Minimum Convex Partition of Degenerate Point Sets is NP-Hard

Nicolas Grelier, ETH Zürich

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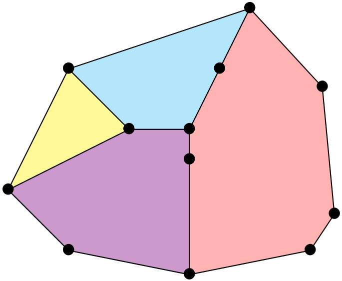
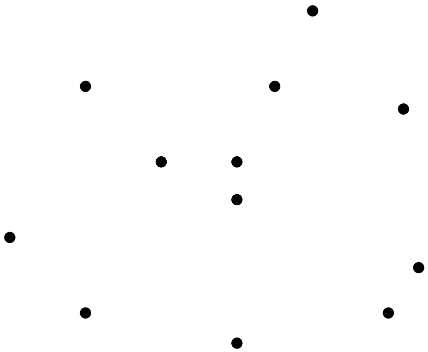


Definition of the problem

Partition into convex polygons:

- The union of the polygons is the convex hull of P,
- The interiors of the polygons are pairwise disjoint,
- No polygon contains a point of P in its interior.



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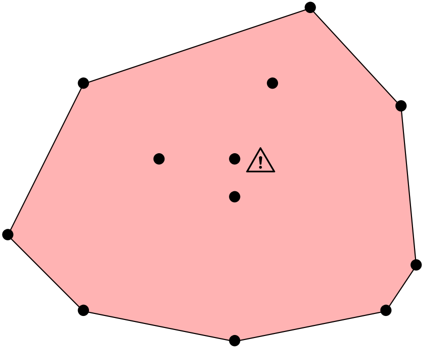

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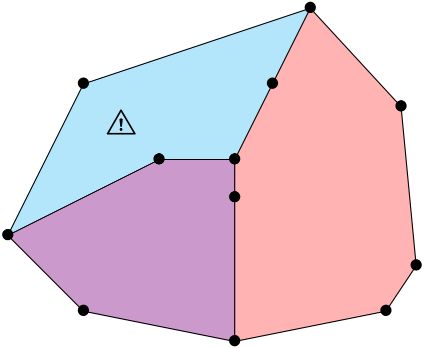

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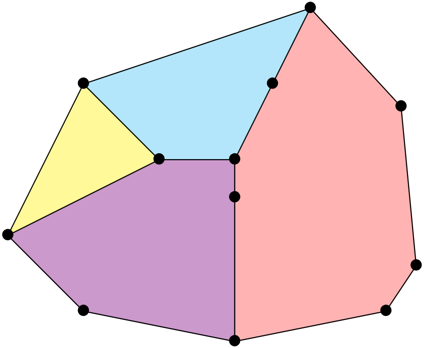
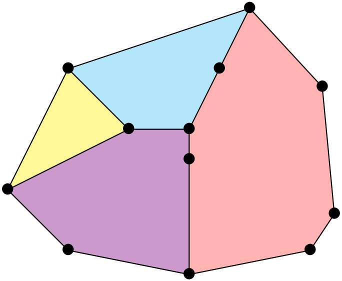


Definition of the problem

Gives a plane graph $G = (V = P, E)$ with set of bounded faces F



Objective is to minimise $|F|$ (or equivalently $|E|$)

Remarks:

- $|F| = 1 \Leftrightarrow$ points in convex position
- $|F| < 2|P| - 4$ (take a triangulation)

Related work

Assume general position: no three points on a line

Lemma (Knauer and Spillner '06)

If one can compute a convex partition with at most $\lambda|P|$ faces, then there exists a 2λ-approximation algorithm.

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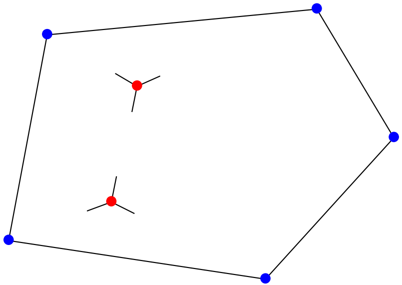

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- Sakai and Urrutia '19: same but with at most $\frac{4|P|}{3} - 2$ faces.

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Theorem (García-Lopez and Nicolás '13)

There exists point sets s.t. any convex partition has at least $\frac{35}{32}|P| - \frac{3}{2}$ convex faces.

Results

Theorem

Minimum convex partition of degenerate point sets is NP-hard.

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The two following problems are NP-hard:

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Lingas' proof: Reduction from a modified version of *Planar 3SAT*

- Given a Boolean formula $F \rightarrow$ construct a polygon with holes Π
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Our proof: Use Lingas' construction and transform Π into a point set

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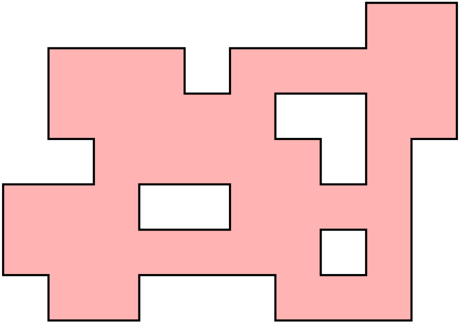

Lingas' reduction → axis-parallel segments

Reduction from a modified version of *Planar 3SAT*

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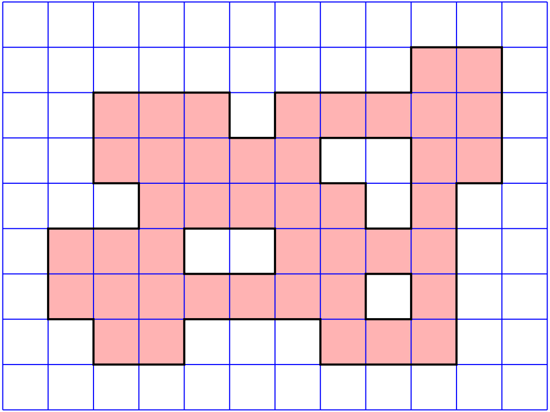

