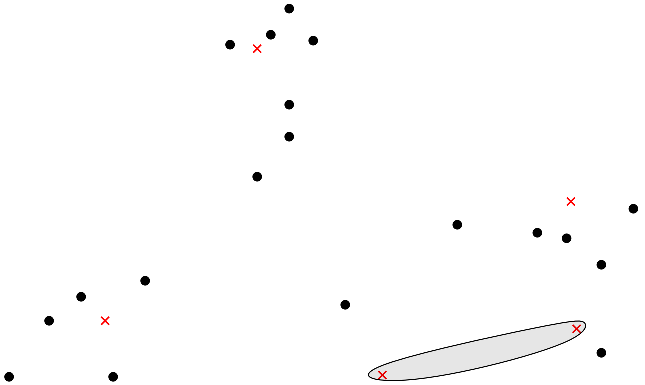
Motivation

Examples



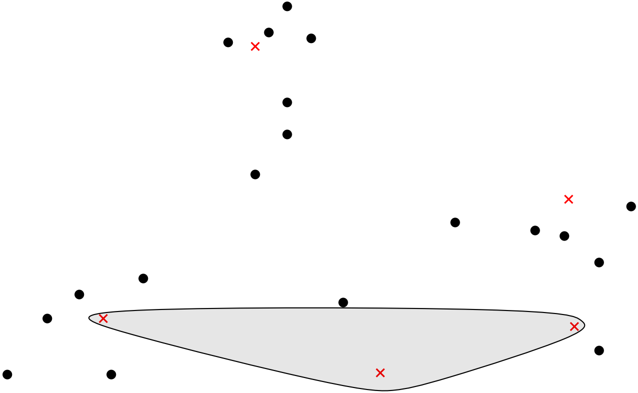
Motivation

Examples



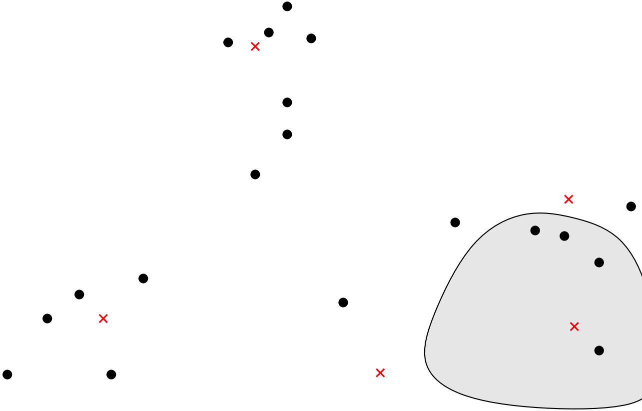
Motivation

Examples



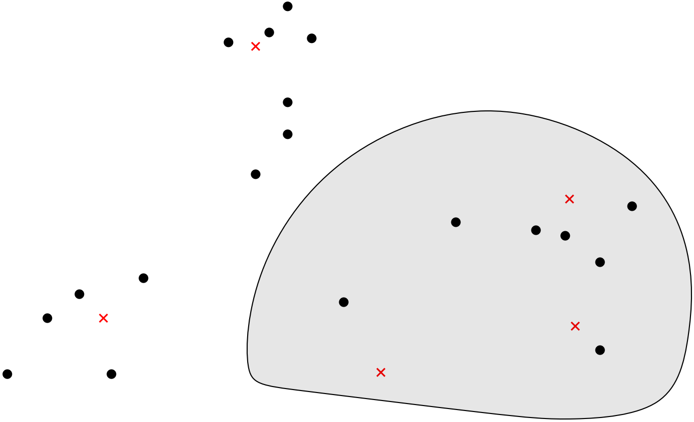
Motivation

Examples



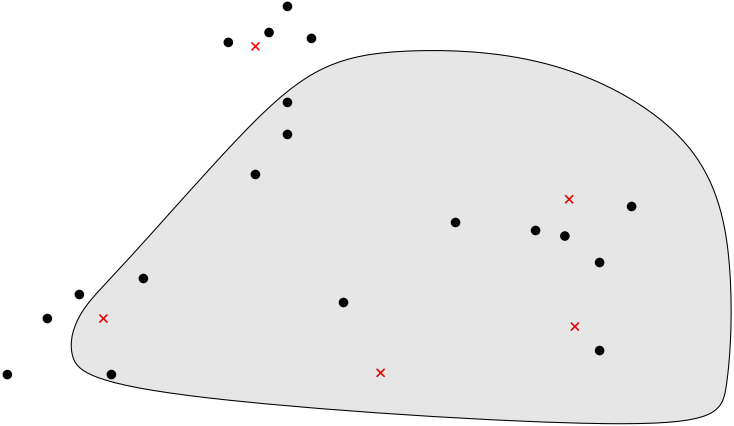
Motivation

Examples



Motivation

Examples



Motivation

Definition [B., Schnider; '20]

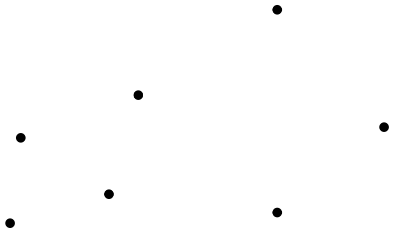
Given any point set $P \subset \mathbb{R}^d$ of size n, a *weighted ε-net* is defined as a set of points $q_1, \ldots, q_k$ and some values $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_k)$ such that every set in the range space containing more than $\varepsilon_i n$ points of P contains at least i of the points $q_1, \ldots, q_k$.

Motivation

Weighted ε-Nets

Definition [B., Schnider; '20]

Given any point set $P \subset \mathbb{R}^d$ of size n, a *weighted ε-net* is defined as a set of points $q_1, \ldots, q_k$ and some values $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_k)$ such that every set in the range space containing more than $\varepsilon_i n$ points of P contains at least i of the points $q_1, \ldots, q_k$.

