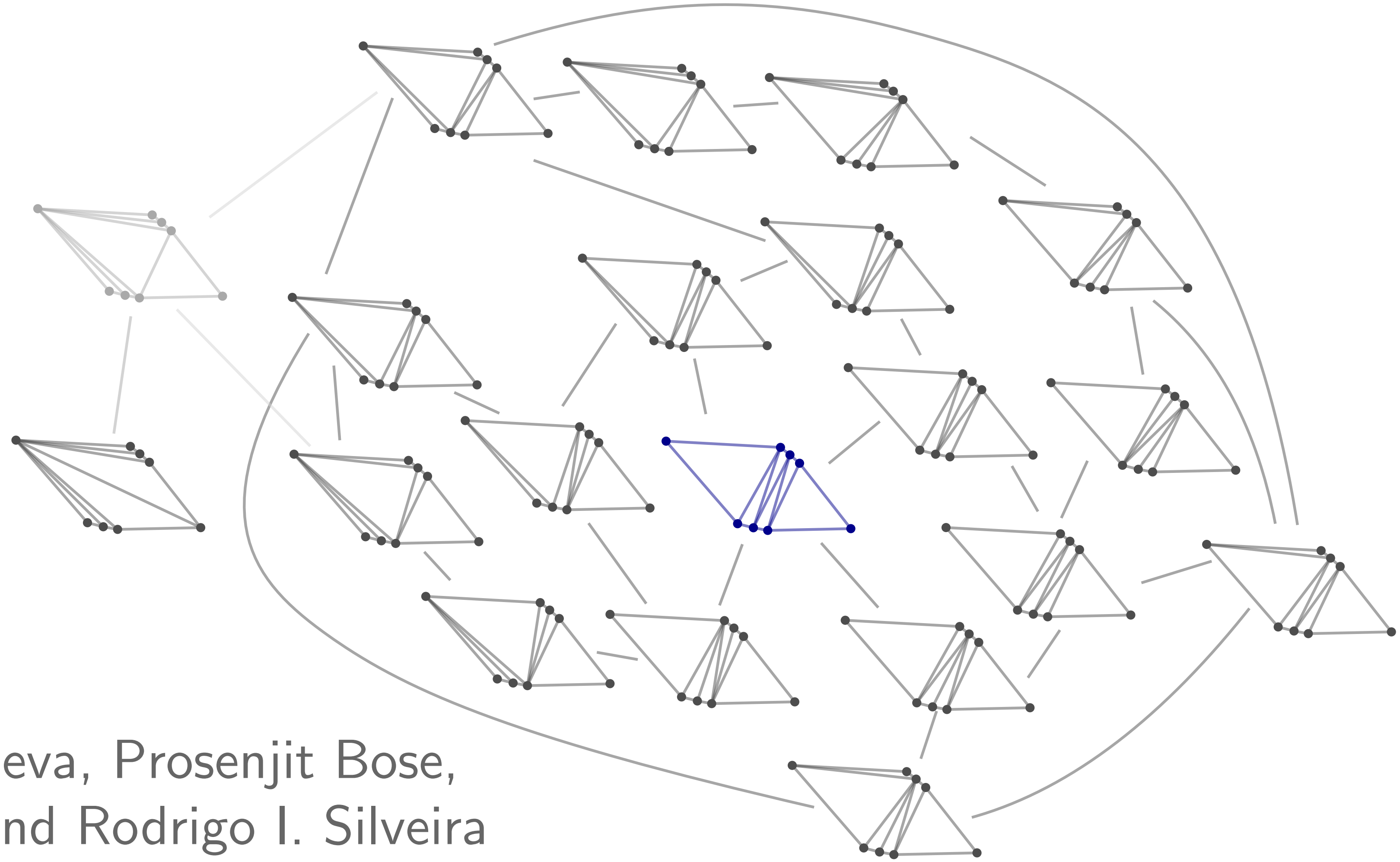
Flips in higher order Delaunay triangulations



