# Smallest Universal Covers for Families of Triangles

Ji-won Park, Otfried Cheong

# Introduction

Def. A *universal cover* for a given family of objects is a *convex* set that contains a *congruent copy* of each object in the family. That is, translations, rotations, and reflections are allowed.

Aim. Given a family of objects, find a smllest universal cover, i.e., a universal cover of smallest *area*.

In general, finding a smallest universal cover is hard:

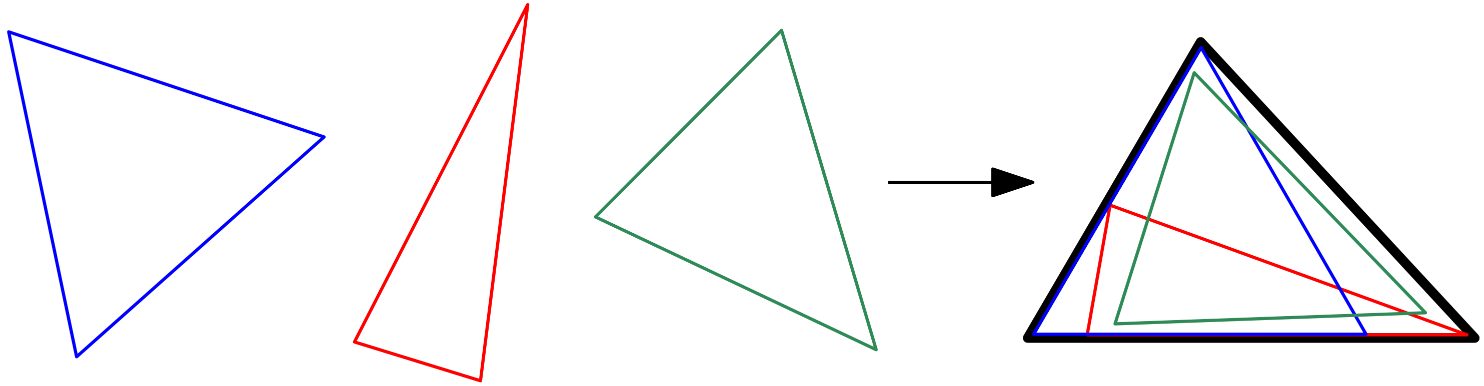
- Sets of unit diameter (a.k.a. Lebesgue's Universal Cover Problem)
- Unit curves
- Unit convex curves
- Sets of unit perimeter

# Introduction

Thm. The smallest universal cover for the family of all triangles of unit diameter is a triangle and it is unique. [K83]

Thm. The same is true for the family of all triangles of unit perimeter. [FW00]

Conj. For any family $\mathcal{T}$ of triangles of bounded diameter, there is a triangle $Z$ that is a smallest universal cover for $\mathcal{T}$.



# Results

Conj. For any family $\mathcal{T}$ of triangles of bounded diameter, there is a triangle $Z$ that is a smallest universal cover for $\mathcal{T}$.

Thm. For any two triangles, there is a triangle that is a smallest universal cover.

Thm. For triangles of unit circumradius, the unique smallest universal cover is a triangle.

Thm. There exist three triangles whose smallest universal cover is not determined by any two of them.

# Two Triangles

Thm. Let $S$ and $T$ be triangles. Then there is a triangle $Z$ that is a smallest universal cover for the family $\{S, T\}$.

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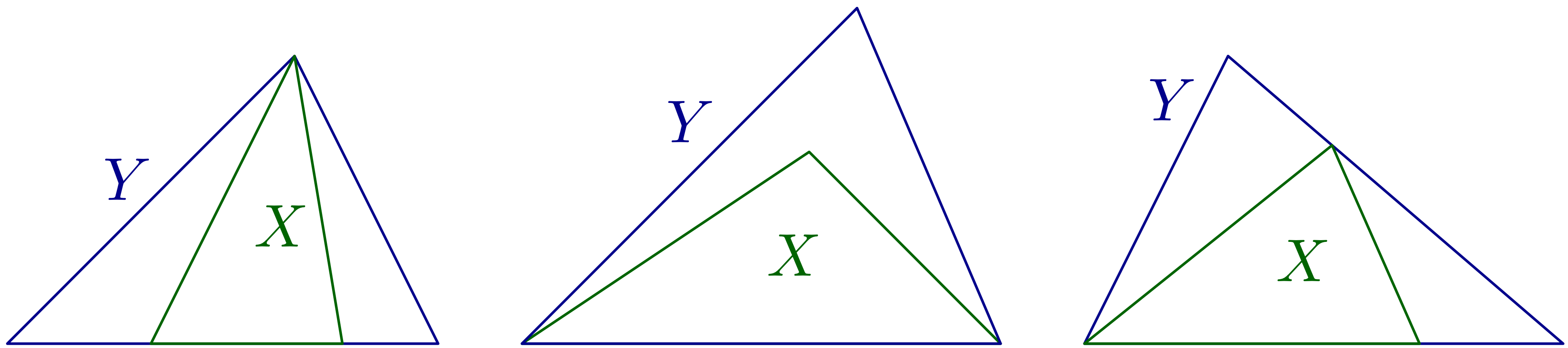
Pf. $S'$ = the smallest triangle similar to $S$ s.t. $T$ fits into $S'$.

