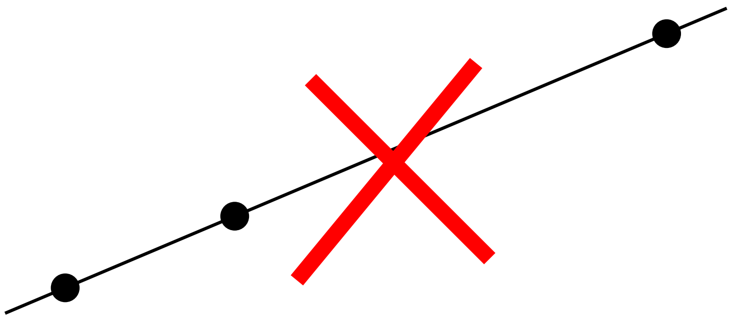
Holes and Islands in Random Point Sets

Martin Balko, Manfred Scheucher, Pavel Valtr

k-Gons

a finite point set S in the plane is
in general position if $\nexists$ collinear points in S



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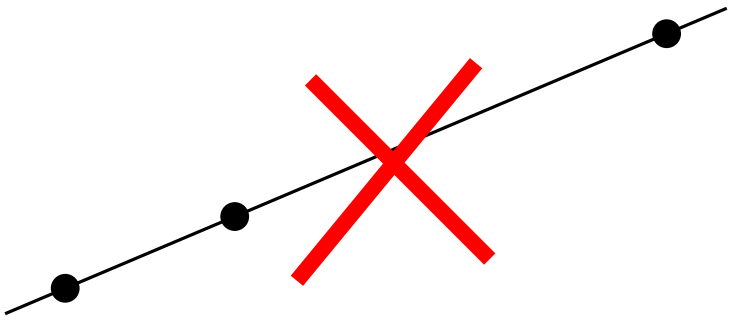

throughout this presentation, every set is in general position

k-Gons

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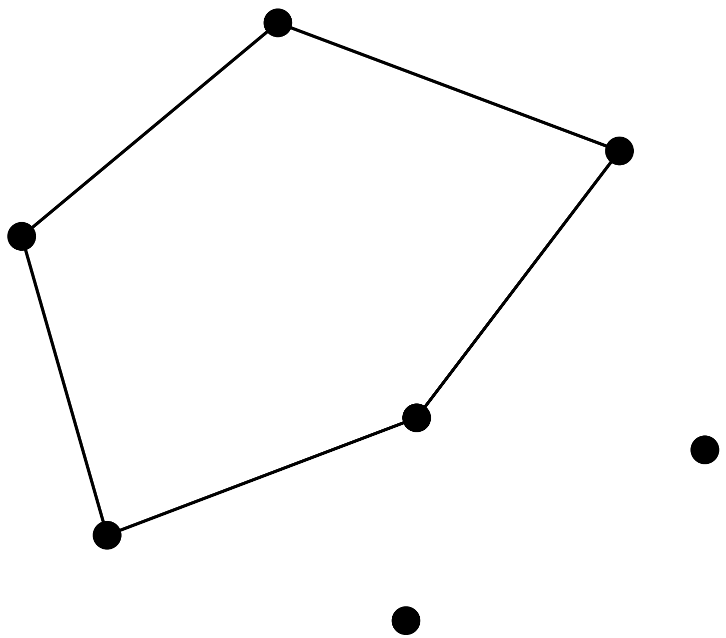
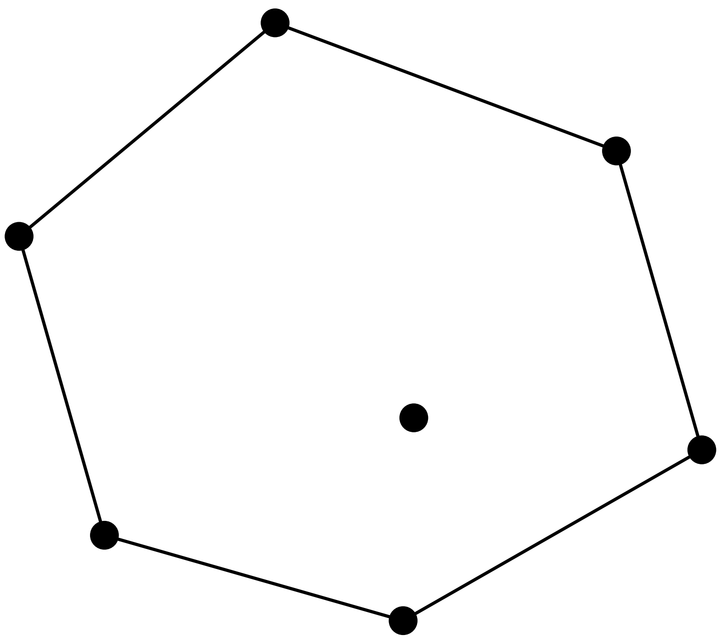
a k-gon (in S) is the vertex set of a convex k-gon



