On the Average Complexity of the k-Level

EuroCG 2020, Würzburg

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joint work with
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Institute of Mathematics, TU Berlin

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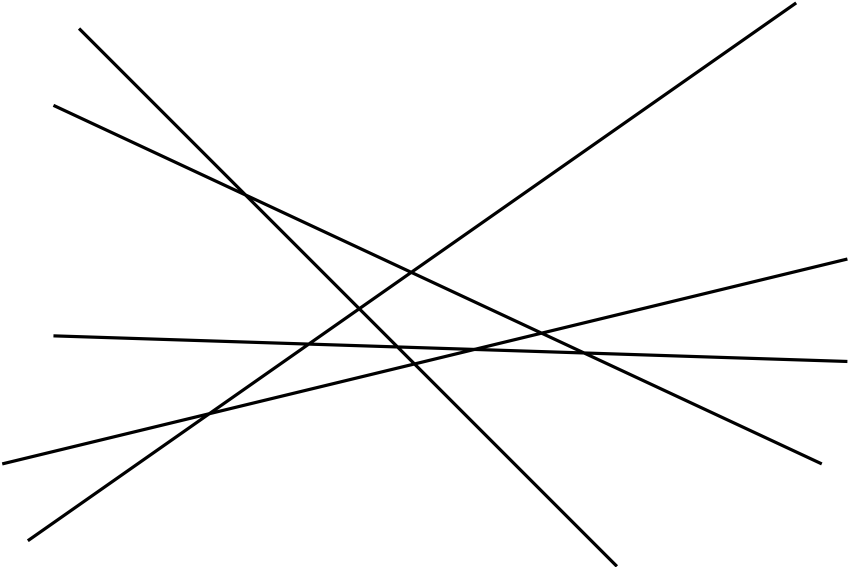
k-Levels in Line-Arrangements



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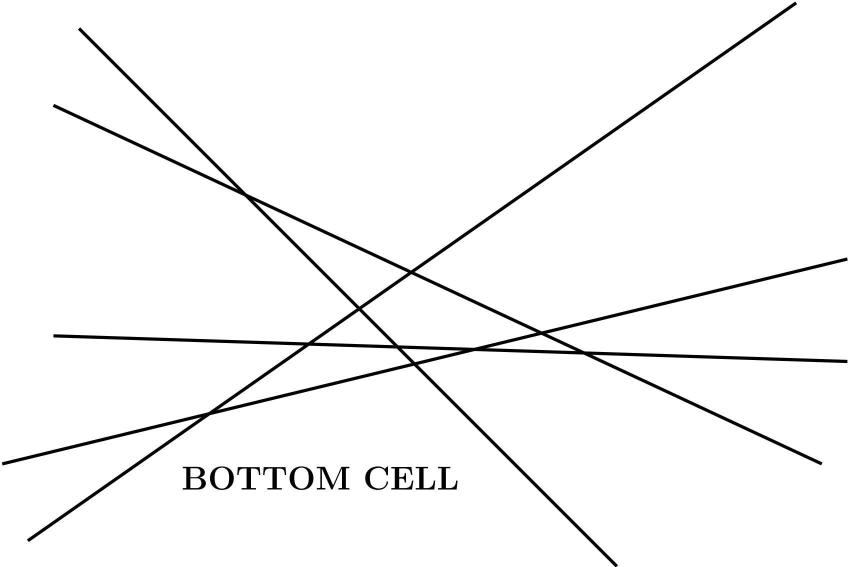

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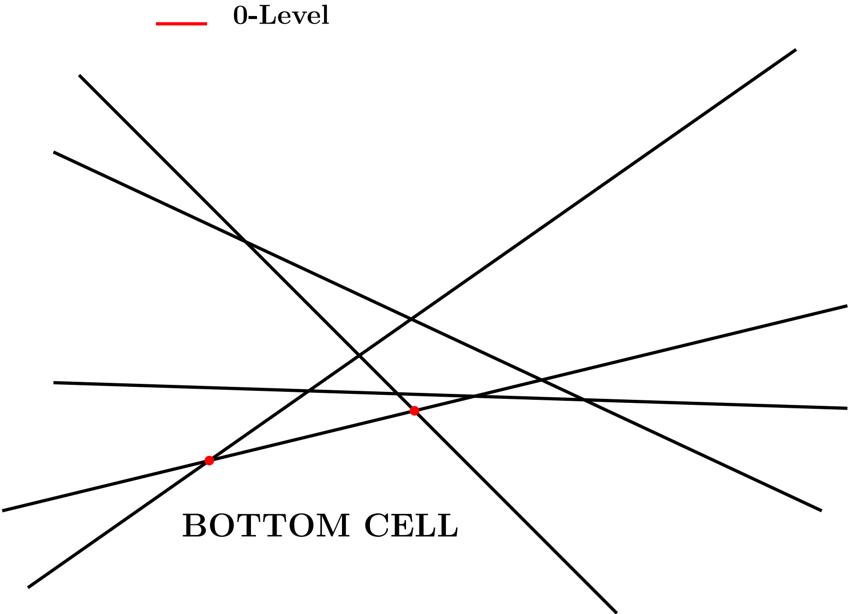

