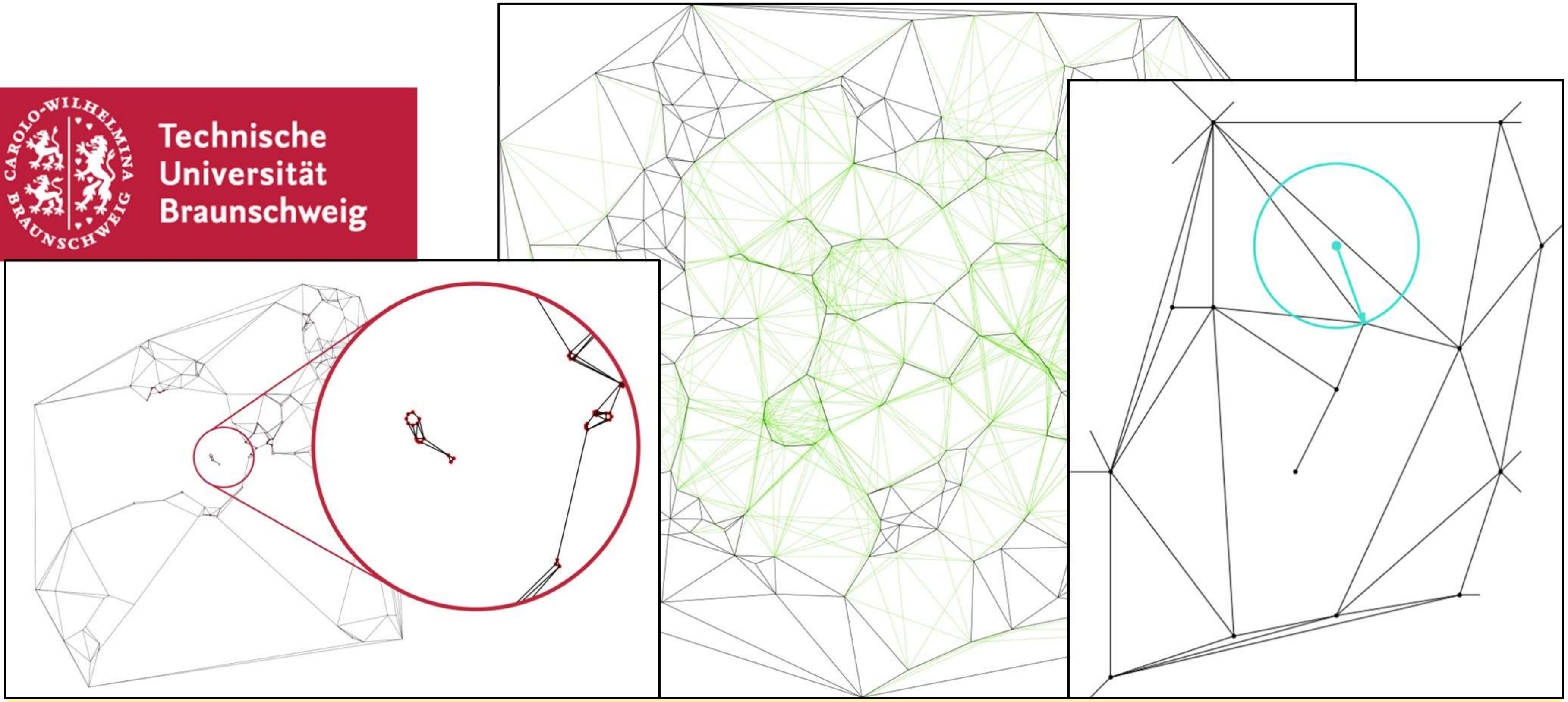


On Hard Instances of the Minimum-Weight Triangulation Problem

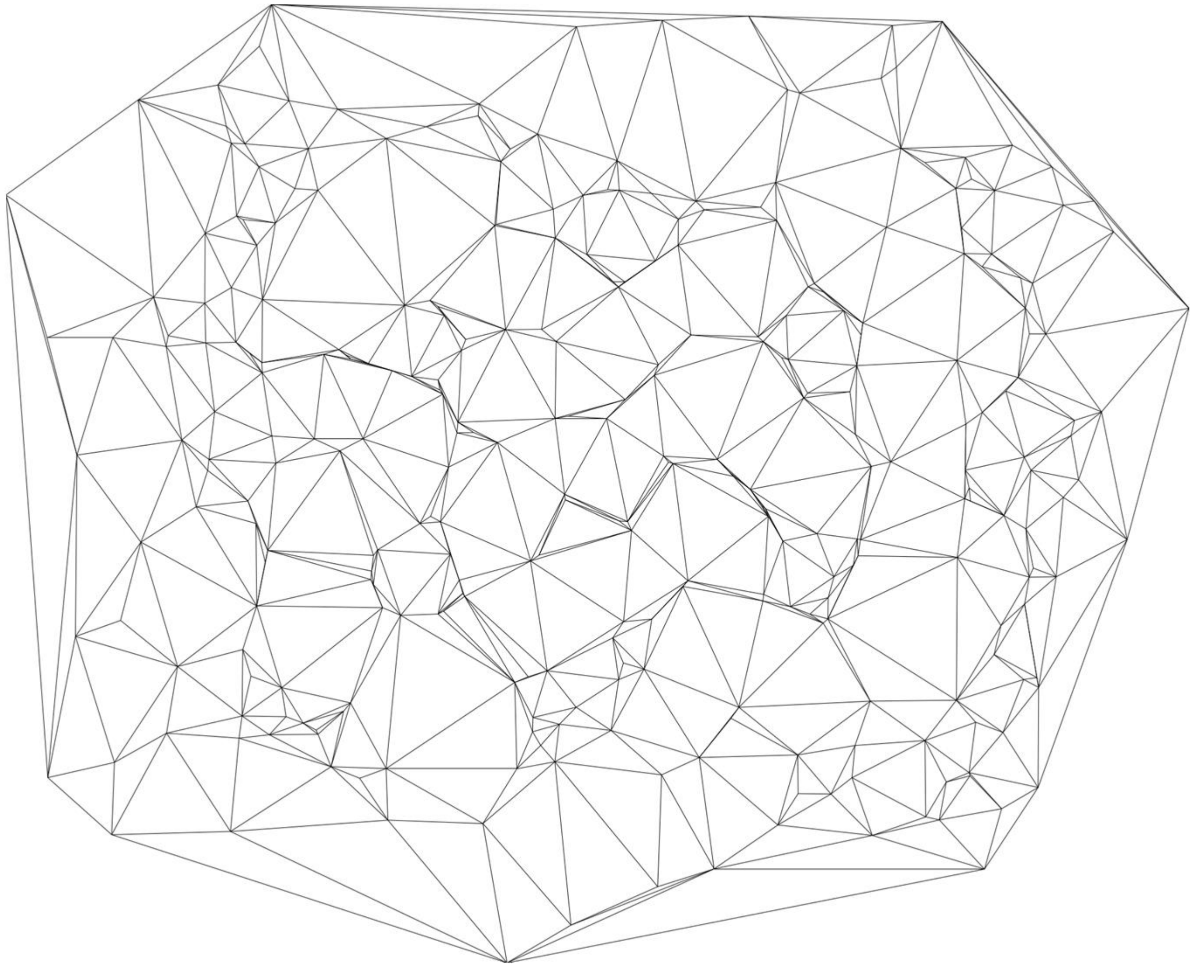
Sándor P. Fekete, Andreas Haas, Yannic Lieder, Eike Niehs, Michael Perk, **Victoria Sack**, and Christian Scheffer

Department of Computer Science
TU Braunschweig

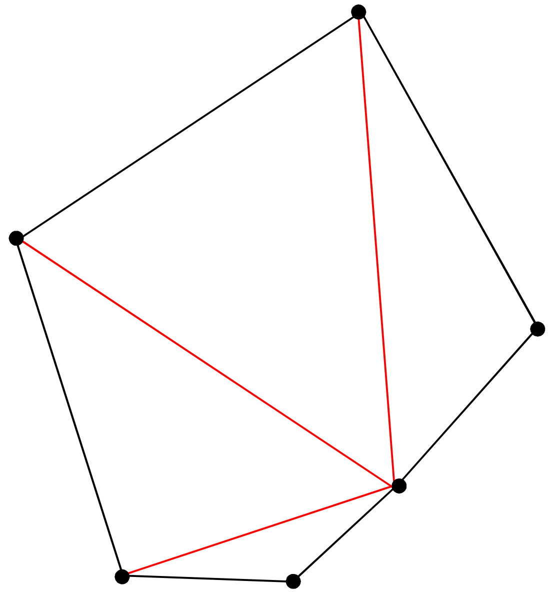
What is a Minimum-Weight Triangulation?

Given: A set P of n points in $\mathbb{R}^2$

Wanted: A triangulation T of P that minimizes the length of its edges

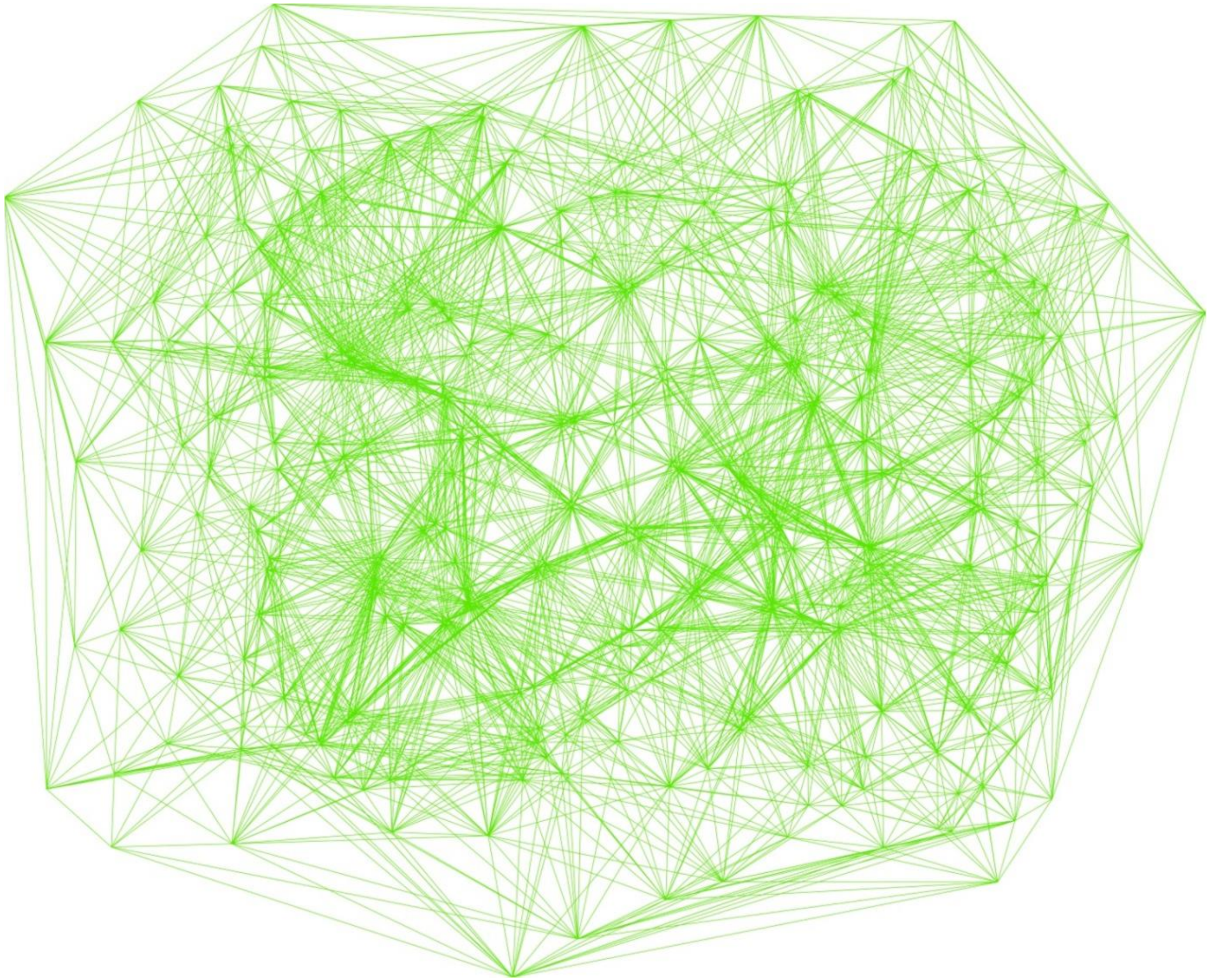


Solving an MWT-Instance Between 1979 and 2008



- 1979/80: simple polygons (DP)
  $O(n^3)$ [Gil79,Kli80]