5-gon



6-gon

k-Gons

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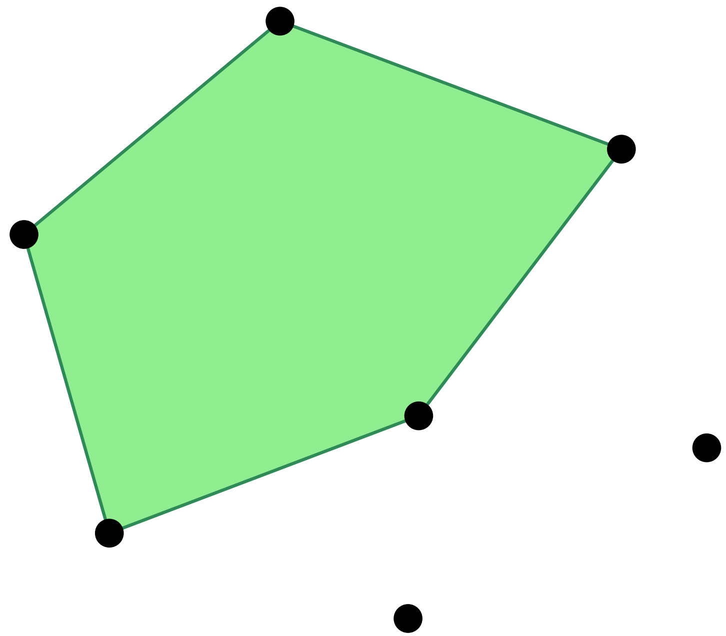
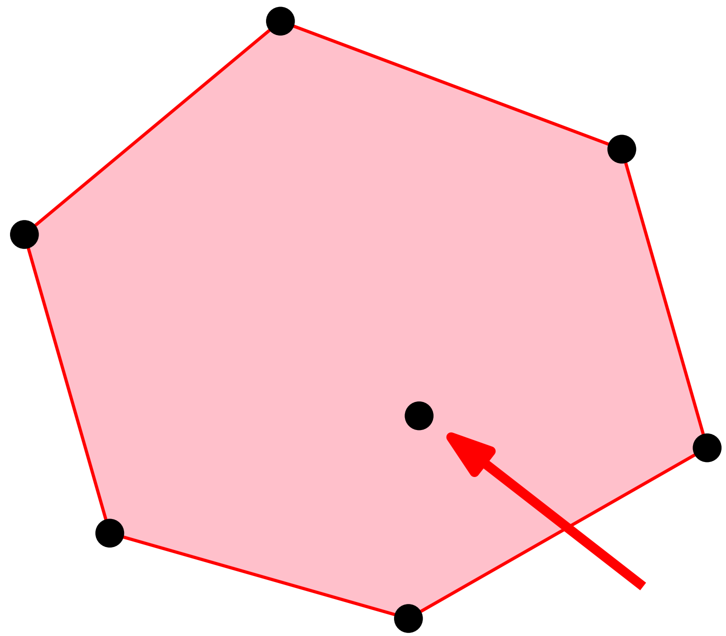
Theorem (Erdős and Szekeres 1935).
$\forall\ k \in \mathbb{N}$, $\exists$ a smallest integer $ES(k)$ such that
every set of $ES(k)$ points contains a k-gon.

k-Holes

a k-hole (in S) is the vertex set of a convex k-gon containing no other points of S



5-hole



not a 6-hole

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Erdős, 1970's: For k fixed, does every sufficiently large point set contain k-holes?

- 3 points $\Rightarrow \exists$ 3-hole
- 5 points $\Rightarrow \exists$ 4-hole
- 10 points $\Rightarrow \exists$ 5-hole [Harborth '78]
- $\exists$ arbitrarily large point sets with no 7-hole [Horton '83]
- Sufficiently large point sets $\Rightarrow \exists$ 6-hole [Gerken '08 and Nicolás '07, independently]

Counting k-Holes

$h_k(n)$:= minimum # of k-holes among all sets of n points

[Bárány and Füredi '87, Bárány and Valtr '04]

- $h_3(n)$ and $h_4(n)$ both in $\Theta(n^2)$
- $h_5(n)$ in $\Omega(n \log^{4/5} n)$ and $O(n^2)$
  [Aichholzer, Balko, Hackl, Kynčl, Parada, S., Valtr, and Vogtenhuber '17]
- $h_6(n)$ in $\Omega(n)$ and $O(n^2)$
  [Gerken '08, Nicolás '07]
- $h_k(n) = 0$ for $k \geq 7$ [Horton '83]

Holes in Higher Dimensions

- $\exists$ d-dimensional Horton sets not containing k-holes for sufficiently large $k = k(d)$ [Valtr '92]

- minimum number of empty simplices $(d+1)$-holes) in n-point set in $\mathbb{R}^d$ is in $\Theta(n^d)$ [Bárány and Füredi '92]

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Random Point Sets

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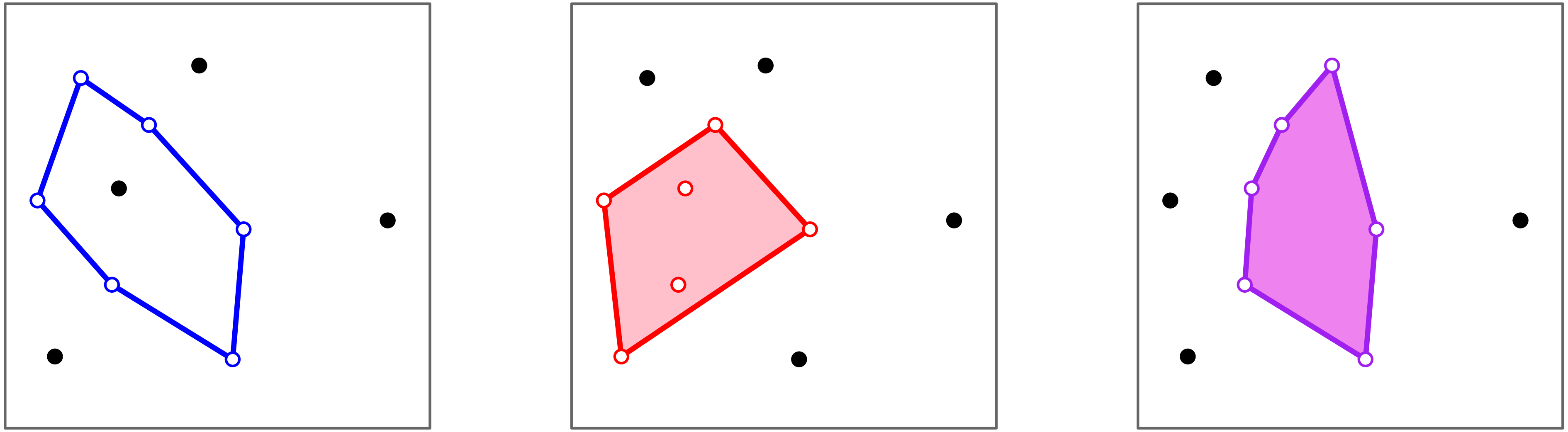
- $EH^K_{d,k}(n)$:= expected number of k-holes in sets of n points chosen independently and uniformly at random from convex shape $K \subset \mathbb{R}^d$
- Bárány and Füredi (1987) showed