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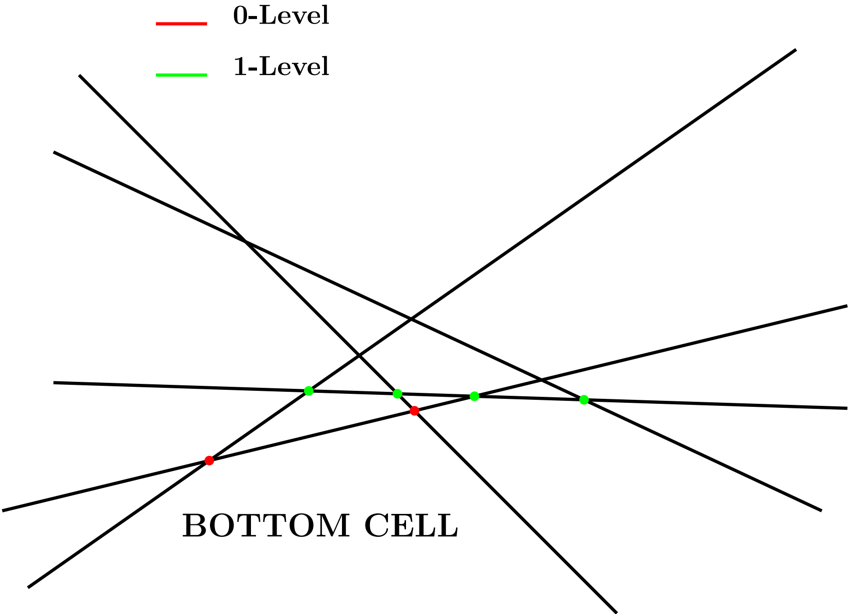

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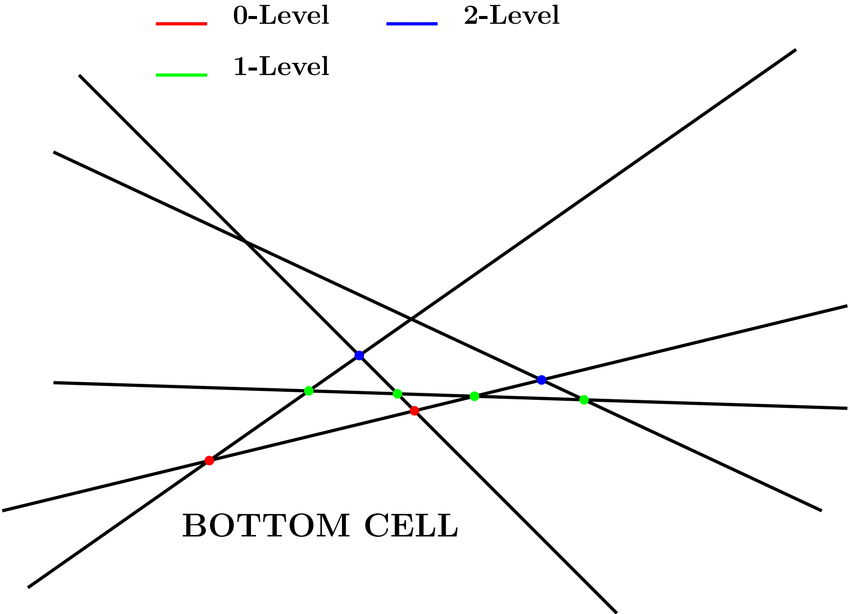