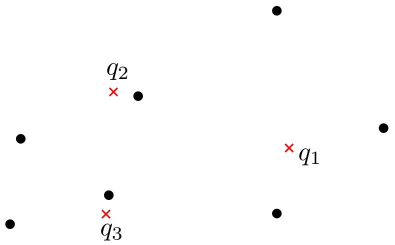


Motivation

Weighted ε-Nets

Definition [B., Schnider; '20]

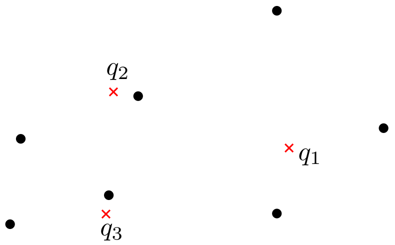
Given any point set $P \subset \mathbb{R}^d$ of size n, a *weighted ε-net* is defined as a set of points $q_1, \ldots, q_k$ and some values $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_k)$ such that every set in the range space containing more than $\varepsilon_i n$ points of P contains at least i of the points $q_1, \ldots, q_k$.



Motivation

Definition [B., Schnider; '20]

Given any point set $P \subset \mathbb{R}^d$ of size n, a *weighted ε-net* is defined as a set of points $q_1, \ldots, q_k$ and some values $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_k)$ such that every set in the range space containing more than $\varepsilon_i n$ points of P contains at least i of the points $q_1, \ldots, q_k$.



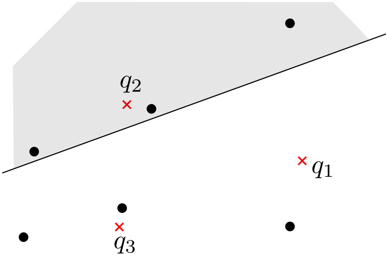
Weighted $(\frac{2}{7}, \frac{4}{7}, \frac{6}{7})$-Net

Motivation

Weighted ε-Nets

Definition [B., Schnider; '20]

Given any point set $P \subset \mathbb{R}^d$ of size n, a *weighted ε-net* is defined as a set of points $q_1, \ldots, q_k$ and some values $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_k)$ such that every set in the range space containing more than $\varepsilon_i n$ points of P contains at least i of the points $q_1, \ldots, q_k$.



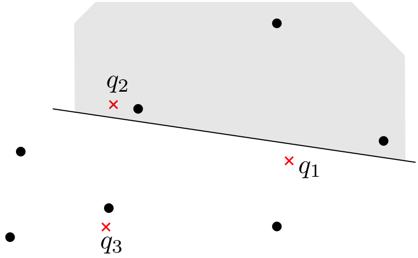
Weighted $(\frac{2}{7}, \frac{4}{7}, \frac{6}{7})$-Net

Motivation

Weighted ε-Nets

Definition [B., Schnider; '20]

Given any point set $P \subset \mathbb{R}^d$ of size n, a *weighted ε-net* is defined as a set of points $q_1, \ldots, q_k$ and some values $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_k)$ such that every set in the range space containing more than $\varepsilon_i n$ points of P contains at least i of the points $q_1, \ldots, q_k$.



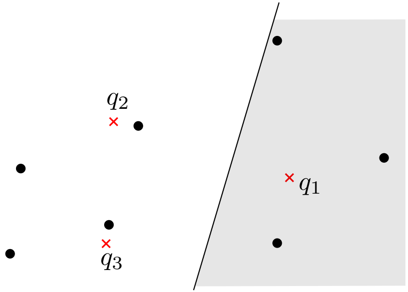
Weighted $(\frac{2}{7}, \frac{4}{7}, \frac{6}{7})$-Net

Motivation

Weighted ε-Nets

Definition [B., Schnider; '20]

Given any point set $P \subset \mathbb{R}^d$ of size n, a *weighted ε-net* is defined as a set of points $q_1, \ldots, q_k$ and some values $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_k)$ such that every set in the range space containing more than $\varepsilon_i n$ points of P contains at least i of the points $q_1, \ldots, q_k$.



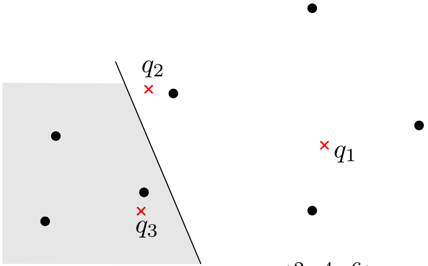
Weighted $(\frac{2}{7}, \frac{4}{7}, \frac{6}{7})$-Net

Motivation

Weighted ε-Nets

Definition [B., Schnider; '20]

Given any point set $P \subset \mathbb{R}^d$ of size n, a *weighted ε-net* is defined as a set of points $q_1, \ldots, q_k$ and some values $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_k)$ such that every set in the range space containing more than $\varepsilon_i n$ points of P contains at least i of the points $q_1, \ldots, q_k$.



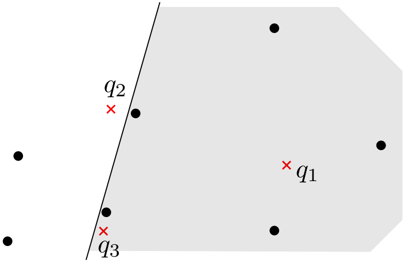
Weighted $(\frac{2}{7}, \frac{4}{7}, \frac{6}{7})$-Net

Motivation

Weighted ε-Nets

Definition [B., Schnider; '20]

Given any point set $P \subset \mathbb{R}^d$ of size n, a *weighted ε-net* is defined as a set of points $q_1, \ldots, q_k$ and some values $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_k)$ such that every set in the range space containing more than $\varepsilon_i n$ points of P contains at least i of the points $q_1, \ldots, q_k$.



