Empty Rainbow Triangles in k-colored Point Sets

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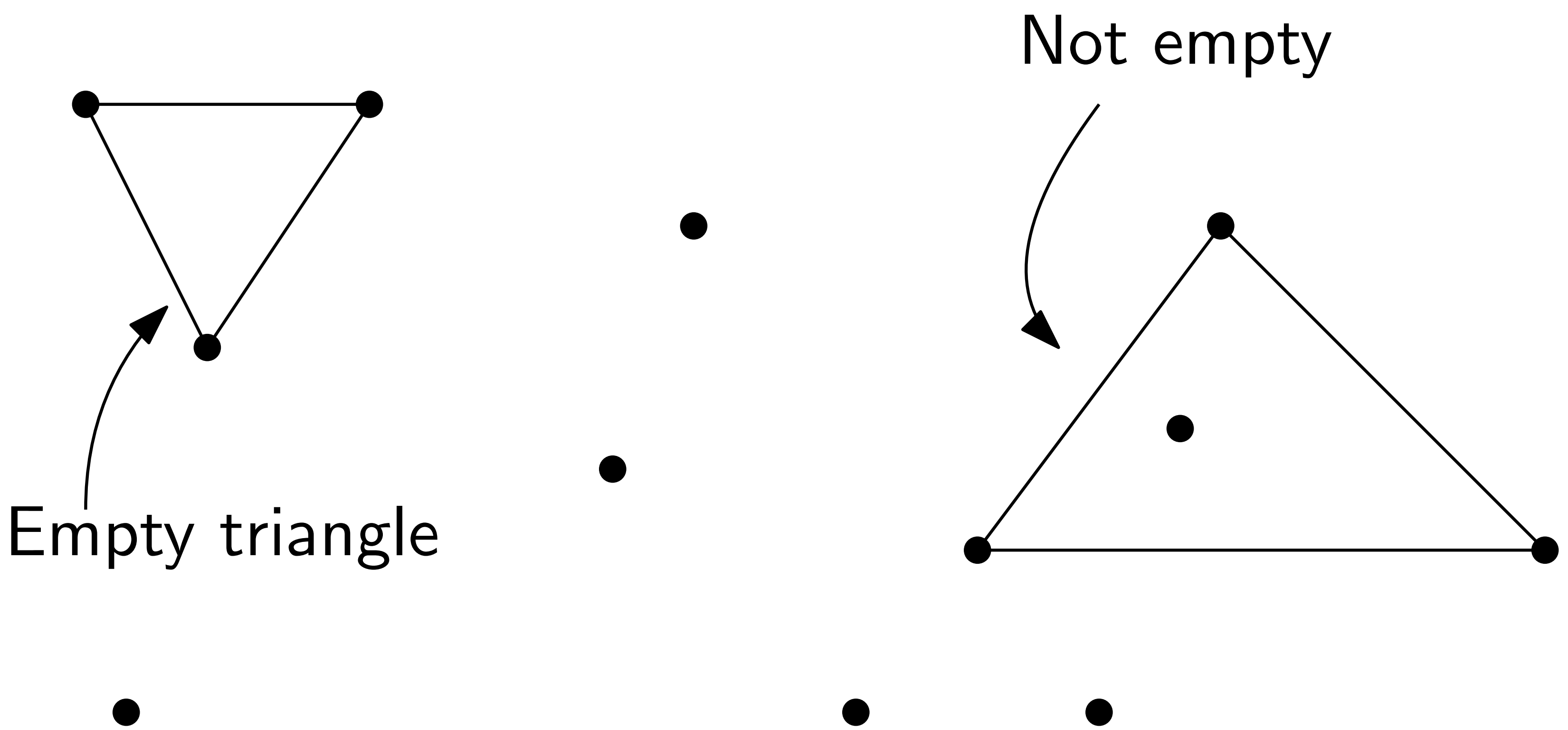
Introduction

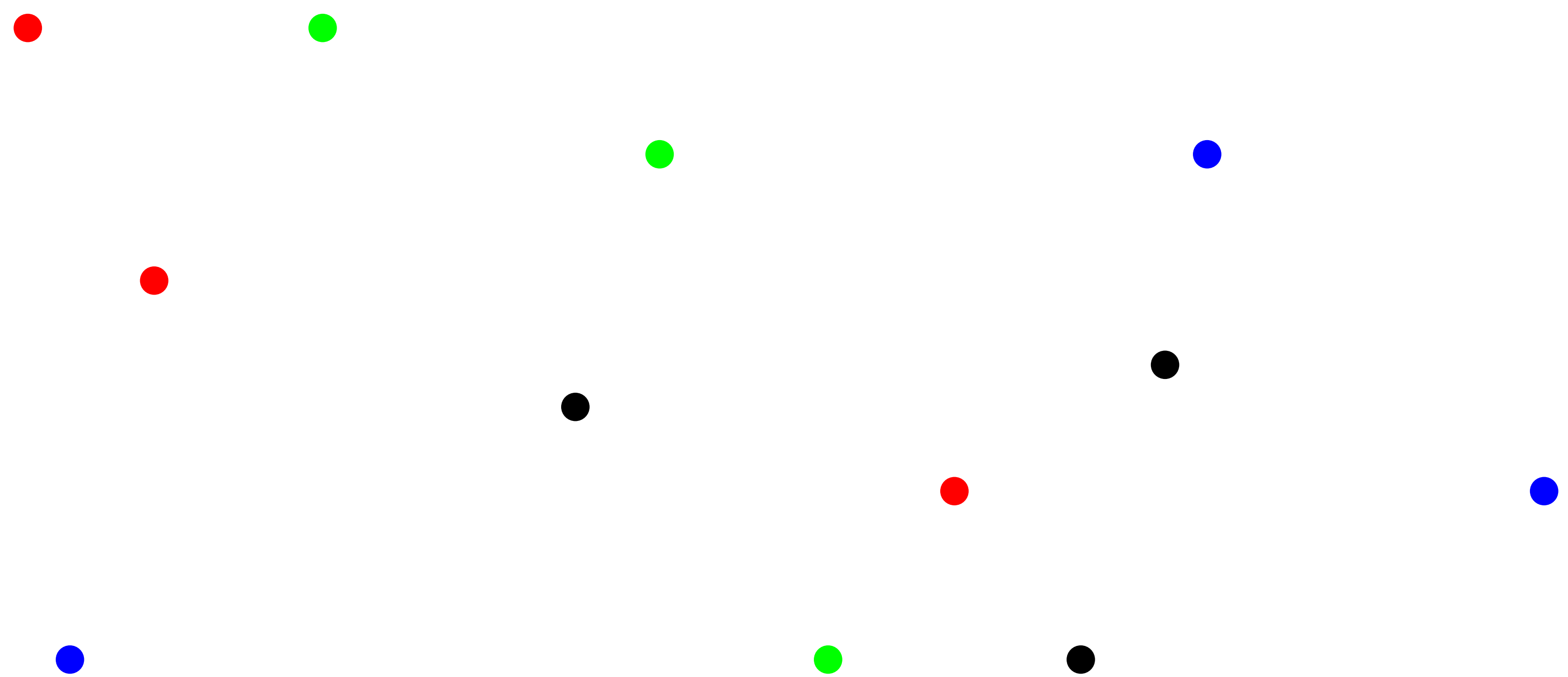
Point set in general position

Empty triangle: triangle with no point inside



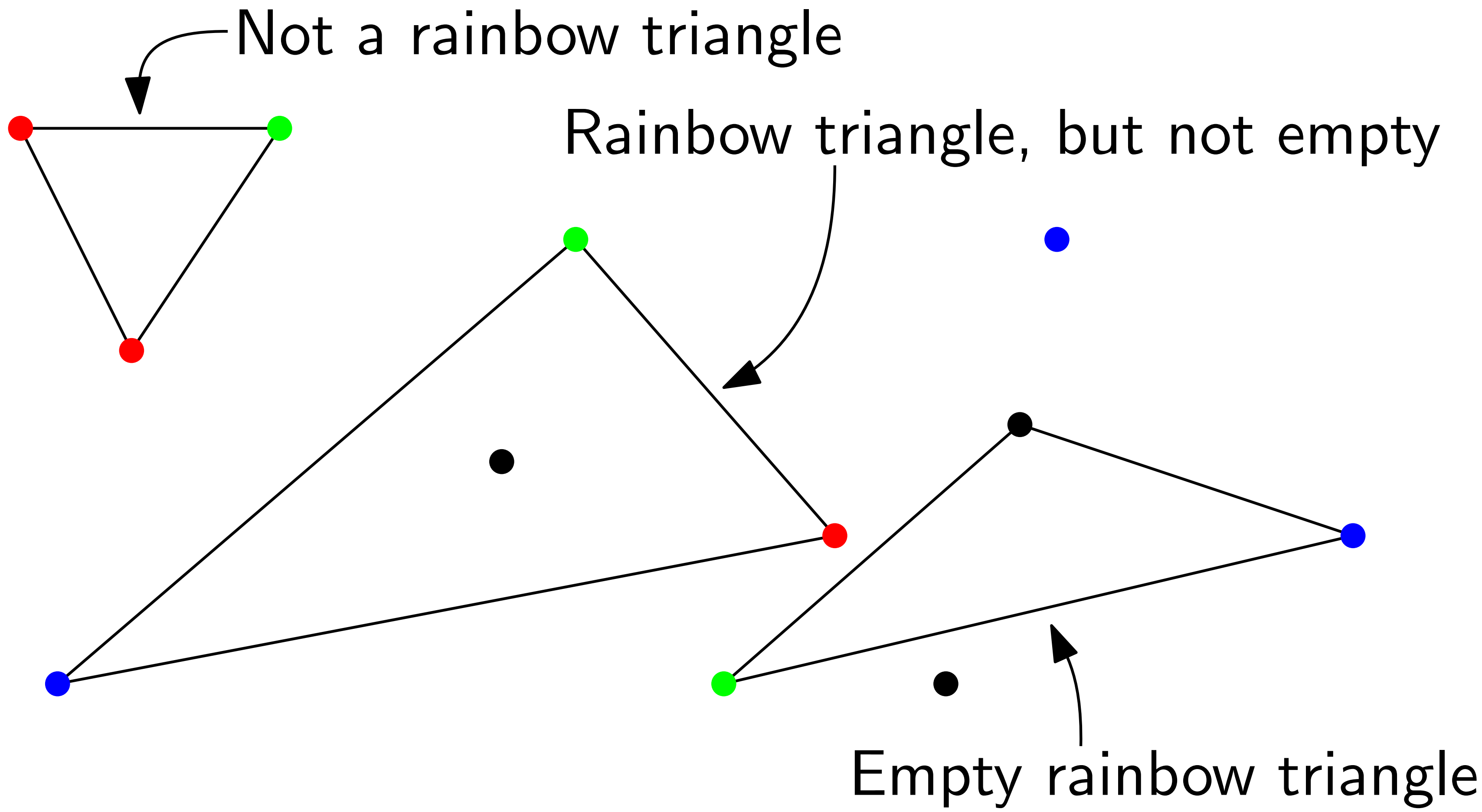
Introduction

k-colored point set: k colors in total
For every color there are m points with this color.