$$EH^K_{d,d+1}(n) \leq \underbrace{(2d)^{2d^2} \cdot \binom{n}{d}}_{O(n^d)}$$

Our Results I

- extend bound to larger holes, and even to islands
- $I \subseteq S$ is an island (in S) if $S \cap \mathrm{conv}(I) = I$
- "hole = gon + island"



gon island hole

Our Results I

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Theorem 1. Let $d \geq 2$ and $k \geq d+1$ be integers, and let K be a convex body in $\mathbb{R}^d$. If S is a set of n points chosen uniformly and independently at random from K, then the expected number of k-islands in S is at most

$$\underbrace{2^{d-1} \cdot \left(2d^{2d-1}\binom{k}{\lfloor d/2 \rfloor}\right)^{k-d-1} \cdot (k-d)}_{} \cdot \underbrace{\frac{n(n-1)\cdots(n-k+2)}{(n-k+1)^{k-d-1}}}_{}$$

$O(n^d)$

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- In particular:
  $\exists$ sets of n points in $\mathbb{R}^d$ with $O(n^d)$ k-islands

Our Results II

- the bound from Theorem 1 is asymptotically optimal, but the leading constant can be improved for k-holes

- for empty simplices in $\mathbb{R}^d$, we have a better bound

$$EH^K_{d,d+1}(n) \leq 2^{d-1} \cdot d! \cdot \binom{n}{d}$$

- for 4-holes in $\mathbb{R}^2$, we have $EH^K_{2,4}(n) \leq 12n^2 + o(n^2)$

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- very recently, Reitzner and Temesvari proved an asymptotically tight bound for $EH^K_{d,d+1}(n)$ if $d = 2$ or if $d \geq 3$ and K is an ellipsoid

Our Results III

- Theorem 1 is the first nontrivial bound for k-islands in $\mathbb{R}^d$ for $d > 2$
- In the plane, the $O(n^2)$ bound is achieved by Horton sets [Fabila-Monroy and Huemer '12]
- however, d-dimensional Horton sets with $d > 2$ do not give the $O(n^d)$ bound on k-islands

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Theorem 3. Let $d \geq 2$ and let k be fixed positive integers.
Then every d-dimensional Horton set H with n points contains at least $\Omega(n^{\min\{2^{d-1},k\}})$ k-islands.
If $k \leq 3 \cdot 2^{d-1}$, then H even contains at least $\Omega(n^{\min\{2^{d-1},k\}})$ k-holes.

Our Results IV

- we cannot have $O(n^d)$ for k-islands if k is not fixed

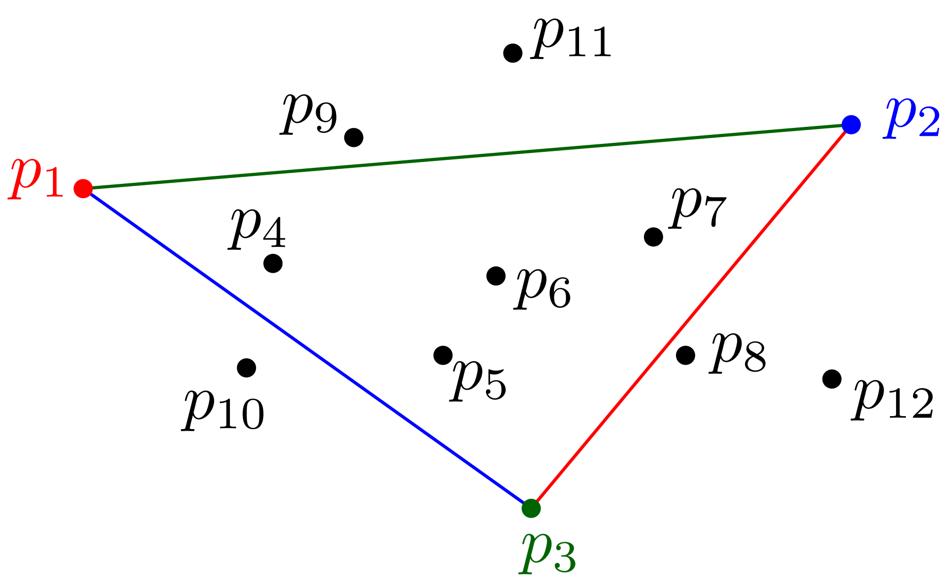
Theorem 3. Let $d \geq 2$ and let K be a convex body in $\mathbb{R}^d$. Then, for every set S of n points chosen uniformly and independently at random from K, the expected number of islands in S is $2^{\Theta(n^{(d-1)/(d+1)})}$.