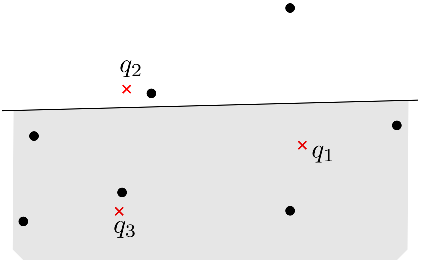
Weighted $(\frac{2}{7}, \frac{4}{7}, \frac{6}{7})$-Net

Motivation

Weighted ε-Nets

Definition [B., Schnider; '20]

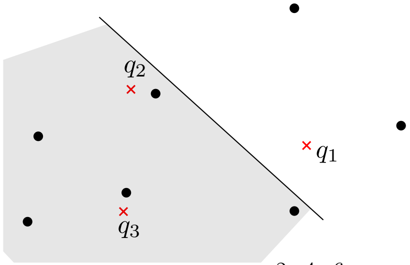
Given any point set $P \subset \mathbb{R}^d$ of size n, a *weighted ε-net* is defined as a set of points $q_1, \ldots, q_k$ and some values $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_k)$ such that every set in the range space containing more than $\varepsilon_i n$ points of P contains at least i of the points $q_1, \ldots, q_k$.



Weighted $(\frac{2}{7}, \frac{4}{7}, \frac{6}{7})$-Net

Definition [B., Schnider; '20]

Given any point set $P \subset \mathbb{R}^d$ of size n, a *weighted ε-net* is defined as a set of points $q_1, \ldots, q_k$ and some values $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_k)$ such that every set in the range space containing more than $\varepsilon_i n$ points of P contains at least i of the points $q_1, \ldots, q_k$.



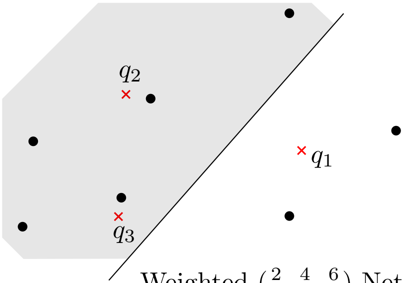
Weighted $(\frac{2}{7}, \frac{4}{7}, \frac{6}{7})$-Net

Motivation

Weighted ε-Nets

Definition [B., Schnider; '20]

Given any point set $P \subset \mathbb{R}^d$ of size n, a *weighted ε-net* is defined as a set of points $q_1, \ldots, q_k$ and some values $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_k)$ such that every set in the range space containing more than $\varepsilon_i n$ points of P contains at least i of the points $q_1, \ldots, q_k$.



Weighted $(\frac{2}{7}, \frac{4}{7}, \frac{6}{7})$-Net

Weighted ε-Nets for Convex Sets

A general result

Theorem [B., Schnider; '20]

Let P be a set of n points in general position in $\mathbb{R}^d$. Let $0 < \varepsilon_1 \leq \varepsilon_2 < 1$ be two constants with

(*i*) $d\varepsilon_1 + \varepsilon_2 \geq d$,

(*ii*) $\varepsilon_1 \geq \frac{2d-1}{2d+1}$.

Then there exists a weighted $(\varepsilon_1, \varepsilon_2)$-Net.

Weighted ε-Nets for Convex Sets

A general result

Theorem [B., Schnider; '20]

Let P be a set of n points in general position in $\mathbb{R}^d$. Let $0 < \varepsilon_1 \leq \varepsilon_2 < 1$ be two constants with

(i) $d\varepsilon_1 + \varepsilon_2 \geq d$,

(ii) $\varepsilon_1 \geq \frac{2d-1}{2d+1}$.

Then there exists a weighted $(\varepsilon_1, \varepsilon_2)$-Net.

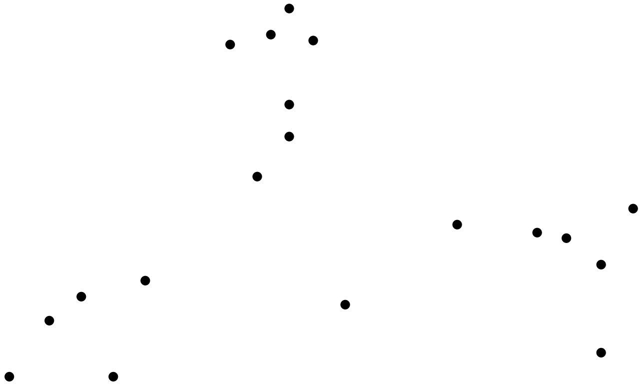
Corollary [B., Schnider; '20]

Let P be a set of n points in general position in the plane. Then there exists a weighted $(\frac{3}{5}, \frac{4}{5})$-Net.

Weighted ε-Nets for Convex Sets

Idea of the proof

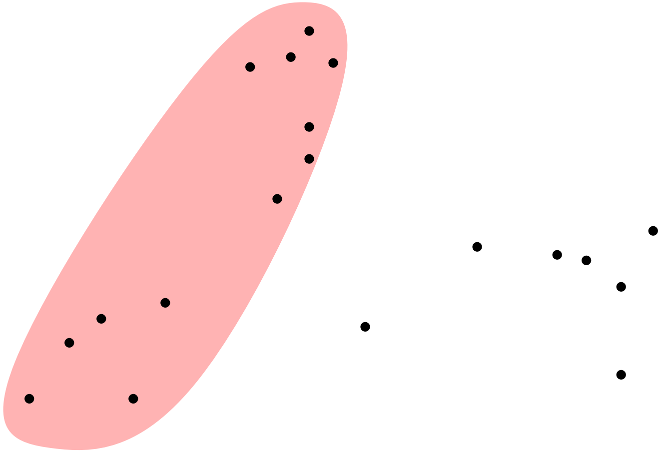
For simplicity, let $\varepsilon_1 = \frac{3}{5}$ and $\varepsilon_2 = \frac{4}{5}$.