If $S' = S$ done; otherwise:

Lemma. If a convex set $X$ maximally fits into a convex set $Y$, then there are at least four incidences between vertices of $X$ and edges of $Y$. [AAS98]



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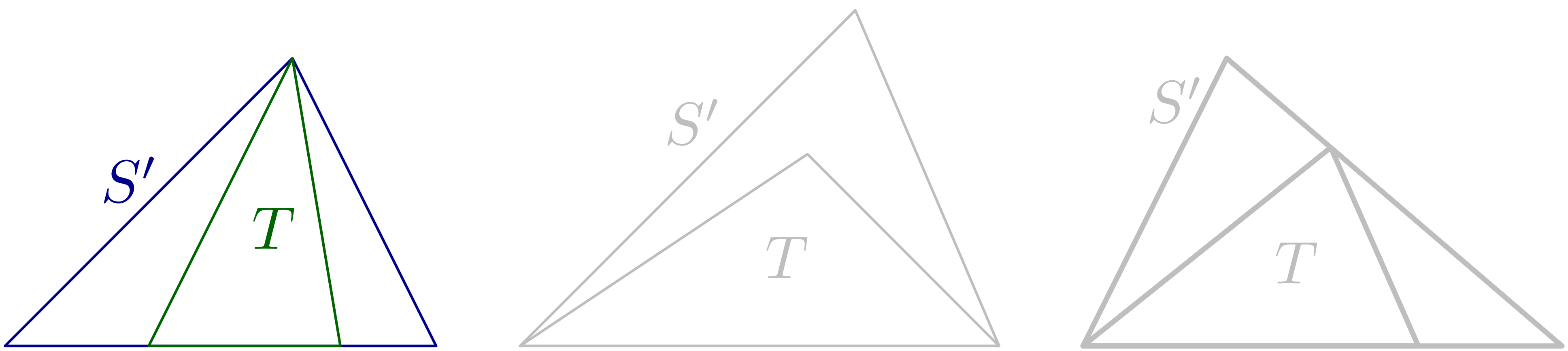


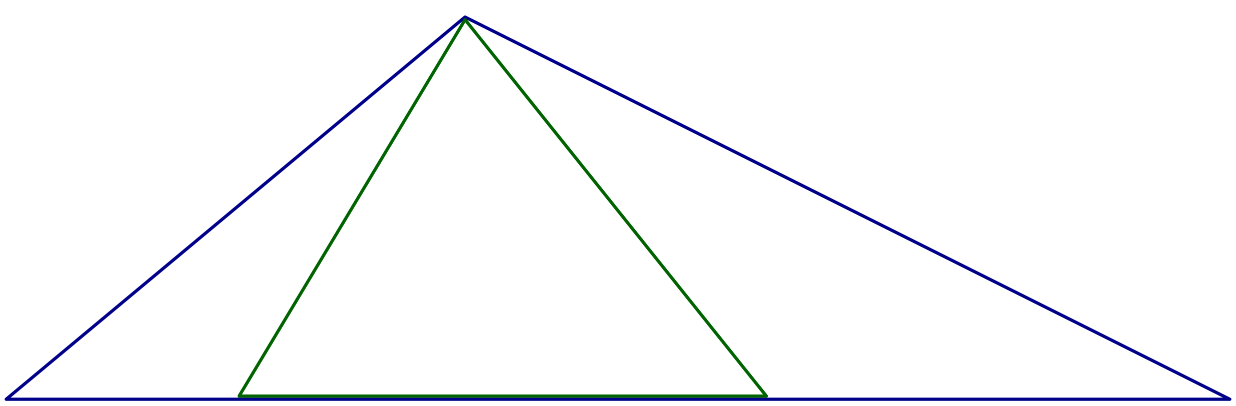

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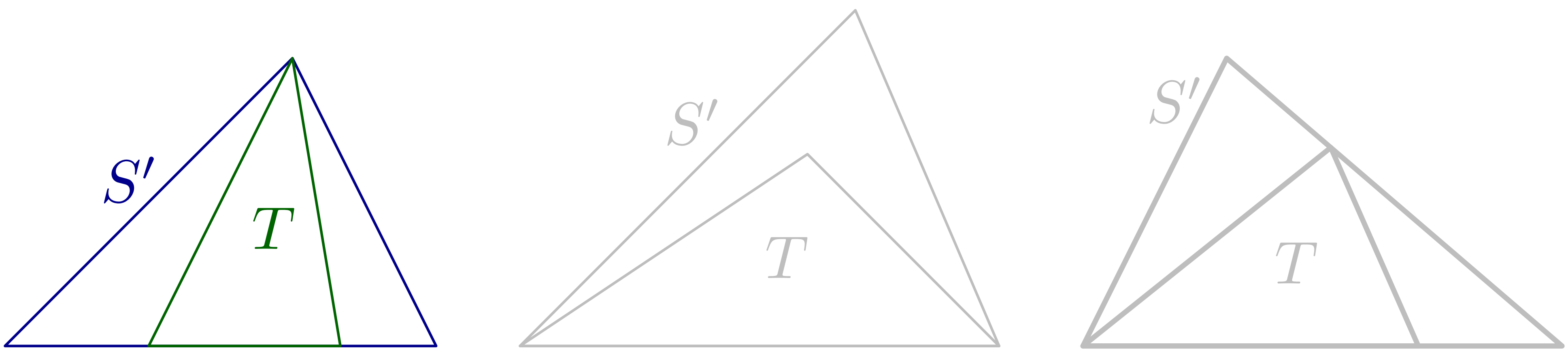