Idea of the proof of Theorem 1

Rest of this presentation:

idea how to prove the bound $O(n^2)$ on the expected number of k-islands in a set S of n points chosen uniformly and independently at random from convex body $K \subset \mathbb{R}^2$ with area $\lambda(K) = 1$

- We prove an $O(1/n^{k-2})$ bound on the probability that a k-tuple $I = (p_1, \ldots, p_k)$ determines k-island with 2 additional properties:
  - (P1) p_1, p_2, p_3 form largest triangle $\triangle$ in I
  - (P2) $p_4, \ldots, p_{3+a}$ inside $\triangle$; rest outside & incr. dist. to $\triangle$



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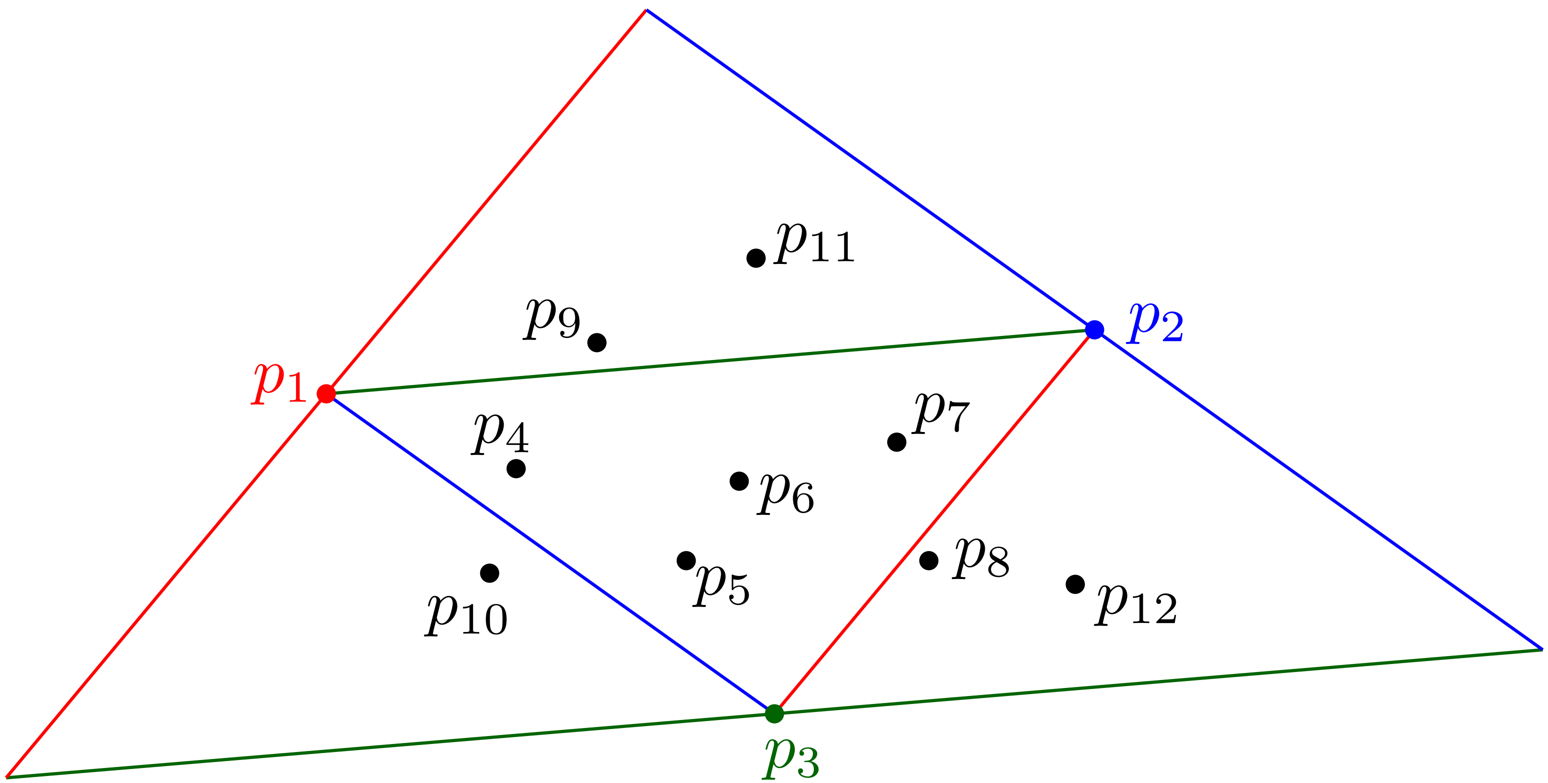