Weighted ε-Nets for Convex Sets

Idea of the proof

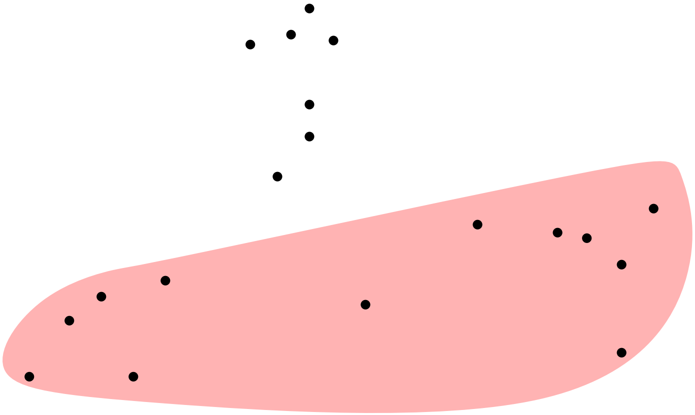
For simplicity, let $\varepsilon_1 = \frac{3}{5}$ and $\varepsilon_2 = \frac{4}{5}$.



Weighted ε-Nets for Convex Sets

Idea of the proof

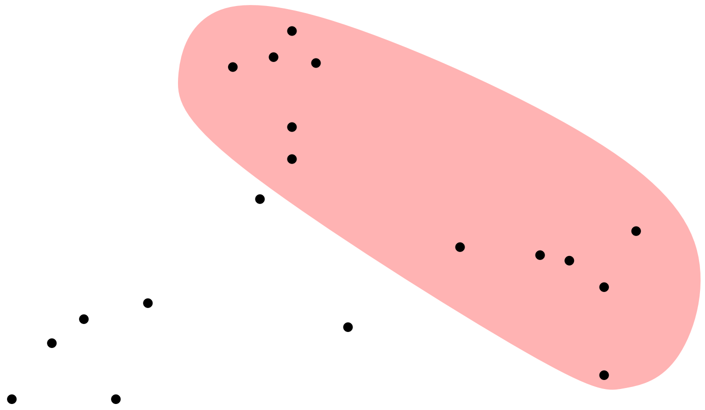
For simplicity, let $\varepsilon_1 = \frac{3}{5}$ and $\varepsilon_2 = \frac{4}{5}$.



Weighted ε-Nets for Convex Sets

Idea of the proof

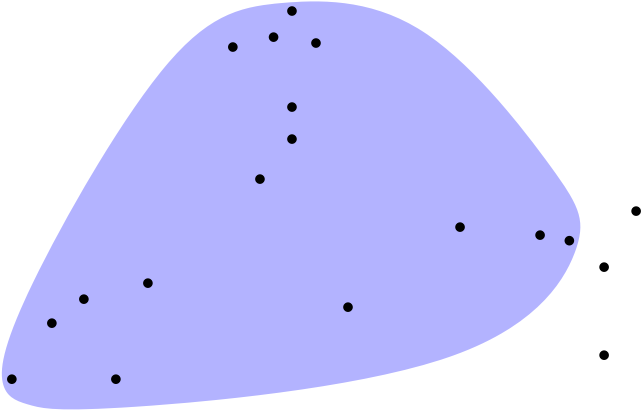
For simplicity, let $\varepsilon_1 = \frac{3}{5}$ and $\varepsilon_2 = \frac{4}{5}$.



Weighted ε-Nets for Convex Sets

Idea of the proof

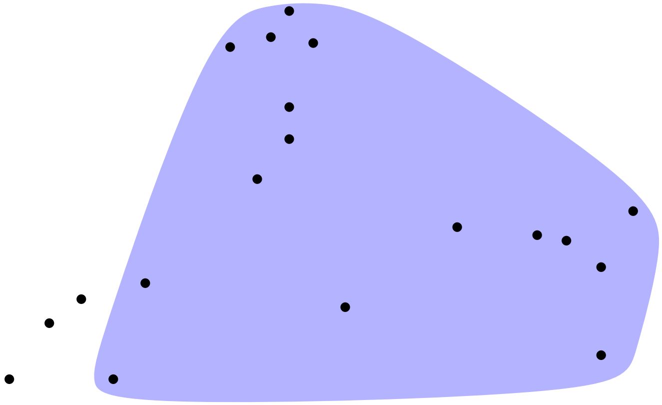
For simplicity, let $\varepsilon_1 = \frac{3}{5}$ and $\varepsilon_2 = \frac{4}{5}$.



Weighted ε-Nets for Convex Sets

Idea of the proof

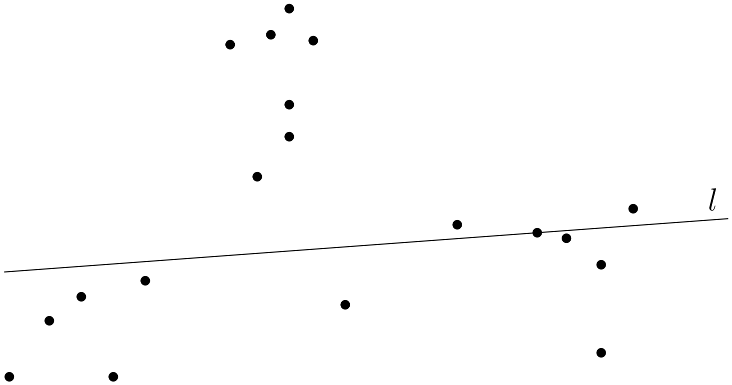
For simplicity, let $\varepsilon_1 = \frac{3}{5}$ and $\varepsilon_2 = \frac{4}{5}$.