Elena Arseneva, Prosenjit Bose,
Pilar Cano, and Rodrigo I. Silveira

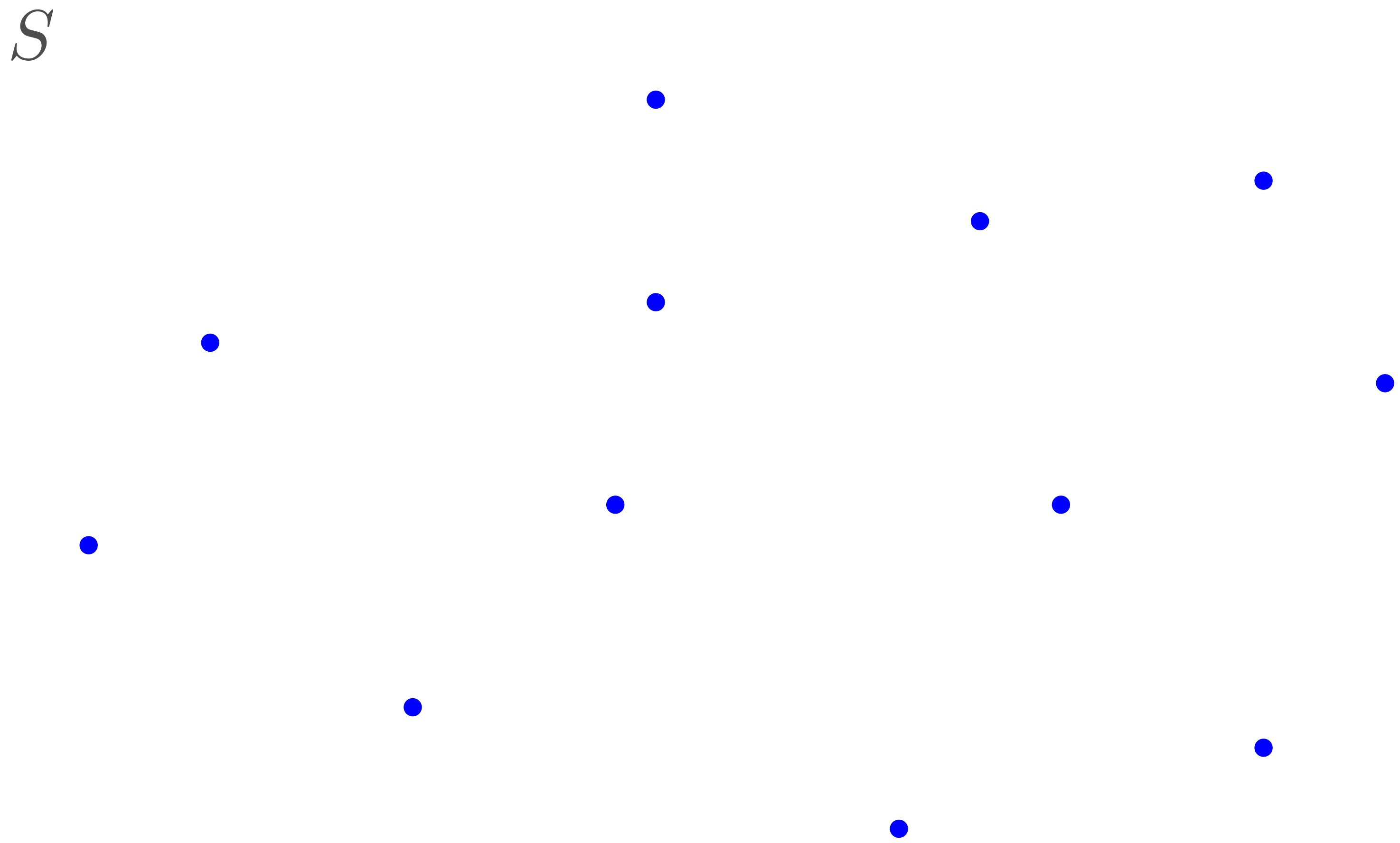
St. Petersburg University

Carleton University

Universitat Politècnica de Catalunya BarcelonaTech

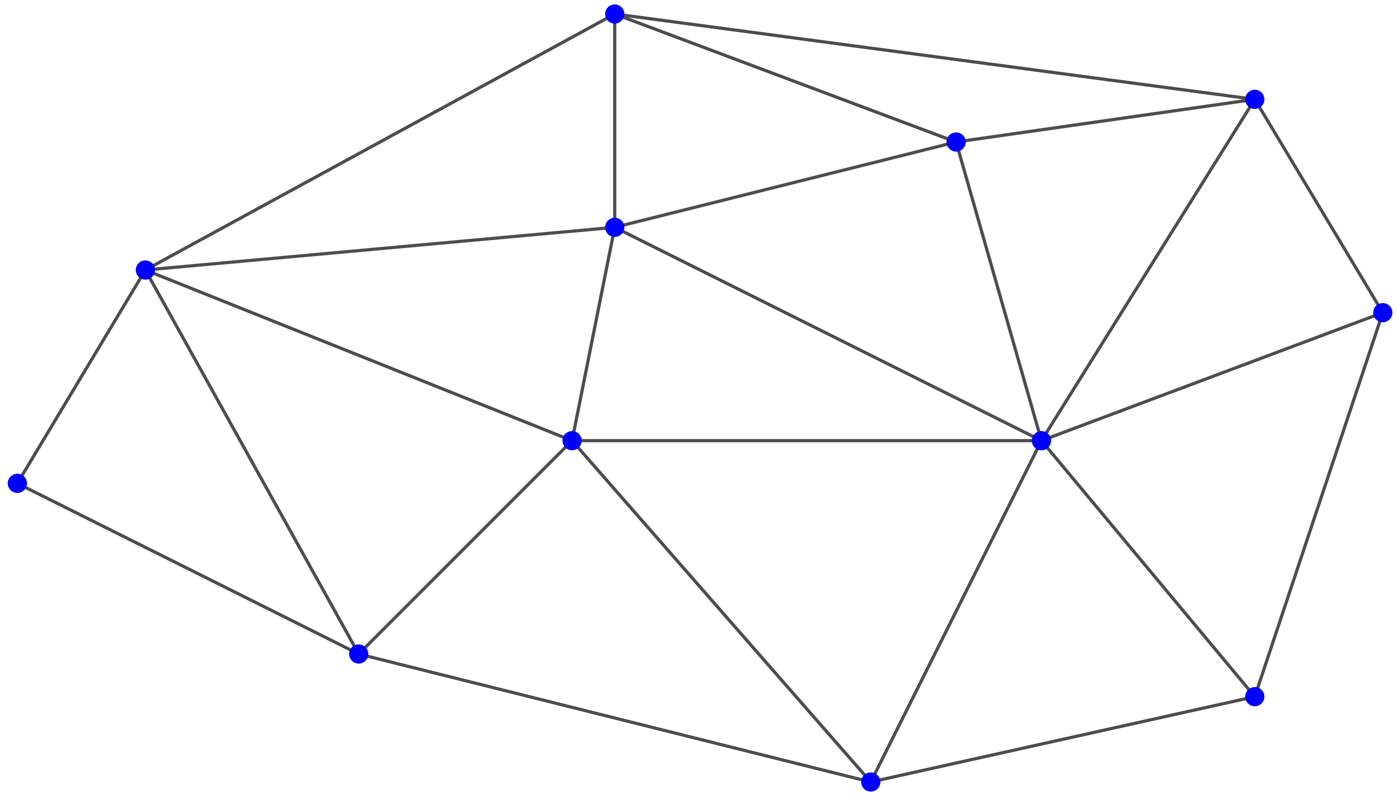
Delaunay triangulations

Delaunay triangulations



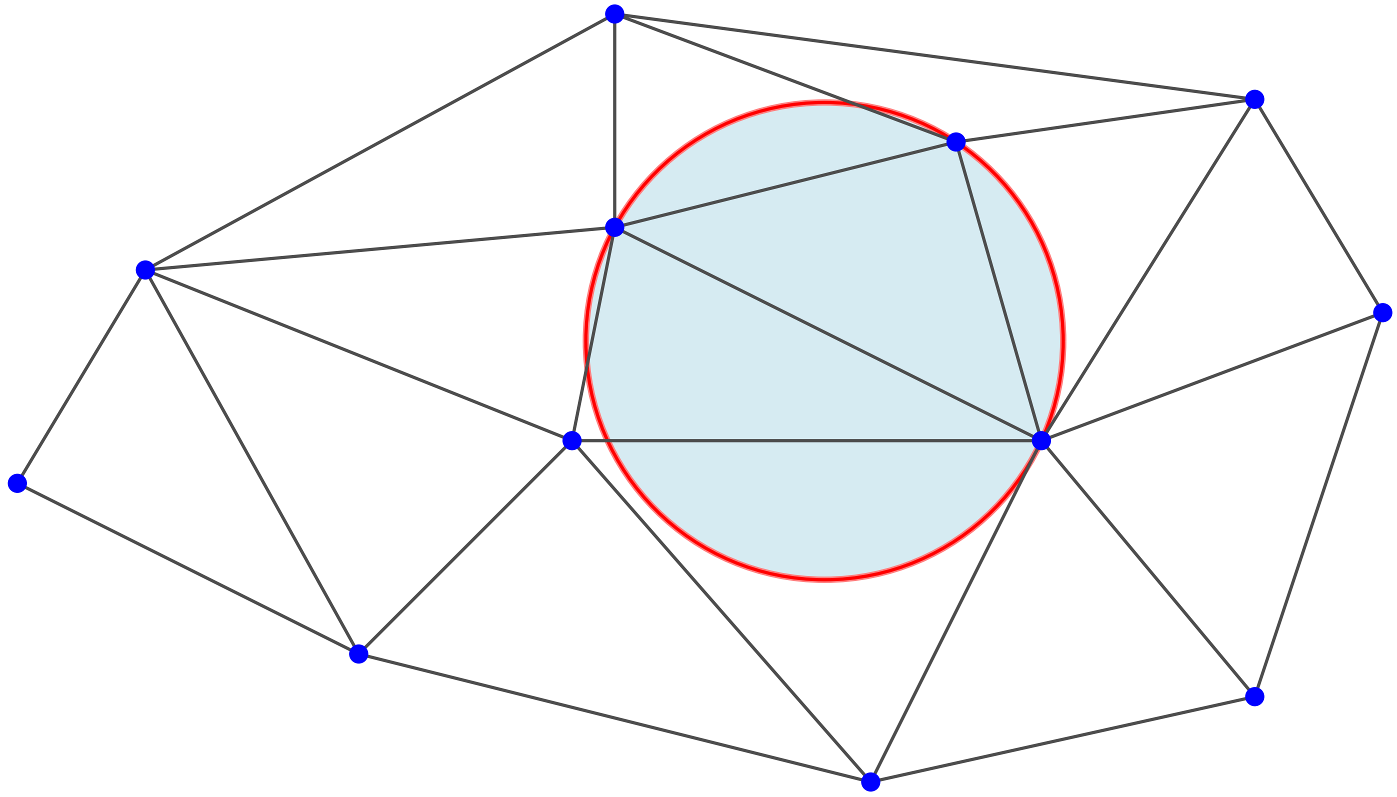
Delaunay triangulations

$DT(S)$



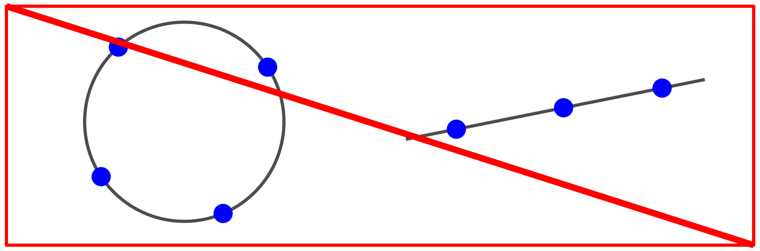
Delaunay triangulations

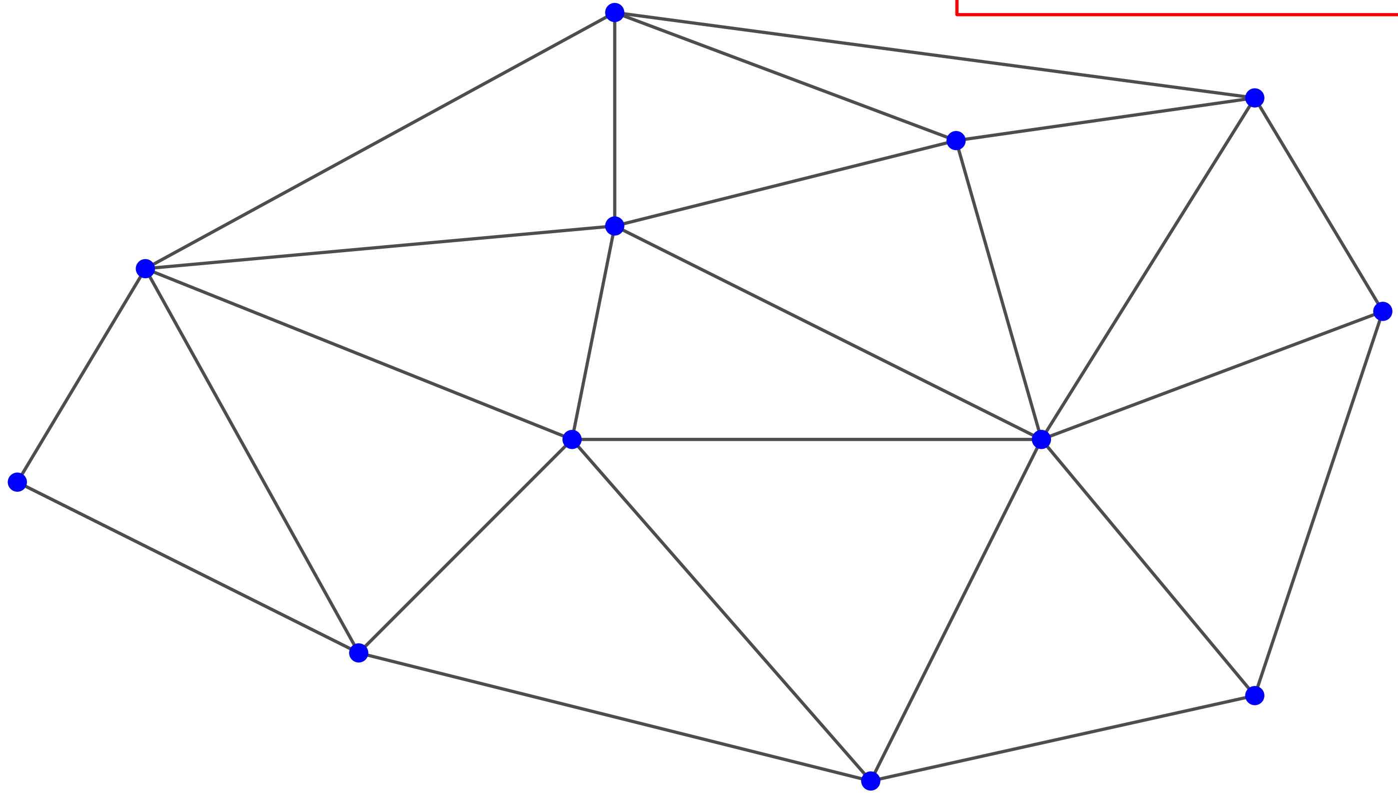
$DT(S)$



Delaunay triangulations

$DT(S)$



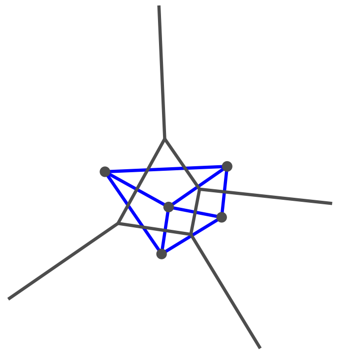


The $DT(S)$ is unique

Delaunay triangulations