Solving an MWT-Instance Between 1979 and 2008



- 1979/80: simple polygons (DP)
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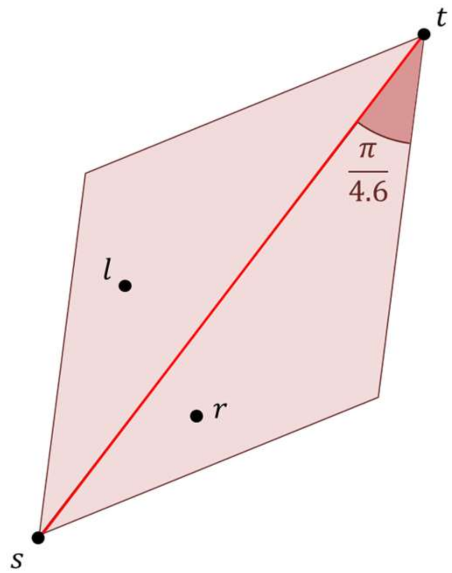
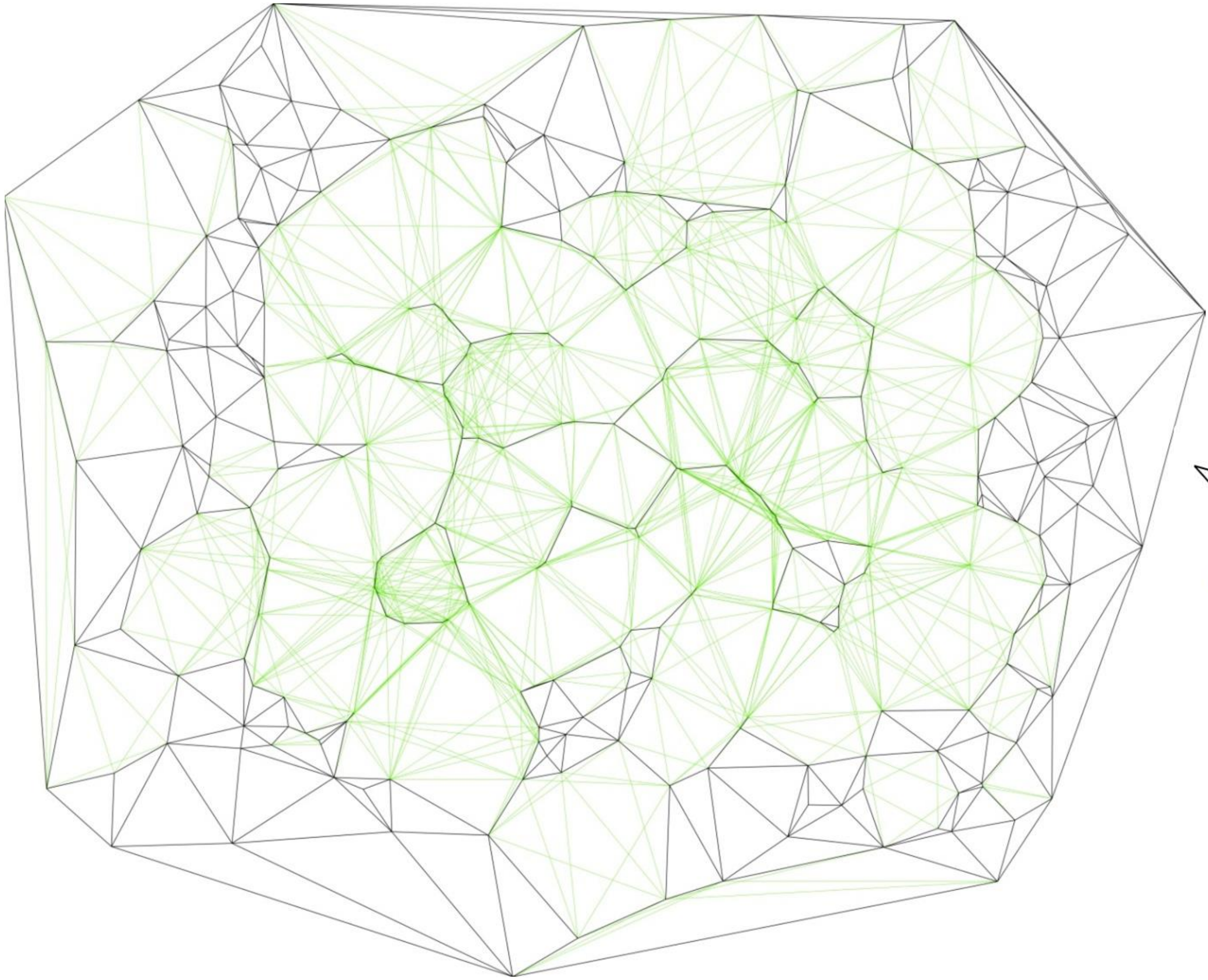
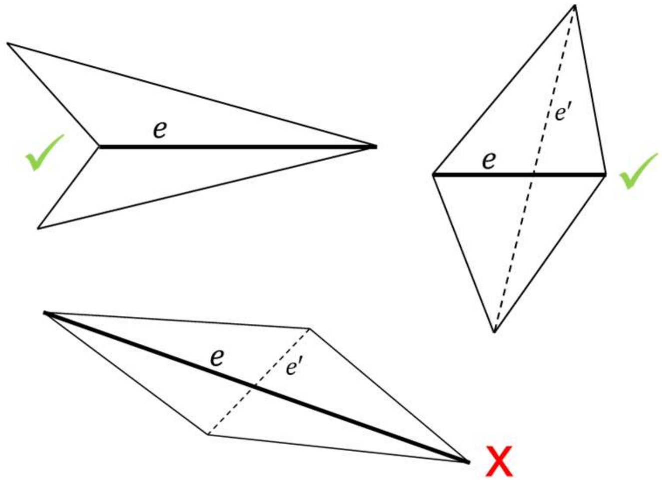
- 1989: diamond property
  [DJ89, DMS01]



Solving an MWT-Instance Between 1979 and 2008



- 1979/80: simple polygons (DP) $O(n^3)$ [Gil79,Kli80]
- 1989: diamond property [DJ89, DMS01]
- 1997: LMT-skeleton [DM96, DKM97]



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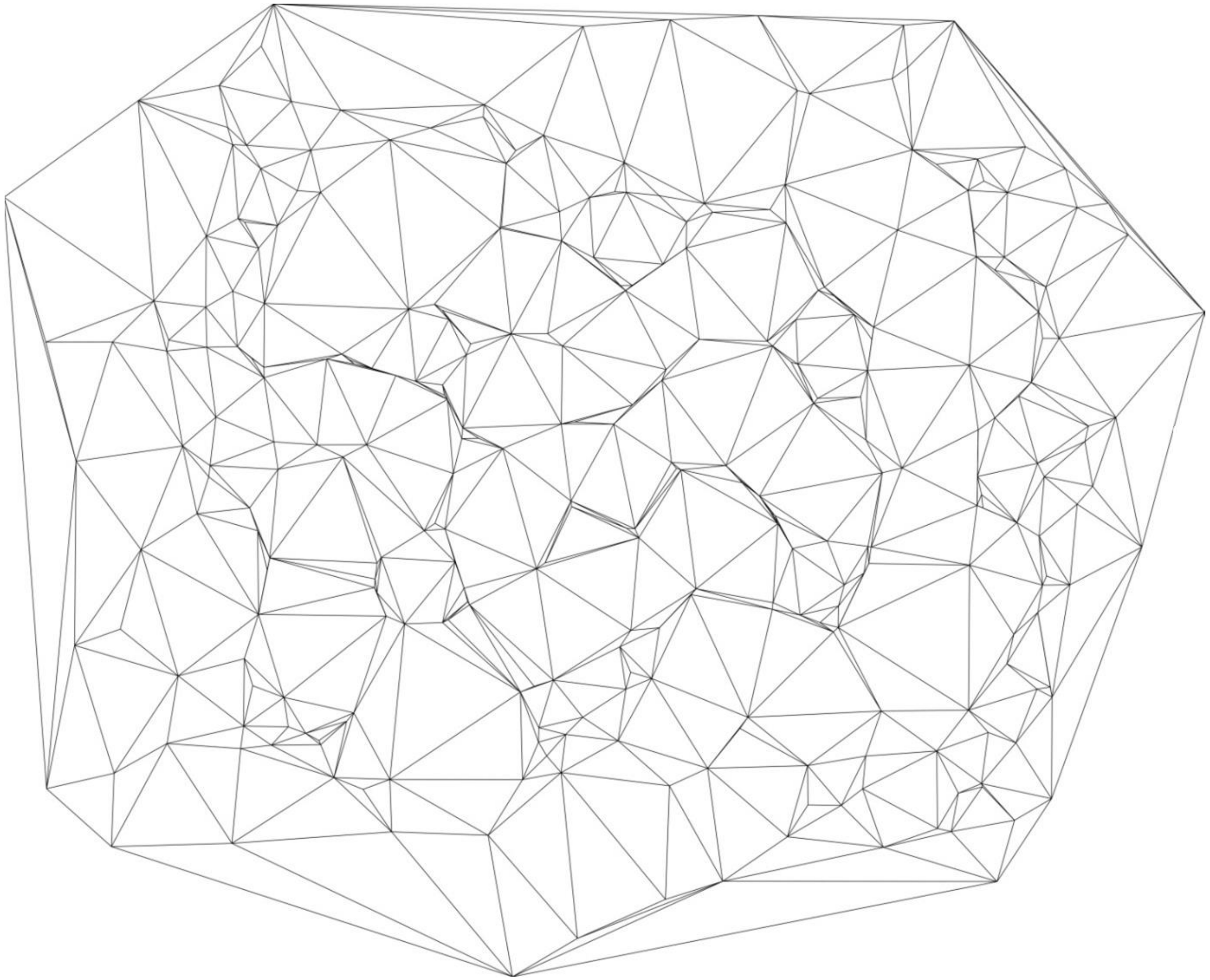
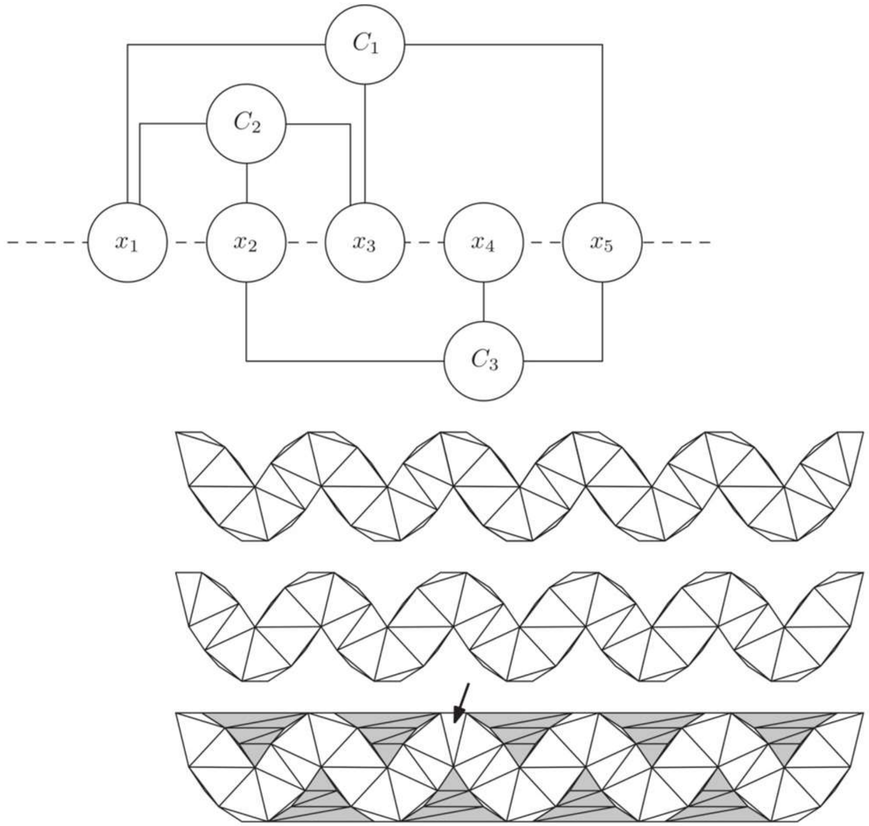
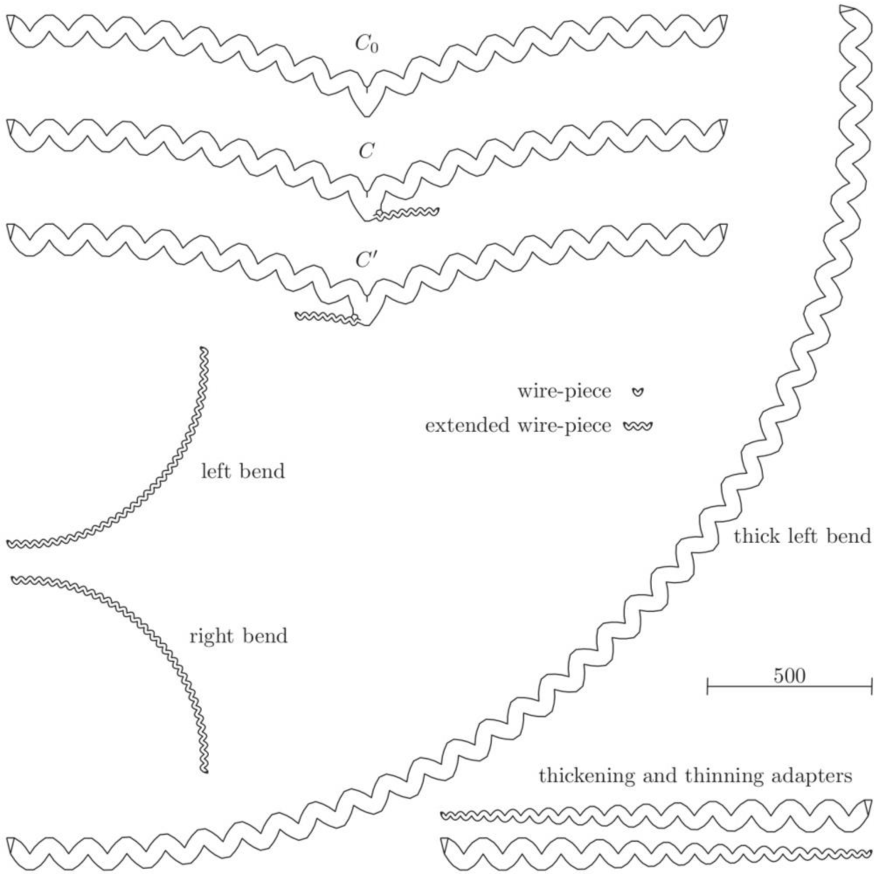


- 1979/80: simple polygons (DP) $O(n^3)$ [Gil79,Kli80]
- 1989: diamond property [DJ89, DMS01]
- 1997: LMT-skeleton [DM96, DKM97]
- 2005/06: DP for polygons with k inner points
  - $O(6^k n^5 \log n)$ [HO06]
  - $O(n^4 4^k k)$ [GBL05a]
  - $O(n^3 k!\, k)$ [GBL05b]

MWT is NP-Hard

- 2008: Proof by Mulzer and Rote [MR08]
- Reduction from PLANAR 1-IN-3-SAT

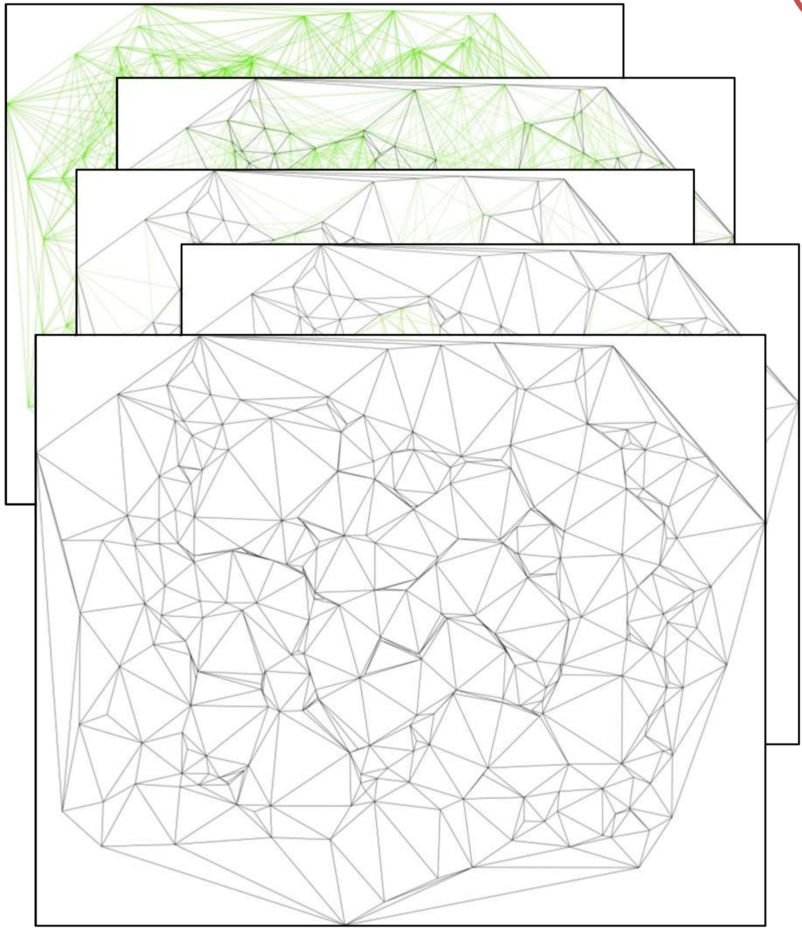




all images from [MR08]