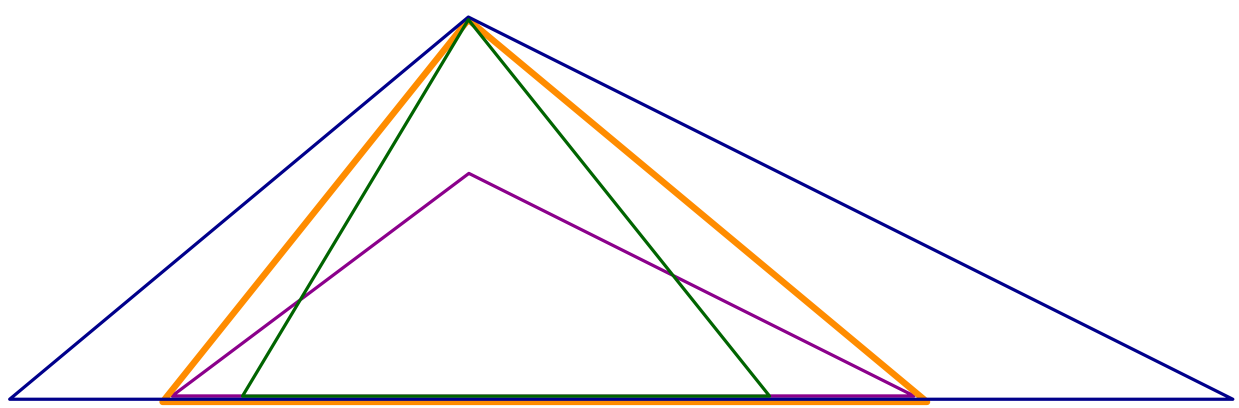

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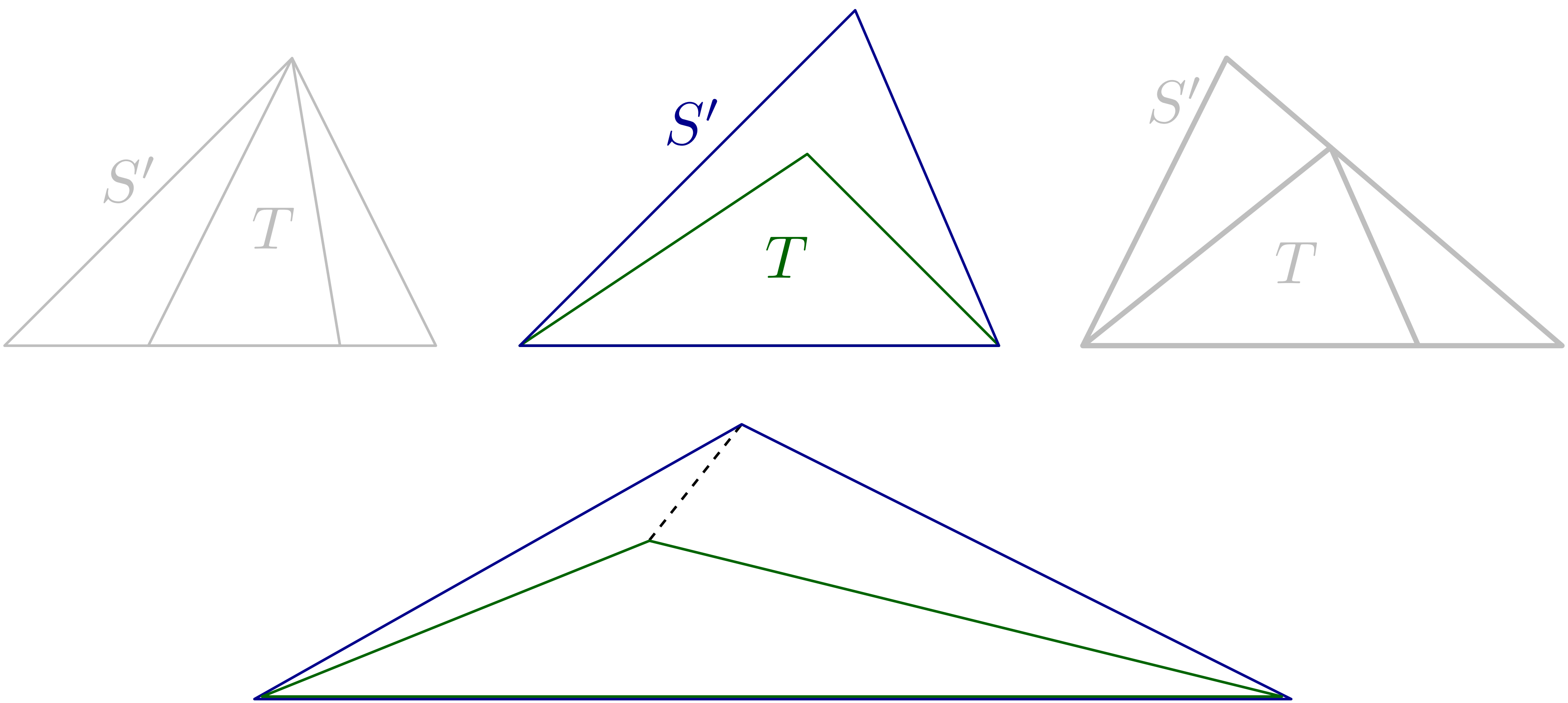