Introduction

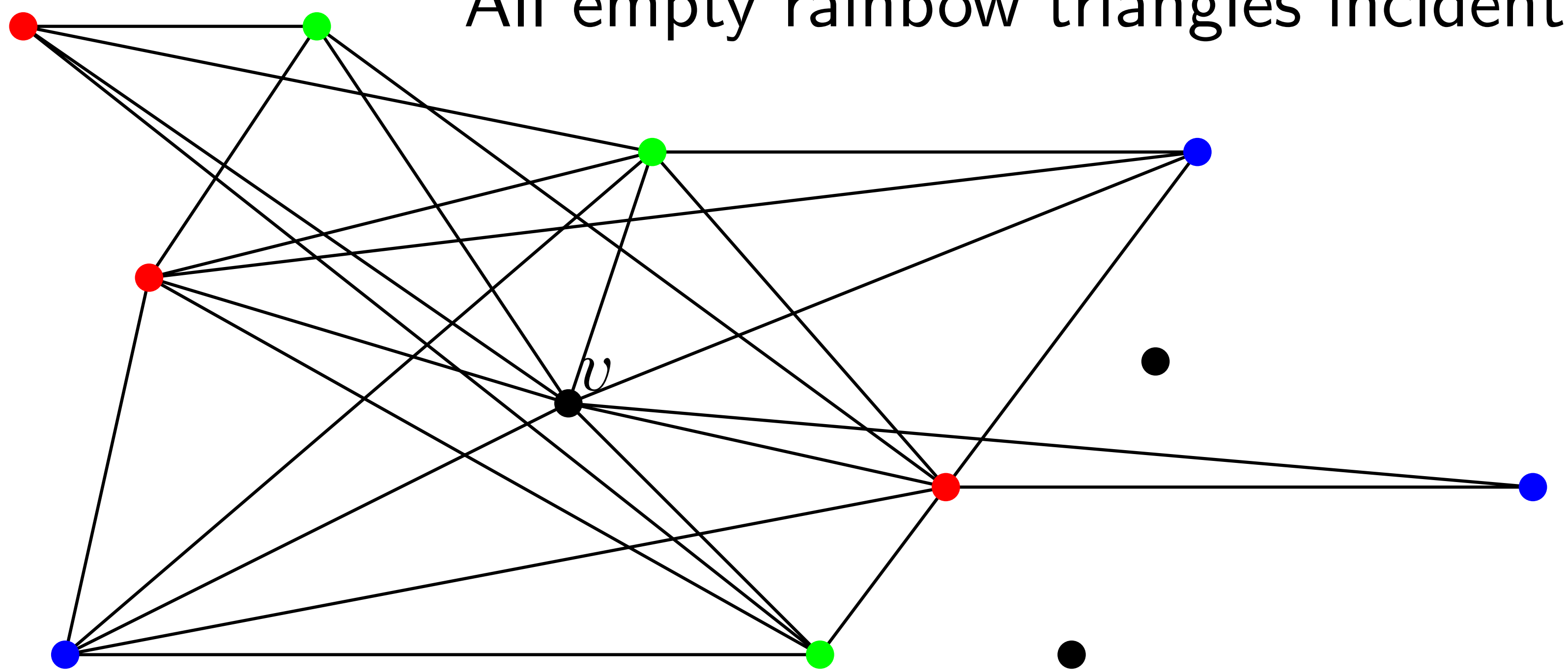
Rainbow triangle: triangle, where all three vertices have different colors.



Introduction

How many empty rainbow triangles are at least in a k-colored point set with n points?

All empty rainbow triangles incident to v



Lower Bound

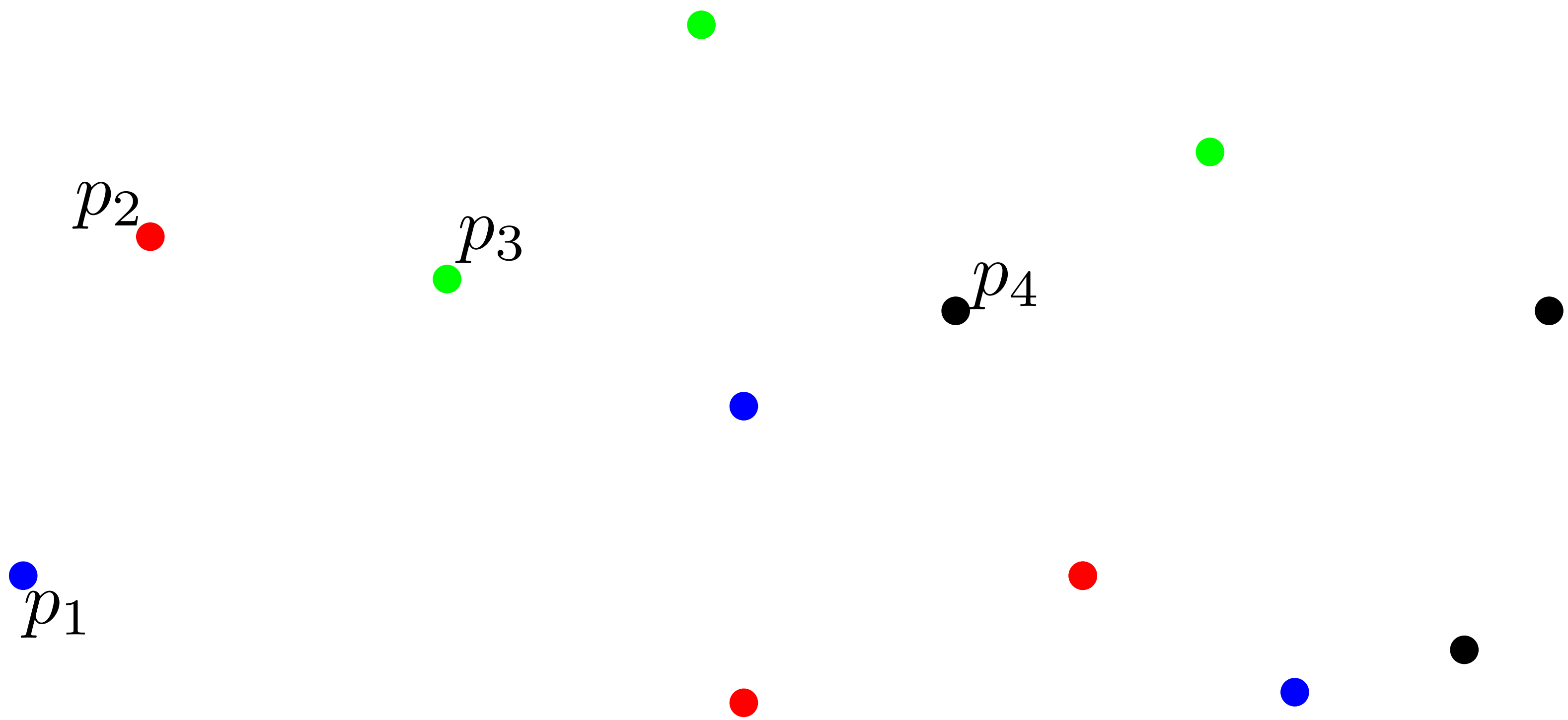
How many empty rainbow triangles are at least in a k-colored point set with n points?

- Take an arbitrary point x.
- Count triangles where x is the rightmost point.
- Sum over all points.
- Happiness

Lower Bound

Label the colors from 1 to k by their leftmost points.

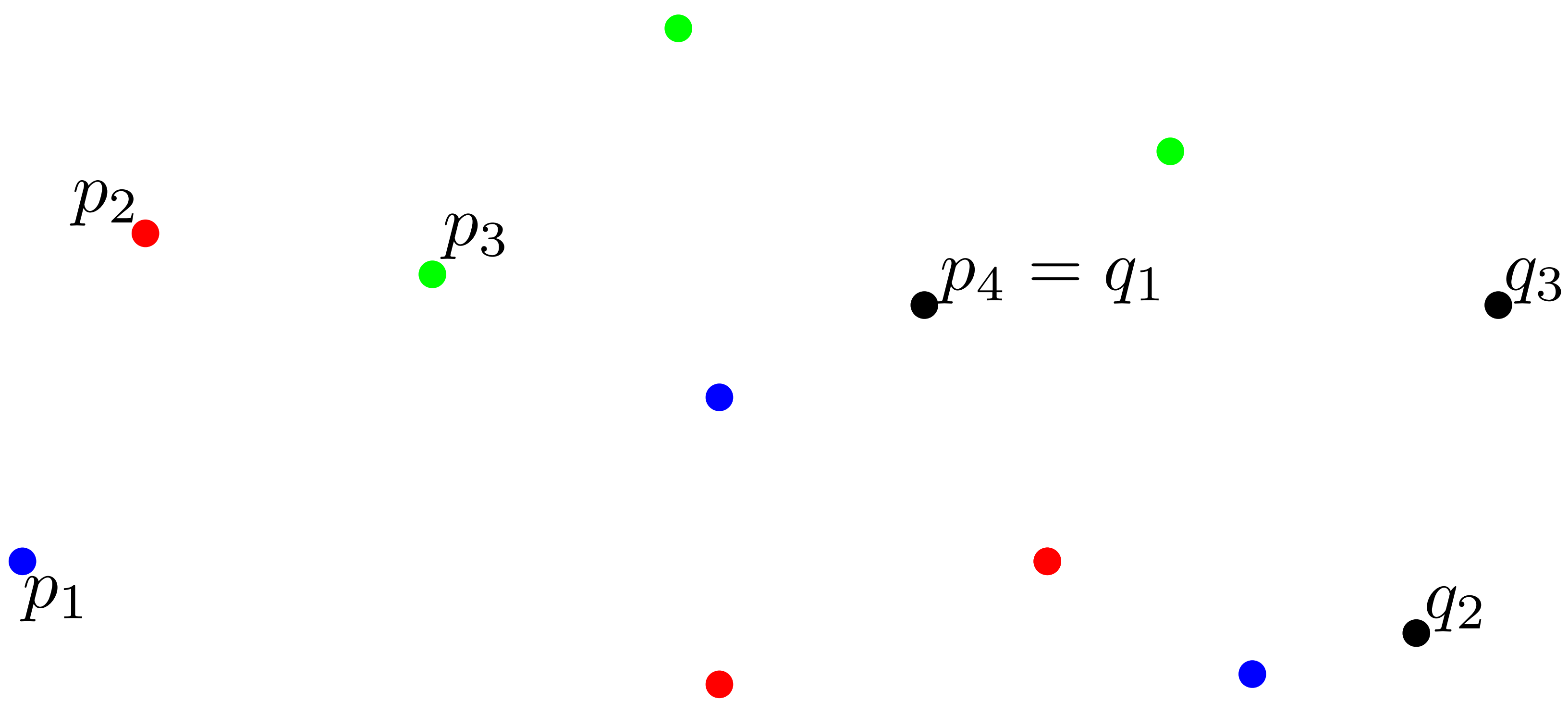
In the 4-colored point set below we have blue as color 1, red as color 2, green as color 3 and black as color 4.