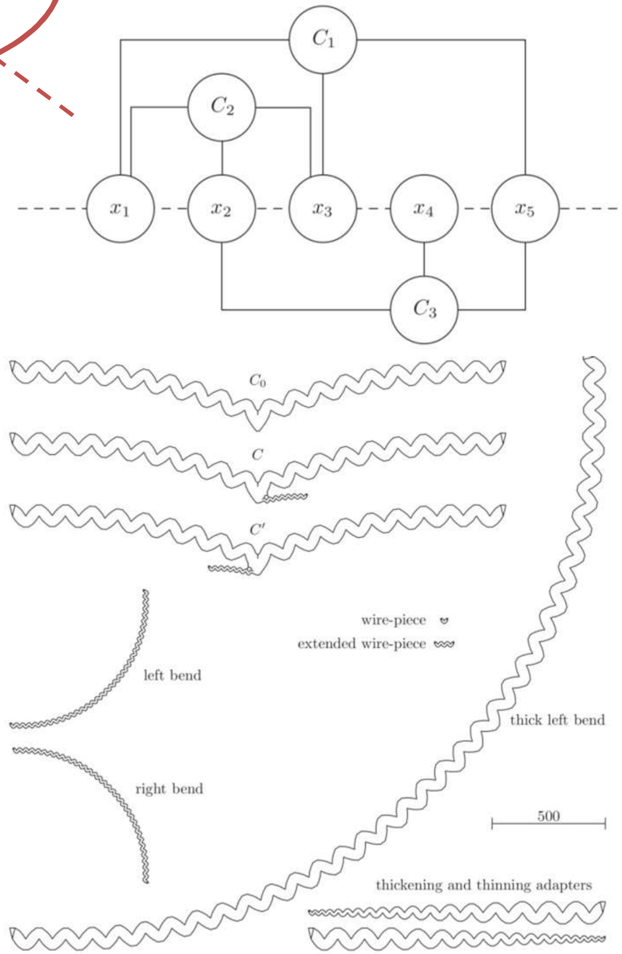
The Gap

hard SAT-instances
→ millions of points
→ hardware as
limiting factor



triangulate up to 30 Mio points in < 4min [Haa18]

?



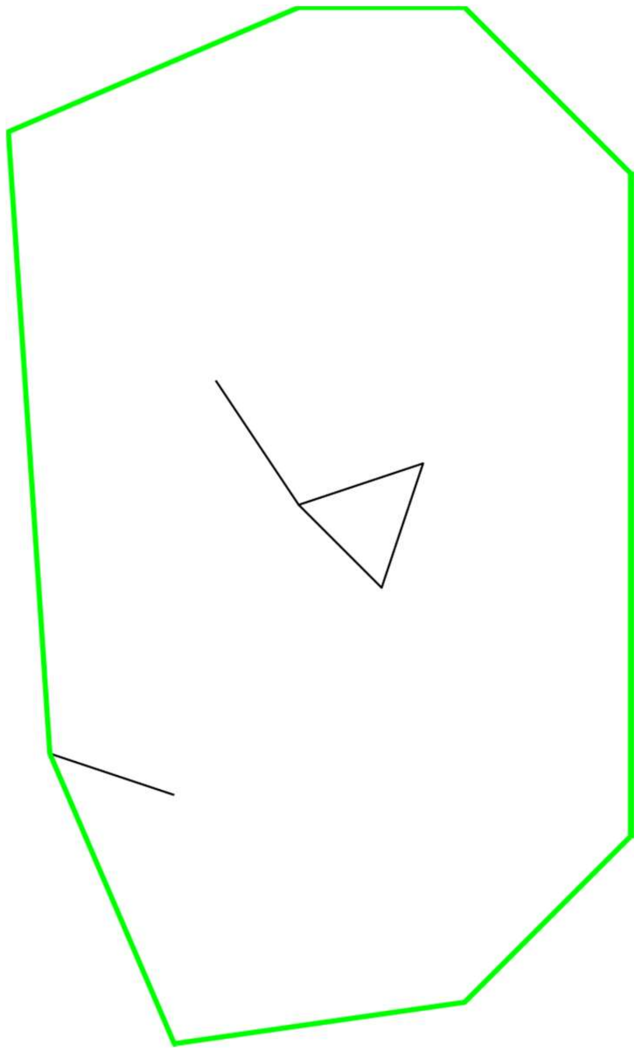
MWT is NP-hard

The IP

$$\min_{x_\triangle} \sum_{\triangle} ||\triangle|| \cdot x_\triangle$$

s.t.

$\sum_{\triangle \in \delta(e)} x_\triangle = 1$ $\forall e \in$ boundary component



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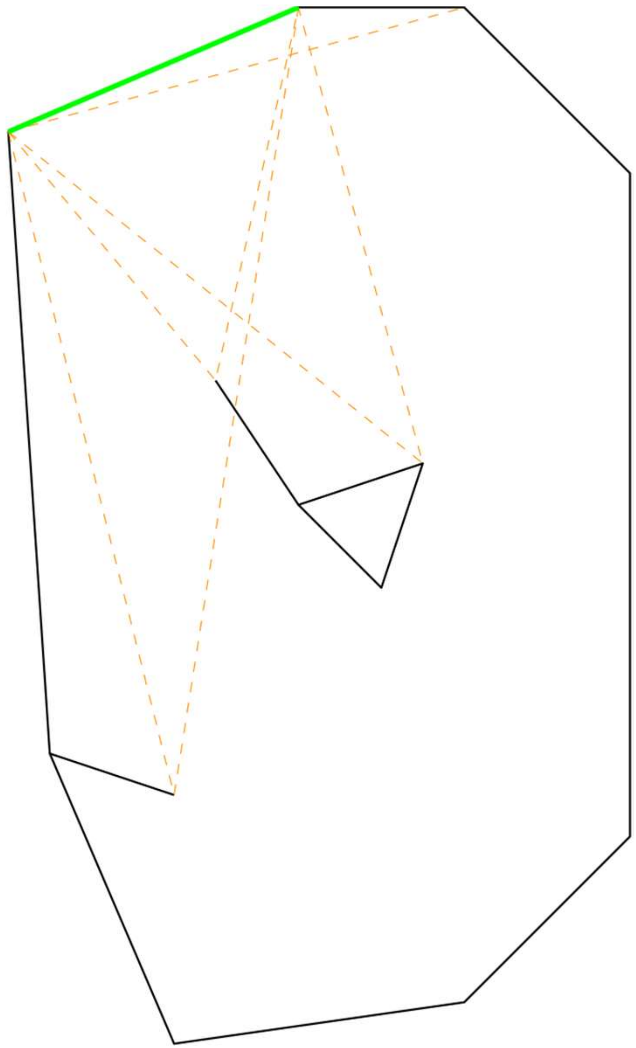

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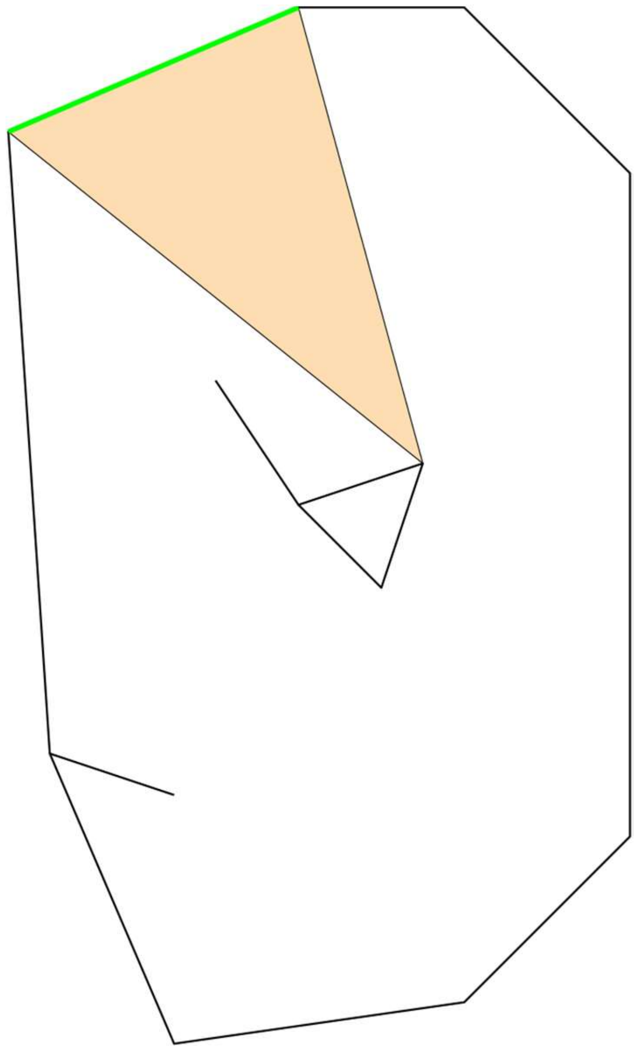

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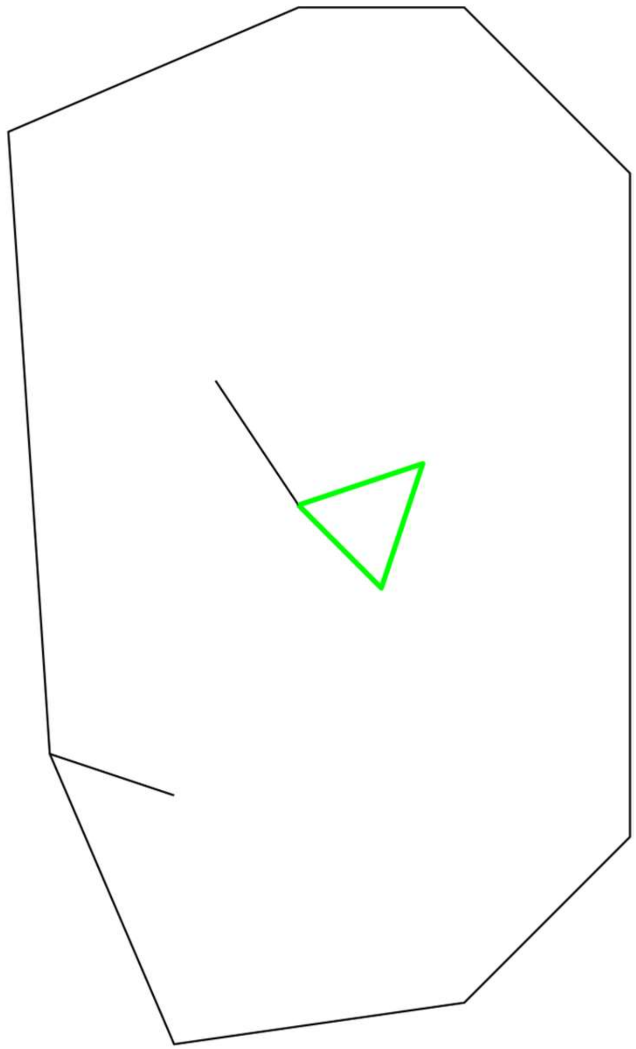

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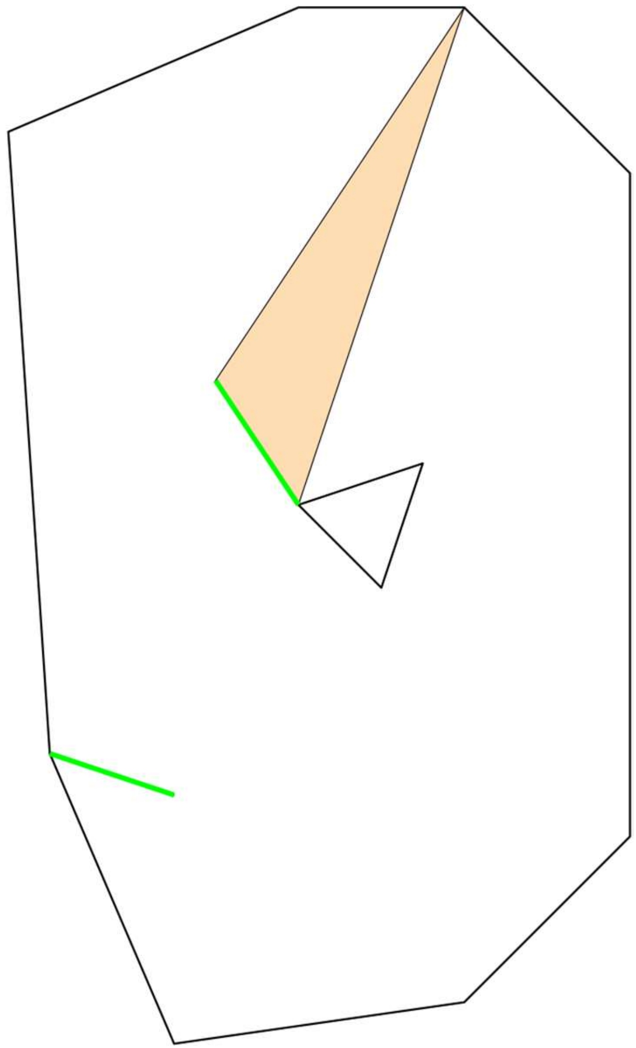
$$\left.\begin{array}{l} \sum_{\triangle \in \delta^+(e)} x_\triangle = 1 \\ \sum_{\triangle \in \delta^-(e)} x_\triangle = 1 \end{array}\right\}$$ $\forall e \in$ antennas