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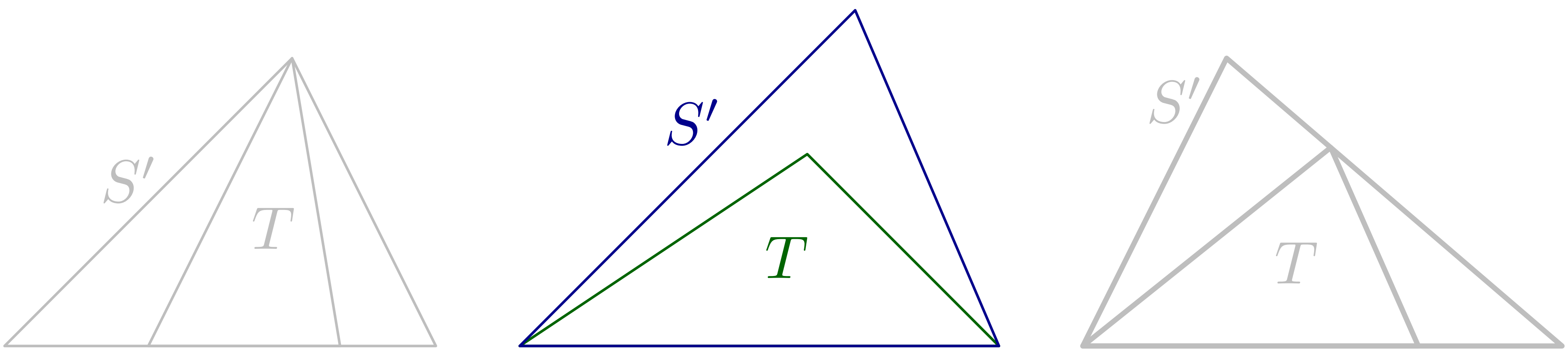