Can even construct Π on a grid

Reduction from a modified version of *Planar 3SAT*

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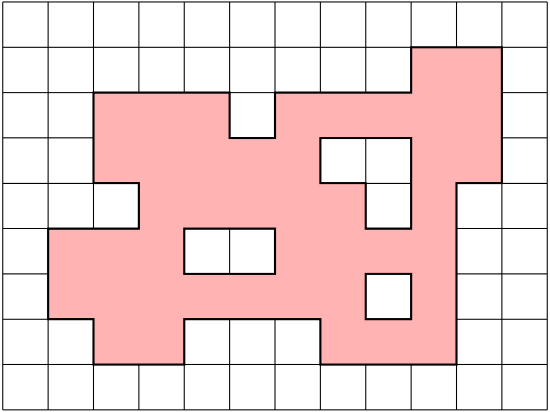

We keep all segments outside of Π

Reduction from a modified version of *Planar 3SAT*

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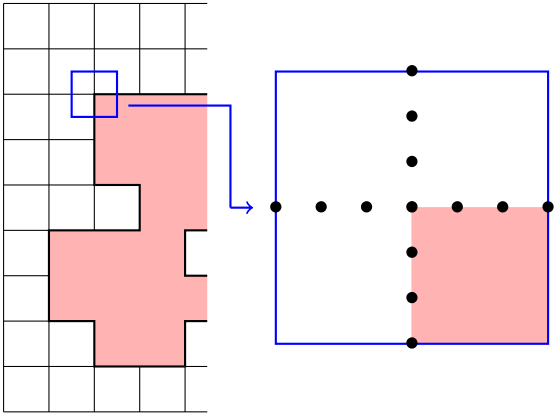
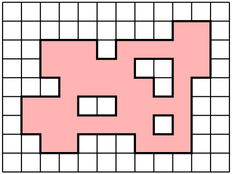


Replace each unit segment by x points

Sketch of proof

Lemma (Lingas '82)

F is satisfiable $\Leftrightarrow$ $\exists$ a partition of Π with at most k convex faces.



Transform Π into a point set P. $k' := \#$ unit squares outside Π.

Lemma

In a minimum convex partition of P, the convex sets do not cross the "segments".

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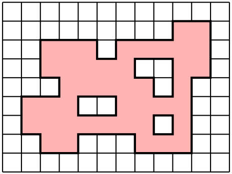

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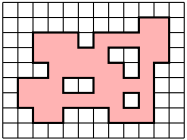
Theorem

F is satisfiable $\Leftrightarrow$ $\exists$ a partition of P with at most $k + k'$ convex faces.

Sketch of proof

Lemma

In a minimum convex partition, the convex sets do not cross the "segments".



Sketch of proof:

- For each unit square u, $\exists$ a convex set $\mathscr{C}$ s.t. $Area(u \cap \mathscr{C})$ is big,
- If $Area(u \cap \mathscr{C})$ is big, then $Area(u' \cap \mathscr{C})$ is small for any other unit square u',

Sketch of proof

Lemma

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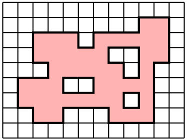


Sketch of proof:

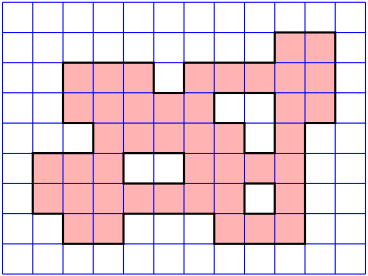
- For each unit square u, $\exists$ a convex set $\mathscr{C}$ s.t. $Area(u \cap \mathscr{C})$ is big,
- If $Area(u \cap \mathscr{C})$ is big, then $Area(u' \cap \mathscr{C})$ is small for any other unit square u',

$\rightarrow$ Each unit square contains its own convex set

Remains to deal with the inside of Π

For each unit square there is a convex set

$x :=$ # of points that replace a unit segment
$U :=$ # of unit squares in the blue grid, take $x > 2U$



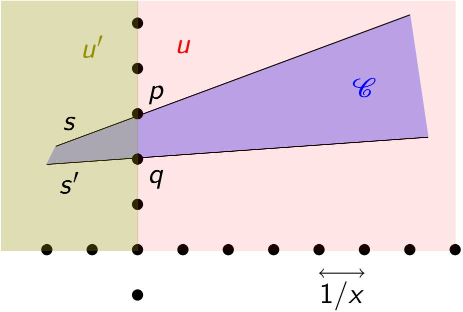
Lemma

In a minimum convex partition, for each unit square u, $\exists$ a convex set $\mathscr{C}$ s.t. Area$(u \cap \mathscr{C}) > \frac{1}{U} > \frac{1}{x}$.

For each convex set there is a unit square

Lemma

If Area$(u \cap \mathscr{C}) > 1/x$ *where* u *is on one side of a "segment", then Area*$(u' \cap \mathscr{C}) \leq 1/x$ *where* u' *is on the other side.*



If $Area(u \cap \mathscr{C}) > 1/x$, the two lines spawned by s and s' intersect on the left side.

Conclusion

Theorem

Minimum convex partition of degenerate point sets is NP-hard.

Open questions:

- What about point sets in general position?
- Is there a good approximation algorithm for degenerate point sets?