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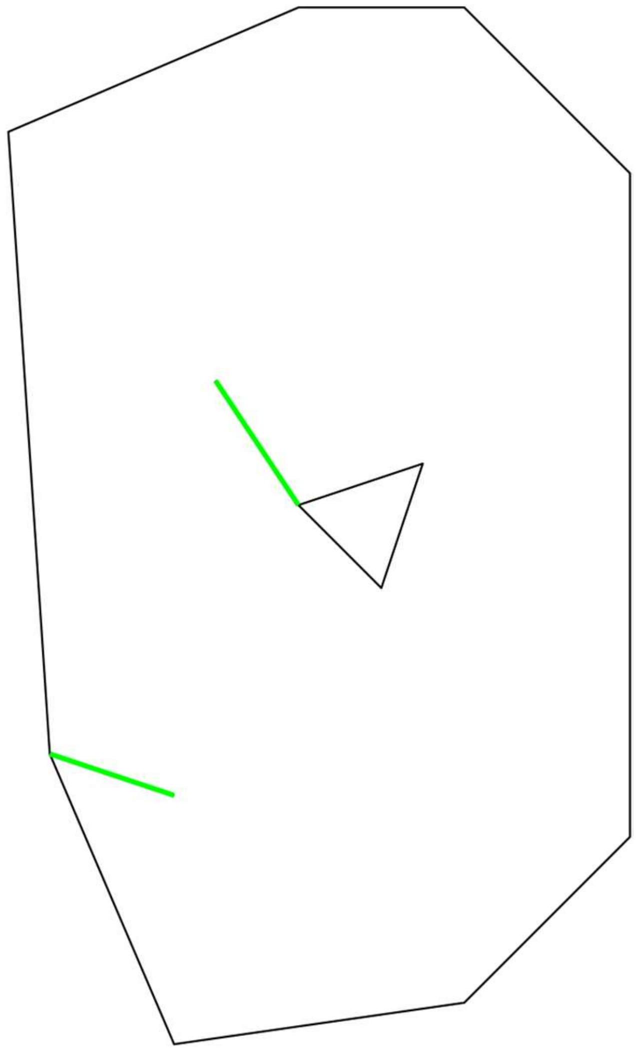

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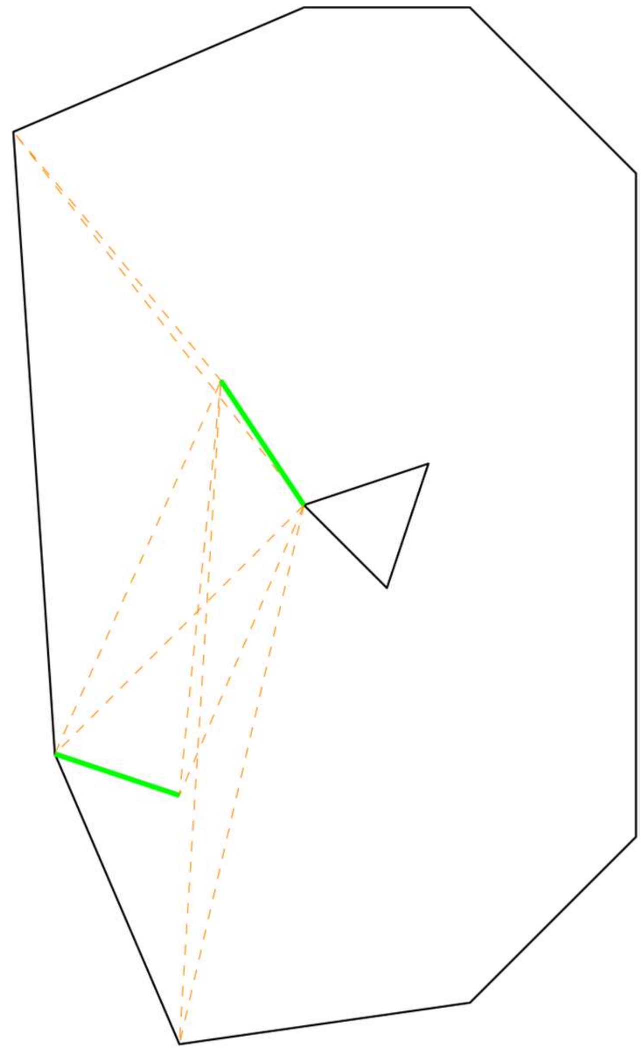

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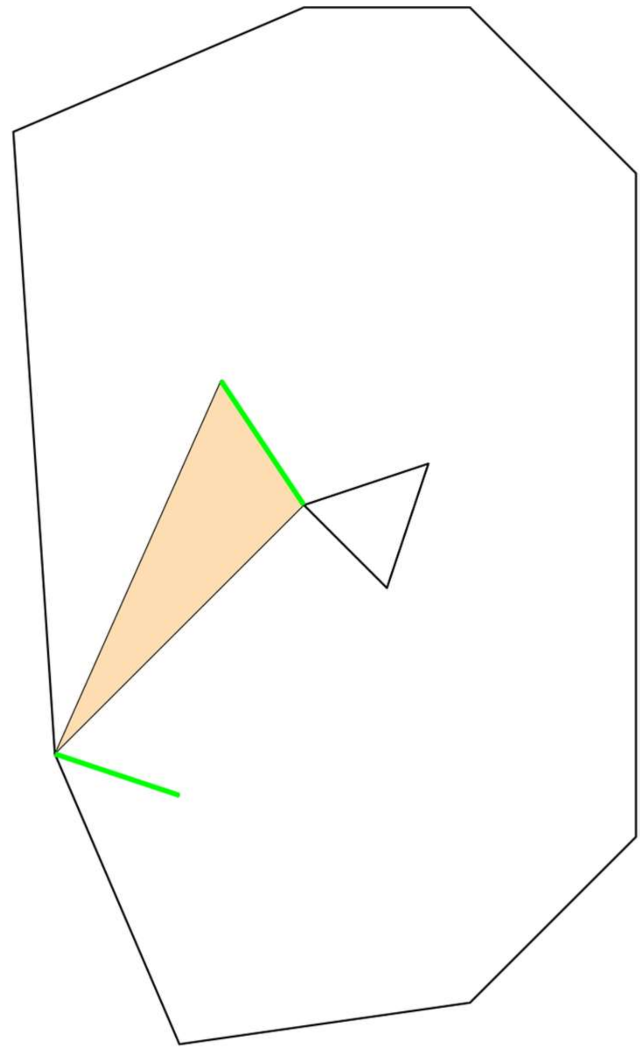

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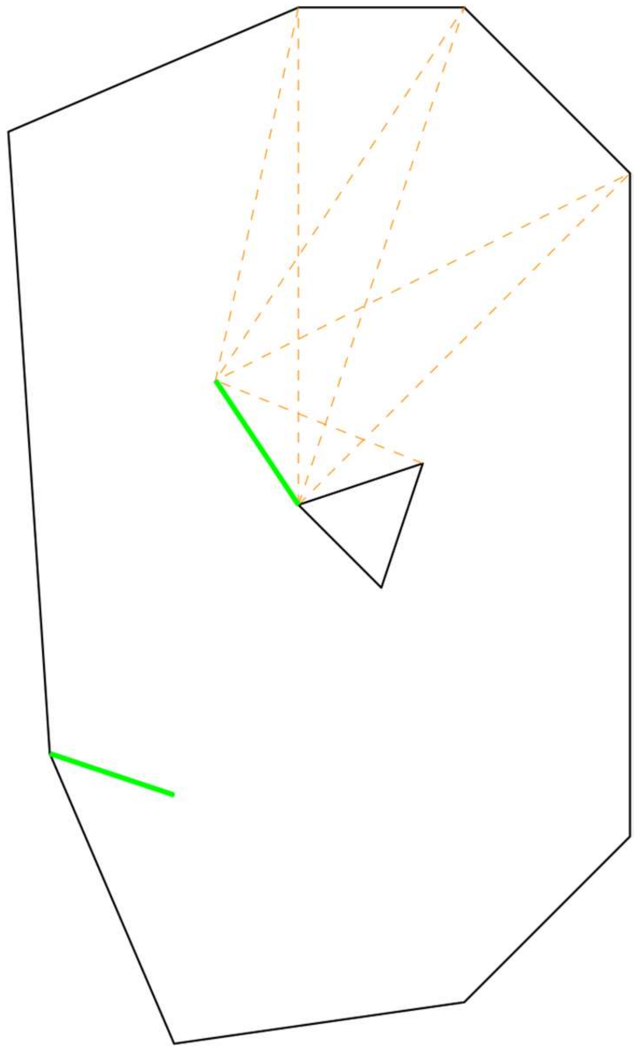

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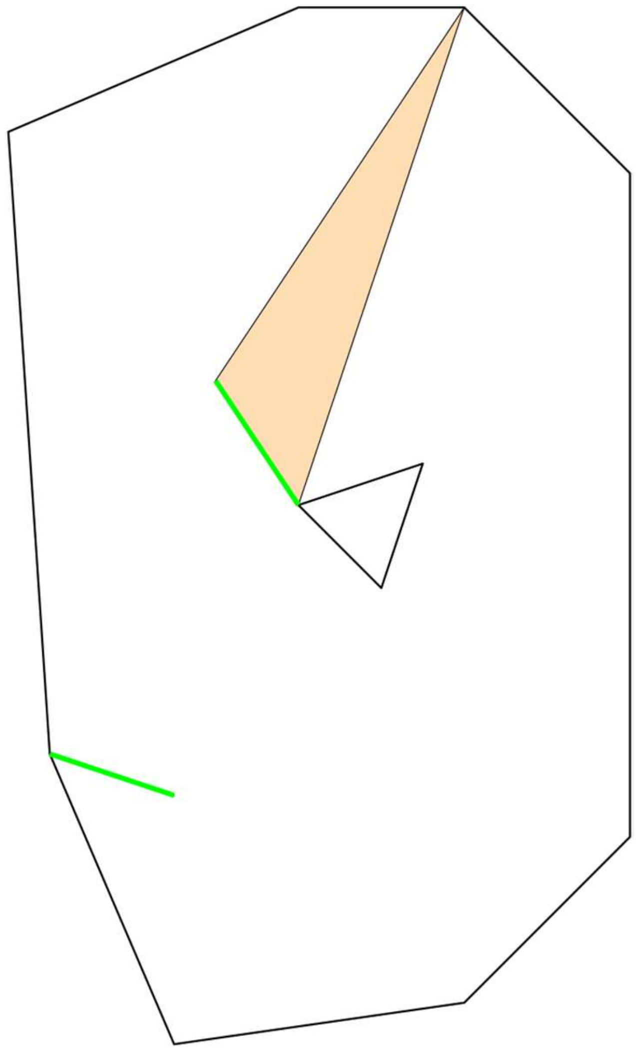

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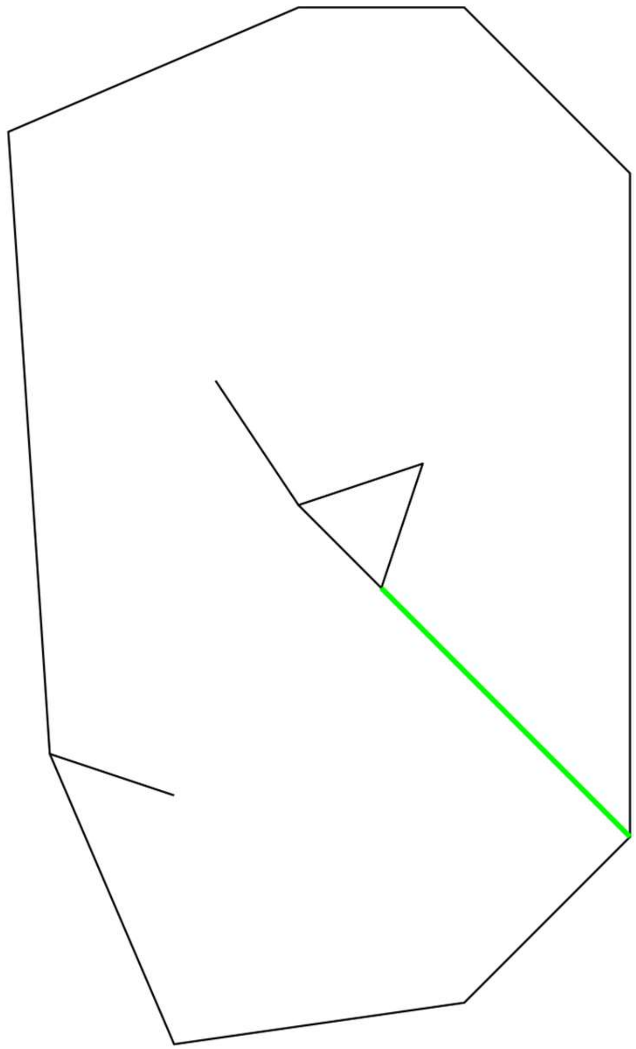
$\sum_{\triangle \in \delta^+(e)} x_\triangle - \sum_{\triangle \in \delta^-(e)} x_\triangle = 0$ $\forall e \in$ inner edges

$x_\triangle \in \{0,1\}$



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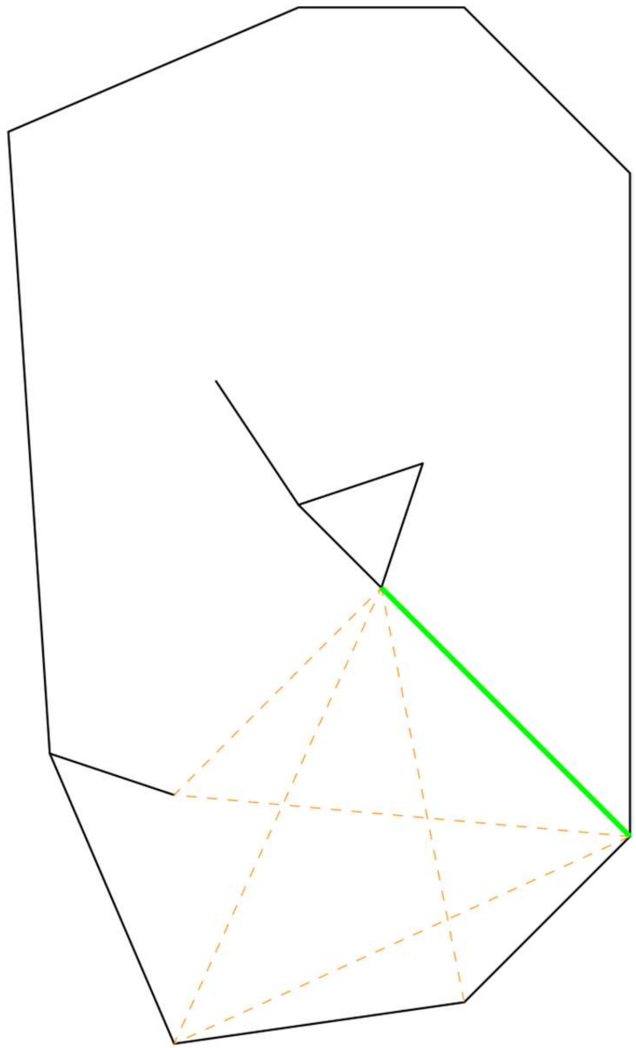