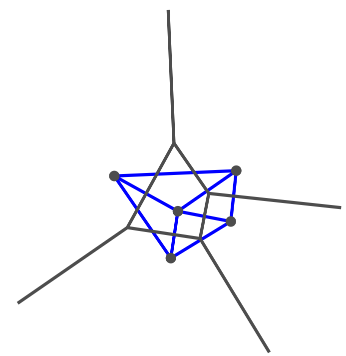
Delaunay triangulations

- Dual of the Voronoi diagram

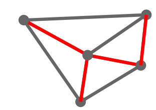


Delaunay triangulations

- Dual of the Voronoi diagram

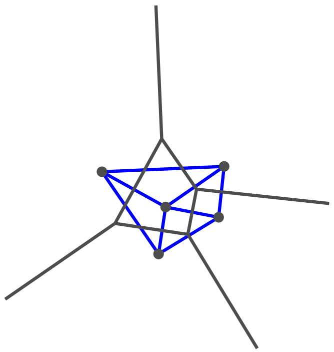


- Contains the minimum spanning tree of a point set

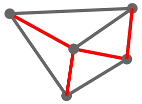


Delaunay triangulations

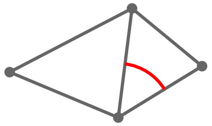
- Dual of the Voronoi diagram



- Contains the minimum spanning tree of a point set

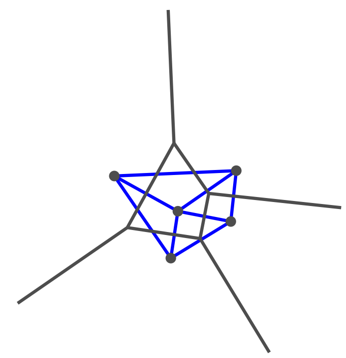


- Maximizes the minimum angle

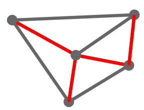