Lower Bound

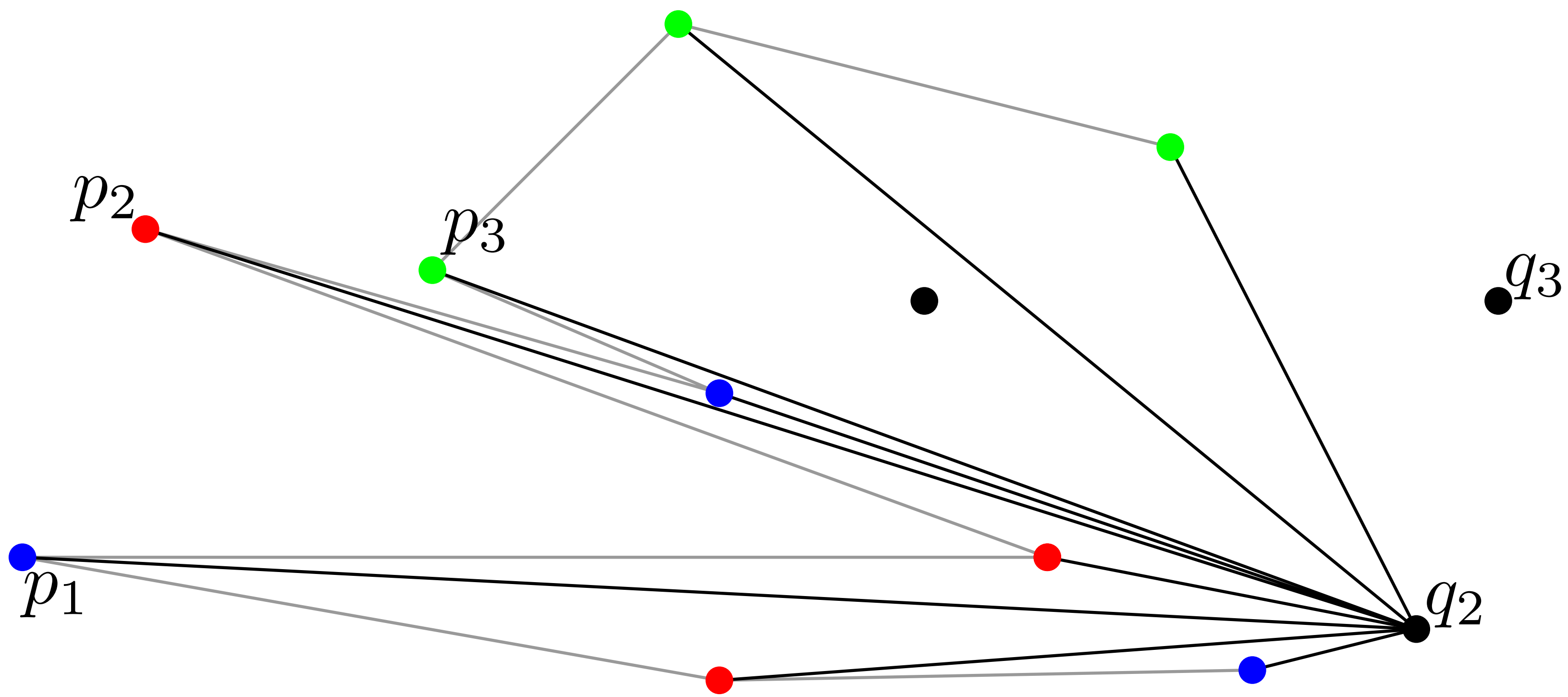
Label points of color i from left to right with q_1 up to q_k.

Count rainbow triangles with q_j as rightmost point.



Lower Bound

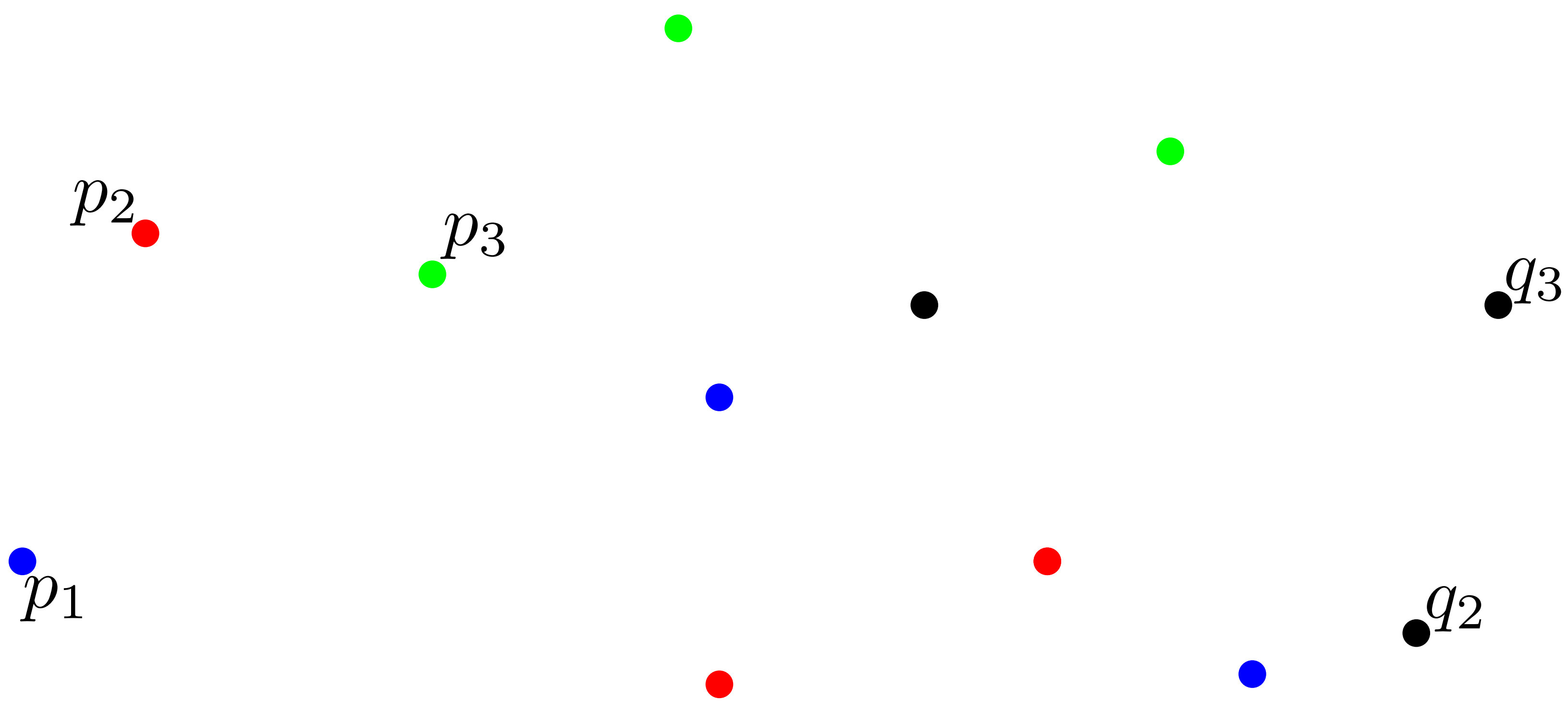
Look at points to the left of q_j in angular order.
There are at least $i-1$ other colors, so at least $i-2$ times two points with different colors are neighbored.
This gives $i-2$ rainbow triangles.



$j-1$ points with color i could lie in these $i-2$ triangles.

Lower Bound

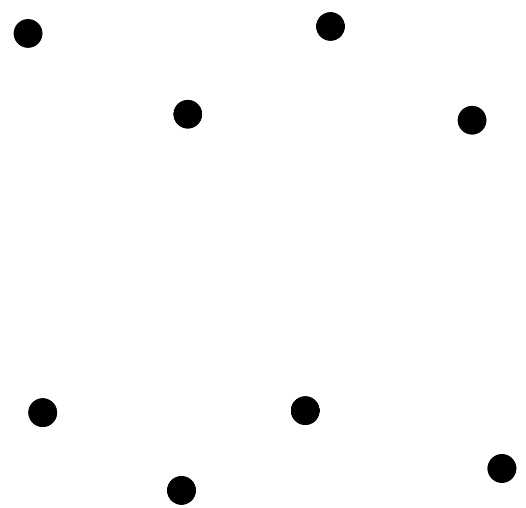
At least $i - j - 1$ empty rainbow triangles with q_j of color i as rightmost point.
Summing over all i and j gives $\frac{1}{6}k^3 - \frac{1}{2}k^2 + \frac{1}{3}k$.



Upper bound

Main ideas:

- ▷ Take a point set on k points with few empty triangles
- ▷ Replace each point with a cluster of same colored points
- ▷ Have few points for each color that are part of an empty rainbow triangle.



Upper bound

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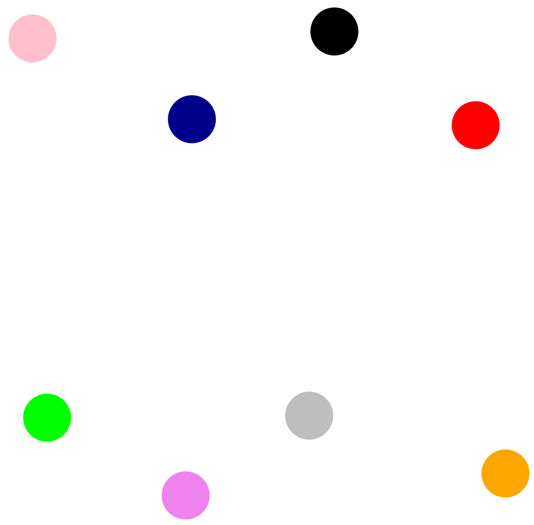

Upper bound

Main ideas:

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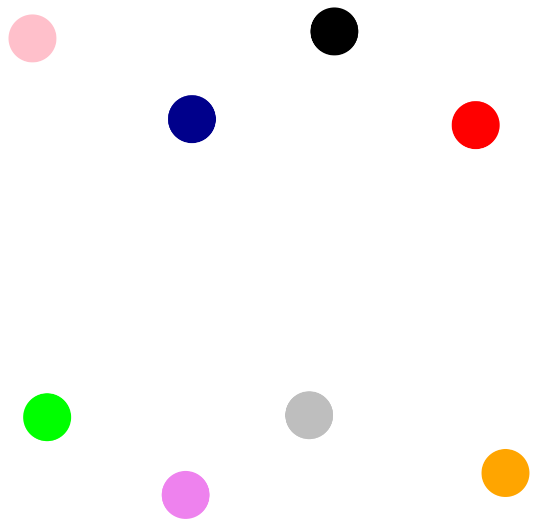

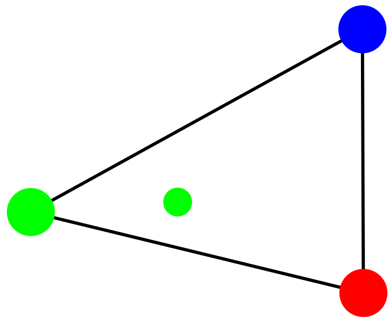

Horton set

$H = H_0 \cup H_1$ where H_0, H_1 are Horton sets, H_1 is high above H_0 and H_0 is far below H_1.

