Topological Drawings meet Classical Theorems of Convex Geometry

Helena Bergold[1], Stefan Felsner[2], Manfred Scheucher[2], Felix Schröder[2], Raphael Steiner[2]

[1]FernUniversität in Hagen
[2]Technische Universität Berlin

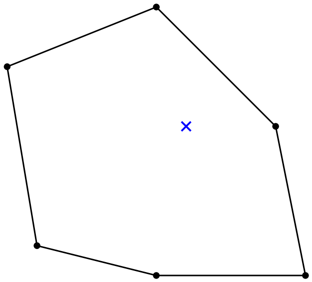
Theorems of Convex Geometry

- Carathéodory's Theorem
- Helly's Theorem
- Kirchberger's Theorem

Carathéodory's Theorem

Theorem

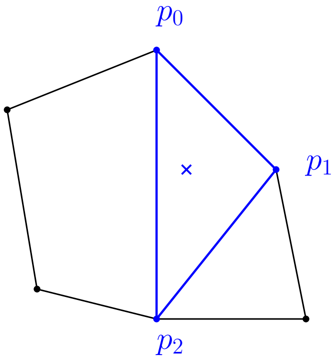
For $P \subseteq \mathbb{R}^d$ and a point $x \in \operatorname{conv} P$, there are points $p_0, \ldots, p_d \in P$ such that $x \in \operatorname{conv}\{p_0, \ldots, p_d\}$.



Carathéodory's Theorem

Theorem

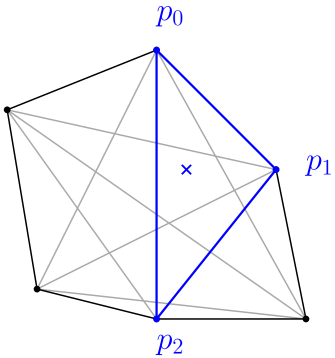
For $P \subseteq \mathbb{R}^d$ and a point $x \in \operatorname{conv} P$, there are points $p_0, \ldots, p_d \in P$ such that $x \in \operatorname{conv}\{p_0, \ldots, p_d\}$.



Carathéodory's Theorem

Theorem

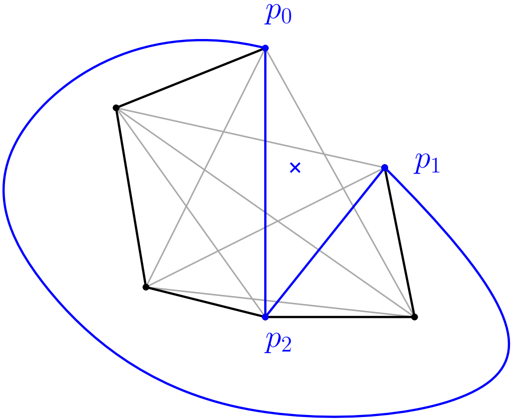
For $P \subseteq \mathbb{R}^d$ and a point $x \in \operatorname{conv} P$, there are points $p_0, \ldots, p_d \in P$ such that $x \in \operatorname{conv}\{p_0, \ldots, p_d\}$.



Carathéodory's Theorem

Theorem

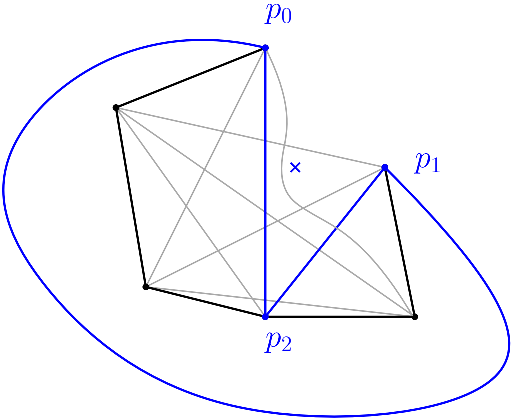
For $P \subseteq \mathbb{R}^d$ and a point $x \in \operatorname{conv} P$, there are points $p_0, \ldots, p_d \in P$ such that $x \in \operatorname{conv}\{p_0, \ldots, p_d\}$.



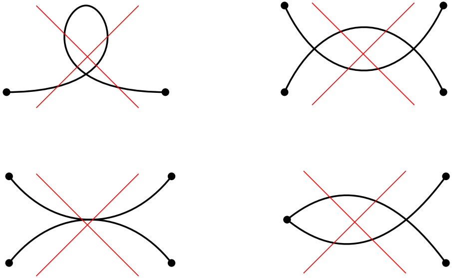
Carathéodory's Theorem

Theorem

For $P \subseteq \mathbb{R}^d$ and a point $x \in \operatorname{conv} P$, there are points $p_0, \ldots, p_d \in P$ such that $x \in \operatorname{conv}\{p_0, \ldots, p_d\}$.