Delaunay triangulations

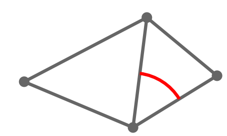
- Dual of the Voronoi diagram



- Contains the minimum spanning tree of a point set



- Maximizes the minimum angle

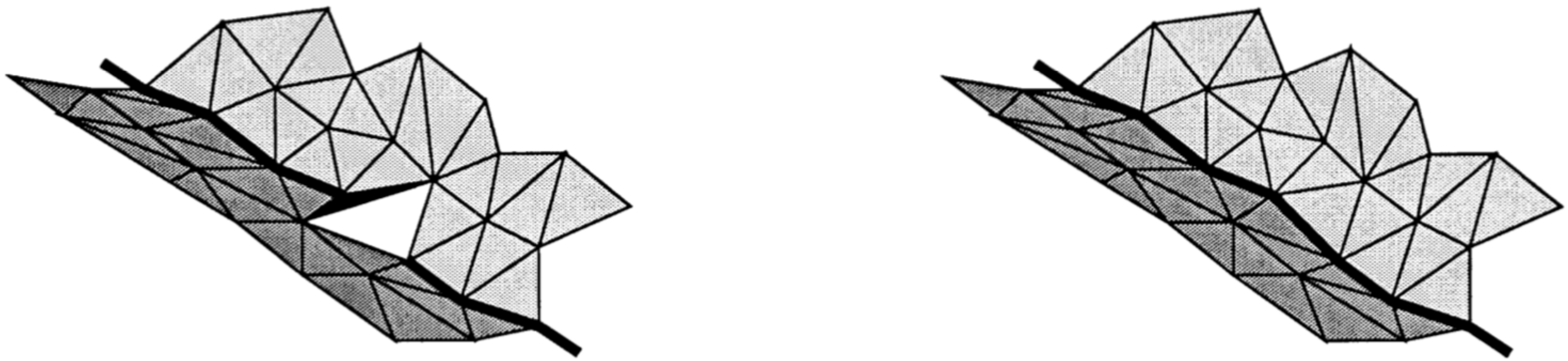


"well-shaped" triangles

Many applications in Graphics, GIS, mesh generation, among others

Higher order Delaunay triangulations

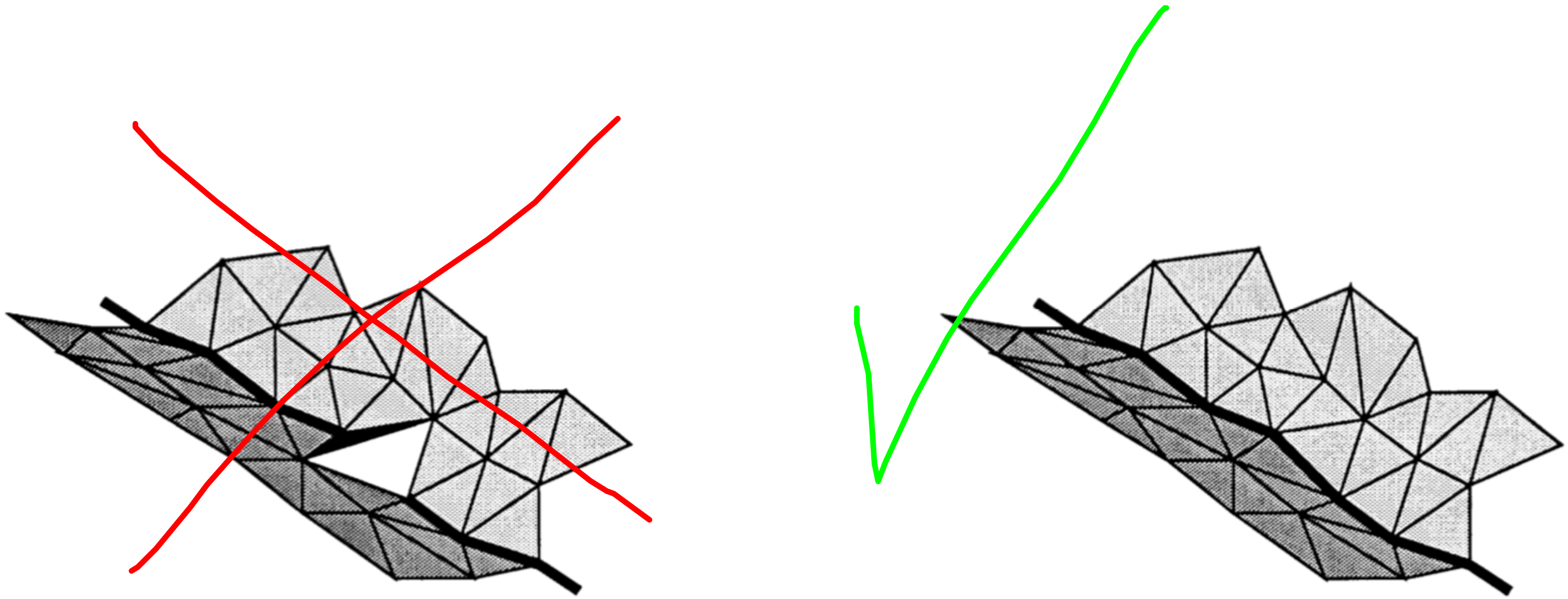
Yet...



Picture from: Gudmundsson, J., Hammar, M., van Kreveld, M.: Higher order Delaunay triangulations. Comput. Geom. 23(1), 8598 (2002)

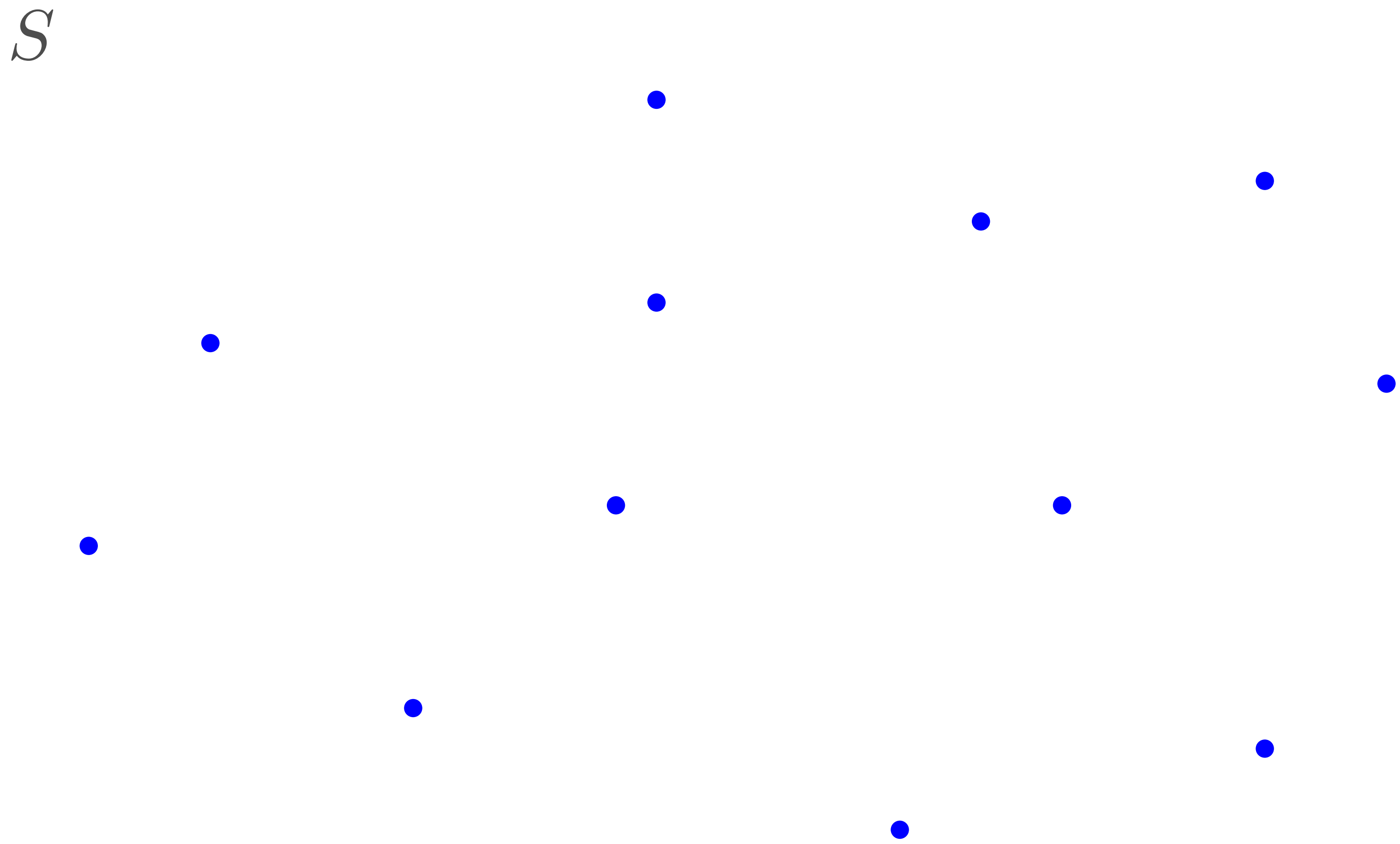
Higher order Delaunay triangulations

Yet...