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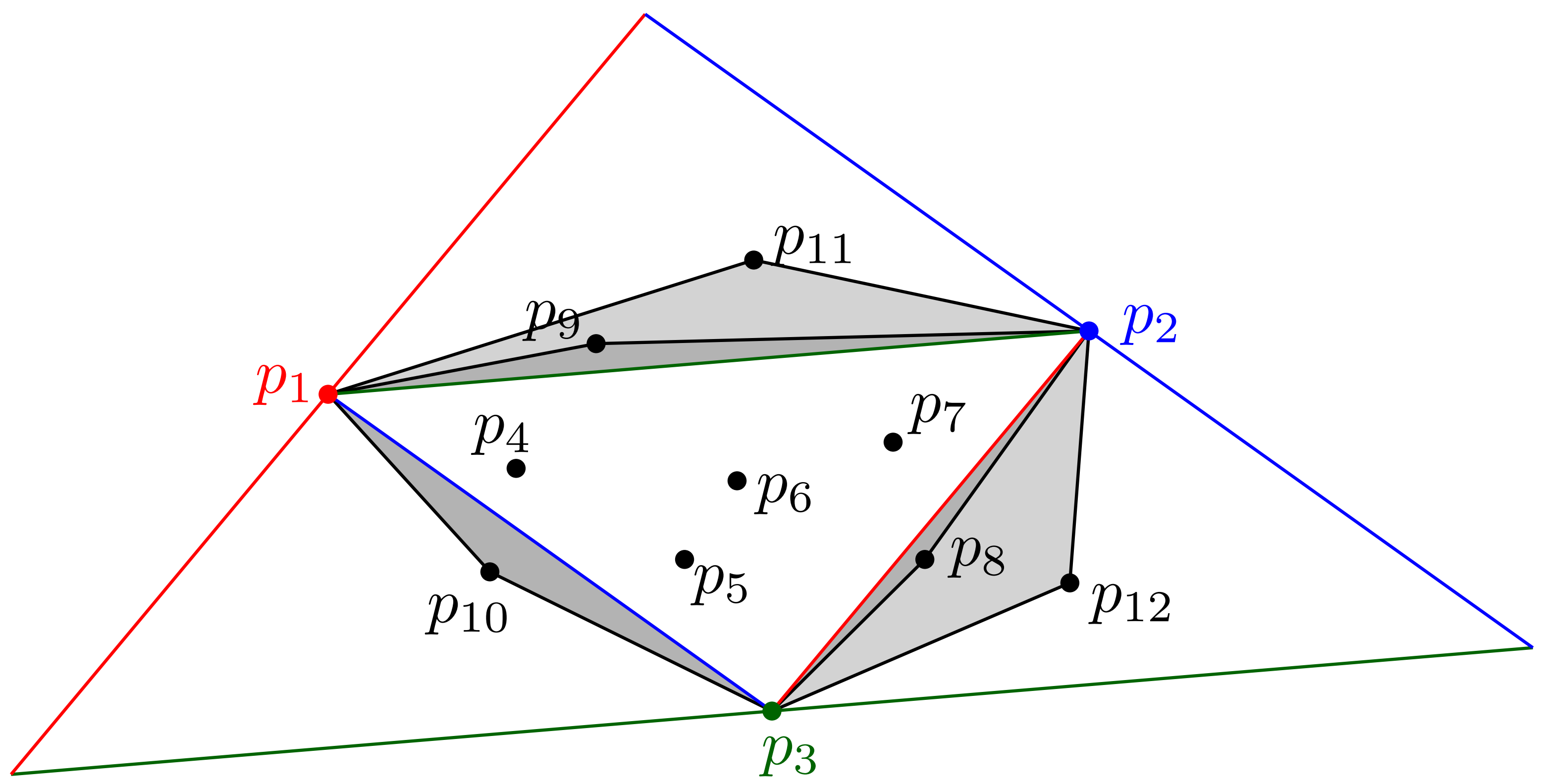

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- First, $\triangle$ contains precisely $p_4, \ldots, p_{3+a}$ with prob. $O(1/n^{a+1})$

$\iff$ $p_1, \ldots, p_{3+a}$ form an island in S satisfying (P1) and (P2)

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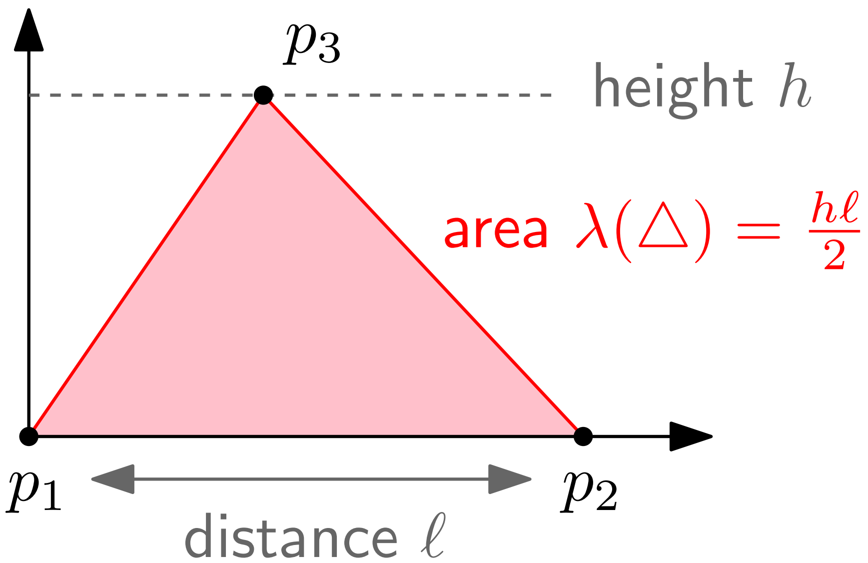