Weighted ε-Nets for Convex Sets

Idea of the proof

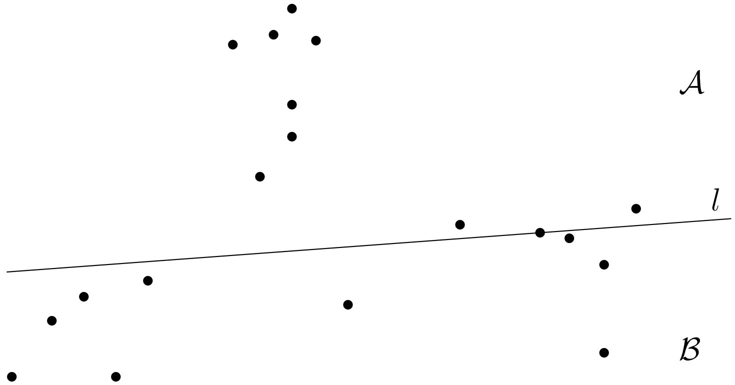
Let l be a halving line of the point set.



Weighted ε-Nets for Convex Sets

Idea of the proof

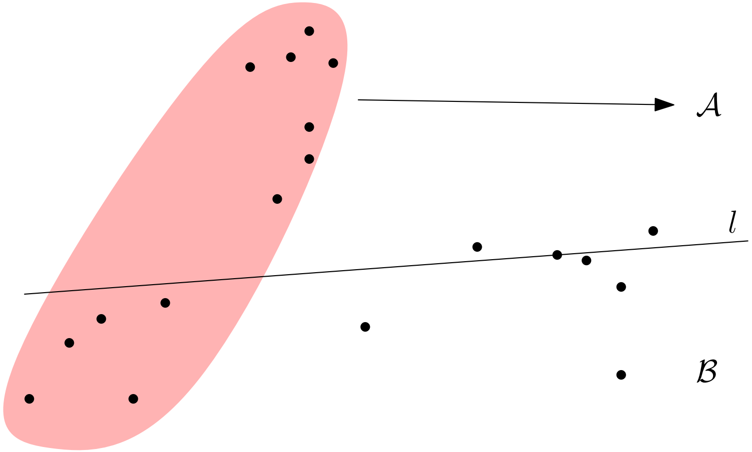
Let l be a halving line of the point set.



Weighted ε-Nets for Convex Sets

Idea of the proof

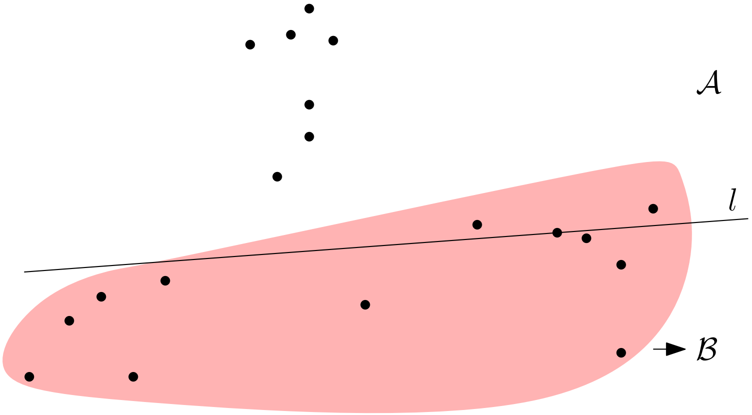
Let l be a halving line of the point set.



Weighted ε-Nets for Convex Sets

Idea of the proof

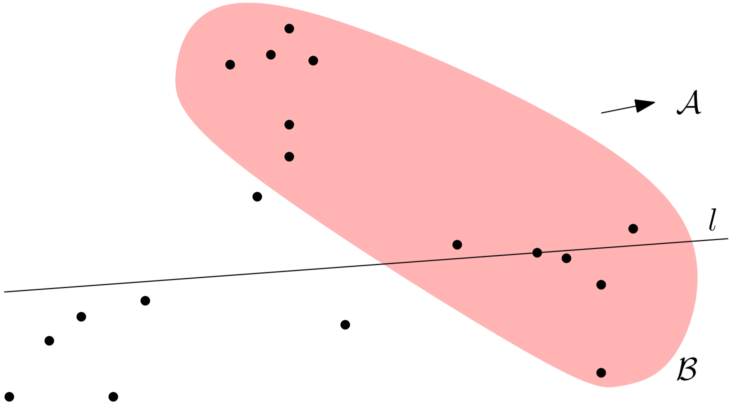
Let l be a halving line of the point set.



Weighted ε-Nets for Convex Sets

Idea of the proof

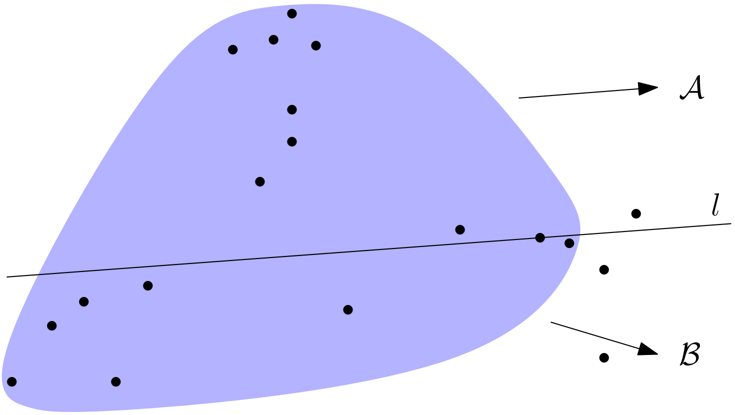
Let l be a halving line of the point set.