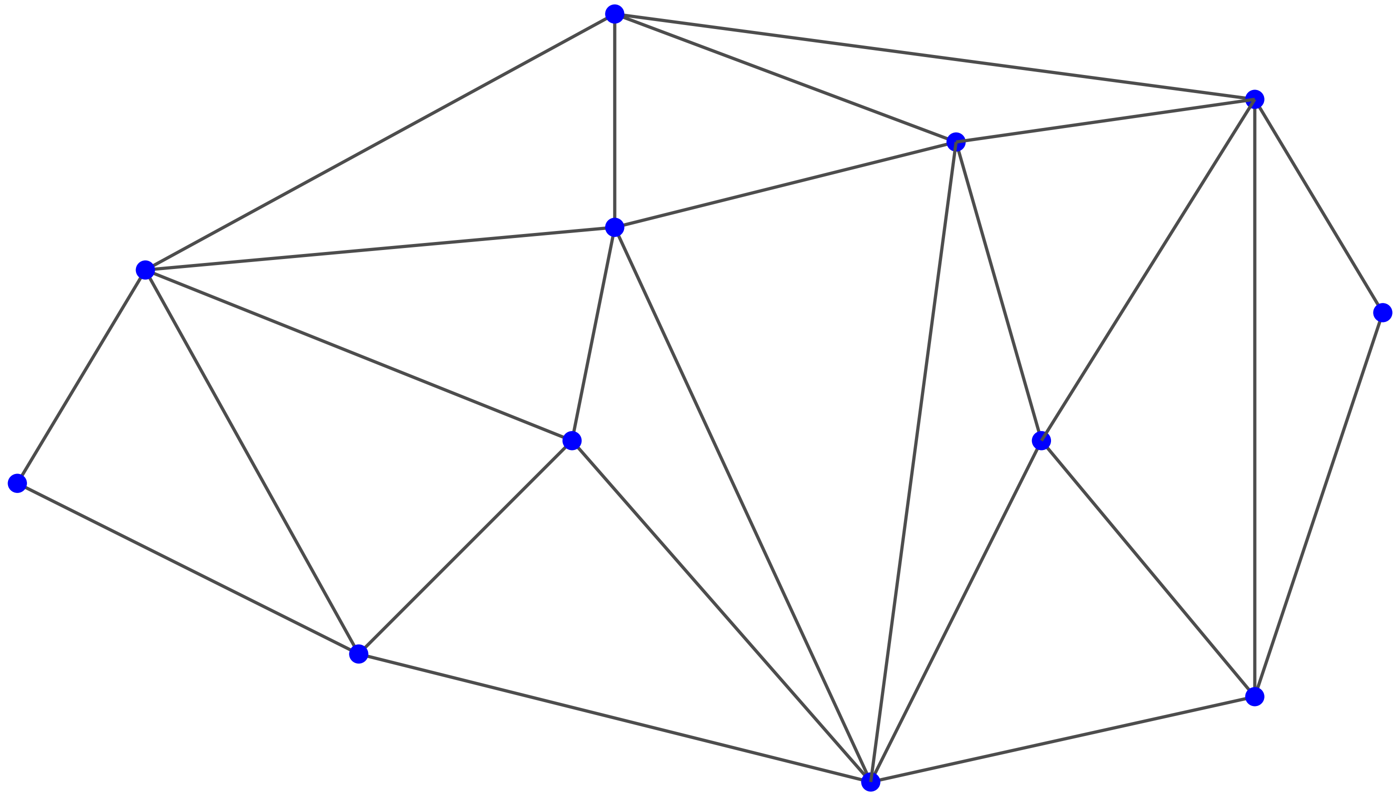


Picture from: Gudmundsson, J., Hammar, M., van Kreveld, M.: Higher order Delaunay triangulations. Comput. Geom. 23(1), 8598 (2002)