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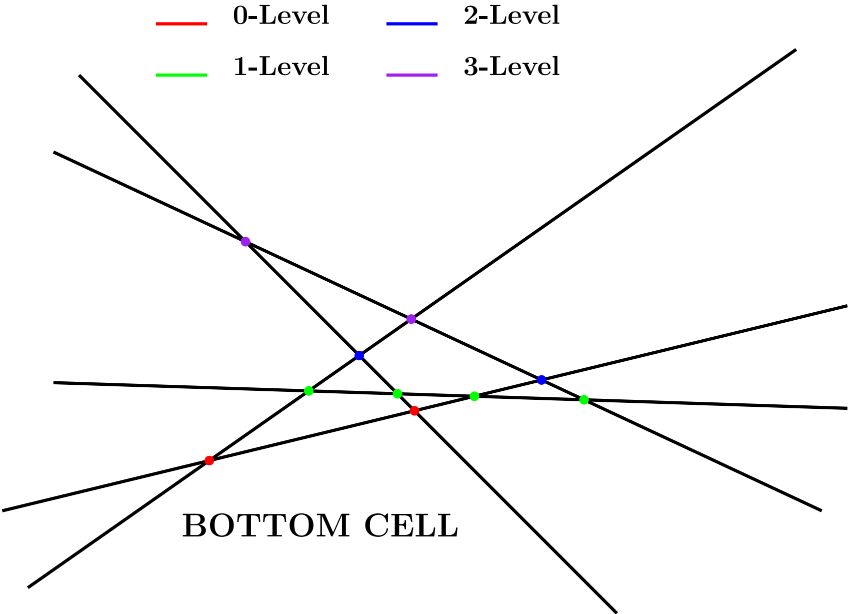
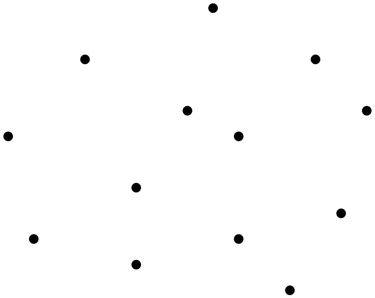


Motivation: k-Sets in Planar Point Sets



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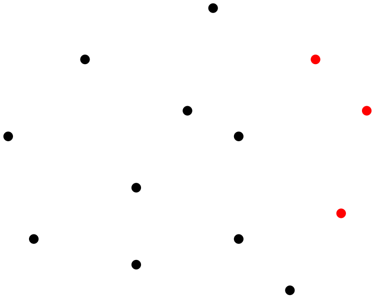

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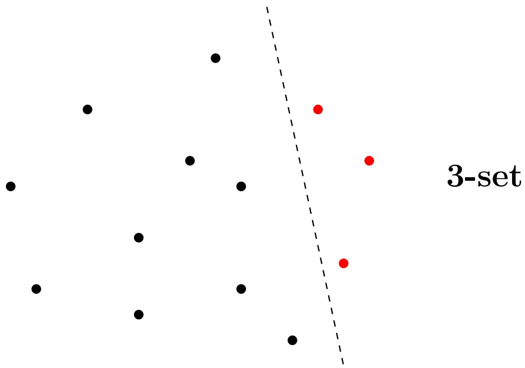

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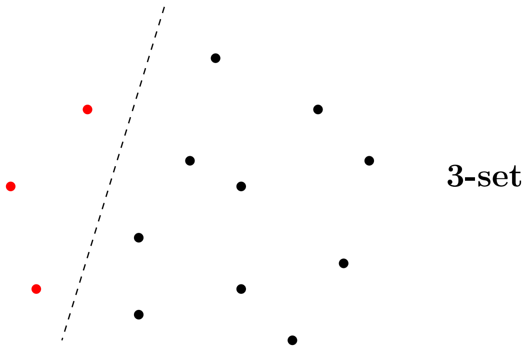

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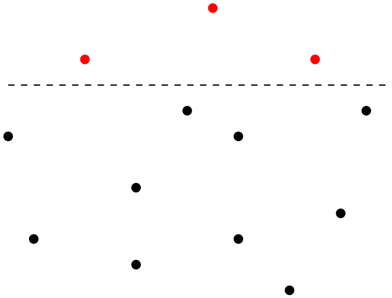