Weighted ε-Nets for Convex Sets

Idea of the proof

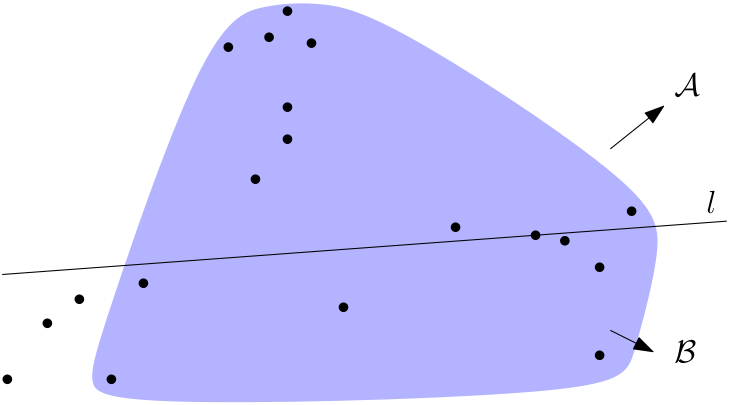
Let l be a halving line of the point set.



Weighted ε-Nets for Convex Sets

Idea of the proof

Let l be a halving line of the point set.



Weighted ε-Nets for Convex Sets

Idea of the proof

Lemma

Any three sets in $\mathcal{A}$ have a common nonempty intersection.

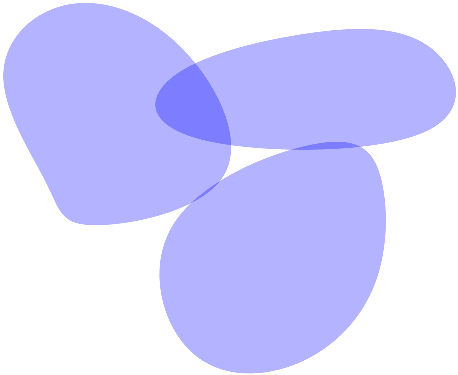
Weighted ε-Nets for Convex Sets

Lemma

Any three sets in $\mathcal{A}$ have a common nonempty intersection.

Proof

Consider three big sets B_1, B_2, B_3 without an intersection.



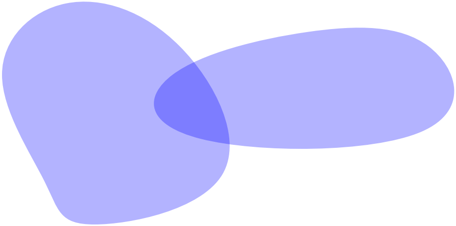
Weighted ε-Nets for Convex Sets

Lemma

Any three sets in $\mathcal{A}$ have a common nonempty intersection.

Proof

Consider three big sets B_1, B_2, B_3 without an intersection.



Weighted ε-Nets for Convex Sets

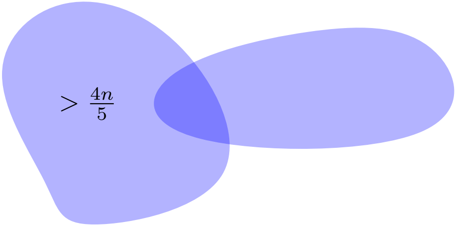
Idea of the proof

Lemma

Any three sets in $\mathcal{A}$ have a common nonempty intersection.

Proof

Consider three big sets B_1, B_2, B_3 without an intersection.



Weighted ε-Nets for Convex Sets

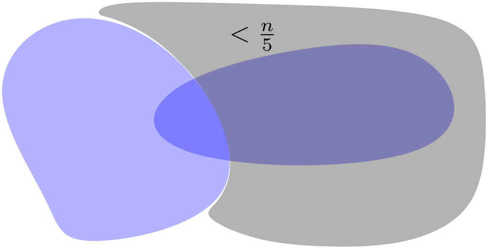
Idea of the proof

Lemma

Any three sets in $\mathcal{A}$ have a common nonempty intersection.

Proof

Consider three big sets B_1, B_2, B_3 without an intersection.



Weighted ε-Nets for Convex Sets

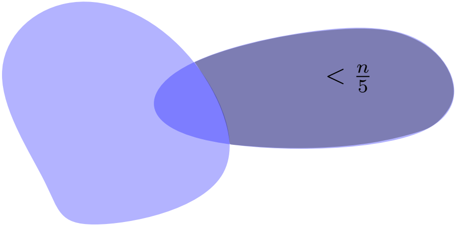
Idea of the proof

Lemma

Any three sets in $\mathcal{A}$ have a common nonempty intersection.

Proof

Consider three big sets B_1, B_2, B_3 without an intersection.



Weighted ε-Nets for Convex Sets

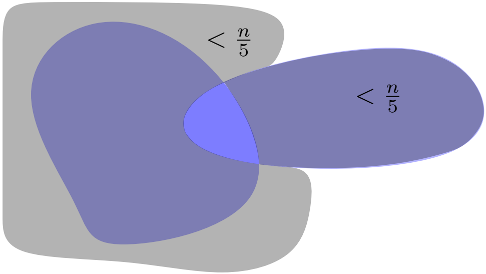
Idea of the proof

Lemma

Any three sets in $\mathcal{A}$ have a common nonempty intersection.

Proof

Consider three big sets B_1, B_2, B_3 without an intersection.