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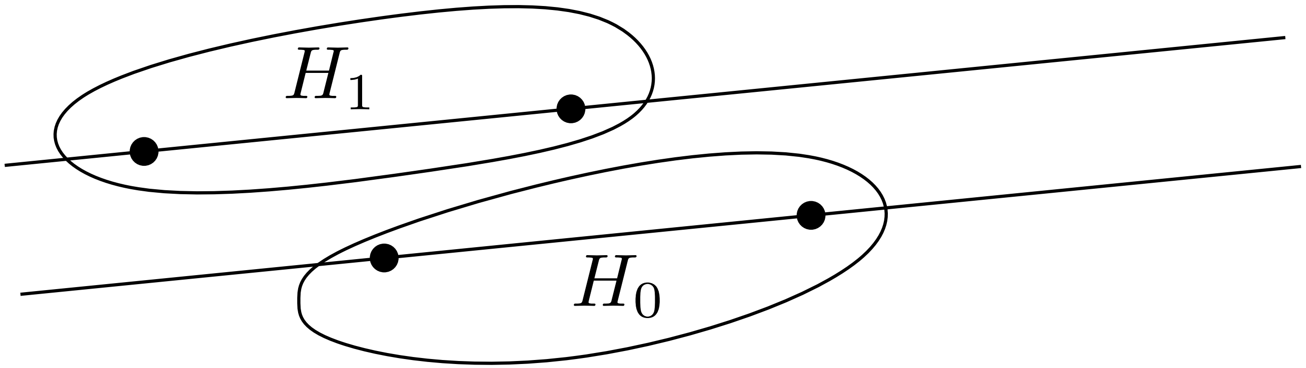

Horton set

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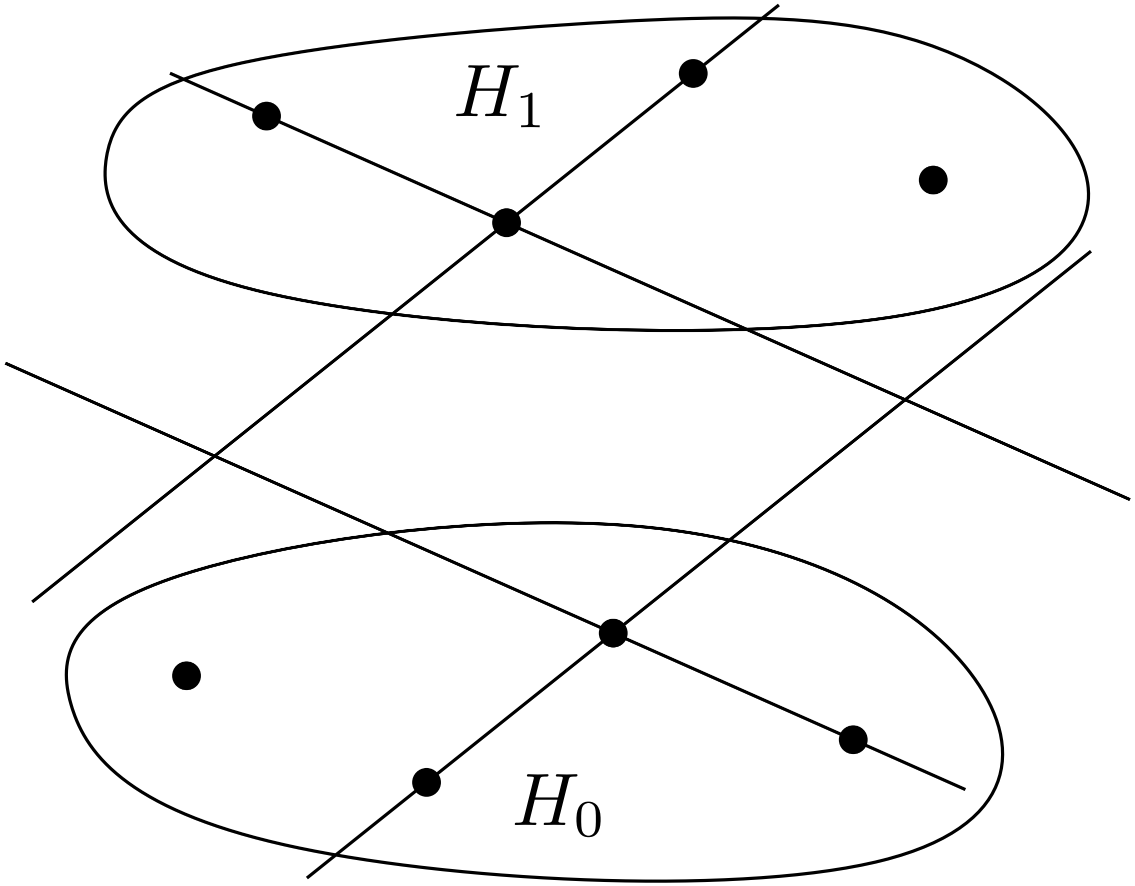
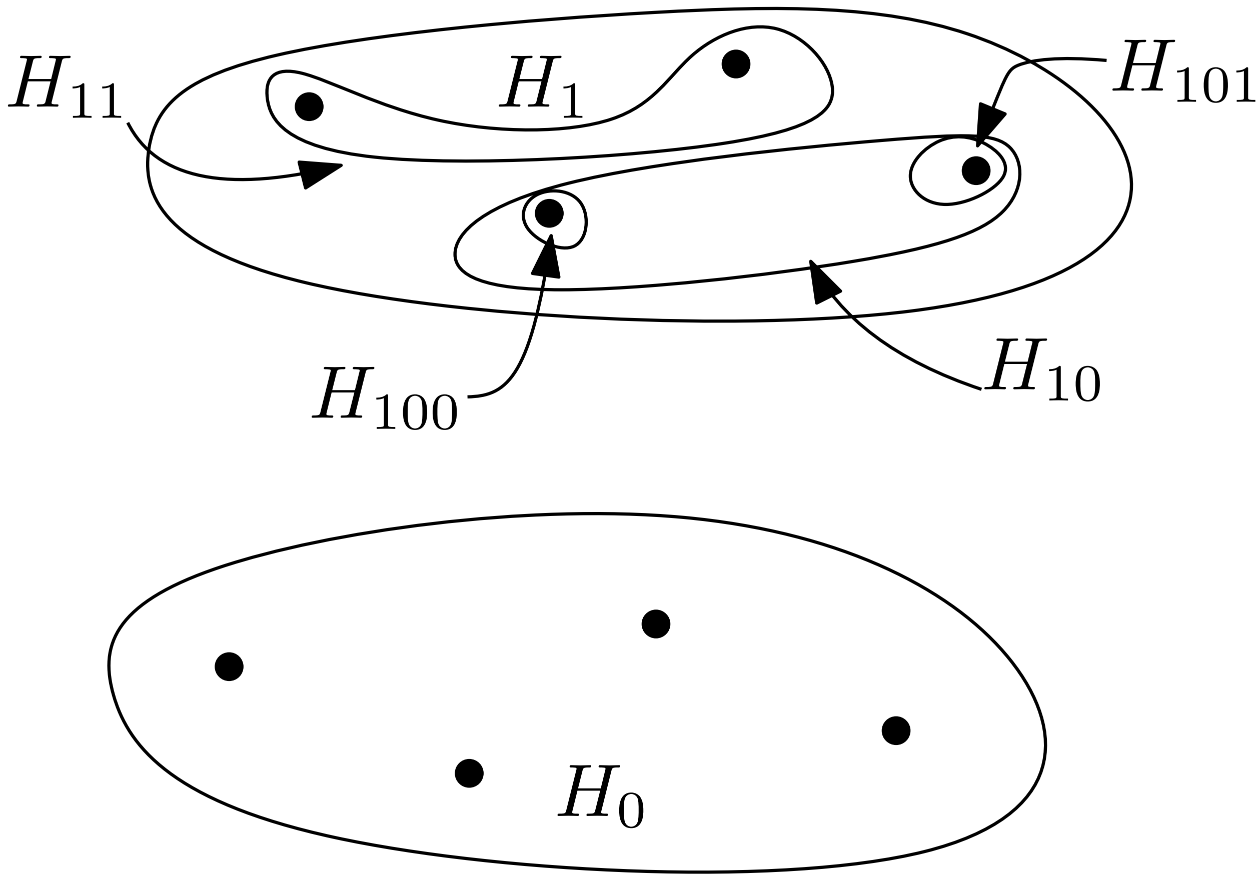


