Geometric bistellar moves relate triangulations of Euclidean, hyperbolic and spherical manifolds

Tejas Kalelkar,
Indian Institute of Science Education and Research, Pune
(Joint work with Advait Phanse)

17th March, 2020

Geometric Flips

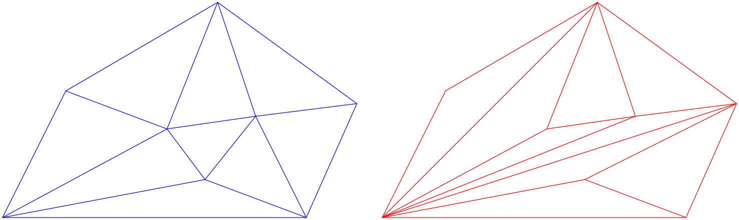


Figure: Flips relate any two triangulations of a 2-polytope with same vertices.

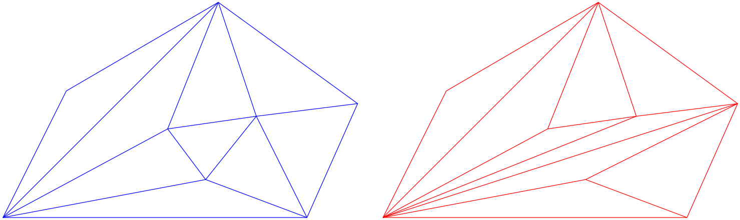


Figure: Flips relate any two triangulations of a 2-polytope with same vertices.

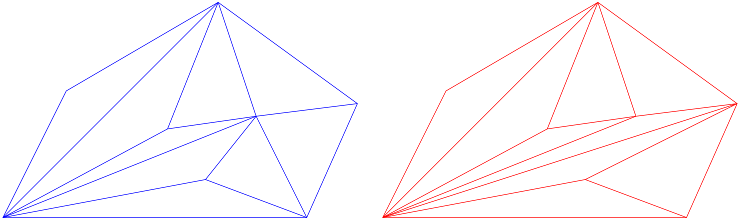


Figure: Flips relate any two triangulations of a 2-polytope with same vertices.

Geometric Flips

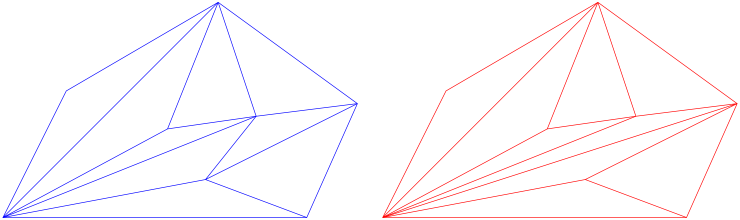


Figure: Flips relate any two triangulations of a 2-polytope with same vertices.

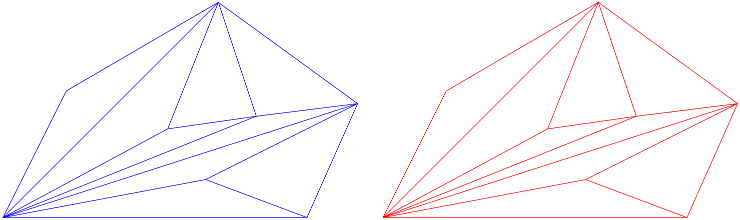


Figure: Flips relate any two triangulations of a 2-polytope with same vertices.

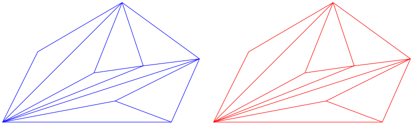


Figure: Flips relate any two triangulations of a 2-polytope with same vertices.

Theorem (Despre - Schlenker - Teillaud)

Let S be either a torus with a Euclidean metric or a closed oriented surface with a hyperbolic metric. Then any two geometric triangulations of S with the same vertex set are related by geometric flips.

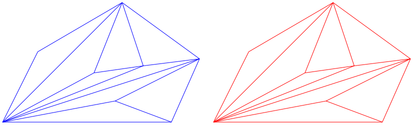


Figure: Flips relate any two triangulations of a 2-polytope with same vertices.

Theorem (Despre - Schlenker - Teillaud)

Let S be either a torus with a Euclidean metric or a closed oriented surface with a hyperbolic metric. Then any two geometric triangulations of S with the same vertex set are related by geometric flips.