Weighted ε-Nets for Convex Sets

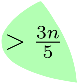
Idea of the proof

Lemma

Any three sets in $\mathcal{A}$ have a common nonempty intersection.

Proof

Consider three big sets B_1, B_2, B_3 without an intersection.

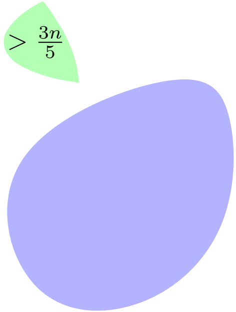


Weighted ε-Nets for Convex Sets

Idea of the proof

Lemma
Any three sets in $\mathcal{A}$ have a common nonempty intersection.

Proof
Consider three big sets B_1, B_2, B_3 without an intersection.



Weighted ε-Nets for Convex Sets

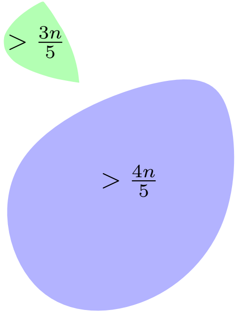
Idea of the proof

Lemma

Any three sets in $\mathcal{A}$ have a common nonempty intersection.

Proof

Consider three big sets B_1, B_2, B_3 without an intersection.



Weighted ε-Nets for Convex Sets

Claim

Any three small sets in $\mathcal{A}$ have a common nonempty intersection.

Proof

Consider three small sets $A_1, A_2, A_3 \in \mathcal{A}$ without an intersection.

Weighted ε-Nets for Convex Sets

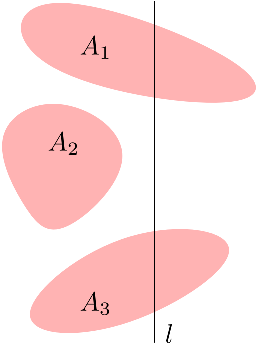
Idea of the proof

Claim

Any three small sets in $\mathcal{A}$ have a common nonempty intersection.

Proof

Consider three small sets $A_1, A_2, A_3 \in \mathcal{A}$ without an intersection.



Weighted ε-Nets for Convex Sets

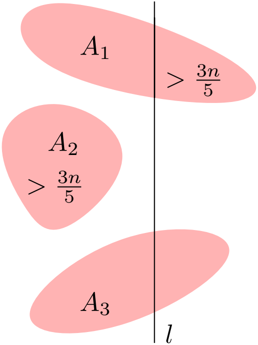
Idea of the proof

Claim

Any three small sets in $\mathcal{A}$ have a common nonempty intersection.

Proof

Consider three small sets $A_1, A_2, A_3 \in \mathcal{A}$ without an intersection.



Weighted ε-Nets for Convex Sets

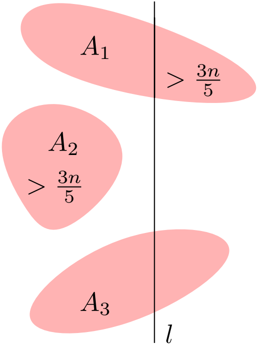
Idea of the proof

Claim

Any three small sets in $\mathcal{A}$ have a common nonempty intersection.

Proof

Observation: These sets intersect pairwise.



Weighted ε-Nets for Convex Sets

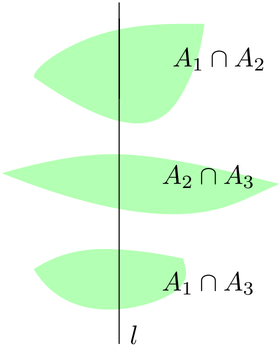
Idea of the proof

Claim

Any three small sets in $\mathcal{A}$ have a common nonempty intersection.

Proof

Observation: These sets intersect pairwise.



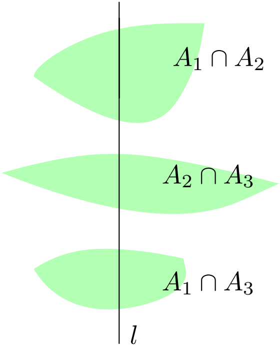
Weighted ε-Nets for Convex Sets

Claim

Any three small sets in $\mathcal{A}$ have a common nonempty intersection.

Proof

Observation: One intersection does not intersect l.



Weighted ε-Nets for Convex Sets

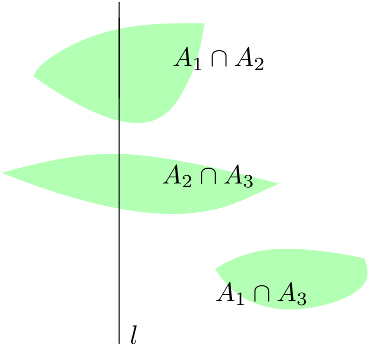
Idea of the proof

Claim

Any three small sets in $\mathcal{A}$ have a common nonempty intersection.

Proof

Observation: One intersection does not intersect l.

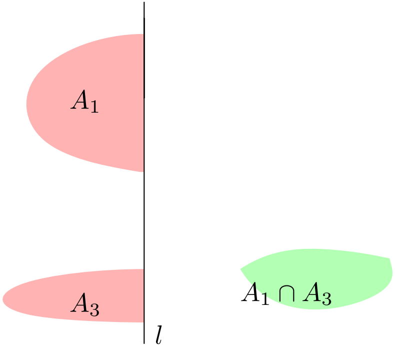


Claim

Any three small sets in $\mathcal{A}$ have a common nonempty intersection.

Proof

Observation: One intersection does not intersect l.