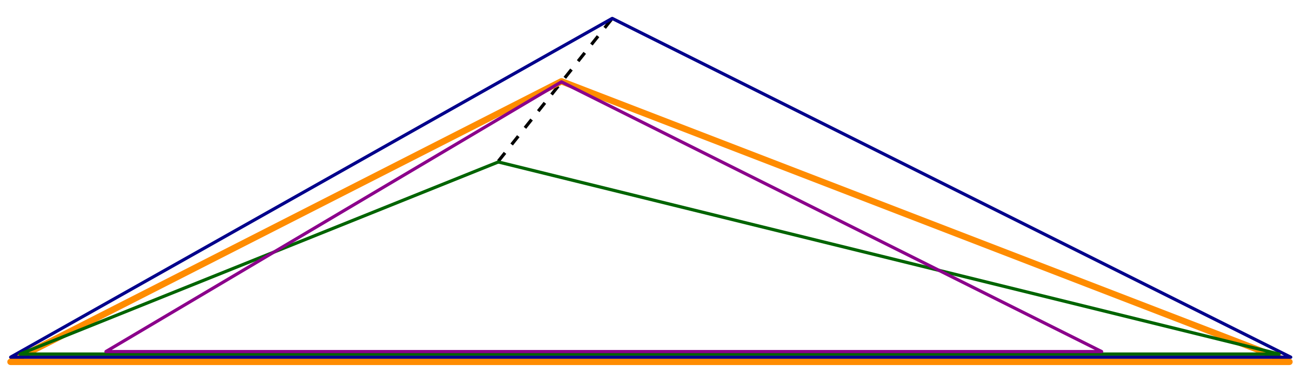

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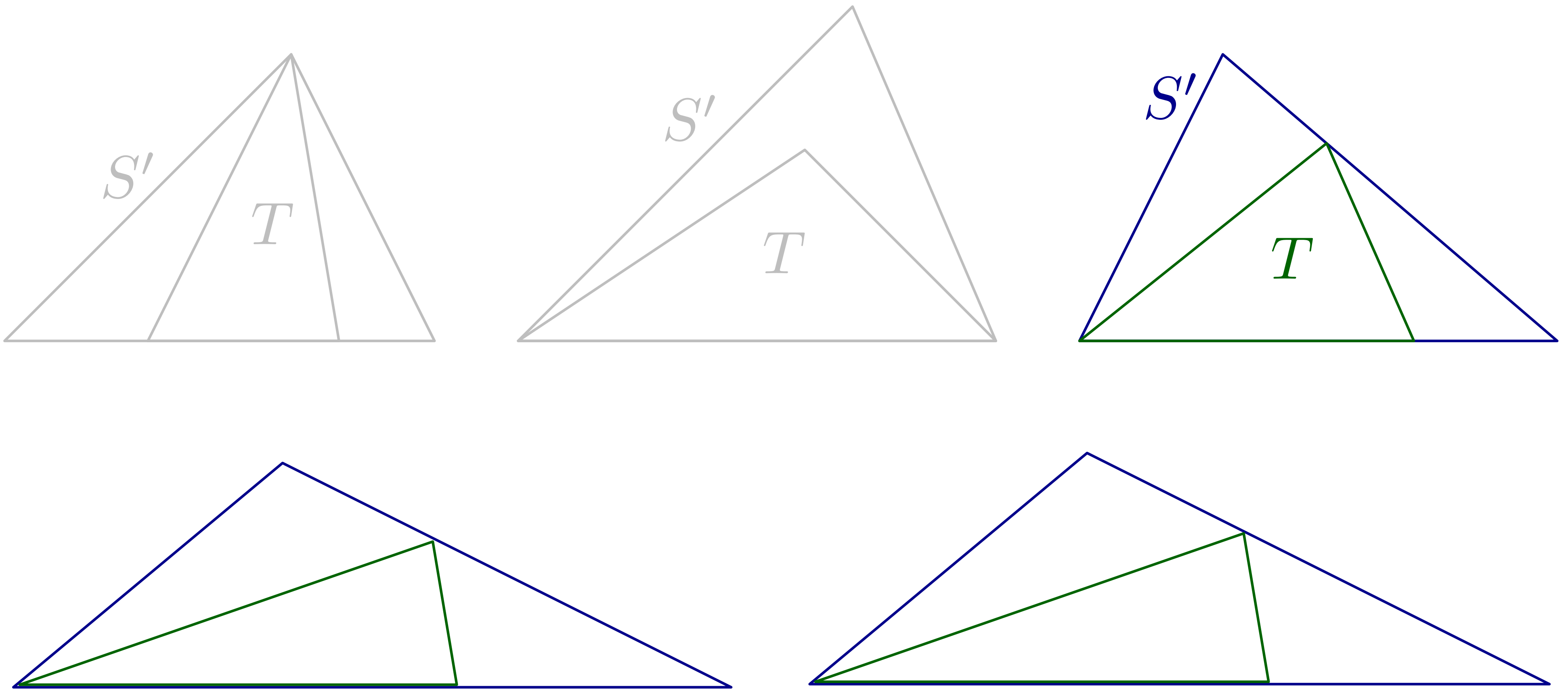

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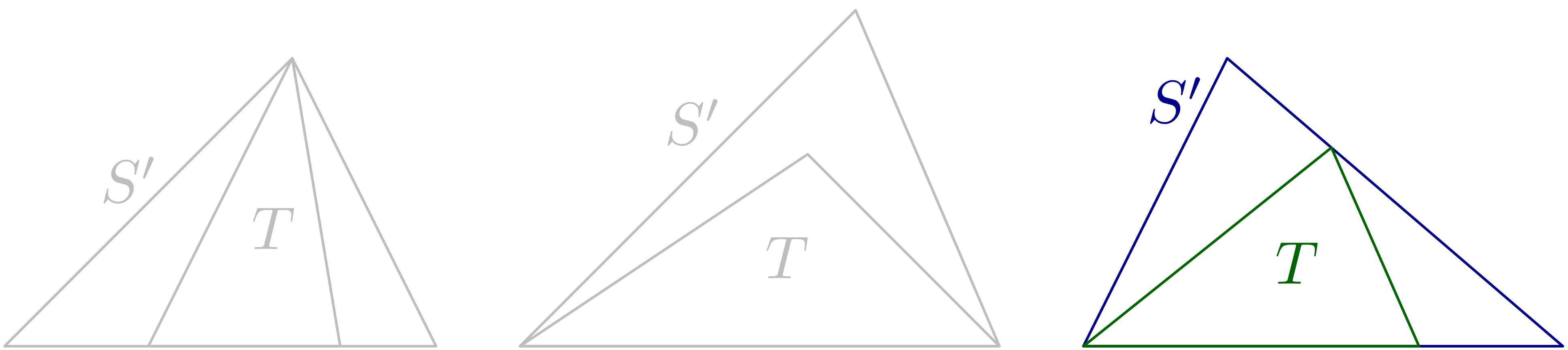


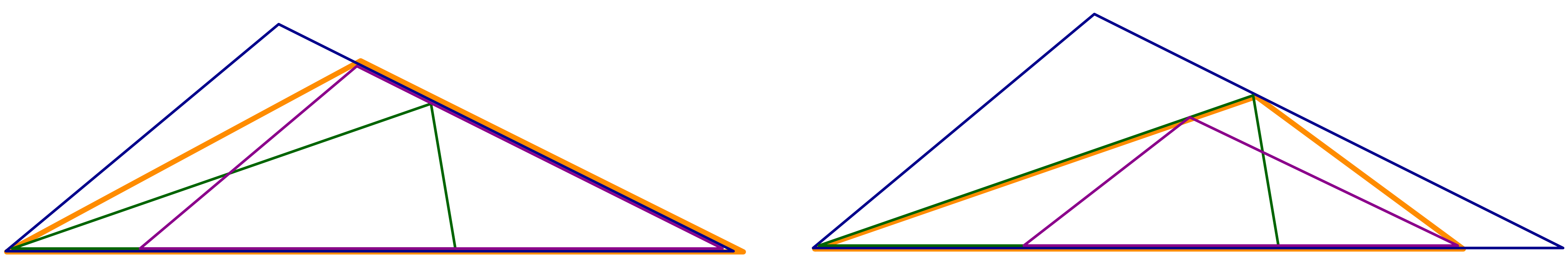

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Lemma. Let $\mathcal{T}$ be a family of triangles, and let $Z$ be a universal cover for $\mathcal{T}$. Let $S \in \mathcal{T}$, and let $S'$ be the smallest universal cover for $\mathcal{T}$ that is similar to $S$. If $\frac{|S'|}{|S|} = \left(\frac{|Z|}{|S|}\right)^2$, then $Z$ is a smallest universal cover for $\mathcal{T}$.