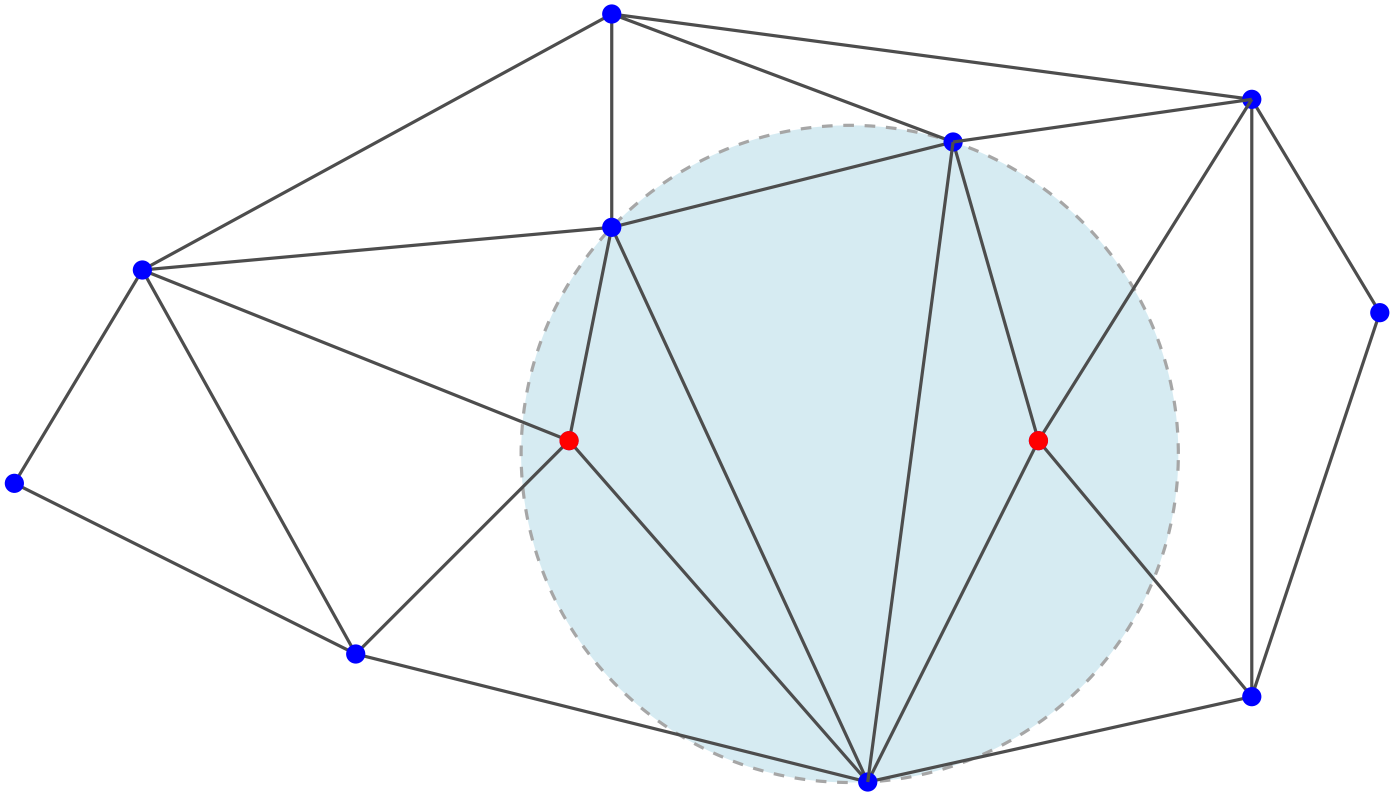
Higher order Delaunay triangulations



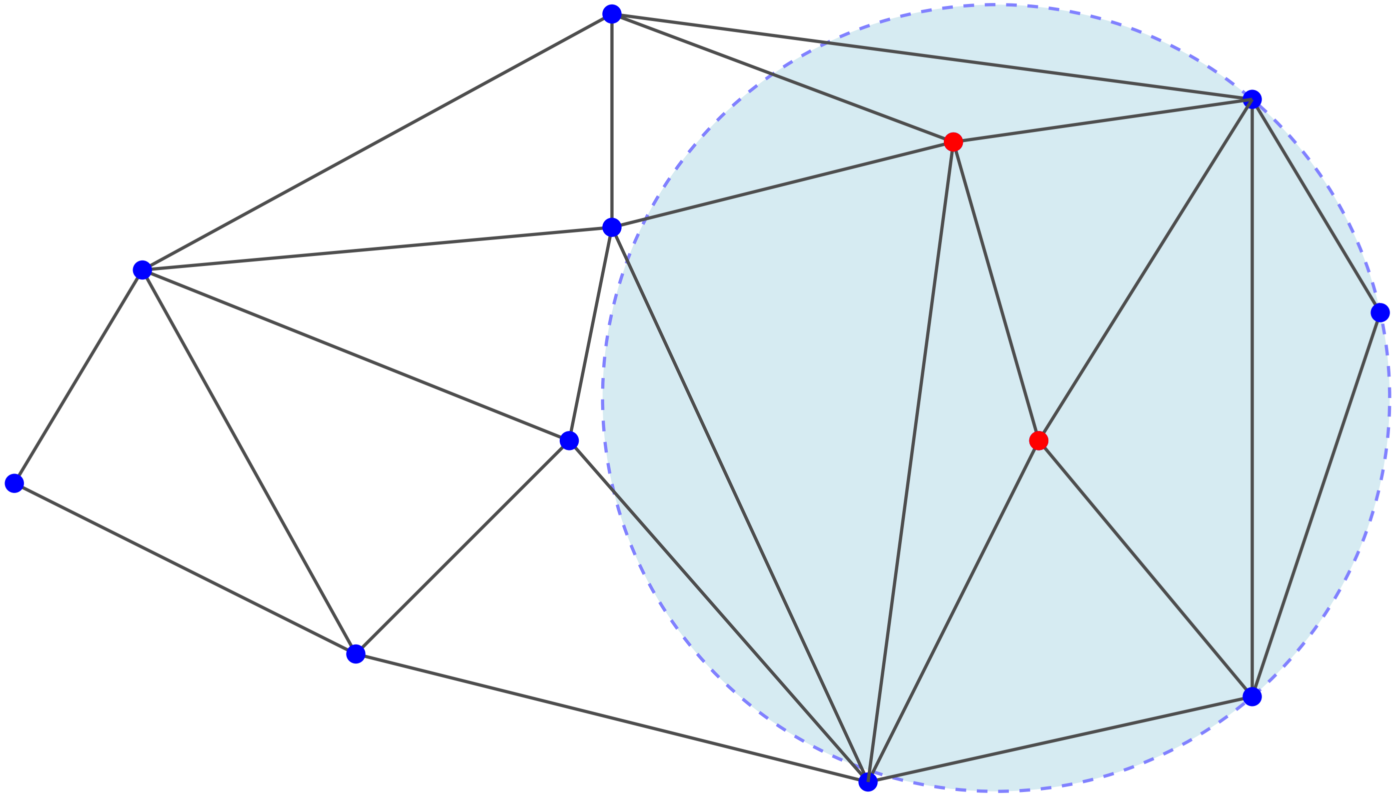
Higher order Delaunay triangulations



Higher order Delaunay triangulations

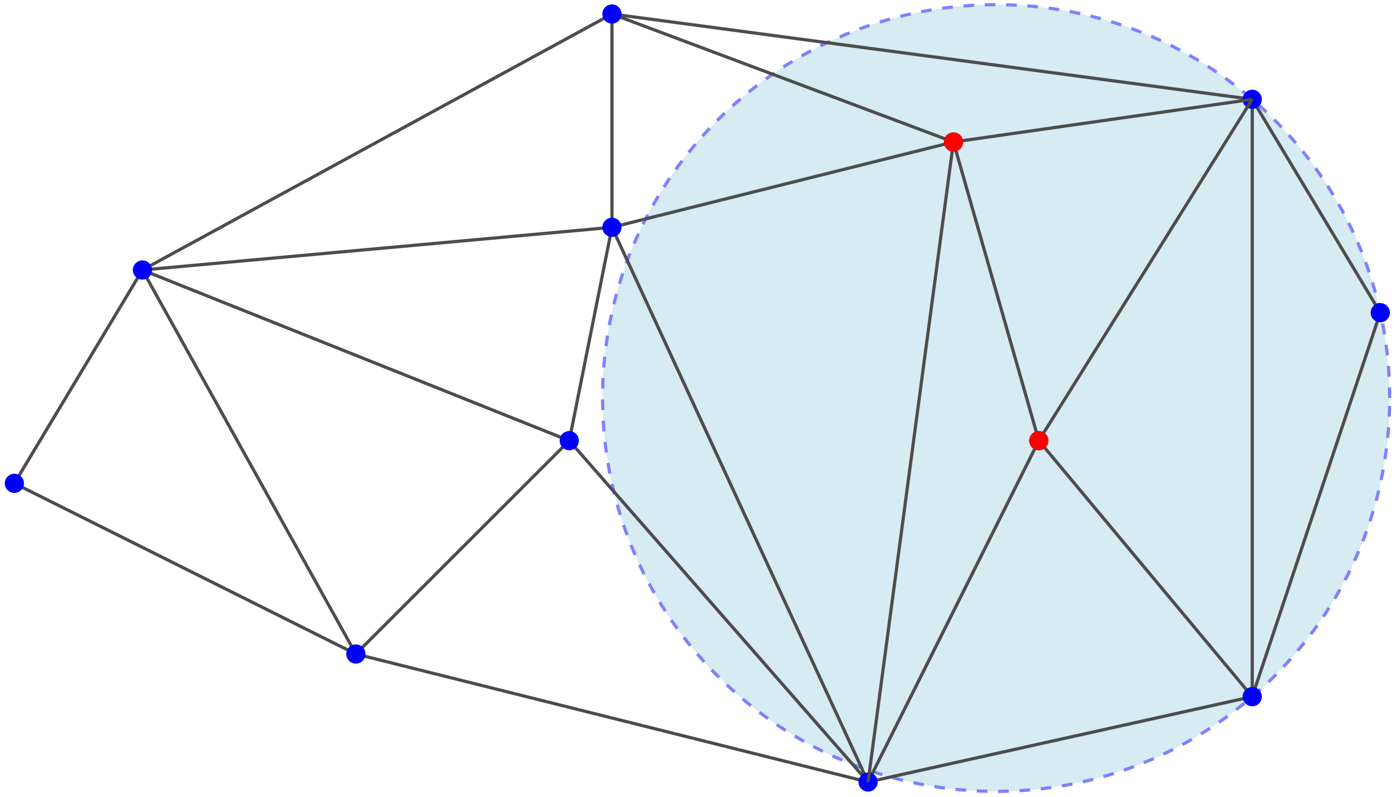


Higher order Delaunay triangulations