Weighted ε-Nets for Convex Sets

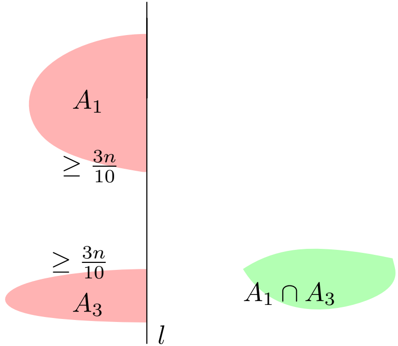
Idea of the proof

Claim

Any three small sets in $\mathcal{A}$ have a common nonempty intersection.

Proof

Observation: One intersection does not intersect l.



Weighted ε-Nets for Convex Sets

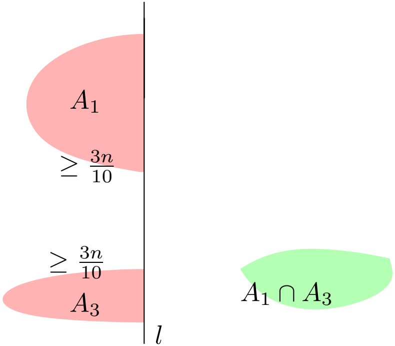
Idea of the proof

Claim

Any three small sets in $\mathcal{A}$ have a common nonempty intersection.

Proof

Observation: One intersection lies on the left side of l.



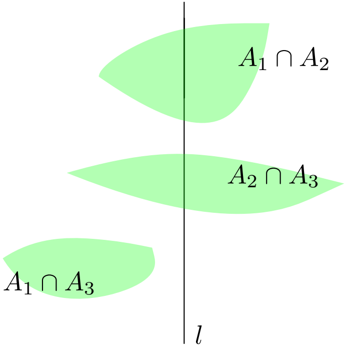
Weighted ε-Nets for Convex Sets

Claim

Any three small sets in $\mathcal{A}$ have a common nonempty intersection.

Proof

Observation: One intersection lies on the left side of l.



Weighted ε-Nets for Convex Sets

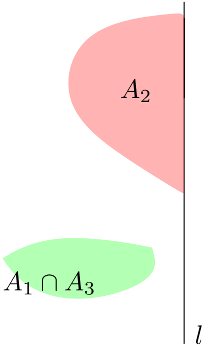
Idea of the proof

Claim

Any three small sets in $\mathcal{A}$ have a common nonempty intersection.

Proof

Observation: One intersection lies on the left side of l.



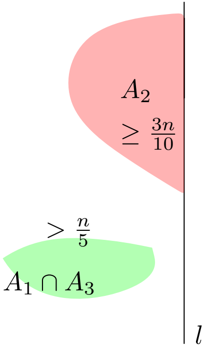
Weighted ε-Nets for Convex Sets

Claim

Any three small sets in $\mathcal{A}$ have a common nonempty intersection.

Proof

Observation: One intersection lies on the left side of l.



Summary

1. Every convex set containing more than $\frac{3n}{5}$ points of P was put into $\mathcal{A}$ or $\mathcal{B}$.

Summary

1. Every convex set containing more than $\frac{3n}{5}$ points of P was put into $\mathcal{A}$ or $\mathcal{B}$.
2. Every convex set containing more than $\frac{4n}{5}$ points of P was put into both $\mathcal{A}$ and $\mathcal{B}$.

Weighted ε-Nets for Convex Sets

Summary

1. Every convex set containing more than $\frac{3n}{5}$ points of P was put into $\mathcal{A}$ or $\mathcal{B}$.
2. Every convex set containing more than $\frac{4n}{5}$ points of P was put into both $\mathcal{A}$ and $\mathcal{B}$.
3. Any three sets in $\mathcal{A}$ have a common intersection.

Summary

1. Every convex set containing more than $\frac{3n}{5}$ points of P was put into $\mathcal{A}$ or $\mathcal{B}$.
2. Every convex set containing more than $\frac{4n}{5}$ points of P was put into both $\mathcal{A}$ and $\mathcal{B}$.
3. Any three sets in $\mathcal{A}$ have a common intersection.
4. By Helly's Theorem there exists a point p in the intersection of all sets in $\mathcal{A}$.

Summary

1. Every convex set containing more than $\frac{3n}{5}$ points of P was put into $\mathcal{A}$ or $\mathcal{B}$.
2. Every convex set containing more than $\frac{4n}{5}$ points of P was put into both $\mathcal{A}$ and $\mathcal{B}$.
3. Any three sets in $\mathcal{A}$ have a common intersection.
4. By Helly's Theorem there exists a point p in the intersection of all sets in $\mathcal{A}$.
5. Define $p_1 := p$.

Summary

1. Every convex set containing more than $\frac{3n}{5}$ points of P was put into $\mathcal{A}$ or $\mathcal{B}$.
2. Every convex set containing more than $\frac{4n}{5}$ points of P was put into both $\mathcal{A}$ and $\mathcal{B}$.
3. Any three sets in $\mathcal{A}$ have a common intersection.
4. By Helly's Theorem there exists a point p in the intersection of all sets in $\mathcal{A}$.
5. Define $p_1 := p$.
6. We do the same with sets in $\mathcal{B}$ to get p_2.

Conclusion and further directions

- Introduced weighted ε-nets.

Conclusion and further directions

- Introduced weighted ε-nets.
- Upper bounds for two points.

Conclusion and further directions

- Introduced weighted ε-nets.
- Upper bounds for two points.

- Upper bounds for two and three points and axis-parallel boxes.

Conclusion and further directions

- Introduced weighted ε-nets.
- Upper bounds for two points.

- Upper bounds for two and three points and axis-parallel boxes.
- Lower bounds for two points.

Conclusion and further directions

  - Introduced weighted ε-nets.
  - Upper bounds for two points.
- Upper bounds for two and three points and axis-parallel boxes.
- Lower bounds for two points.

Thanks for your attention!