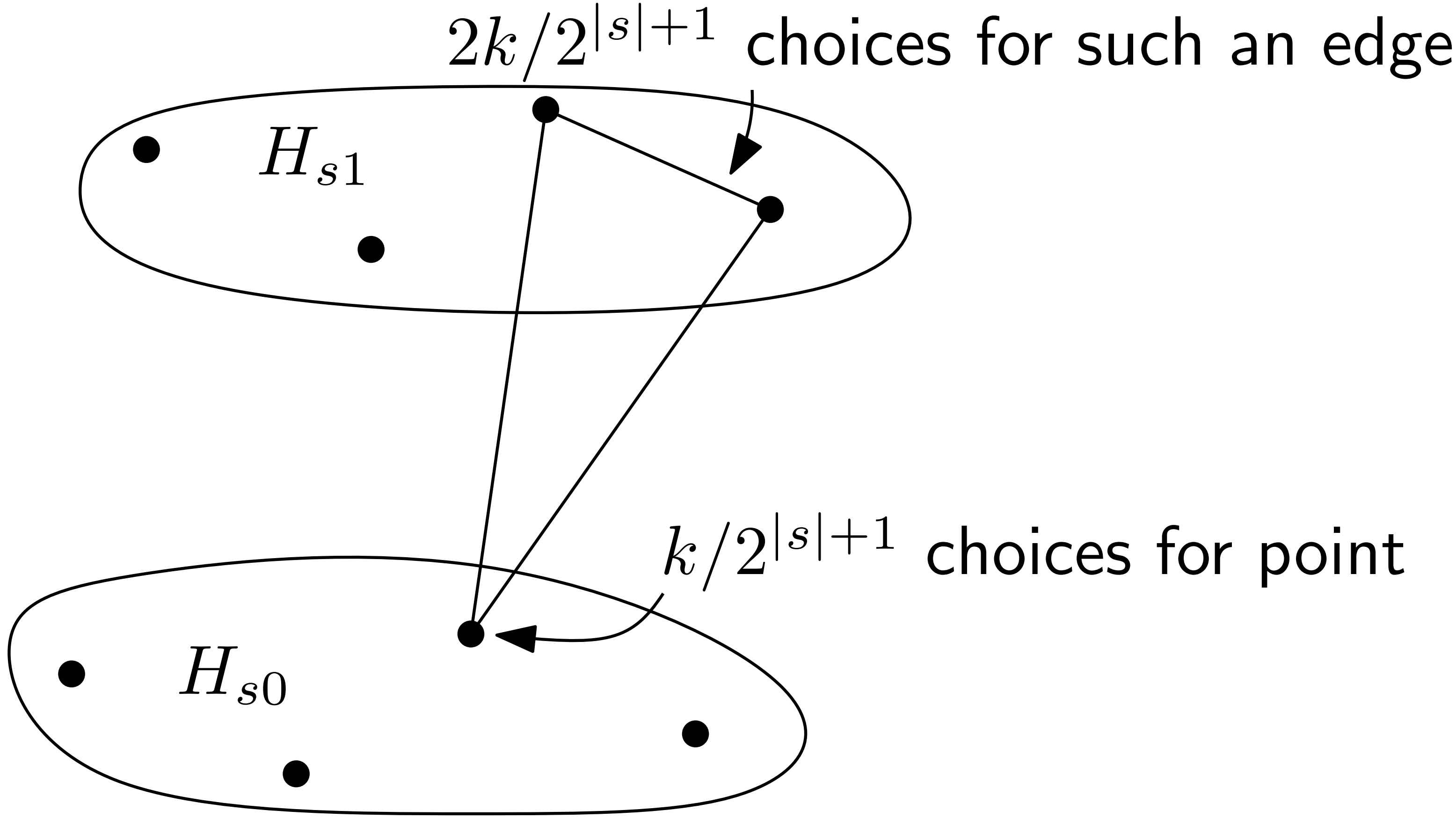
Horton set

Let s be a 01-string, $H_\emptyset = H$ a Horton set on k points.
$H_s = H_{s0} \cup H_{s1}$, H_{s1} high above H_{s0}, H_{s0} far below H_{s1}.



Horton set

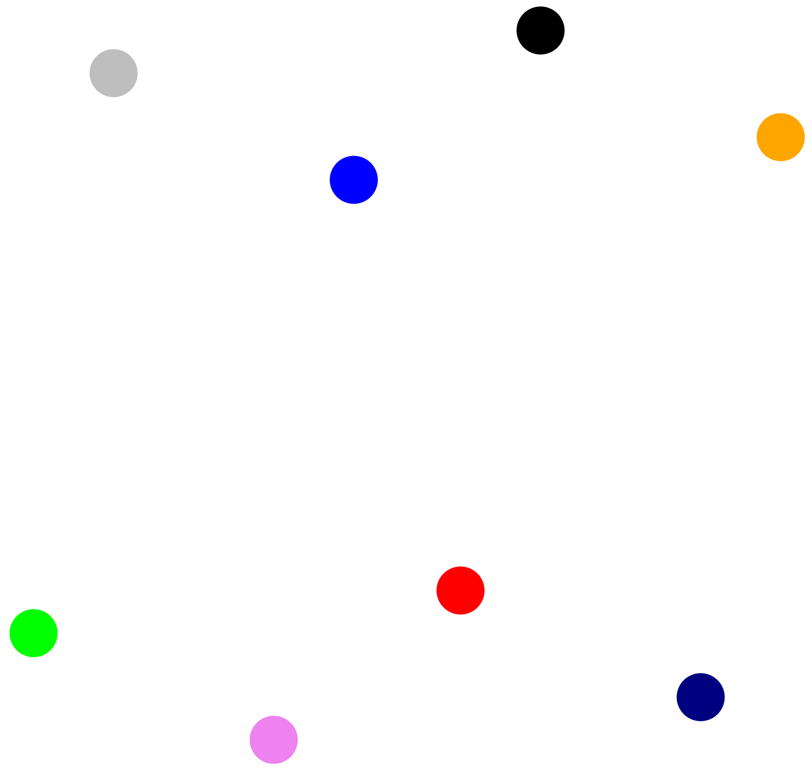
Number of points in H_s: $k/2^{|s|}$



$k^2/2^{|s|}$ empty triangles in H_s, not in H_{s0} or H_{s1}.

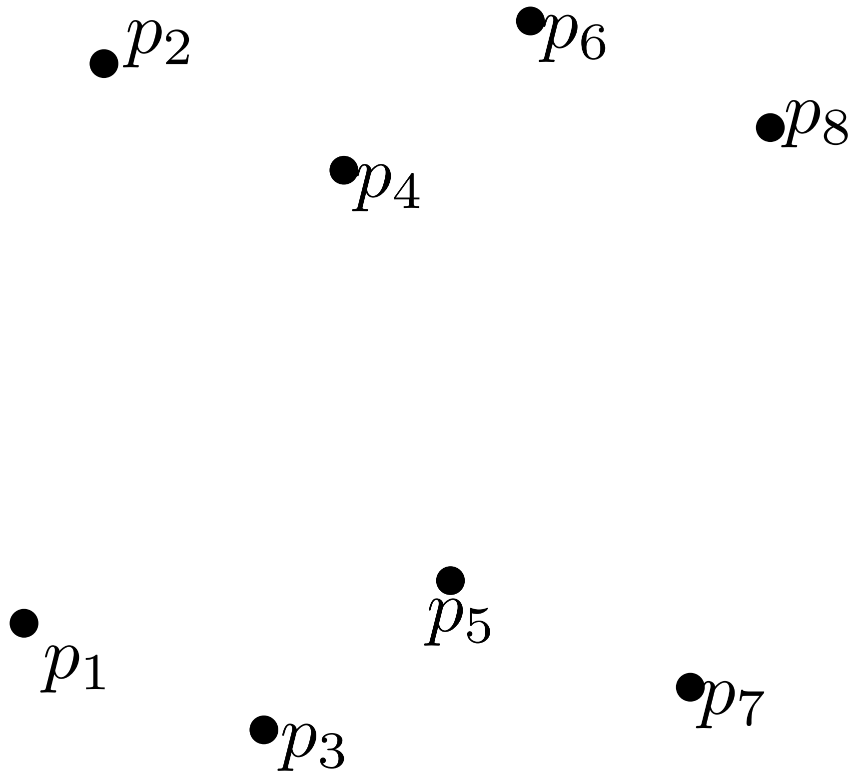
Colored point set

Idea: Take a Horton set on k points and replace each point with a cluster of same colored points.



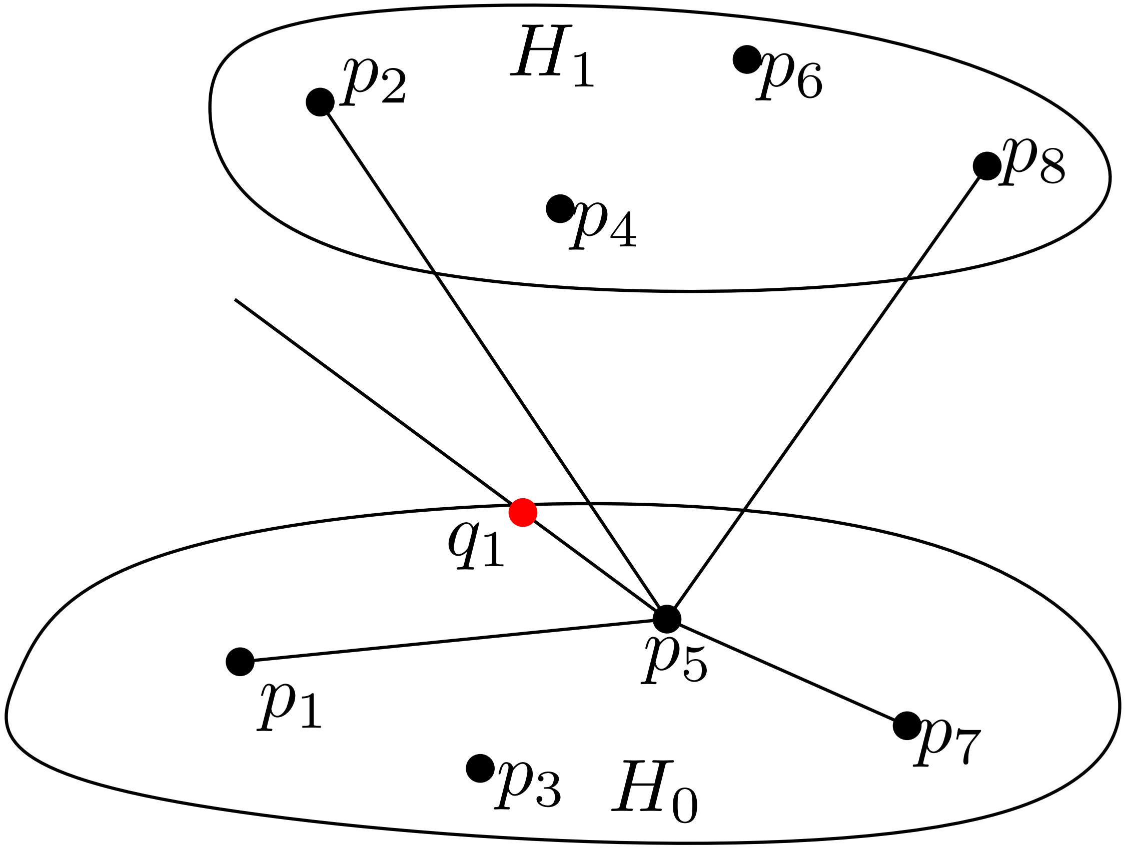
Colored point set

Construction of cluster: Every cluster is small and behaves like a point.