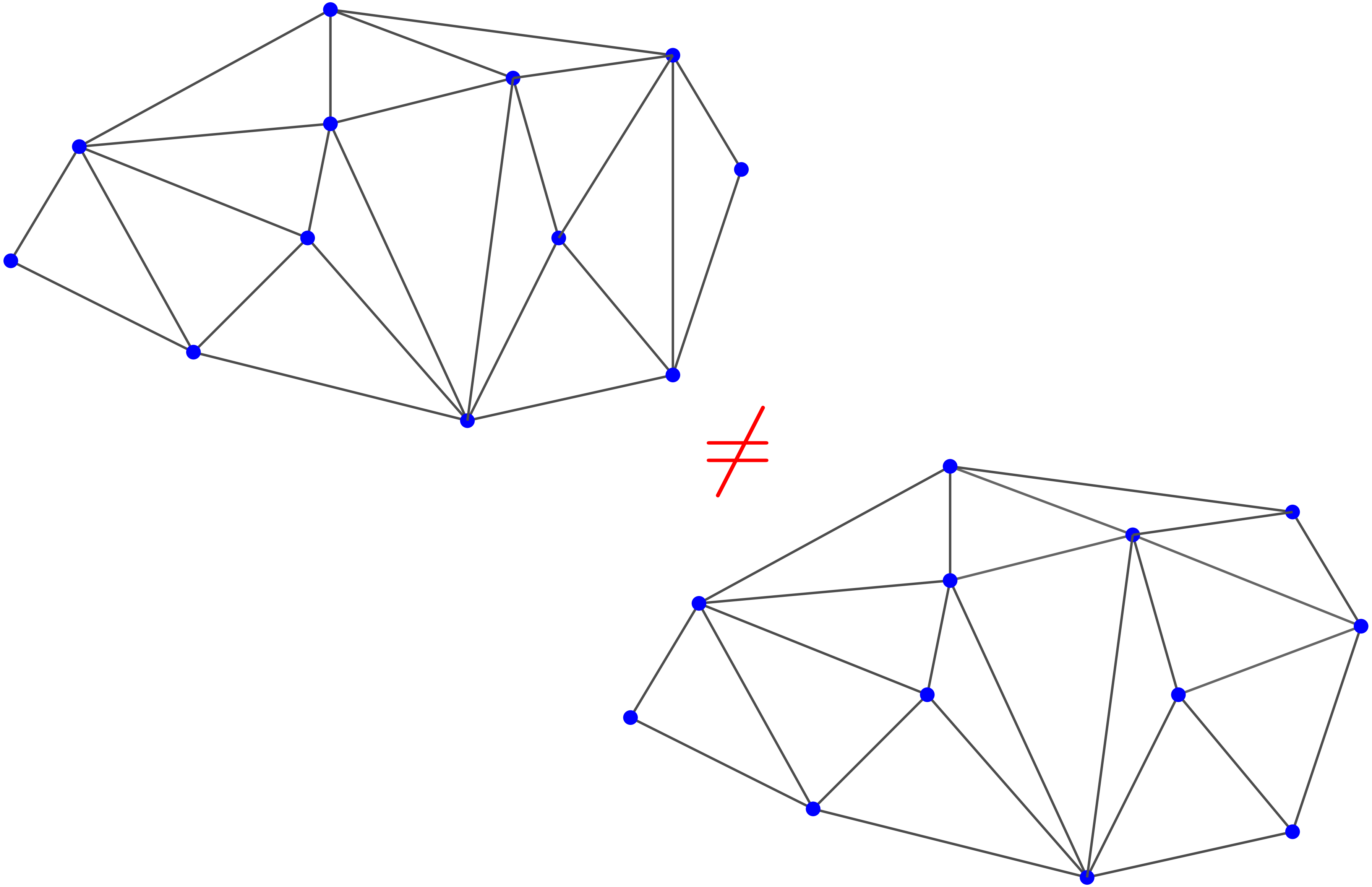


Higher order Delaunay triangulations

Also called order-k triangulations



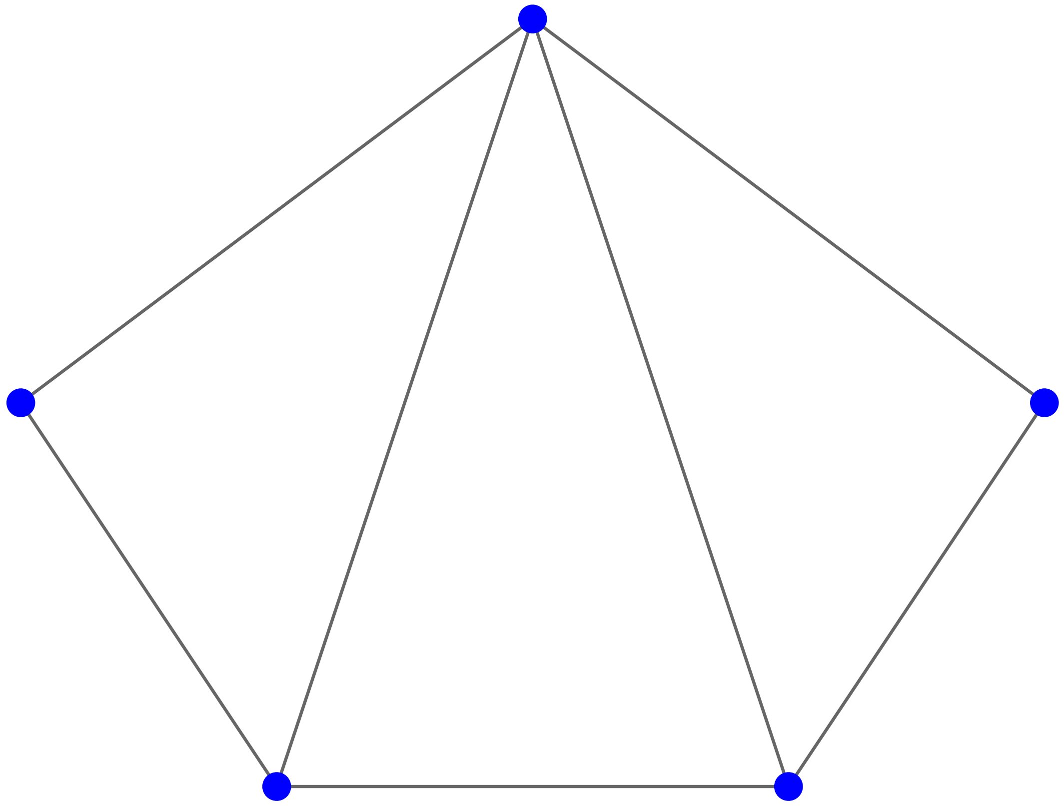
Higher order Delaunay triangulations



The flip graph

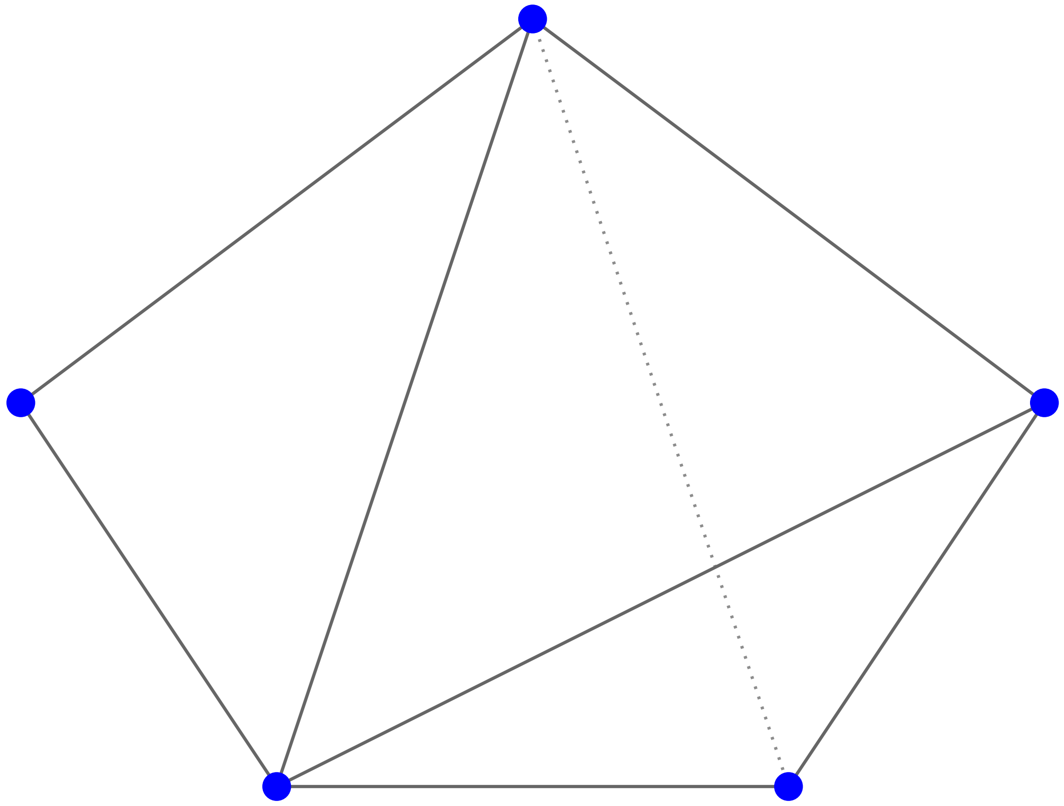