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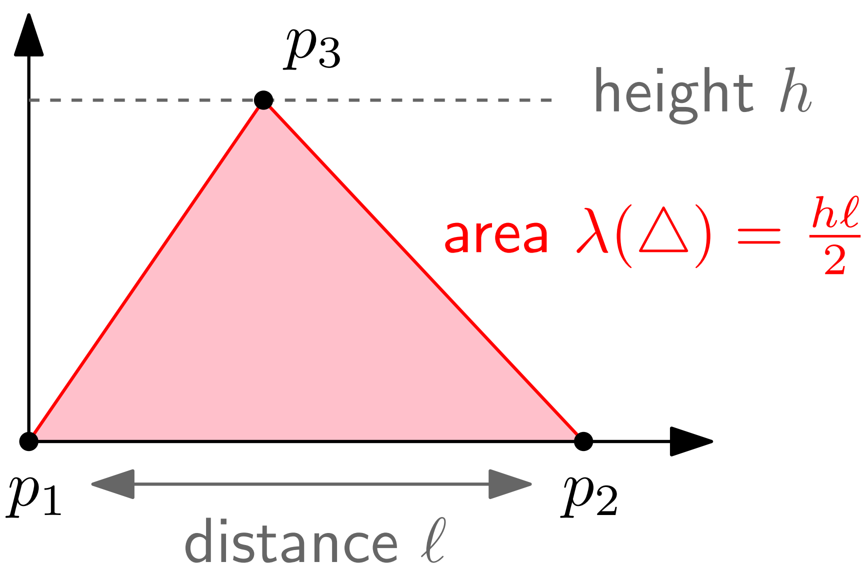

because $\lambda(\triangle) \leq \lambda(K) = 1$

$$\int_{h=0}^{2/\ell} \left(\frac{h\ell}{2}\right)^{a} \left(1 - \frac{h\ell}{2}\right)^{n-3-a} dh$$

a points inside $\quad n-3-a$ outside

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$$\int_{h=0}^{2/\ell} \left(\frac{h\ell}{2}\right)^a \left(1 - \frac{h\ell}{2}\right)^{n-3-a} dh$$

$$\int_{x=0}^{1} x^a (1-x)^{n-3-a} dx = \frac{a! \cdot (n-3-a)!}{(a+n-3-a+1)!} \approx a! \cdot n^{(n-3-a)-(n-2)}$$

(Beta-function)