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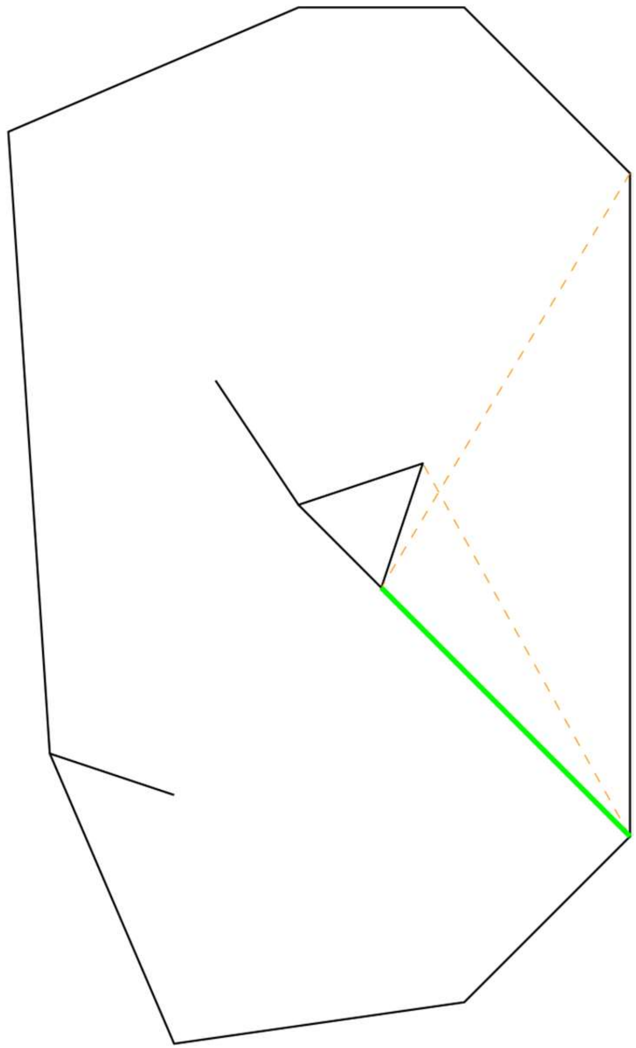

Technische Universität Braunschweig

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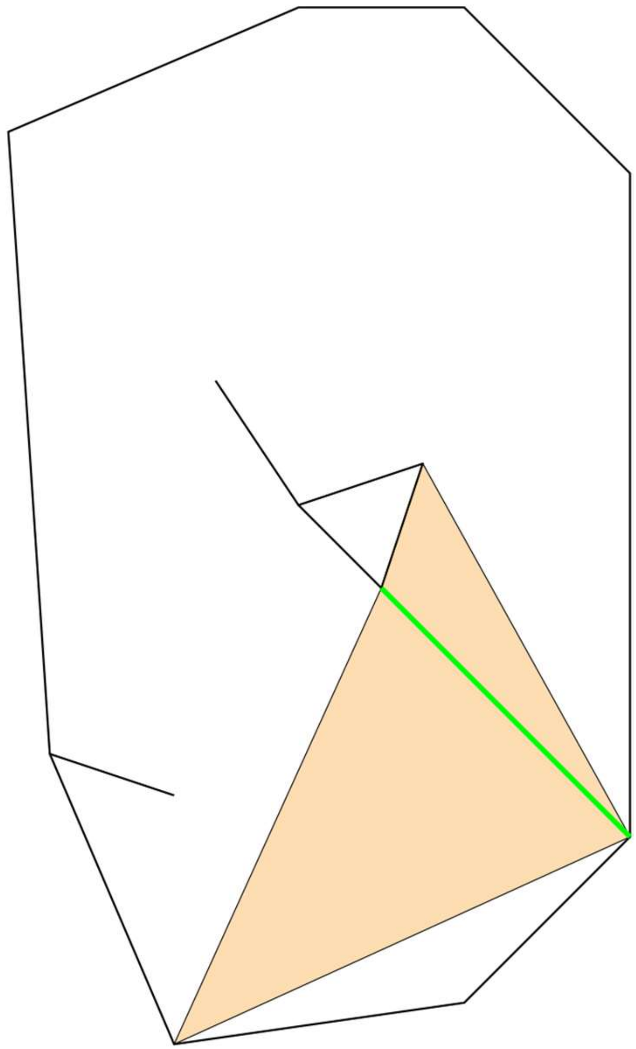

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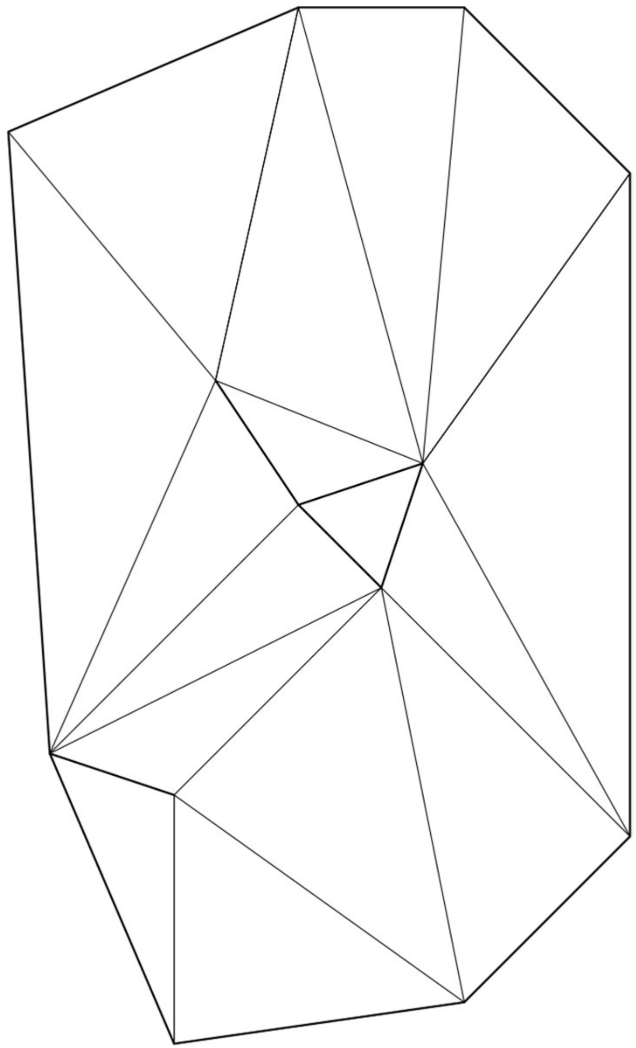
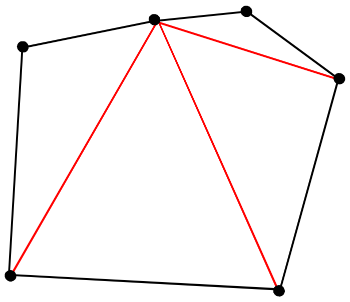
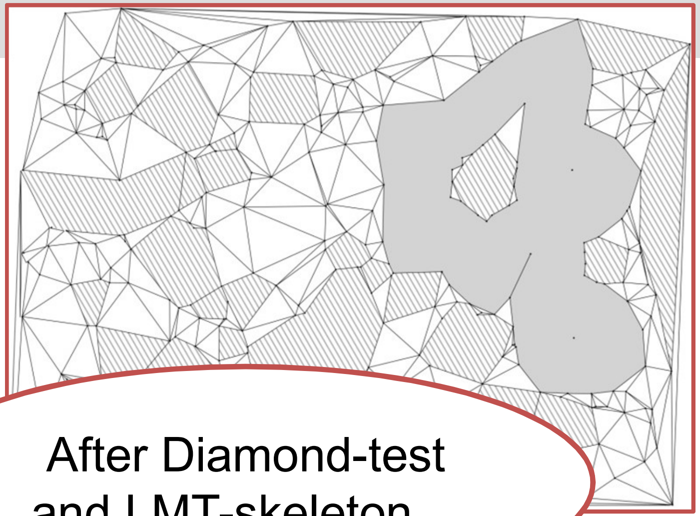


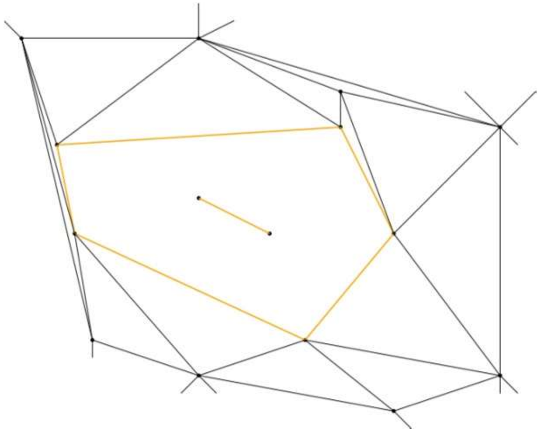
What Makes an MWT-Instance Practically Hard?



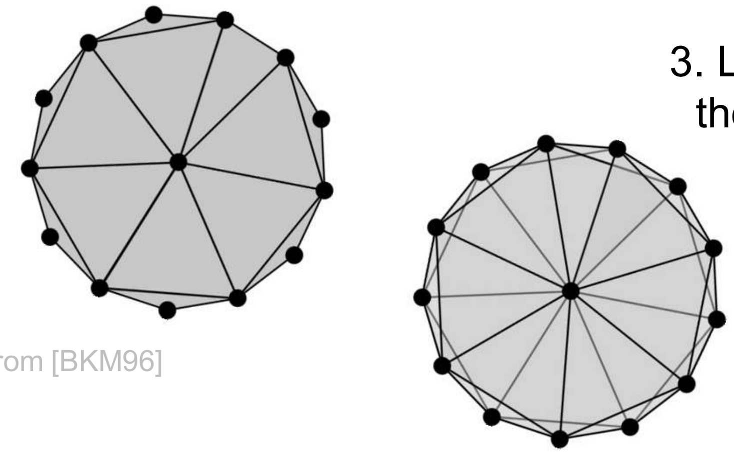
1. $\geq$ 1 complex face, else $O(n^3)$



After Diamond-test and LMT-skeleton…

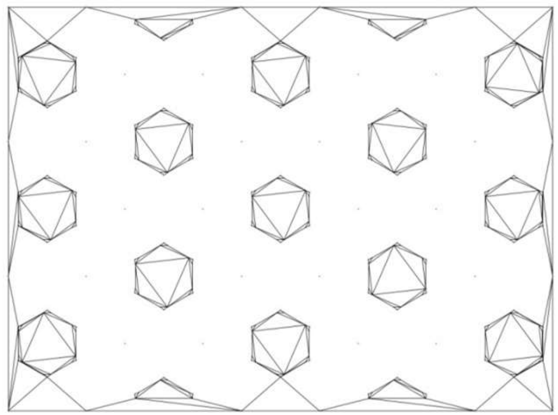


2. faces with many inner points, else $O(6^k n^5 \log n)$/ $O(n^4 4^k k)$ / $O(n^3 k!\, k)$



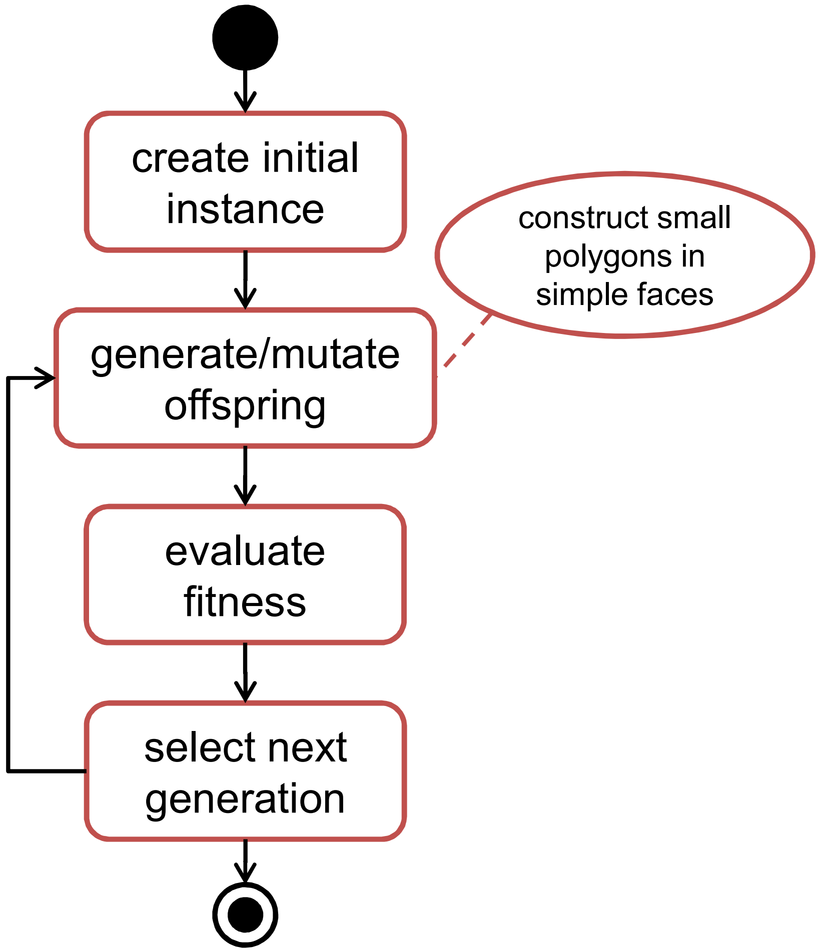
from [BKM96]