A (simple) topological drawing is a drawing of a complete graph such that:



Carathéodory in Topological Drawings

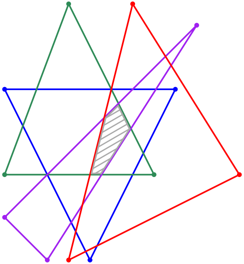
Theorem (Balko, Fulek, Kynčl '15)

Let D be a topological drawing of K_n, $x \in \mathbb{R}^2$ a point in a bounded connected component of $\mathbb{R}^2 - D$.
Then there is a triangle which contains x in the interior.

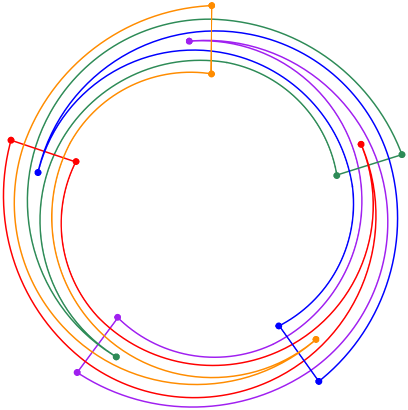
Helly's Theorem

Theorem

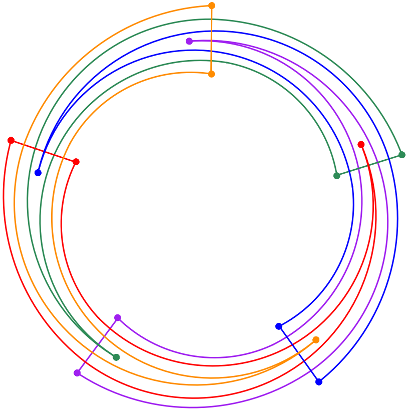
Let $V_1, \ldots, V_n$ **convex** *subsets of $\mathbb{R}^d$. If every $d+1$ subsets have a non-empty intersection, then the intersection of all n subsets is non-empty.*



Helly in topological drawings

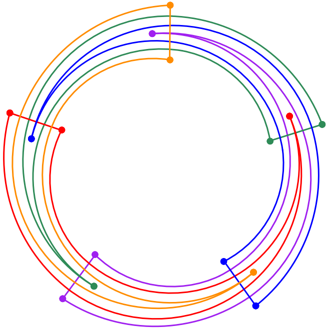


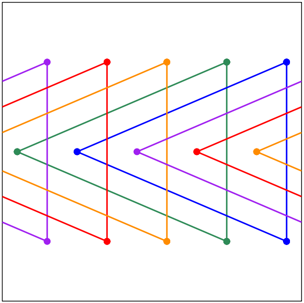
Helly in topological drawings



Complete this drawing to a topological drawing.

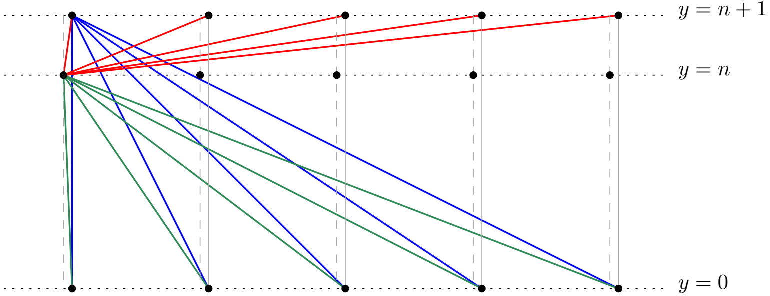
Helly in topological drawings



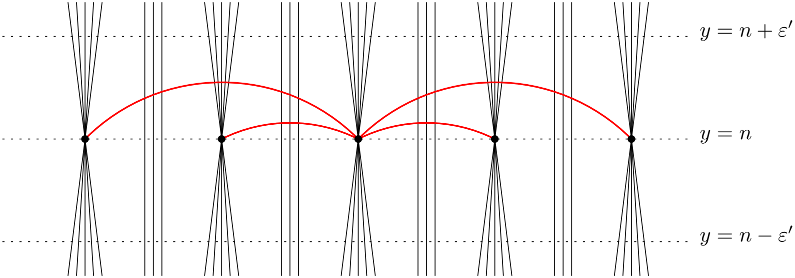


Complete this drawing to a topological drawing.