Theorem (Santos)

There exist 5-dimensional polytopes with triangulations with the same vertex set which are not related by geometric flips.

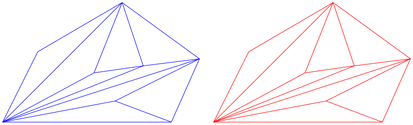


Figure: Flips relate any two triangulations of a 2-polytope with same vertices.

Theorem (Despre - Schlenker - Teillaud)

Let S be either a torus with a Euclidean metric or a closed oriented surface with a hyperbolic metric. Then any two geometric triangulations of S with the same vertex set are related by geometric flips.

Theorem (Santos)

There exist 5-dimensional polytopes with triangulations with the same vertex set which are not related by geometric flips.

Question

When the vertex sets are possibly different, what classes of triangulations are related by n-dimensional geometric bistellar moves?

Definition

Let K be a triangulation of an n-manifold M and let D be a disk-subcomplex of K simplicially isomorphic to an n-disk in $\partial\Delta^{n+1}$. Then a bistellar move on D replaces D with the disk isomorphic to $\partial\Delta^{n+1} \setminus int(D)$.

Definition

Let K be a triangulation of an n-manifold M and let D be a disk-subcomplex of K simplicially isomorphic to an n-disk in $\partial\Delta^{n+1}$. Then a bistellar move on D replaces D with the disk isomorphic to $\partial\Delta^{n+1} \setminus int(D)$.

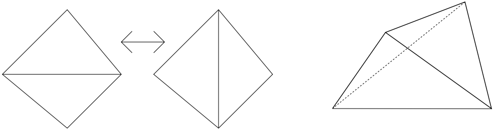


Figure: A 2-2 bistellar move

Geometric bistellar moves

Definition

Let K be a triangulation of an n-manifold M and let D be a disk-subcomplex of K simplicially isomorphic to an n-disk in $\partial\Delta^{n+1}$. Then a bistellar move on D replaces D with the disk isomorphic to $\partial\Delta^{n+1} \setminus int(D)$.

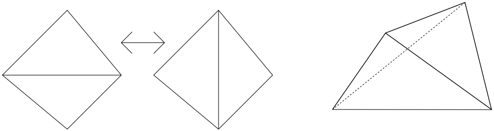


Figure: A 2-2 bistellar move

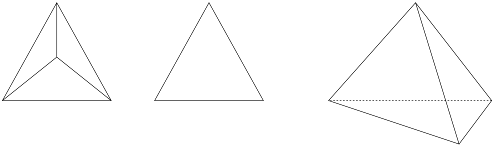


Figure: A 3-1 and 1-3 bistellar move

Definition

Let K be a triangulation of an n-manifold M and let D be a disk-subcomplex of K simplicially isomorphic to an n-disk in $\partial\Delta^{n+1}$. Then a bistellar move on D replaces D with the disk isomorphic to $\partial\Delta^{n+1} \setminus int(D)$.

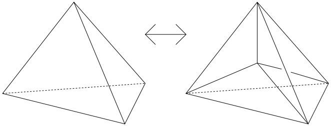


Figure: A 1-4 and 4-1 bistellar move

Definition

Let K be a triangulation of an n-manifold M and let D be a disk-subcomplex of K simplicially isomorphic to an n-disk in $\partial\Delta^{n+1}$. Then a bistellar move on D replaces D with the disk isomorphic to $\partial\Delta^{n+1} \setminus int(D)$.

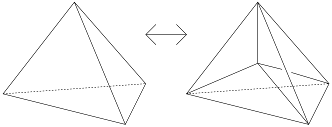


Figure: A 1-4 and 4-1 bistellar move

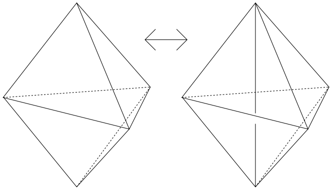


Figure: A 2-3 and 3-2 bistellar move

Question

When the vertex sets are possibly different, what classes of triangulations are related by n-dimensional geometric bistellar moves?

Question

When the vertex sets are possibly different, what classes of triangulations are related by n-dimensional geometric bistellar moves?

Theorem (Izmestiev - Schlenker)

Any two triangulations of a convex polytope in $\mathbb{R}^3$ can be connected by a sequence of geometric bistellar moves, boundary geometric stellar moves and continuous displacements of the interior vertices.