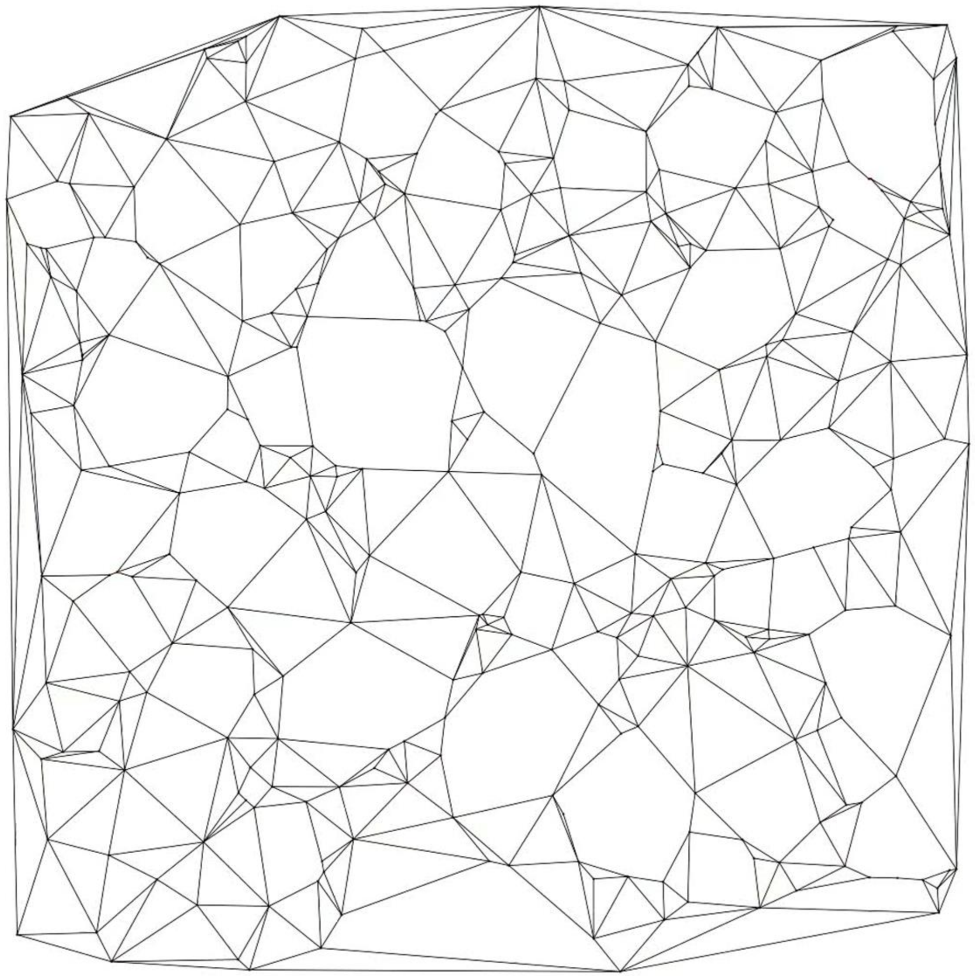
3. LP-relaxation of the IP must have fractional solution, else easy to solve



from [YY14]

Experimental Setup: Evolver





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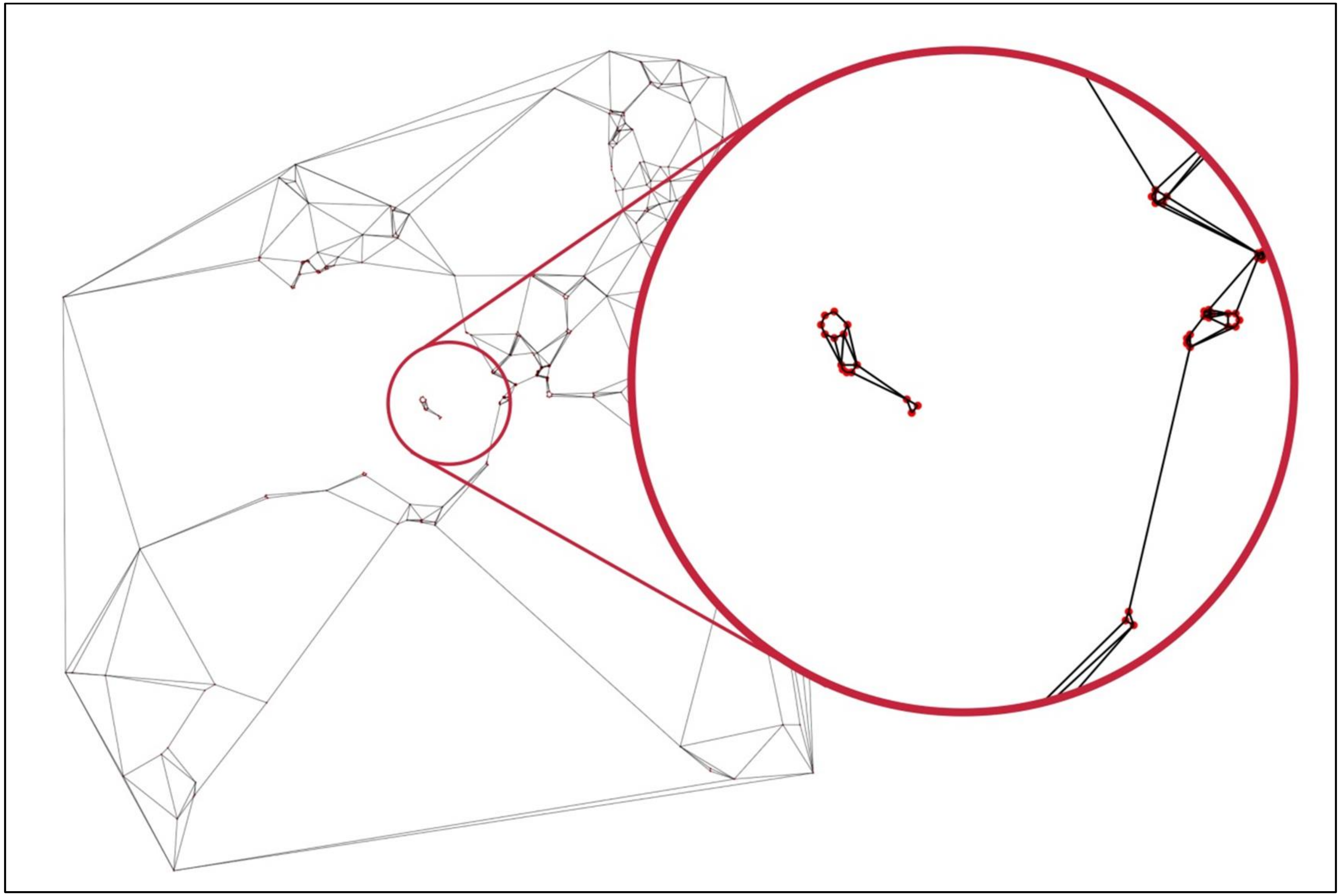
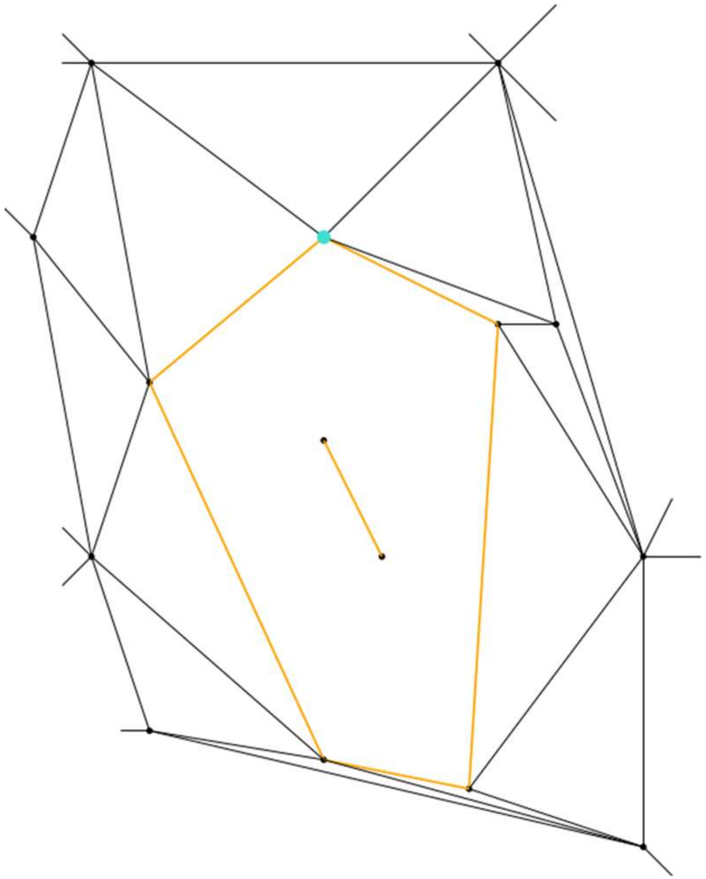
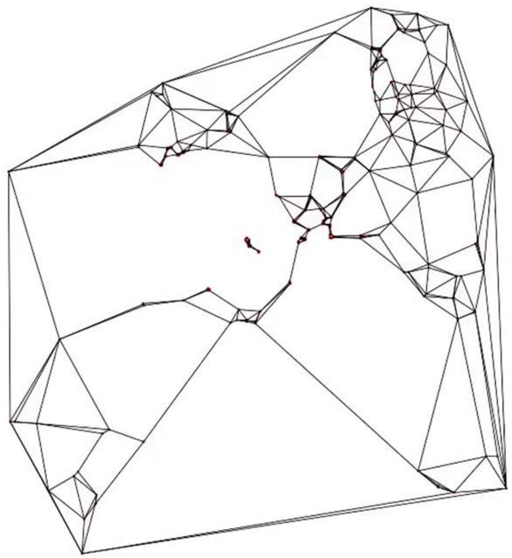


Experimental Setup: Pertubation





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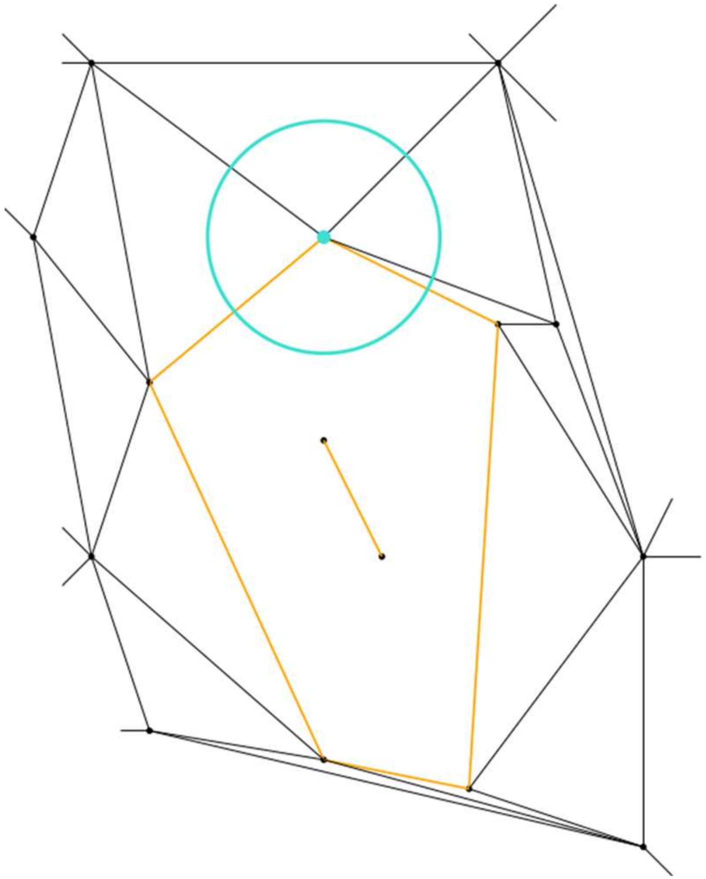

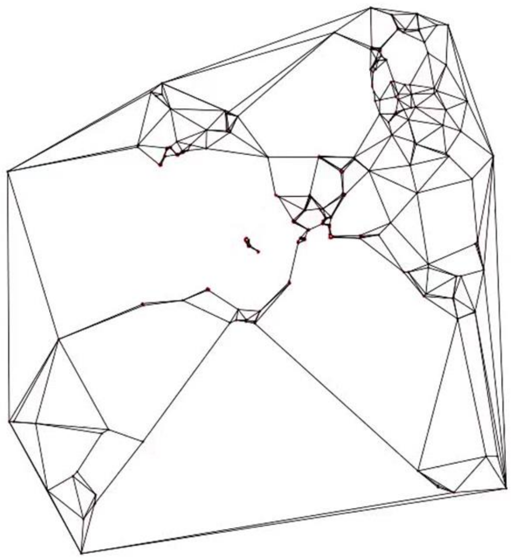

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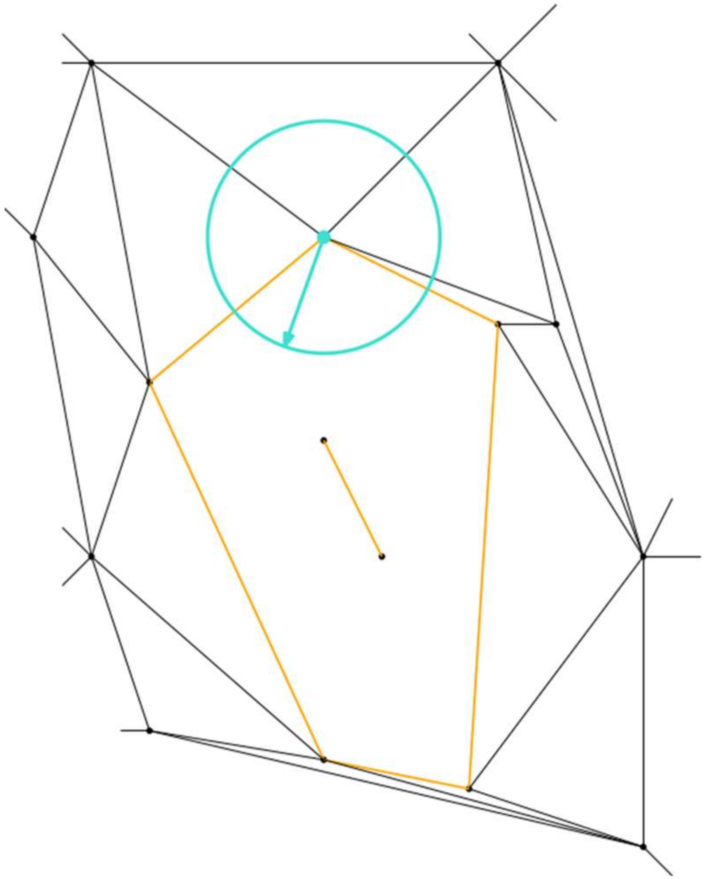