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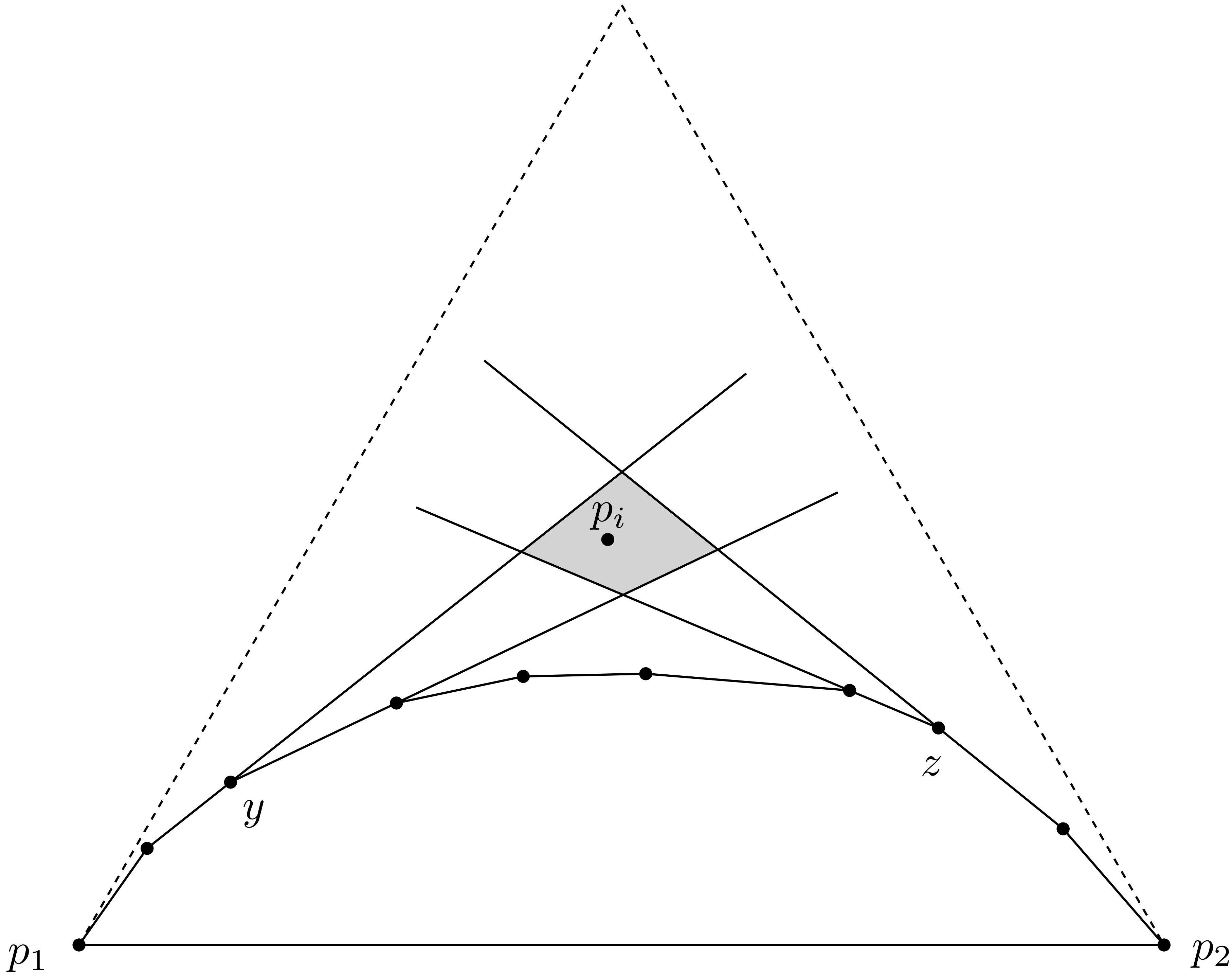
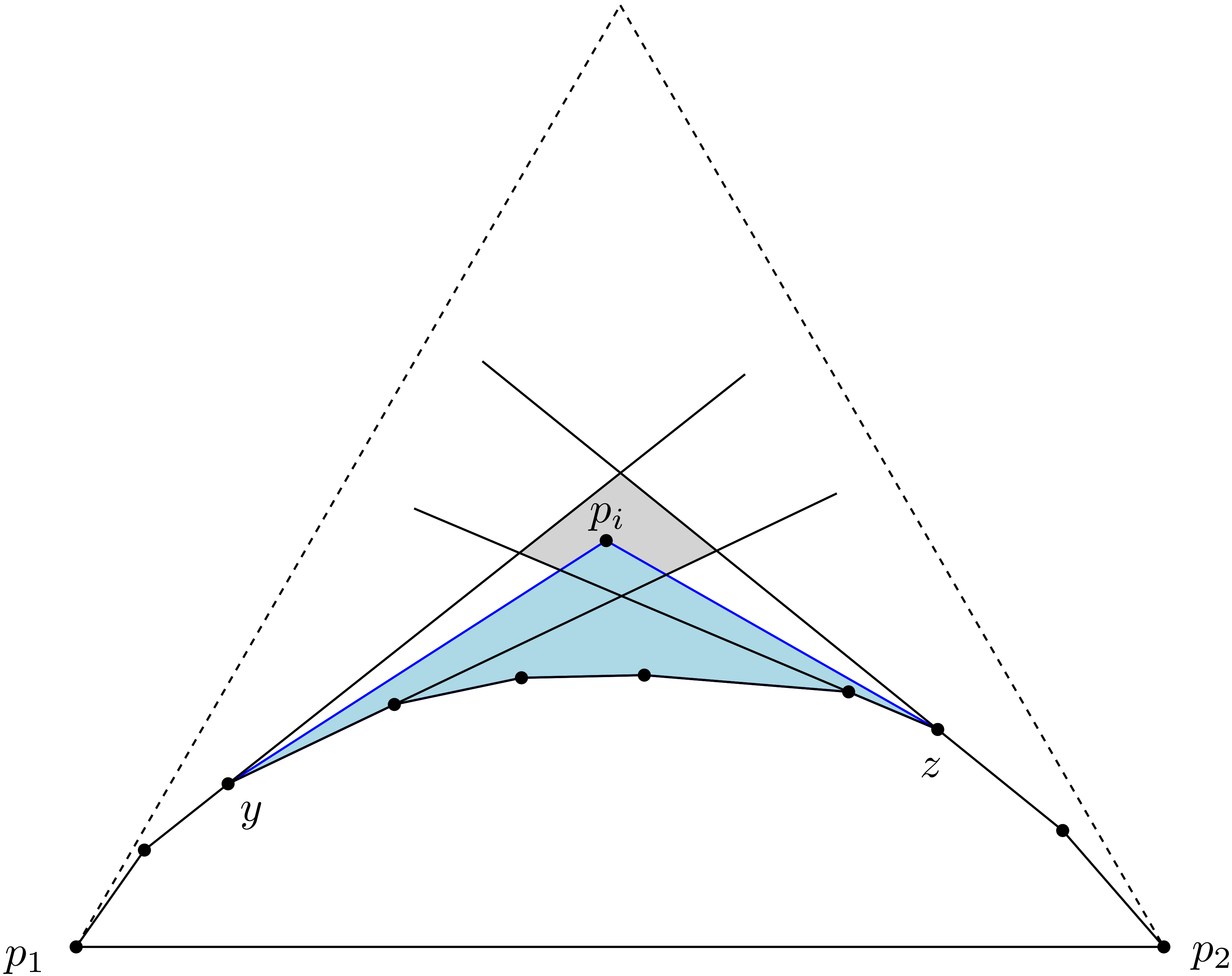