Question

When the vertex sets are possibly different, what classes of triangulations are related by n-dimensional geometric bistellar moves?

Theorem (Izmestiev - Schlenker)

Any two triangulations of a convex polytope in $\mathbb{R}^3$ can be connected by a sequence of geometric bistellar moves, boundary geometric stellar moves and continuous displacements of the interior vertices.

Theorem

Let K_1 and K_2 be geometric simplicial triangulations (with possibly different vertex sets) of a compact Euclidean, hyperbolic or spherical n-manifold M. If M is spherical, we assume that the star of each simplex has diameter less than π. Let L be a possibly empty common subcomplex of K_1 and K_2. If M has boundary then we insist that K_1 and K_2 agree on ∂M, i.e., $|L| \supset \partial M$.

- *When n is 2 or 3, then K_1 and K_2 are related by geometric bistellar moves which keep L fixed.*
- *When $n > 3$, then some s-th iterated derived subdivisions $\beta^s K_1$ and $\beta^s K_2$ are related by geometric bistellar moves which keep $\beta^s L$ fixed.*

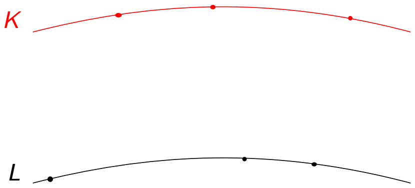


Figure: Two triangulations K and L of a hyperbolic manifold M

Simplicial cobordism

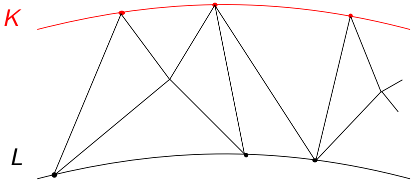


Figure: A geometric triangulation of $M \times I$ from K to L

Simplicial cobordism

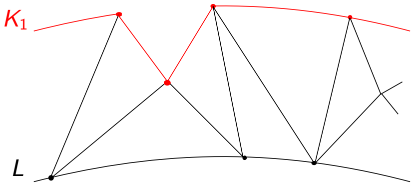


Figure: Removing an n-simplex from the top and then projecting the upper boundary down to $M \times 0$ gives a bistellar move from K to K_1

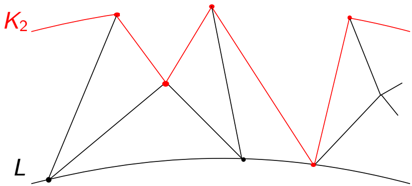


Figure: Removing an n-simplex from the top and then projecting the upper boundary down to $M \times 0$ gives a bistellar move from K_1 to K_2

Simplicial cobordism

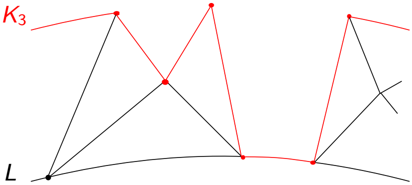


Figure: Removing an n-simplex from the top and then projecting the upper boundary down to $M \times 0$ gives a bistellar move from K_2 to K_3

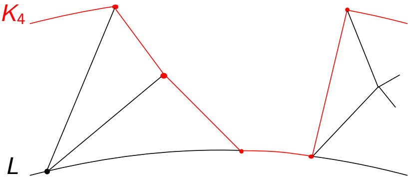


Figure: Removing an n-simplex from the top and then projecting the upper boundary down to $M \times 0$ gives a bistellar move from K_3 to K_4

Question

Is there a geometric triangulation of $M \times I$ in $\mathbb{H}^n \times \mathbb{R}$ geometry, with the given triangulations K and L on $M \times 0$ and $M \times 1$?

Question

Is there a geometric triangulation of $M \times I$ in $\mathbb{H}^n \times \mathbb{R}$ geometry, with the given triangulations K and L on $M \times 0$ and $M \times 1$?

Theorem (Cartan)

If at every point $p \in M$ and for every subspace V of T_pM there exists a totally geodesic surface S through p with $T_pS = V$ then M has constant curvature.

Question

Is there a geometric triangulation of $M \times I$ in $\mathbb{H}^n \times \mathbb{R}$ geometry, with the given triangulations K and L on $M \times 0$ and $M \times 1$?

Theorem (Cartan)

If at every point $p \in M$ and for every subspace V of T_pM there exists a totally geodesic surface S through p with $T_pS = V$ then M has constant curvature.