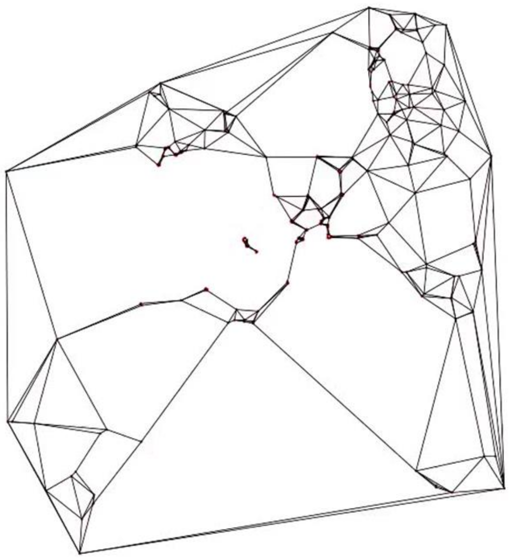

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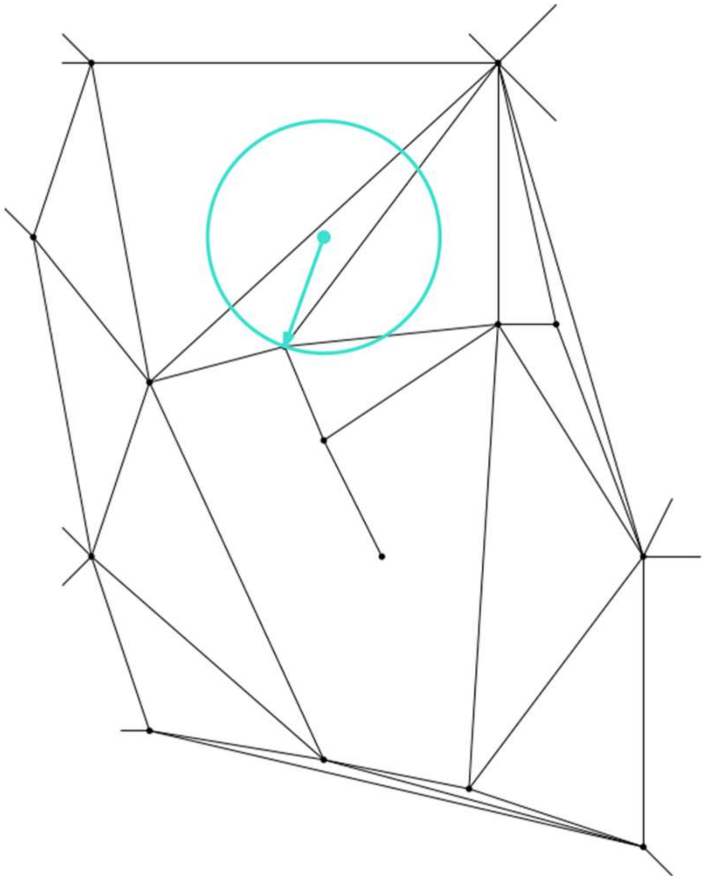

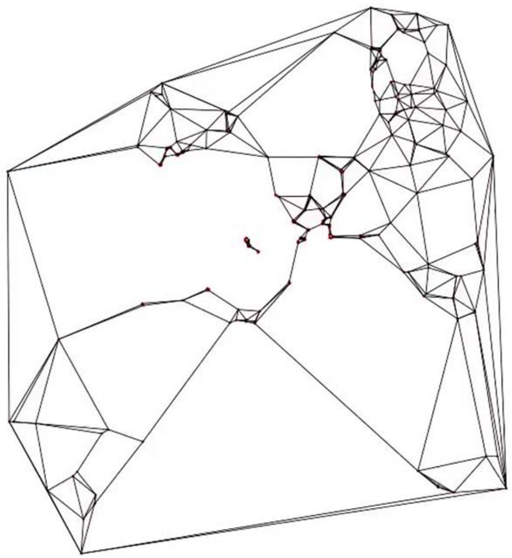

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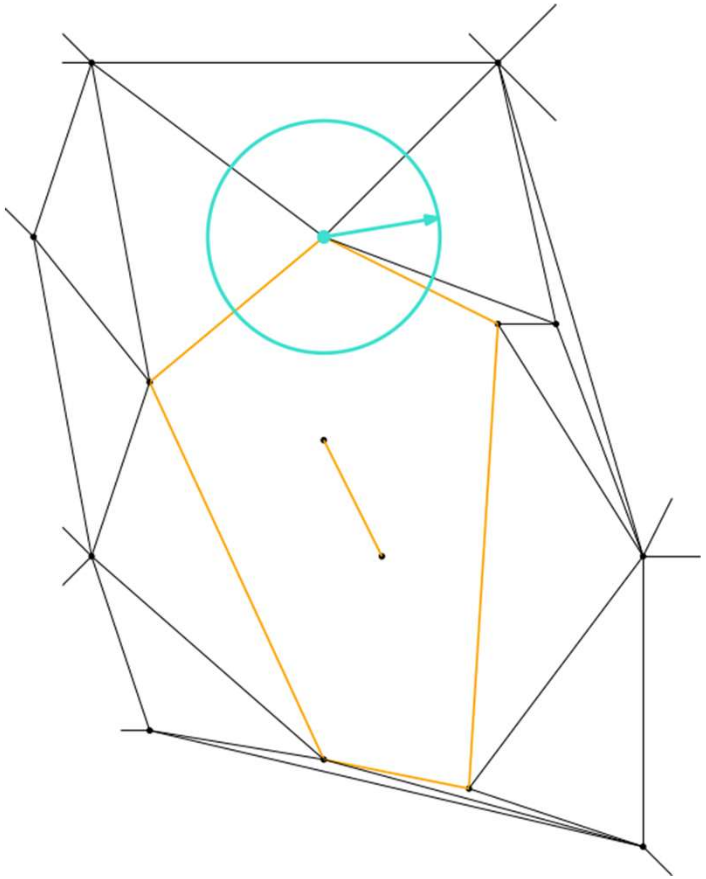

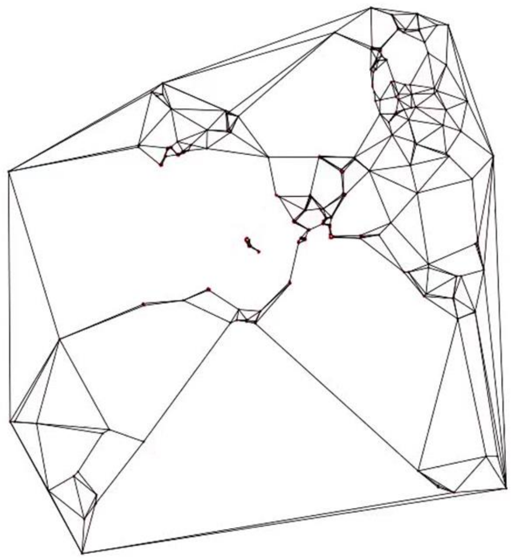

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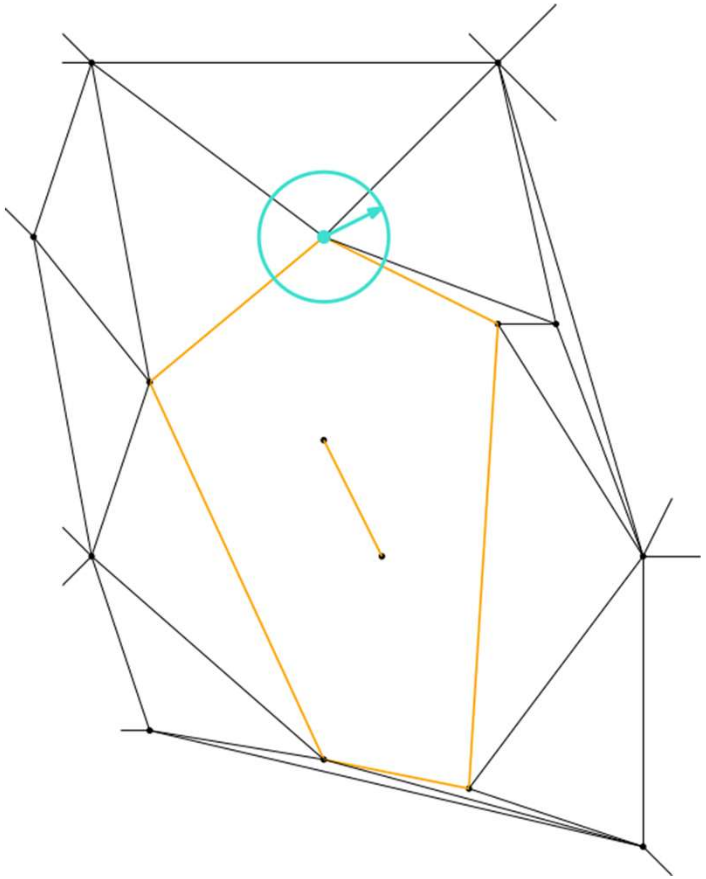

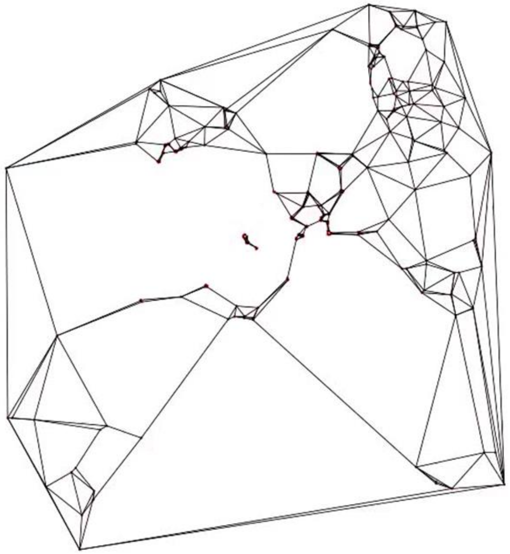

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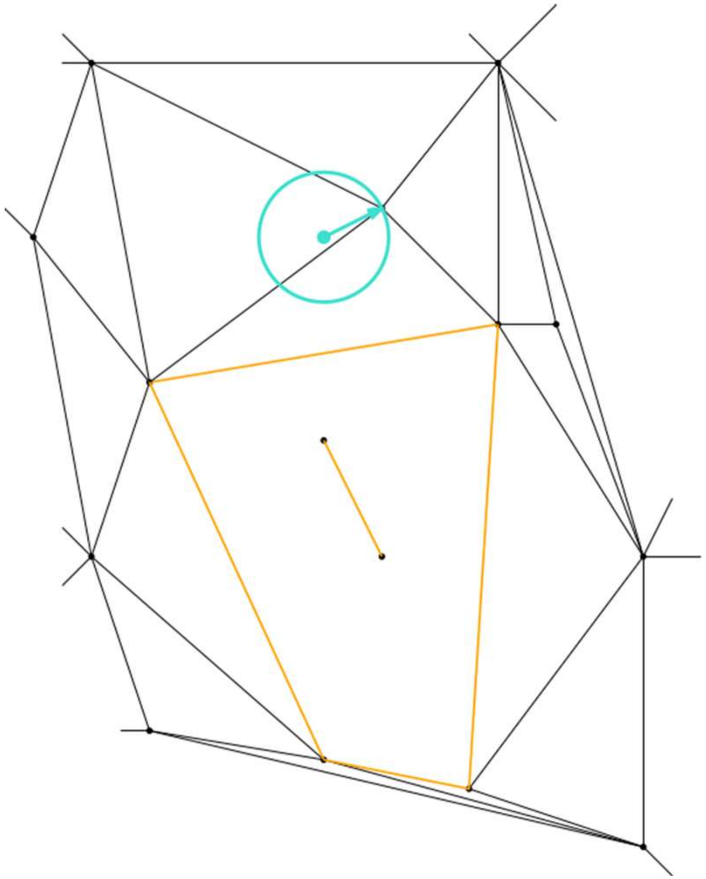
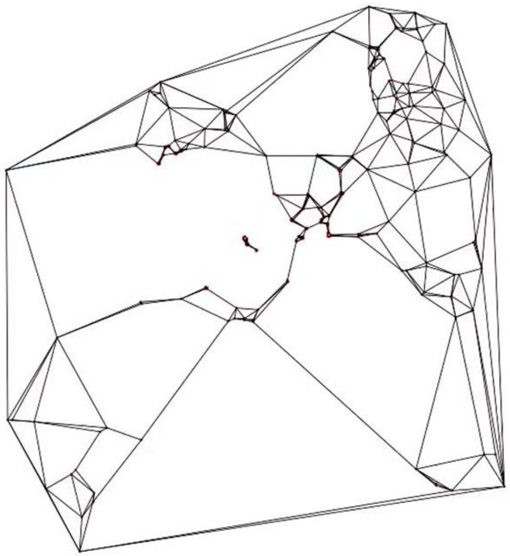
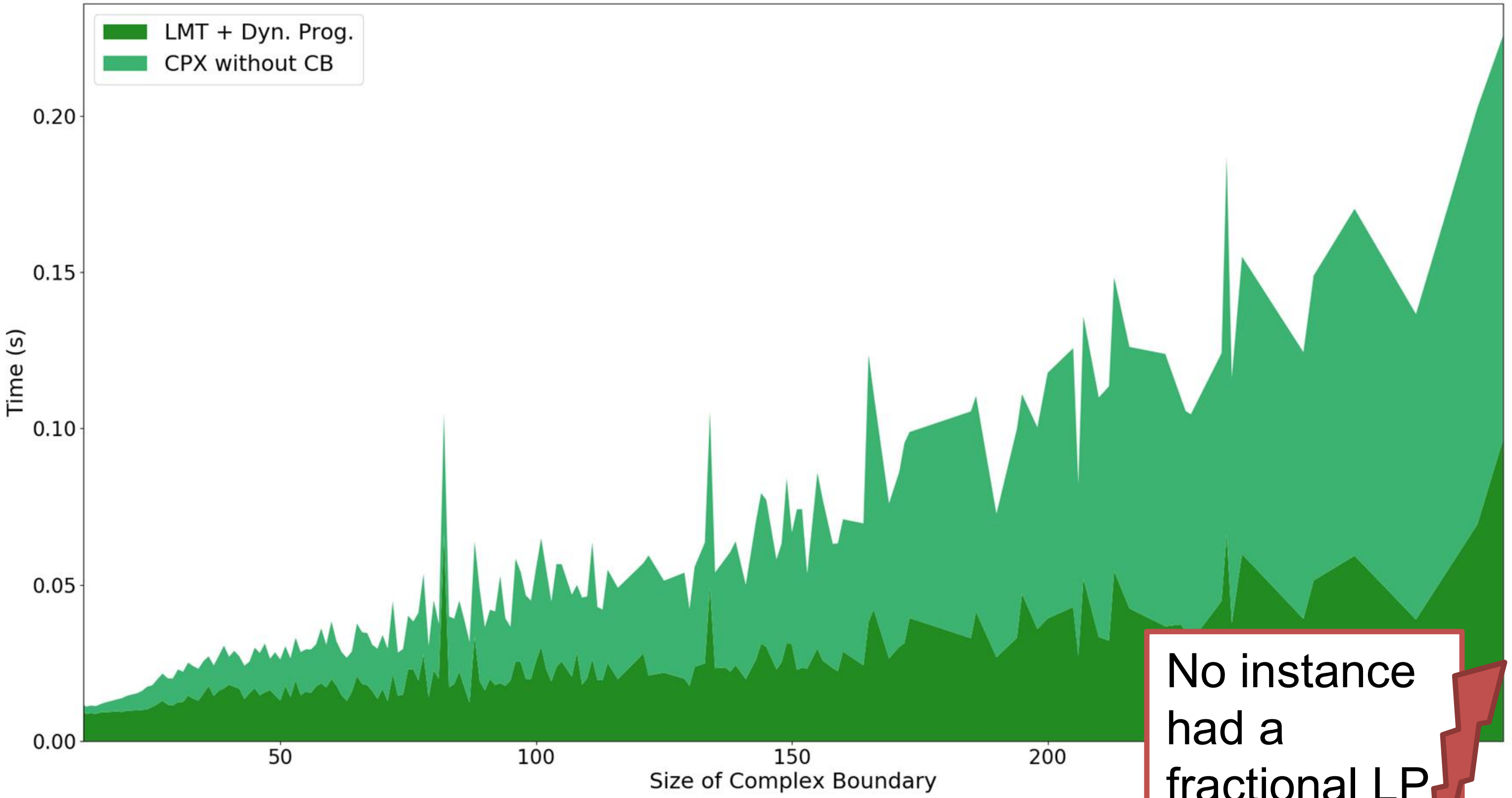