3-set

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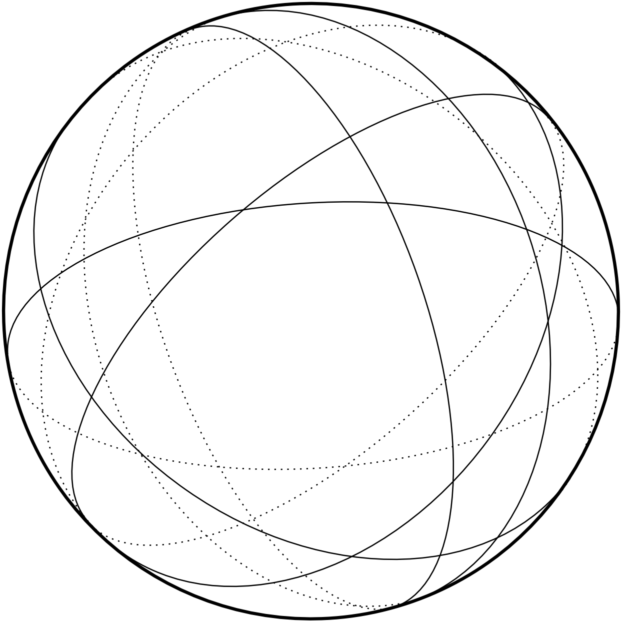
Theorem (Nivasch, 2008)

$f_k(n) = n2^{\Omega(\sqrt{\log k})}$.

What is the 'usual' complexity of the k-level?

Setting: k-Levels on the Sphere



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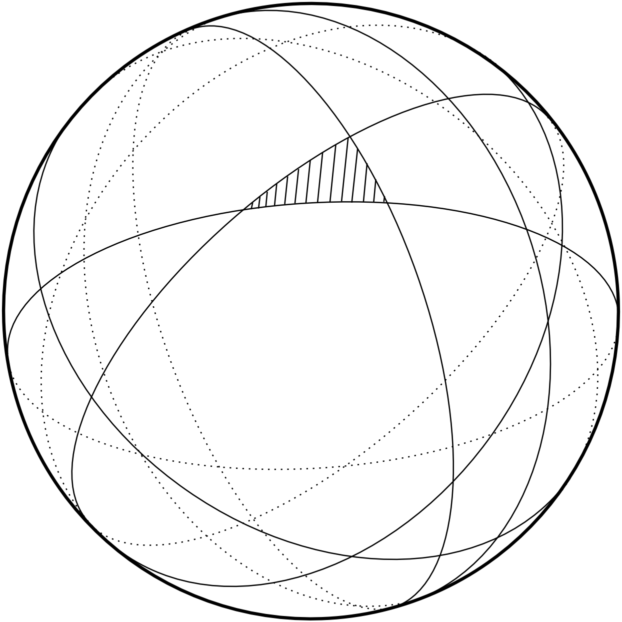

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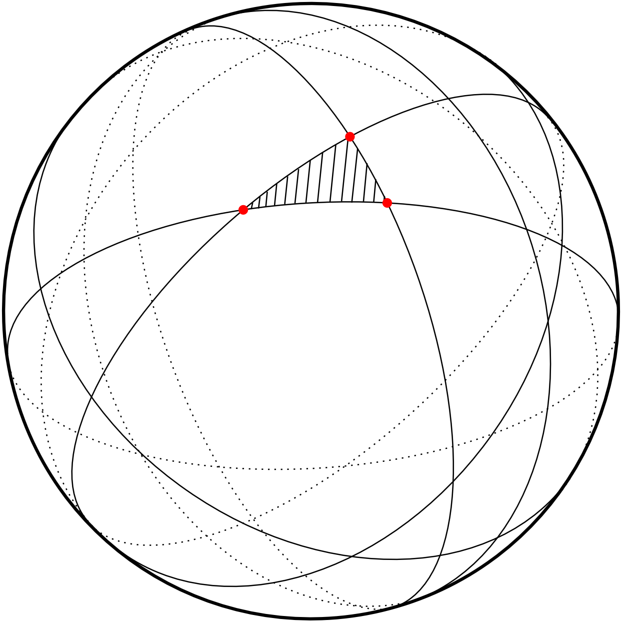