Question

Is it possible to get an enumeration Δ_0, Δ_1, ..., Δ_m of the n-simplexes such that the projection $pr : \cup_{j=i}^{m} \Delta_j \to M \times 0$ is an injection when restricted to the upper boundary of each $\cup_{j=i}^{m} \Delta_j$?

Question

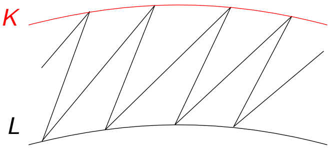
Is there a geometric triangulation of $M \times I$ in $\mathbb{H}^n \times \mathbb{R}$ geometry, with the given triangulations K and L on $M \times 0$ and $M \times 1$?

Theorem (Cartan)

If at every point $p \in M$ and for every subspace V of T_pM there exists a totally geodesic surface S through p with $T_pS = V$ then M has constant curvature.

Question

Is it possible to get an enumeration Δ_0, Δ_1, ..., Δ_m of the n-simplexes such that the projection $pr : \cup_{j=i}^{m} \Delta_j \to M \times 0$ is an injection when restricted to the upper boundary of each $\cup_{j=i}^{m} \Delta_j$?



- Let $K^* = \beta(K_1 \cap K_2)$ be a common geometric subdivision of K_1 and K_2. Any constant curvature manifold M has local maps taking balls in M to $\mathbb{E}^n$ by a homeomorphism taking geodesics to straight lines. So stars of simplexes in K^* are identified with star-convex n-polytopes in $\mathbb{E}^n$.

- Let $K^* = \beta(K_1 \cap K_2)$ be a common geometric subdivision of K_1 and K_2. Any constant curvature manifold M has local maps taking balls in M to $\mathbb{E}^n$ by a homeomorphism taking geodesics to straight lines. So stars of simplexes in K^* are identified with star-convex n-polytopes in $\mathbb{E}^n$.

- We show that K^* is related to βK_1 and to βK_2 by geometric bistellar moves that change the star of each simplex to the cone over its boundary.

- Let $K^* = \beta(K_1 \cap K_2)$ be a common geometric subdivision of K_1 and K_2. Any constant curvature manifold M has local maps taking balls in M to $\mathbb{E}^n$ by a homeomorphism taking geodesics to straight lines. So stars of simplexes in K^* are identified with star-convex n-polytopes in $\mathbb{E}^n$.
- We show that K^* is related to βK_1 and to βK_2 by geometric bistellar moves that change the star of each simplex to the cone over its boundary.
- In dimension 2 and 3, $K \sim \beta K_i$ by geometric bistellar moves.

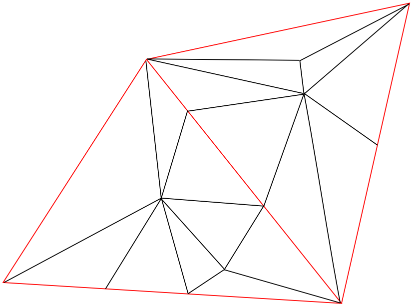


Figure: A simplical complex K and its subdivision K^*

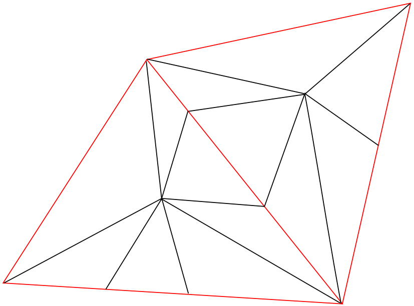


Figure: Complex K' obtained by replacing $Star(A, K)$ with $C(\partial Star(A, K))$, for A varying over 2-dimensional simplexes

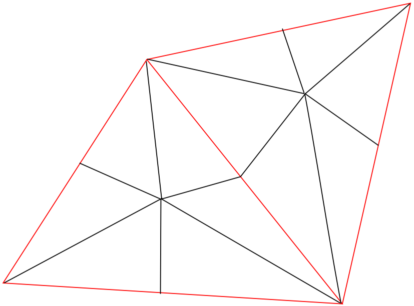


Figure: Complex βK obtained by replacing $Star(A, K')$ with $C(\partial Star(A, K))$, for A varying over 1-dimensional simplexes

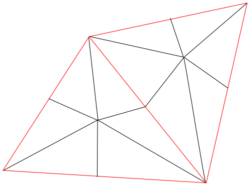


Figure: Complex βK is obtained from K^* by replacing $Star(A, K)$ with $C(\partial Star(A, K))$ inductively over dimension of A

- Enough to show that star-convex polytopes in $\mathbb{E}^n$ can be starred, i.e., any linear triangulation of a star-convex polytope can be changed to a cone over it's boundary by Euclidean bistellar moves. Then $\beta K_1 \sim K^* \sim \beta K_2$.