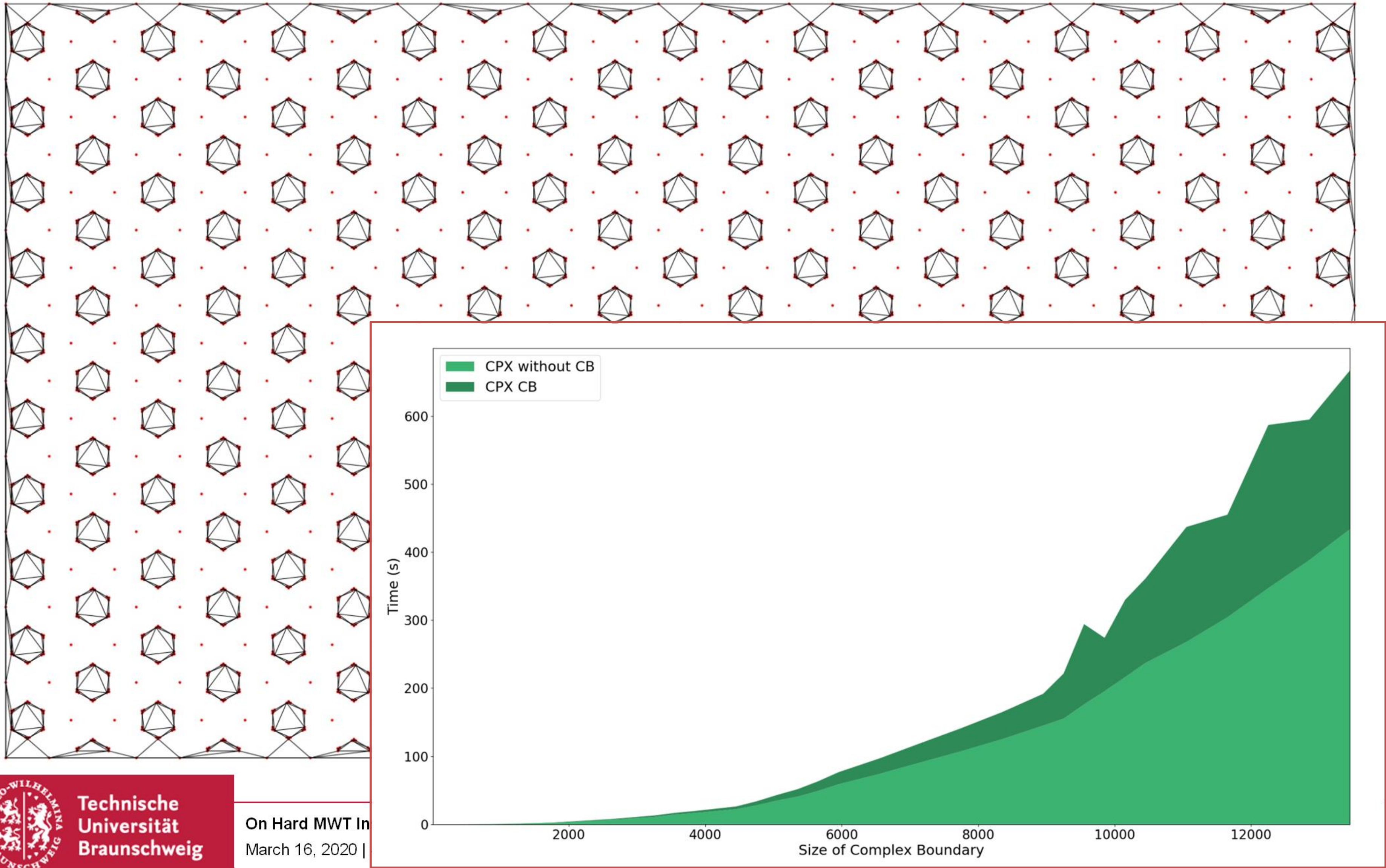
Experimental Results: Runtime

- created ~17,000 instances
- fixed size of 306 points

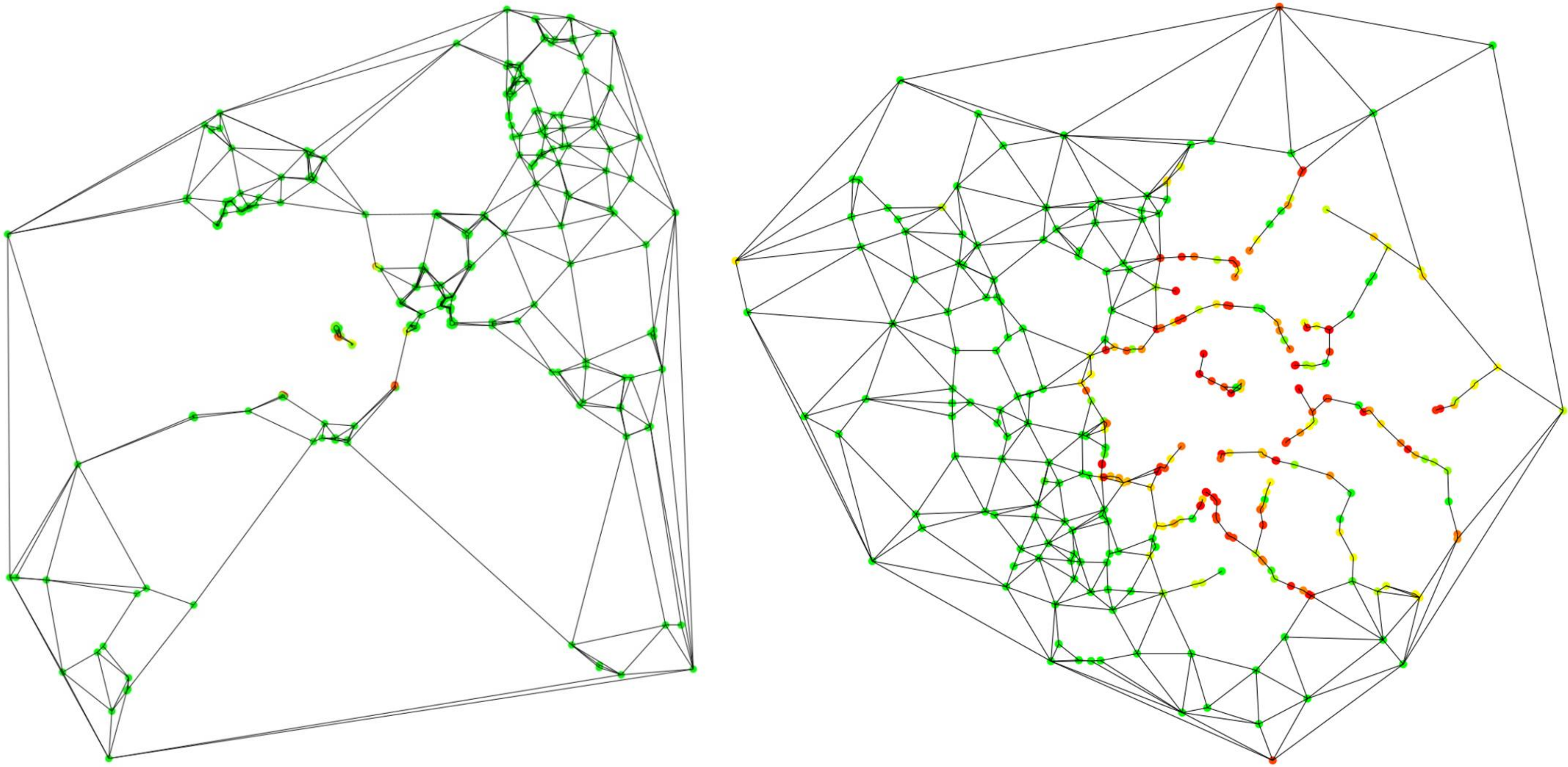


No instance had a fractional LP solution!

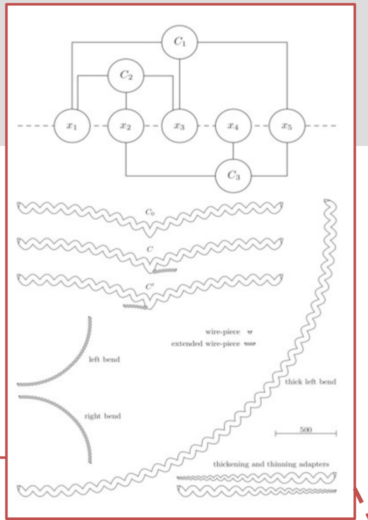
Experimental Results: Scaling

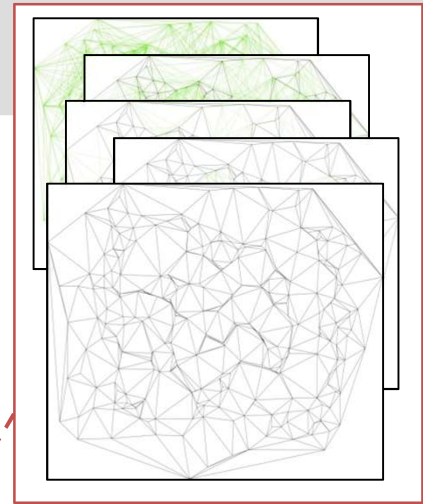


Experimental Results: Stability



Conclusion





$$\min_{x_\triangle} \sum_{\triangle} || \triangle || \cdot x_\triangle$$

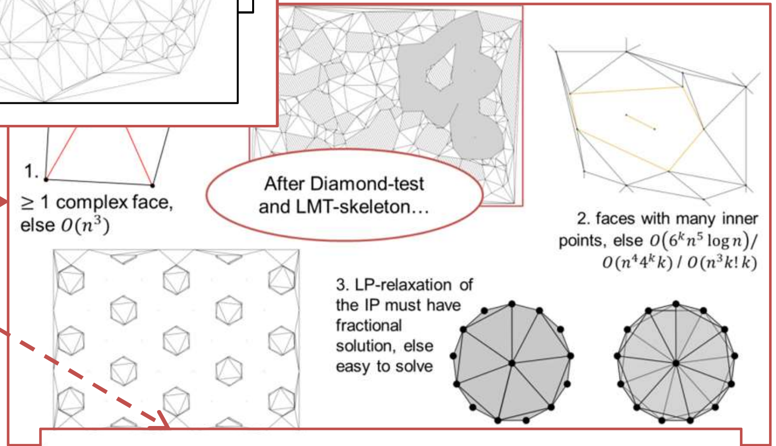
s.t.

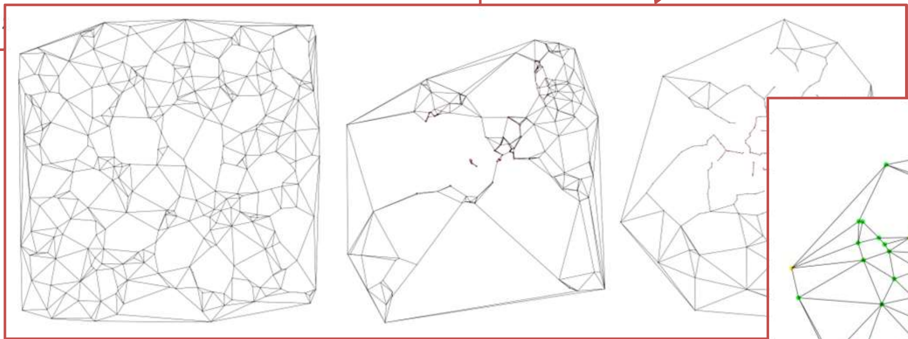
$\sum_{\triangle \in \delta(e)} x_\triangle = 1$ $\forall e \in$ boundary component

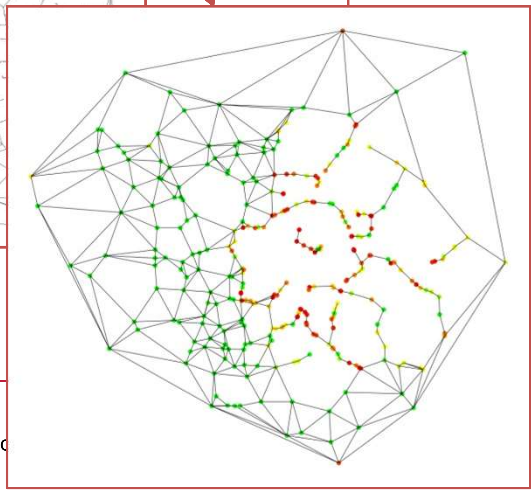
$\left.\begin{array}{l}\sum_{\triangle \in \delta^+(e)} x_\triangle = 1 \\ \sum_{\triangle \in \delta^-(e)} x_\triangle = 1\end{array}\right\}$ $\forall e \in$ antennas

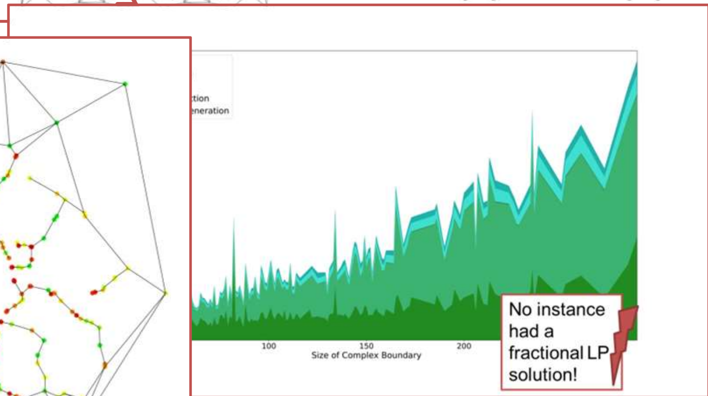
$\sum_{\triangle \in \delta^+(e)} x_\triangle - \sum_{\triangle \in \delta^-(e)} x_\triangle = 0$ $\forall e \in$ inner edges

What makes an MWT instance practically hard?









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