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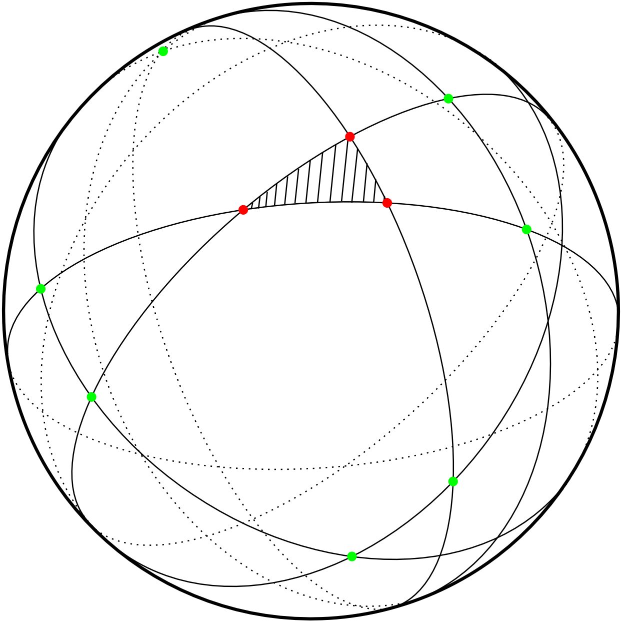

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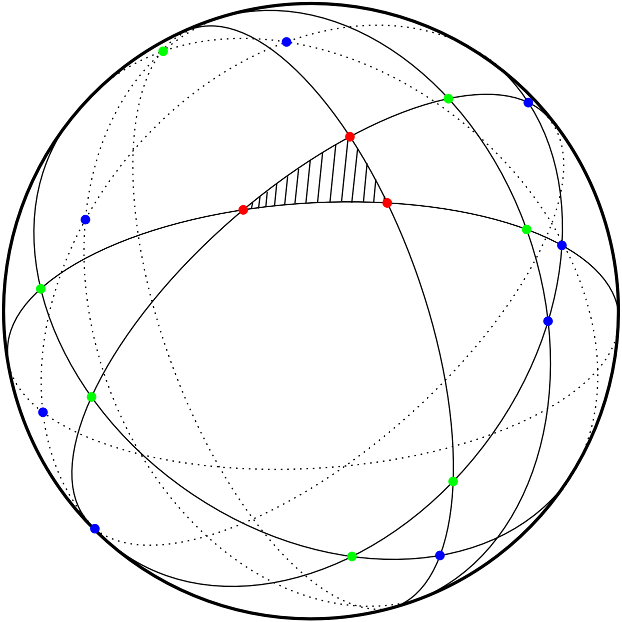

Averaging over all possible choices of cells

Theorem (CFSSSV '19)

If $\mathcal{C}$ is an arrangement of n great-circles, then the expected size of the k-level with respect to a random cell is $O(k^2)$.

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- Independent of n!
- Improves over worst-case bound for $k \ll n^{3/5}$

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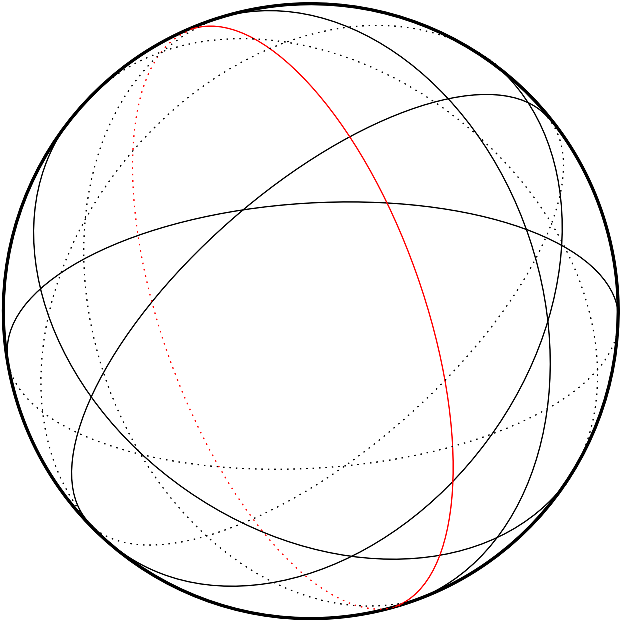
Goal: Show bound $O(k^2 n^2)$ on number of pairs