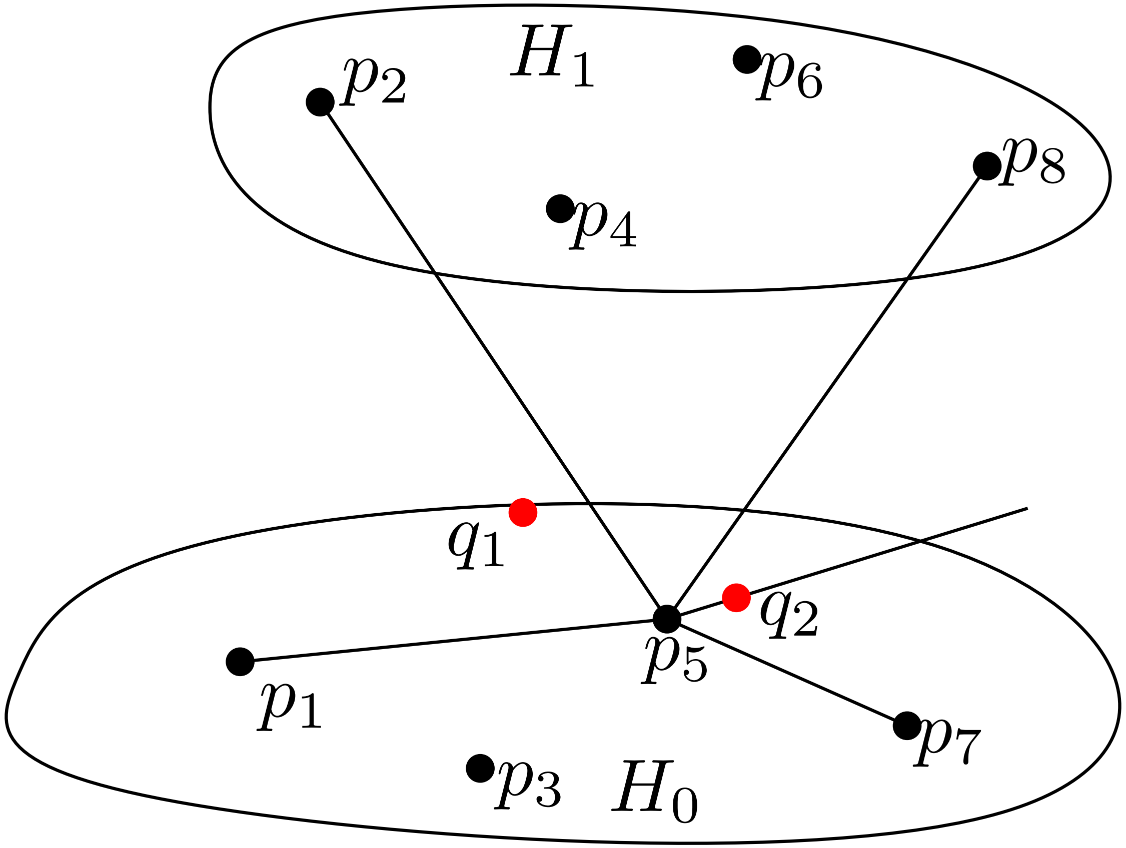
Colored point set

Order points in angular order around p_i.
Place a point q_1 between the last point of H_0 and the first point of H_1 with distance ε_1 from p_i.



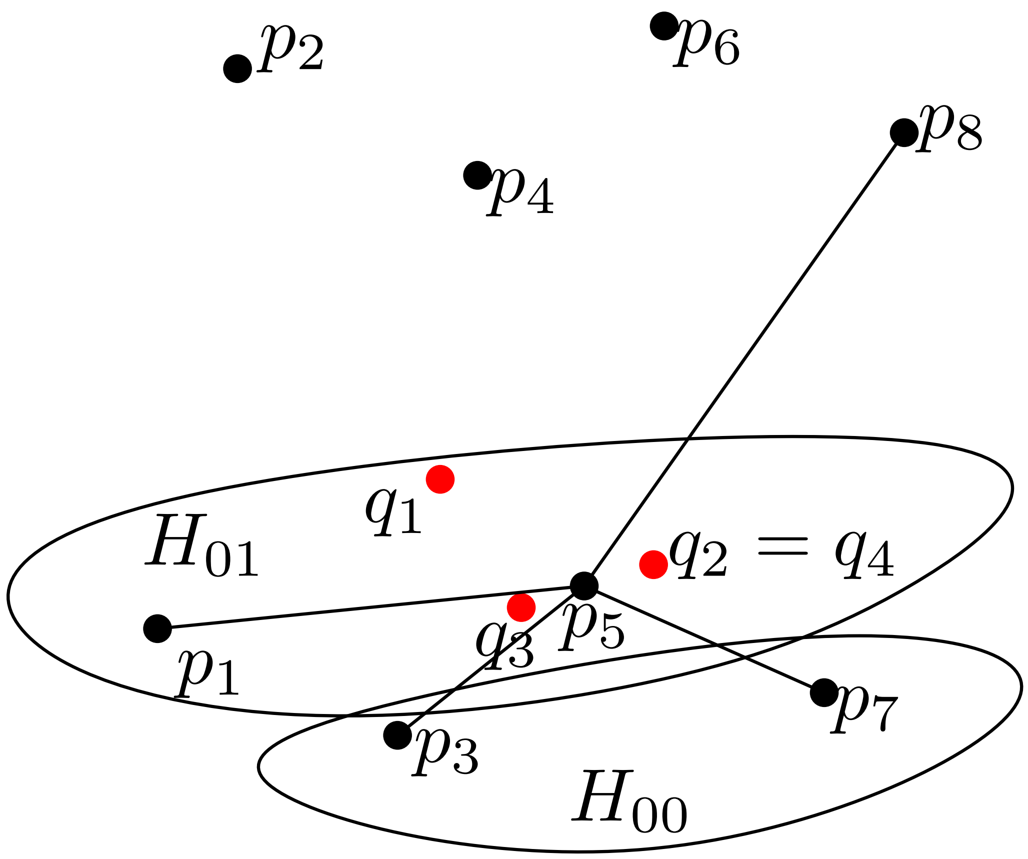
Colored point set

Place a point q_2 between the first point of H_0 and the last point of H_1 with distance ε_2 from p_i, where ε_2 is much smaller than ε_1.



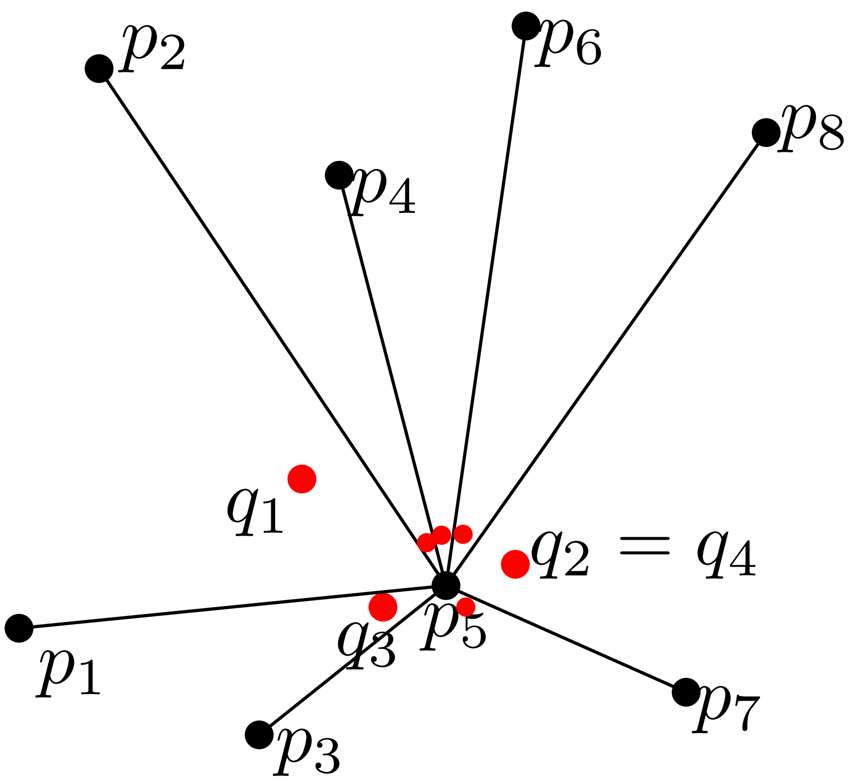
Colored point set

Continue in the same way and place new points closer to p_i than the points before.



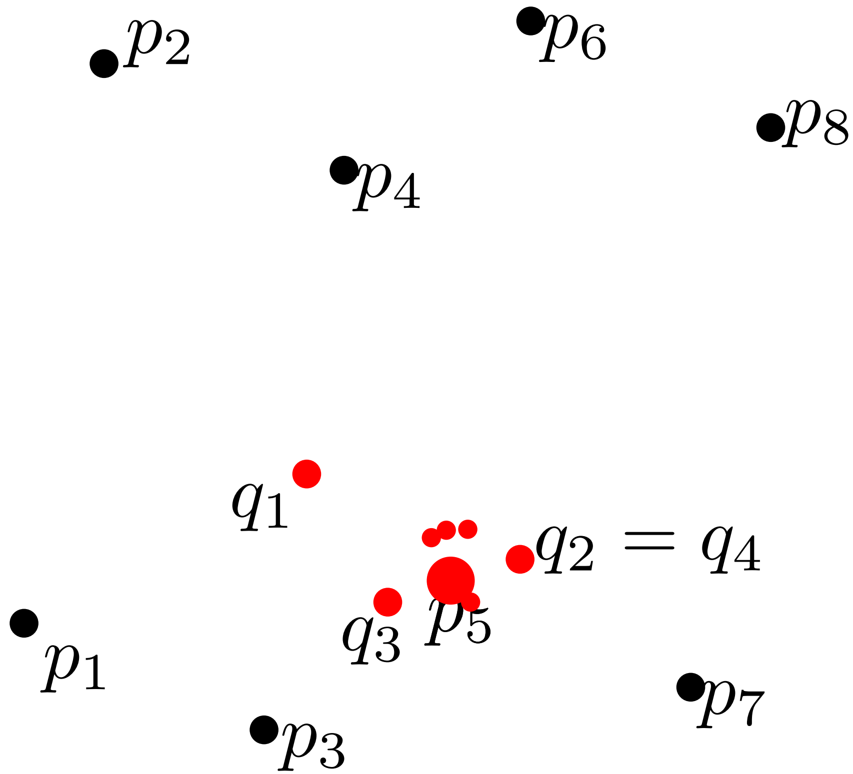
Colored point set

Place points in every angle, which does not contain a point of the cluster.