p_i
y
z
p_1
p_2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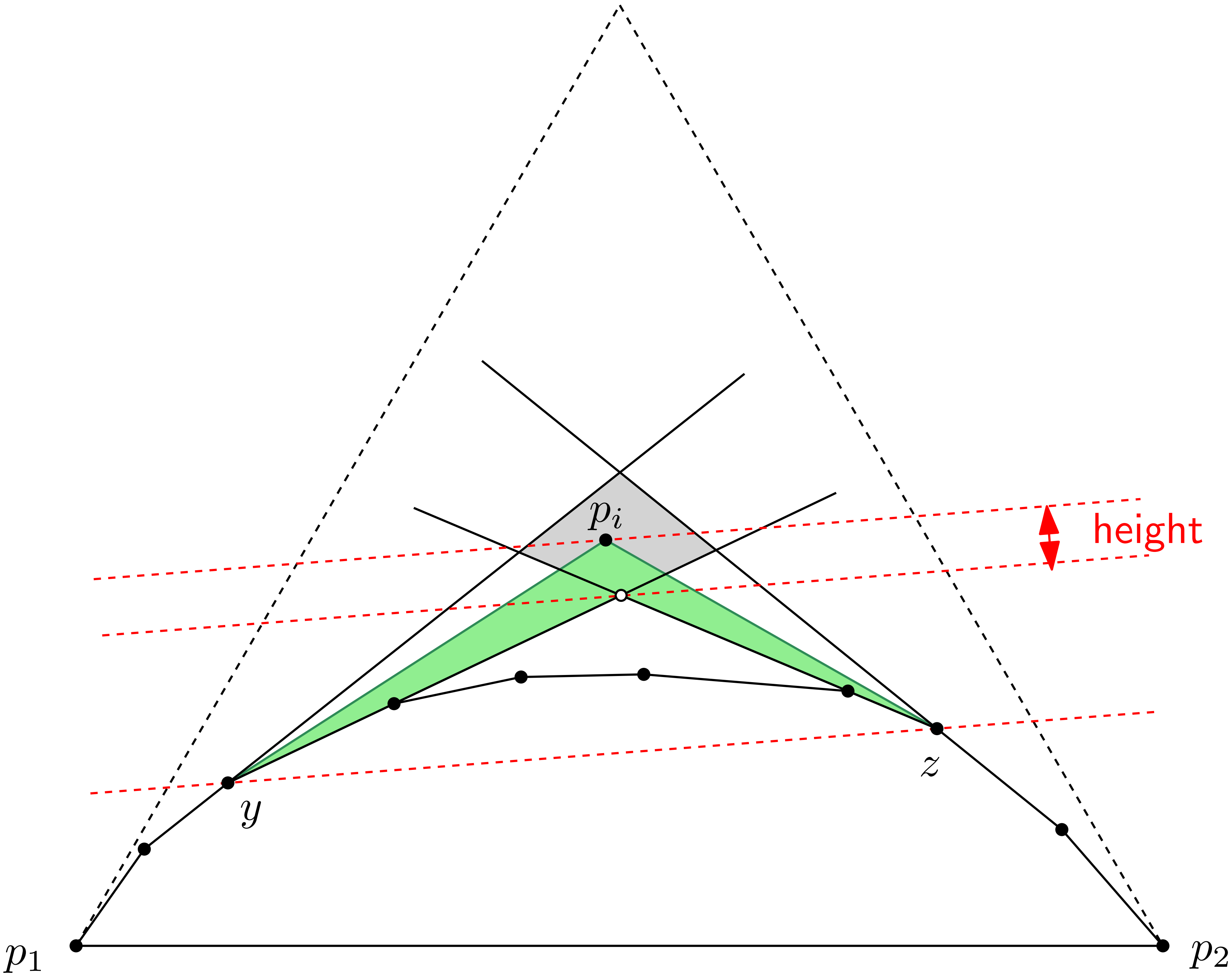

p_i
height
y
z
p_1
p_2

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- $\Rightarrow$ I determines k-island with (P1) and (P2) prob. at most

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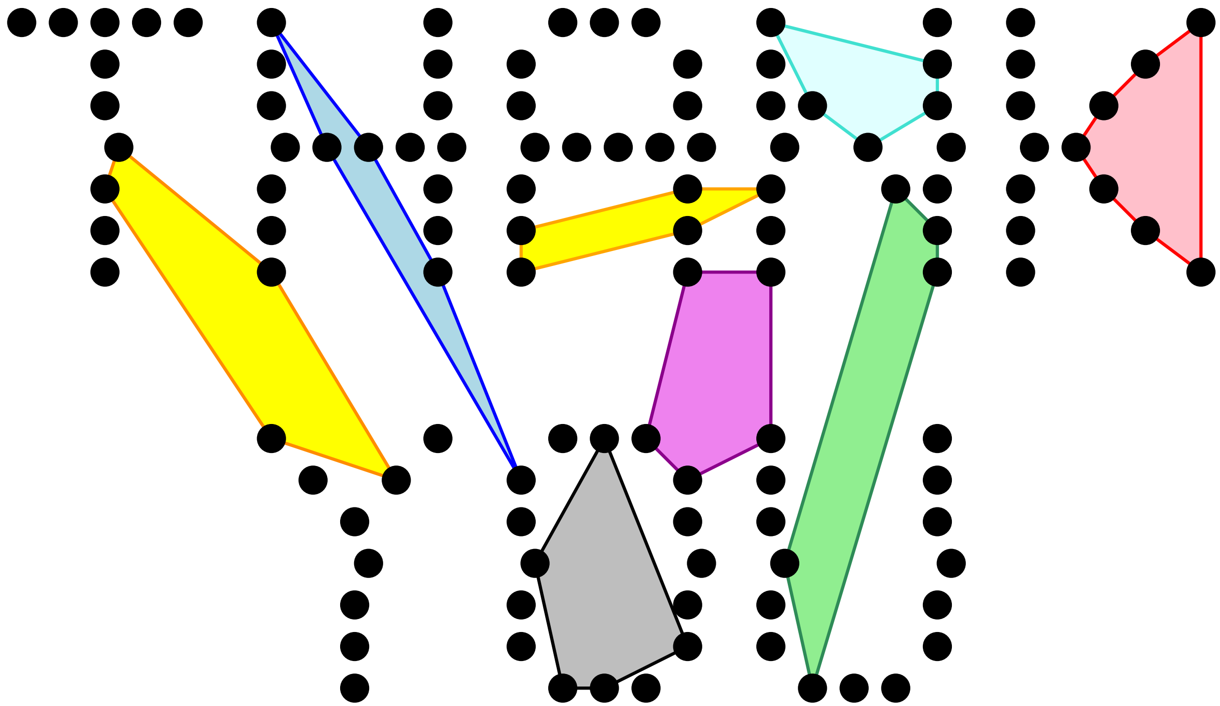
  $$O\left(1/n^{a+1} \cdot (1/n)^{k-(3+a)}\right) = O(1/n^{k-2})$$
- Finally, since there are $n \cdot (n-1) \cdots (n-k+1)$ possibilities to select I, we obtain the desired bound $O(n^k \cdot n^{2-k}) = O(n^2)$ on the expected number of k-islands in S

THANK YOU


THANK
YOU