Counting pairs for a fixed hemisphere



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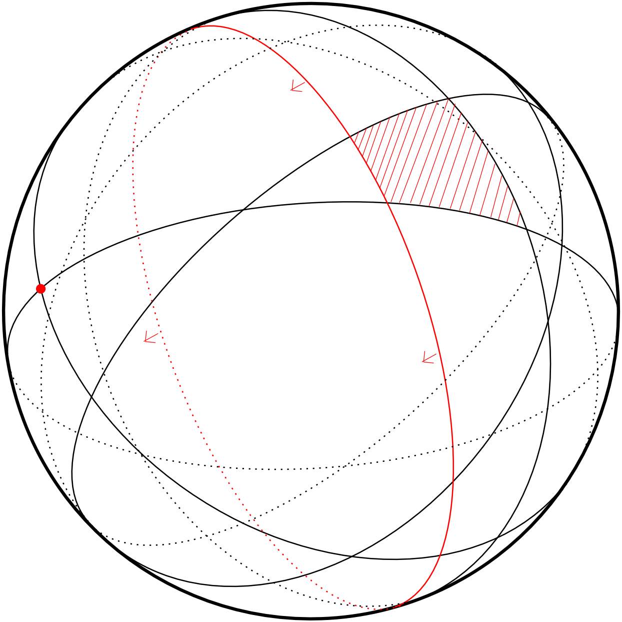
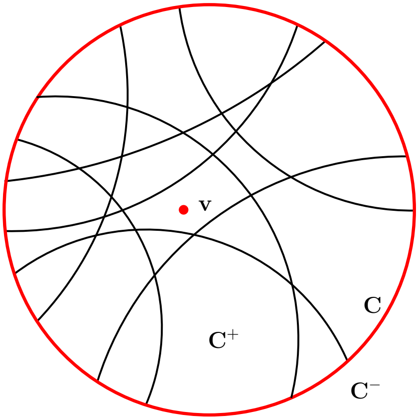


Counting pairs for a fixed hemisphere and vertex



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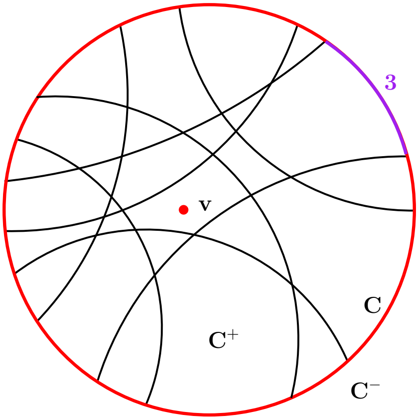

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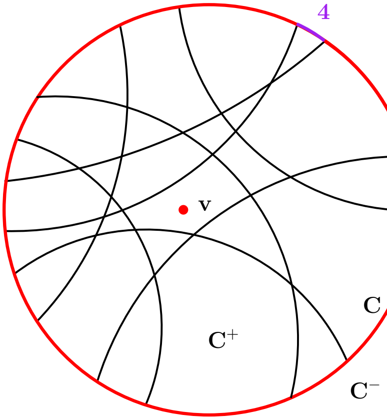

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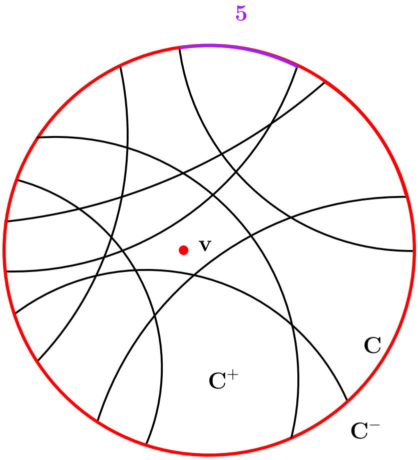

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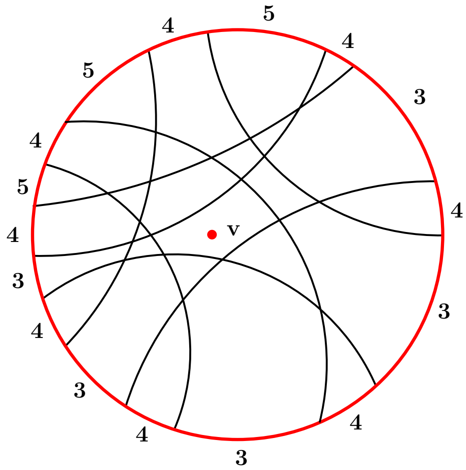

Lemma

The number of k-regions on C is $O(k)$.

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- Number of pairs (F, v) at distance k is $O(k^2 n) \cdot n = O(k^2 n^2)$.

Sampling great-circle arrangements on $\mathbb{S}^2$

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Problem

Is the expected complexity of the k-level for a random cell in an arrangement of great-circles $\Theta(k)$?

The End

Thank you for your attention.