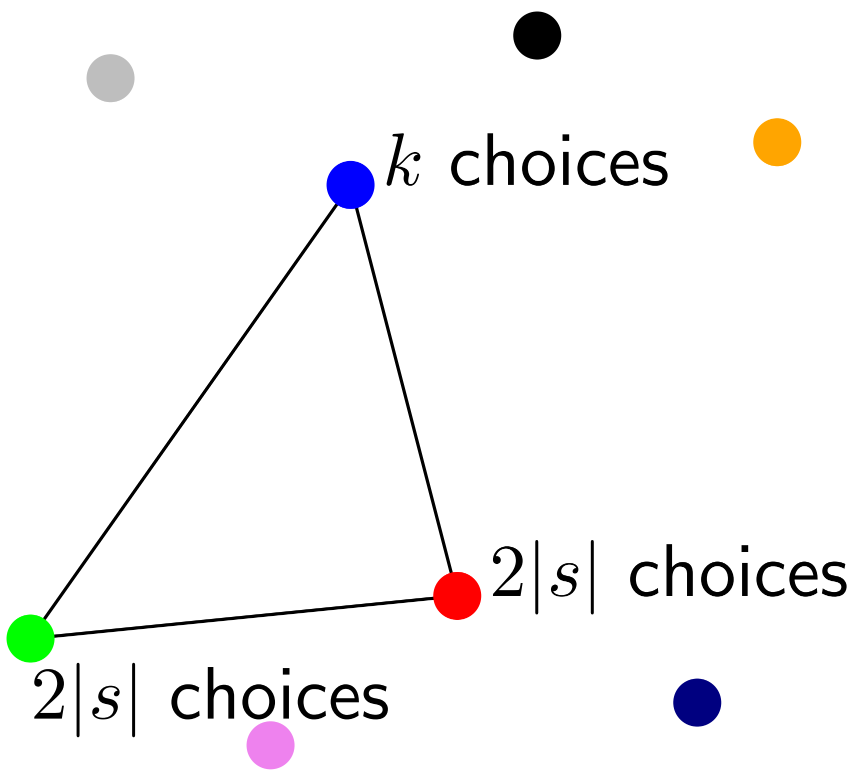
Colored point set

Place the remaining points sufficiently close to p_i.
Only the first k added points can be in an empty rainbow triangle.



Colored point set

How many such triangles exist?



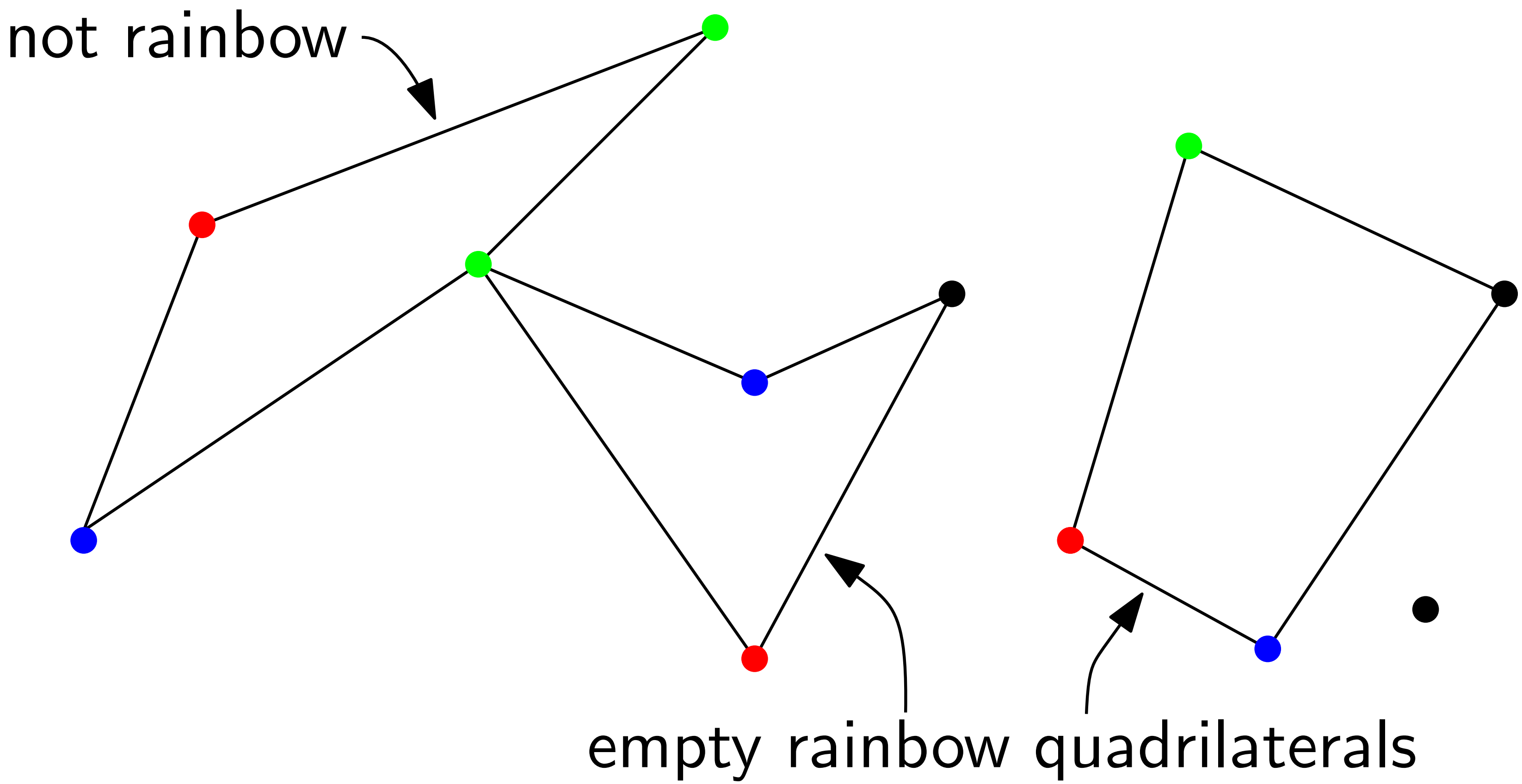
For an empty triangle in H_s there are $4|s|^2k$ empty rainbow triangles in colored point set.

Colored point set

At most $8(k^2/2^{2|s|})$ empty triangles in H_s.
For every such triangle, we have at most $4|s|^2k$ rainbow triangles in P.
$\sum_{|s|\leq \log_2(k)} 8(k^2/2^{2|s|})(4|s|^2k) = 192k^3 + O(k^2)$

Empty rainbow quadrilaterals

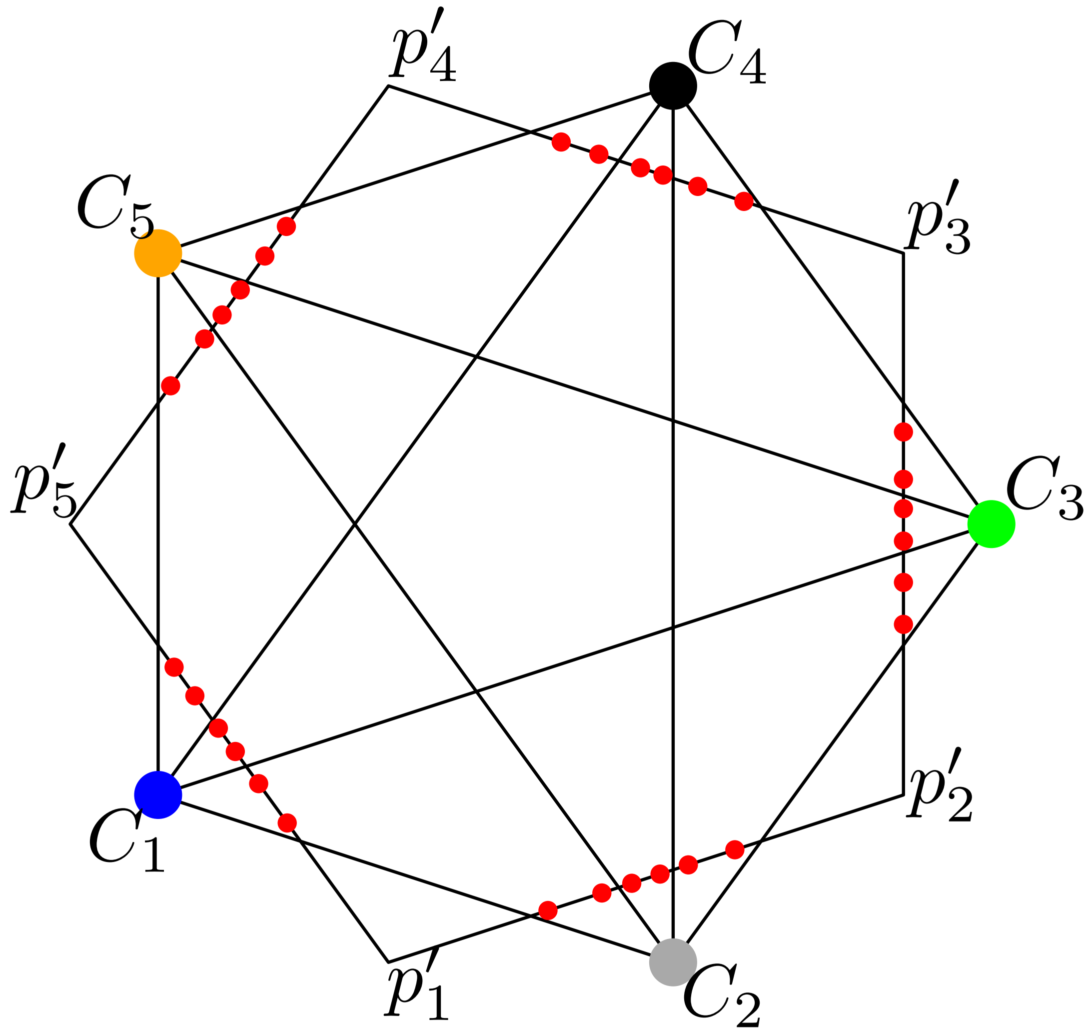
Does there always exist an empty rainbow quadrilateral?
Empty and rainbow like with triangles.
The quadrilateral can be convex or concave.



Empty rainbow quadrilaterals

Does there always exist an empty rainbow quadrilateral?

Points set contains the clusters C_1, C_2, C_3, C_4, C_5 and the red points.