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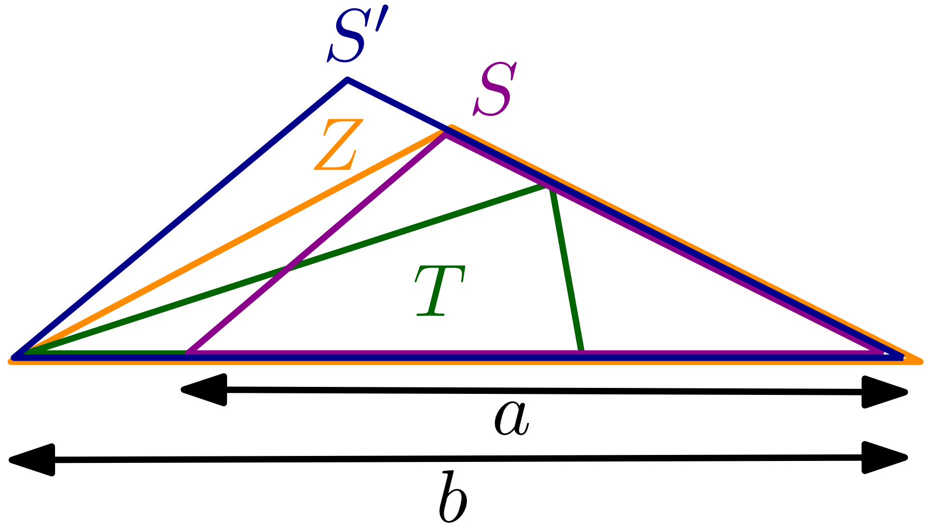


$$\frac{|S'|}{|S|} = \left(\frac{b}{a}\right)^2$$

$$\frac{|Z|}{|S|} = \frac{b}{a}$$

# Triangles of Unit Circumradius

$T_0 =$ the equilateral triangle (i.e., $T_1(60^\circ) = T_0$).

$T_1(\theta) =$ the isosceles triangle of base angle $\theta$.

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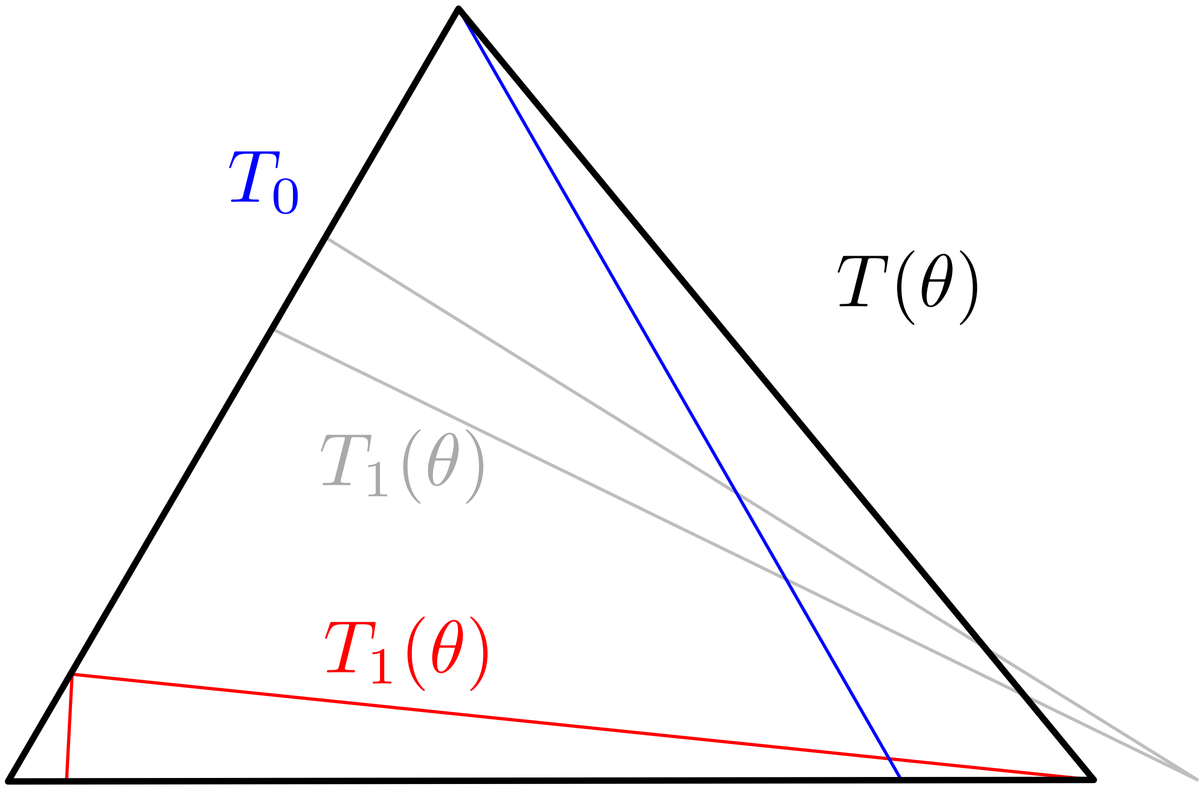
For some $75° < \theta_m < 80°$, if $60° \leq \theta \leq \theta_m$ or $\theta \geq 80°$:



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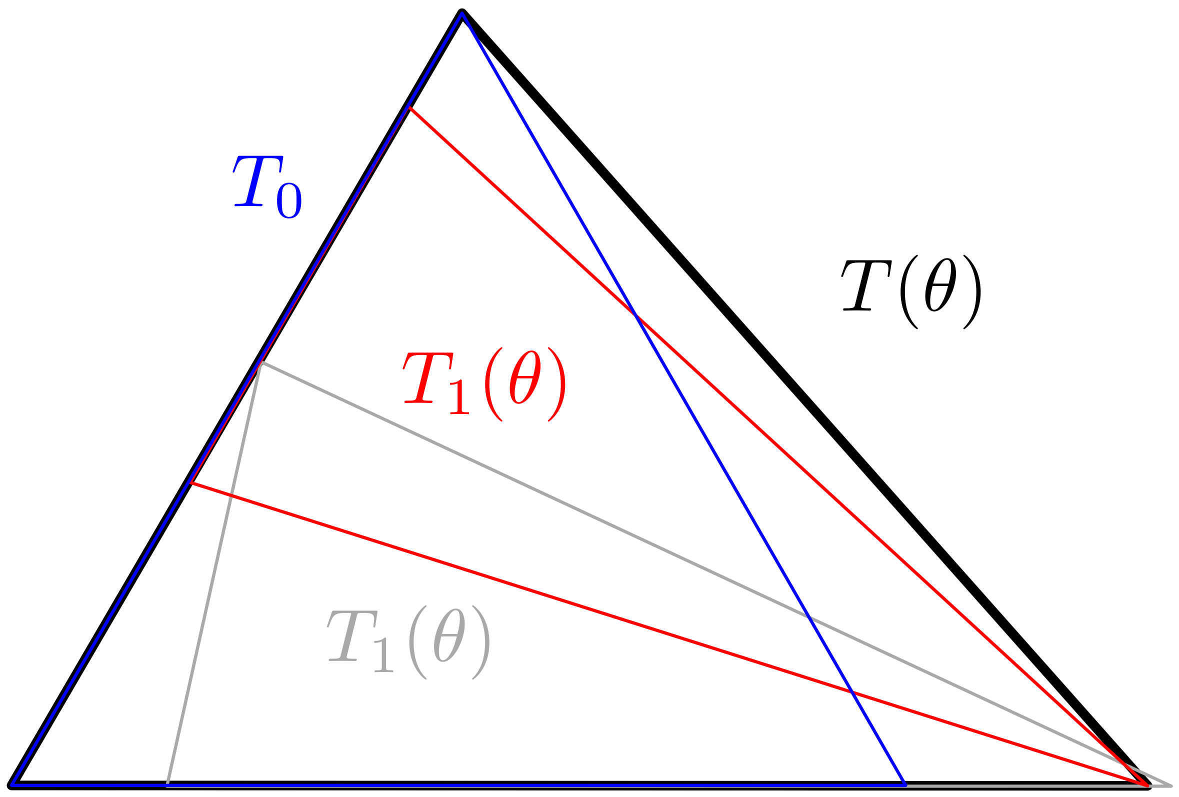