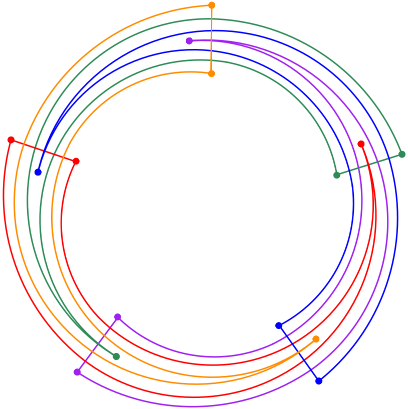
Helly in topological drawings



Helly in topological drawings



Helly in topological drawings

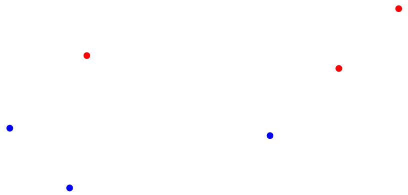


For every n there exists a topological drawing of K_{3n} such that each $n-1$ triangles intersect, but all n do not (unbounded *Helly number*)

Kirchberger's Theorem

Theorem

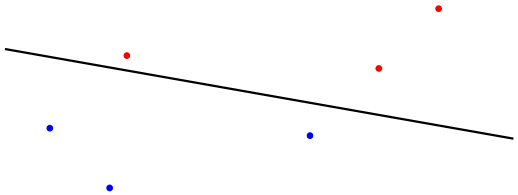
A, B finite point sets of $\mathbb{R}^d$ such that for every $X \subseteq A \cup B$, $|X| = d + 2$ the two sets $A \cap X, B \cap X$ are separable, then A and B are separable.



Kirchberger's Theorem

Theorem

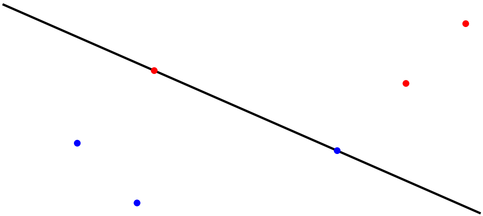
A, B finite point sets of $\mathbb{R}^d$ such that for every $X \subseteq A \cup B$, $|X| = d + 2$ the two sets $A \cap X, B \cap X$ are separable, then A and B are separable.



Kirchberger's Theorem

Theorem

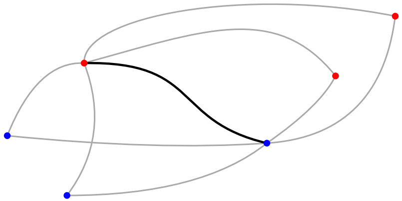
A, B finite point sets of $\mathbb{R}^d$ such that for every $X \subseteq A \cup B$, $|X| = d + 2$ the two sets $A \cap X, B \cap X$ are separable, then A and B are separable.



Kirchberger's Theorem

Theorem

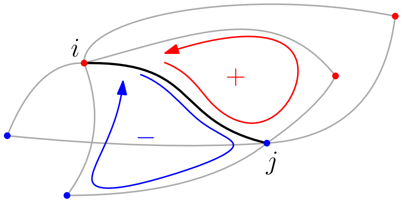
A, B finite point sets of $\mathbb{R}^d$ such that for every $X \subseteq A \cup B$, $|X| = d + 2$ the two sets $A \cap X, B \cap X$ are separable, then A and B are separable.



Kirchberger's Theorem

Theorem

A, B finite point sets of $\mathbb{R}^d$ such that for every $X \subseteq A \cup B$, $|X| = d + 2$ the two sets $A \cap X, B \cap X$ are separable, then A and B are separable.



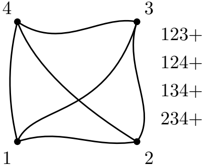
Generalized Signotopes

Definition

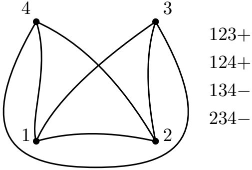
A *generalized signotope* is $\chi : [n]_3 \to \{+,-\}$ s.t.

(i) abc, abd, acd, bcd has at most 2 sign changes
($+-+-/-+-+$ is forbidden)

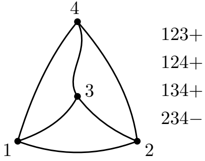
(ii) $abc = -acb = -bac$



type I_a



type I_b



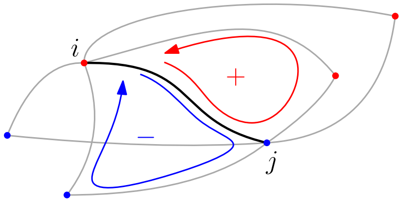
type II

- Crossing: even number of '$-$'
- Non-crossing: odd number of '$-$'

Generalized Kirchberger's Theorem

Theorem

Let $\chi : [n]_3 \to \{+,-\}$ be a generalized signotope.
If for every $X \subseteq A \cup B$ with $|X| = 4$ there are $i \in A \cap Y, j \in B \cap X$ with $\chi(i,j,A \cap X) = +$ and $\chi(i,j,B \cap X) = -$,
then there are $i \in A, j \in B$ with $\chi(i,j,A) = +$ and $\chi(i,j,B) = -$.



Other Classical Theorems

- Colorful Carathéodory
- Ramsey-Type
- Tverberg's Theorem
- Radon's Theorem
- Birch's Theorem
- ...

Thank you for your attention!