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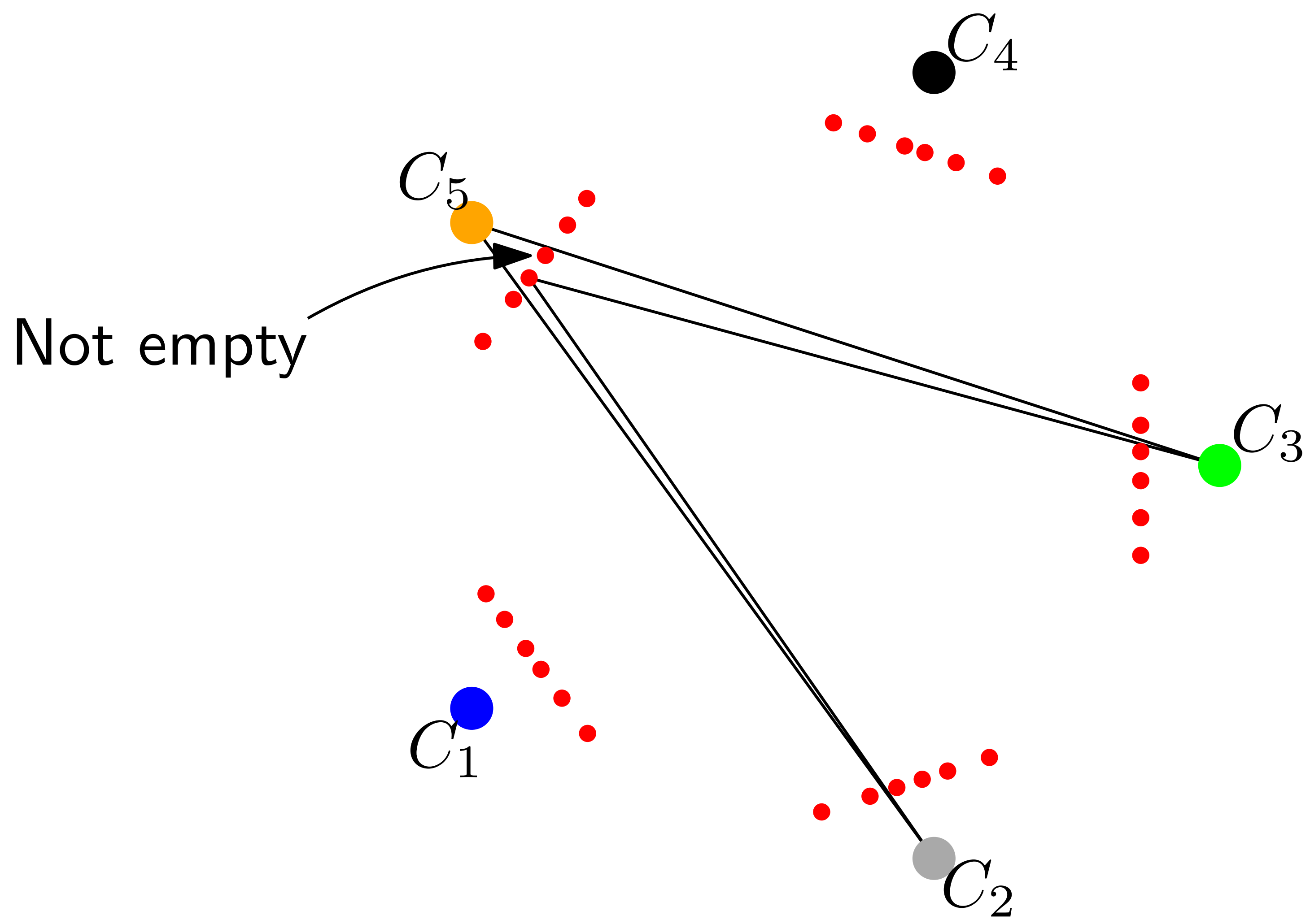

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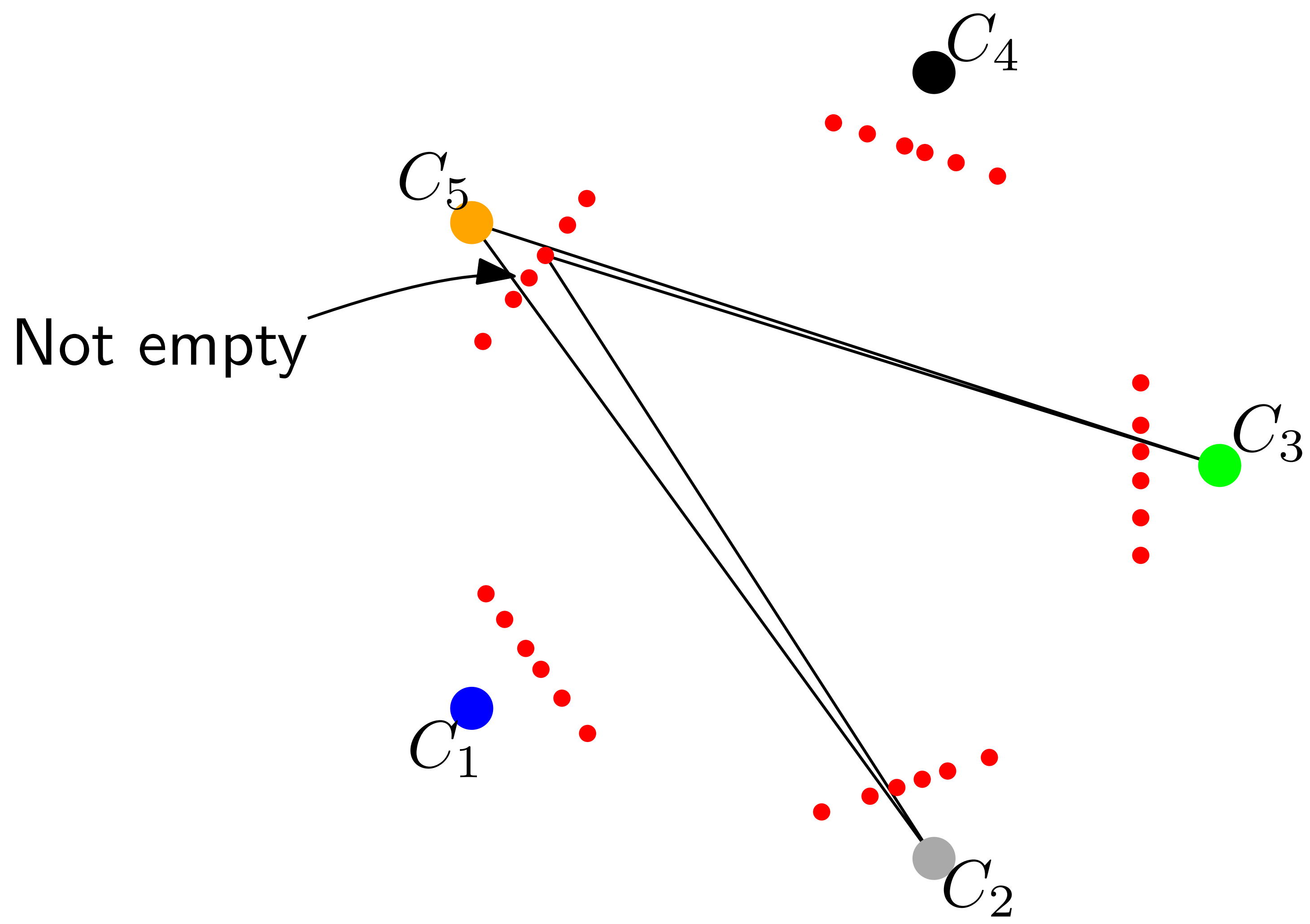
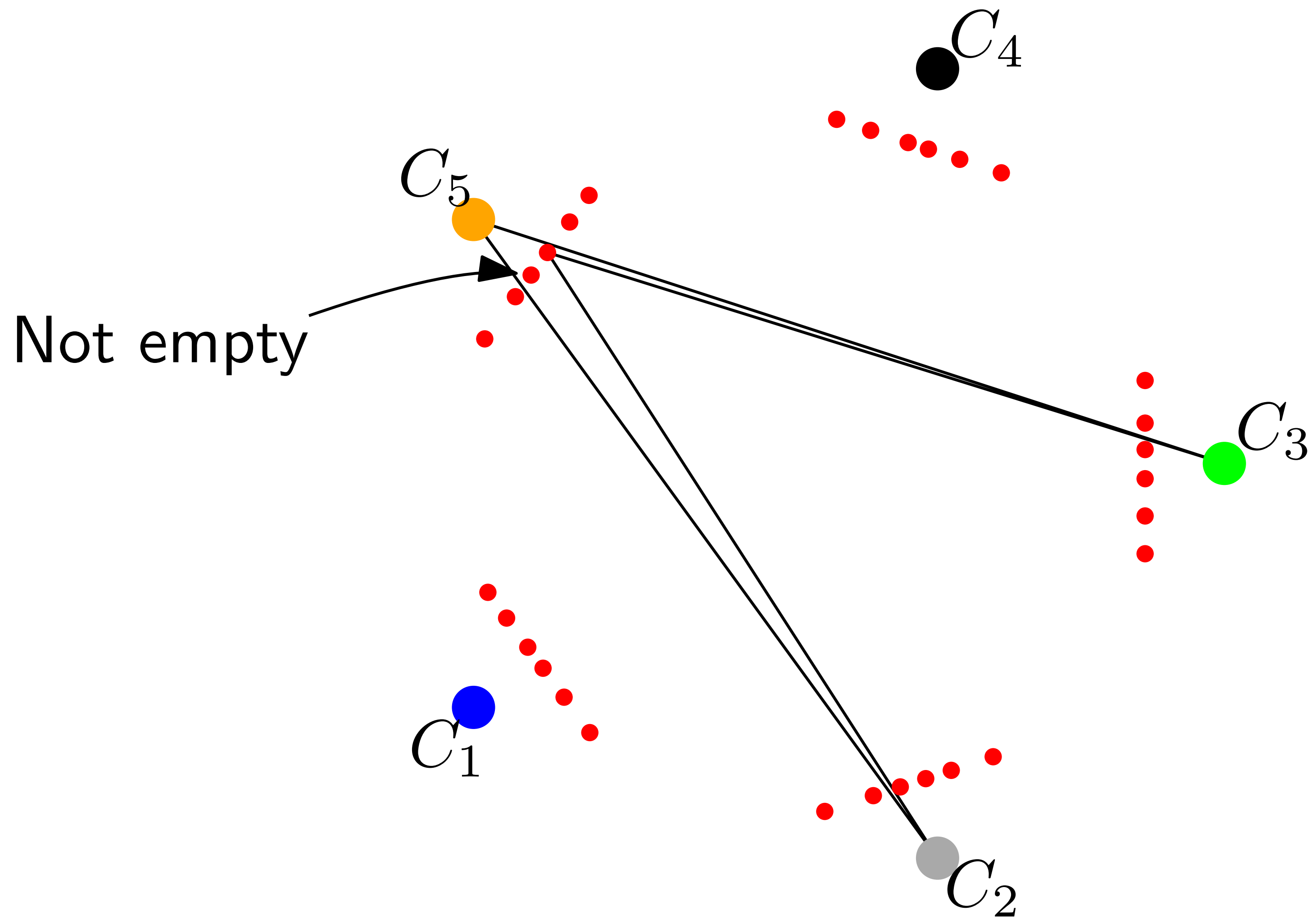


Empty rainbow quadrilaterals

Does there always exist an empty rainbow quadrilateral?

No



Summary and open problems

Number of empty rainbow triangles in a k-colored point set: $\Theta(k^3)$, for $n \geq k^2$.

Problem 1: What happens if $n < k^2$?

There exist point sets without empty rainbow quadrilaterals.

Problem 2: Does every sufficiently large k-colored ($k \geq 4$) point set with the same number of points in each color class contain an empty rainbow 4-gon or an empty monochromatic 4-gon?