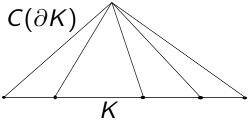


Figure: Cone over a star-convex n-polytope K

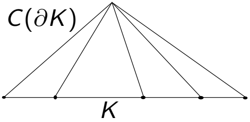


Figure: Cone over a star-convex n-polytope K

- We call a triangulation K of a polytope P regular if there is a function $h : |K| \to \mathbb{R}$ that is linear on each simplex of K and strictly convex across codimension one simplexes of K, i.e., if points x and y are in neighboring top-dimensional simplexes of K then the segment connecting $h(x)$ and $h(y)$ is above the graph of h (except at the end points).

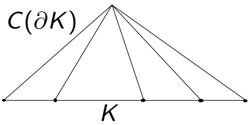


Figure: Cone over a star-convex n-polytope K

- We call a triangulation K of a polytope P regular if there is a function $h : |K| \to \mathbb{R}$ that is linear on each simplex of K and strictly convex across codimension one simplexes of K, i.e., if points x and y are in neighboring top-dimensional simplexes of K then the segment connecting $h(x)$ and $h(y)$ is above the graph of h (except at the end points).
- Let K denote a triangulated n-polytope in $\mathbb{E}^n$. We show that for some $s \in \mathbb{N}$ the s-th derived subdivision $\beta^s K$ is regular.

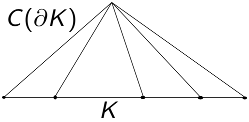


Figure: Cone over a star-convex n-polytope K

- We call a triangulation K of a polytope P regular if there is a function $h : |K| \to \mathbb{R}$ that is linear on each simplex of K and strictly convex across codimension one simplexes of K, i.e., if points x and y are in neighboring top-dimensional simplexes of K then the segment connecting $h(x)$ and $h(y)$ is above the graph of h (except at the end points).
- Let K denote a triangulated n-polytope in $\mathbb{E}^n$. We show that for some $s \in \mathbb{N}$ the s-th derived subdivision $\beta^s K$ is regular.
- (Assume $s = 0$ for simplicity) Let $h : |K| \to \mathbb{R}$ be a regular function. Enumerating n-simplexes in decreasing order of $\partial h / \partial x_{n+1}$, we get the required sequence Δ_0, ..., Δ_m such that the projection $pr : \cup_{j=i}^{m} \Delta_j \to M \times 0$ is injective on the upper boundary of $\cup_{j=i}^{m} \Delta_j$ and therefore $K \sim C(\partial K)$ by geometric bistellar moves.

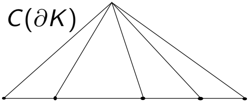


Figure: Cone over a star-convex n-polytope K

- We call a triangulation K of a polytope P regular if there is a function $h : |K| \to \mathbb{R}$ that is linear on each simplex of K and strictly convex across codimension one simplexes of K, i.e., if points x and y are in neighboring top-dimensional simplexes of K then the segment connecting $h(x)$ and $h(y)$ is above the graph of h (except at the end points).
- Let K denote a triangulated n-polytope in $\mathbb{E}^n$. We show that for some $s \in \mathbb{N}$ the s-th derived subdivision $\beta^s K$ is regular.
- (Assume $s = 0$ for simplicity) Let $h : |K| \to \mathbb{R}$ be a regular function. Enumerating n-simplexes in decreasing order of $\partial h / \partial x_{n+1}$, we get the required sequence Δ_0, ..., Δ_m such that the projection $pr : \cup_{j=i}^{m} \Delta_j \to M \times 0$ is injective on the upper boundary of $\cup_{j=i}^{m} \Delta_j$ and therefore $K \sim C(\partial K)$ by geometric bistellar moves.
- Inductively starring the stars of simplexes in decreasing order of their dimension, we get a sequence of geometric bistellar moves $\beta^{s+1} K_1 \sim \beta^s K^* \sim \beta^{s+1} K_2$ as required.

Thank you

Danke Schoen!