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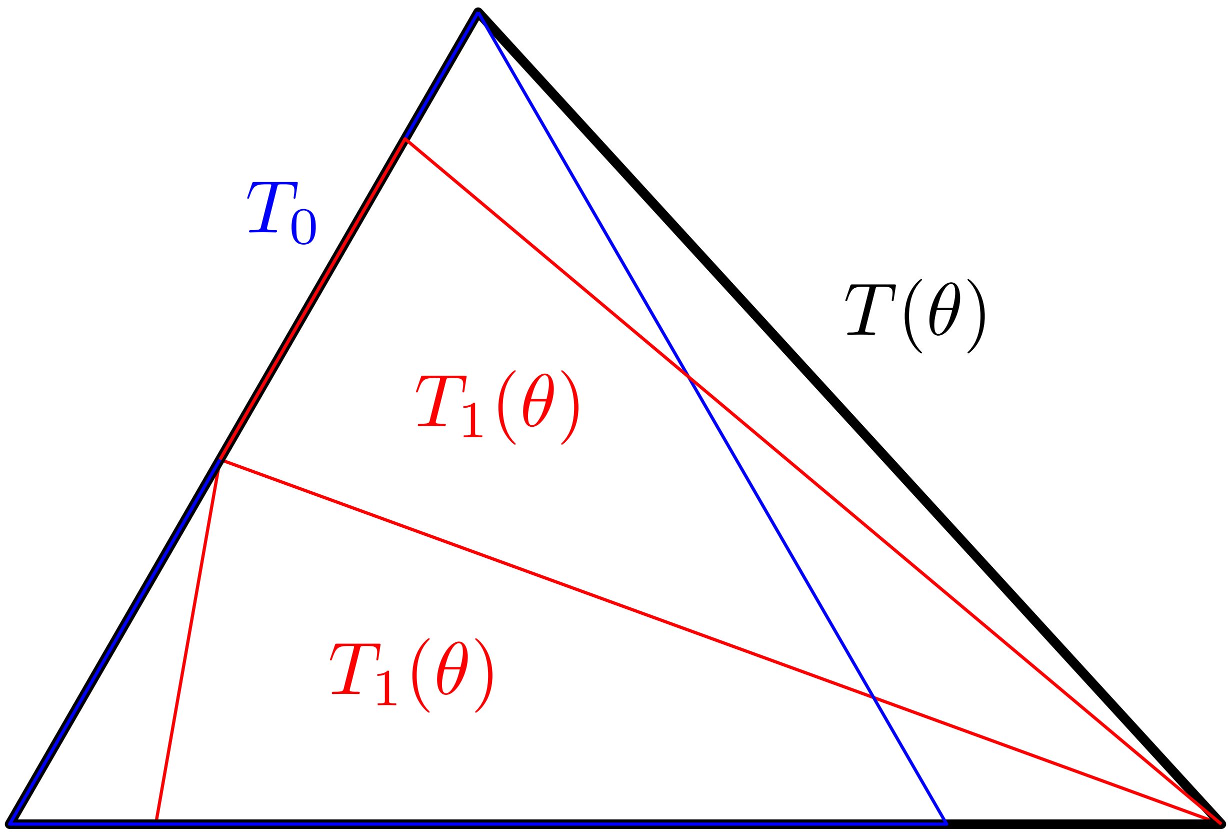
When $\theta = 80^\circ$:



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$T^* = T(80^\circ)$ is the largest one.
Thm. $T^*$ is the smallest universal cover for the family $\mathcal{T}$ of triangles of unit ciricumradius.

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Sketch. 1) $T^*$ covers every triangle of unit circumradius.

2) $T^*$ is a smallest universal cover for $\mathcal{T}$.
$\because$ $T^*$ is the smallest universal cover for $T_0$ and $T_1(80^\circ)$.

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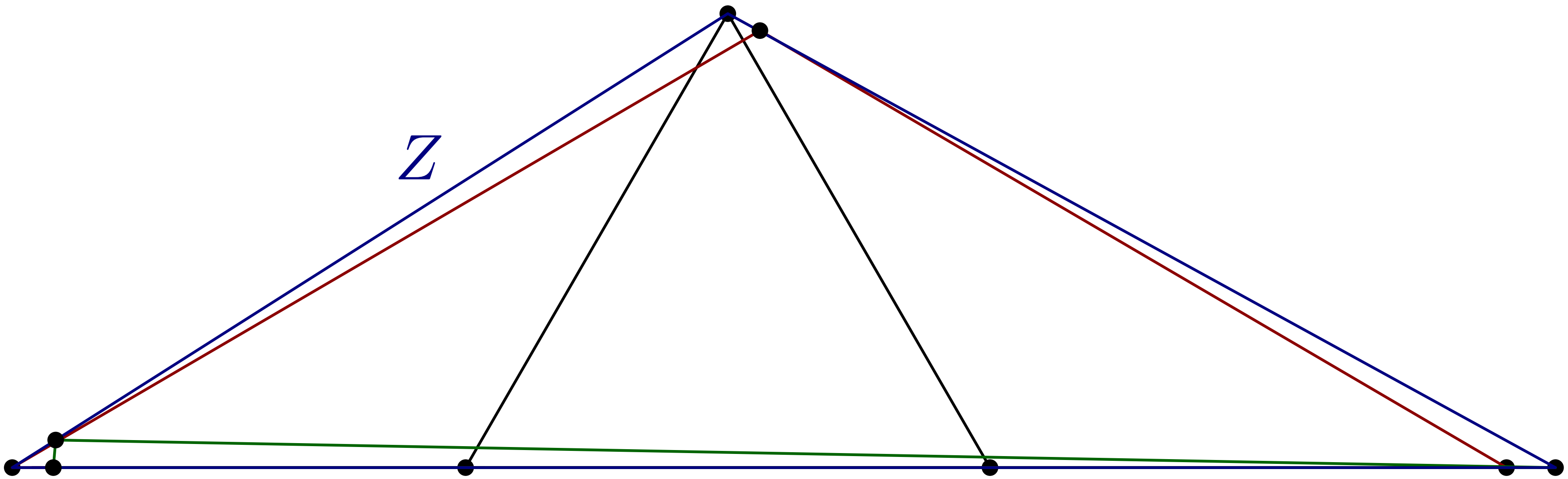
Sketch. 1) $T^*$ covers every triangle of unit circumradius.

2) $T^*$ is a smallest universal cover for $\mathcal{T}$.
$\because T^*$ is the smallest universal cover for $T_0$ and $T_1(80^\circ)$.

3) $T^*$ is the unique smallest universal cover for $\mathcal{T}$.
$\because$ Any smallest universal cover for $\{T_1(\theta)\}$ should be congruent to $T^*$.

# Three Triangles

Thm. There exist three triangles whose universal cover is not determined by any two of them.




Conj. $Z$ is the smallest universal cover for these three triangles.