The flip graph

Flip



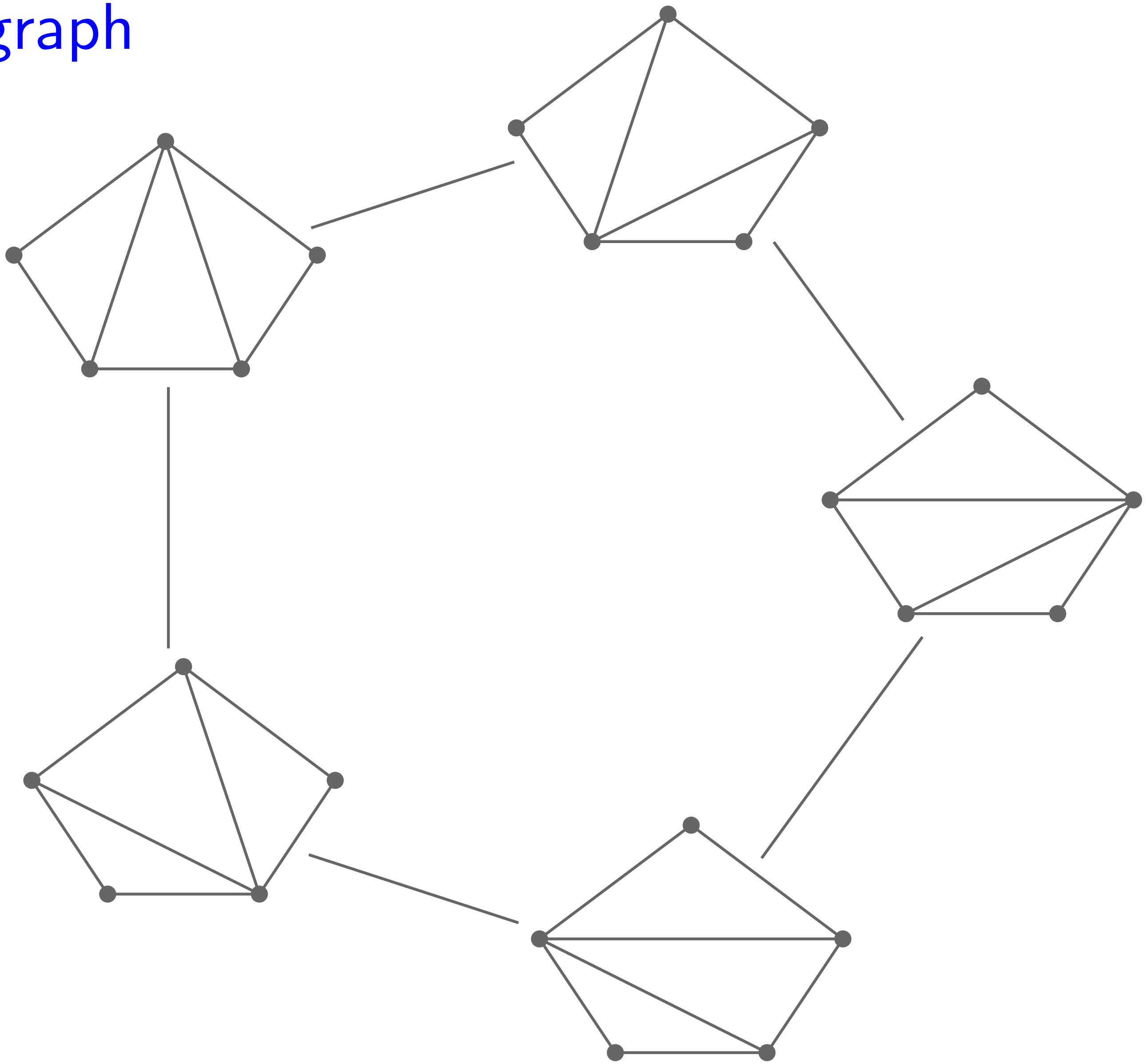
The flip graph

Flip



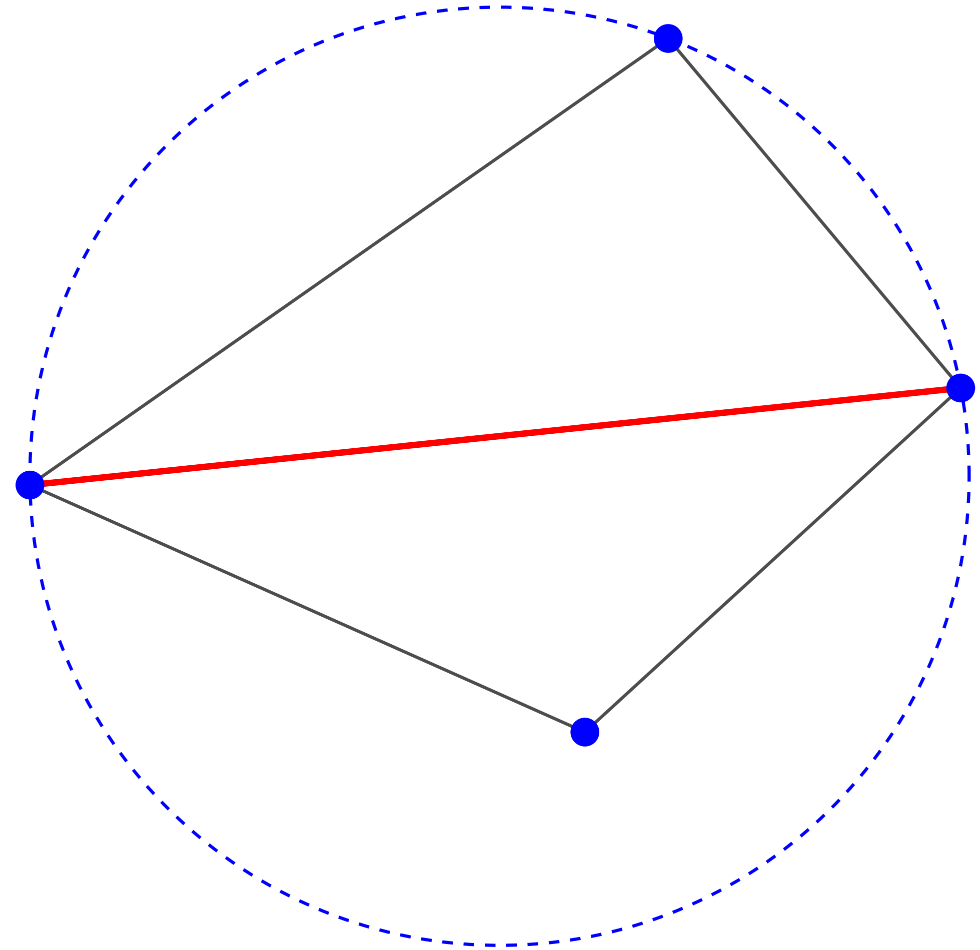
The flip graph

Flip graph



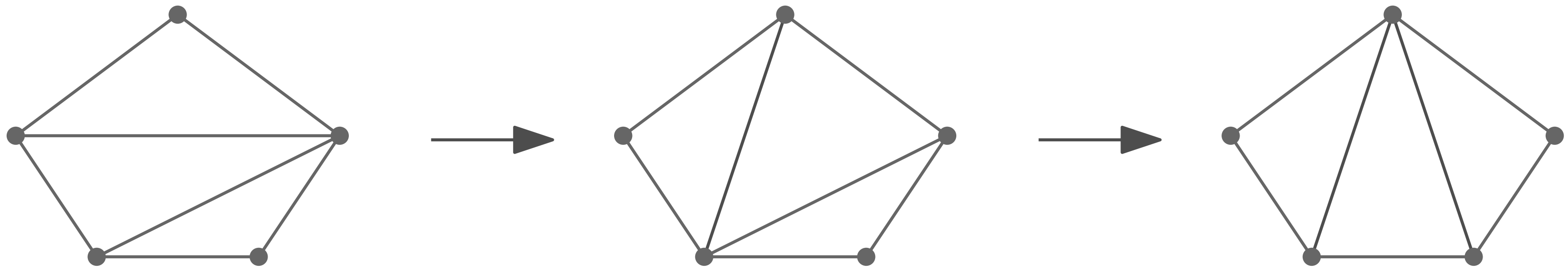
The flip graph and the Delaunay triangulation

Illegal edge



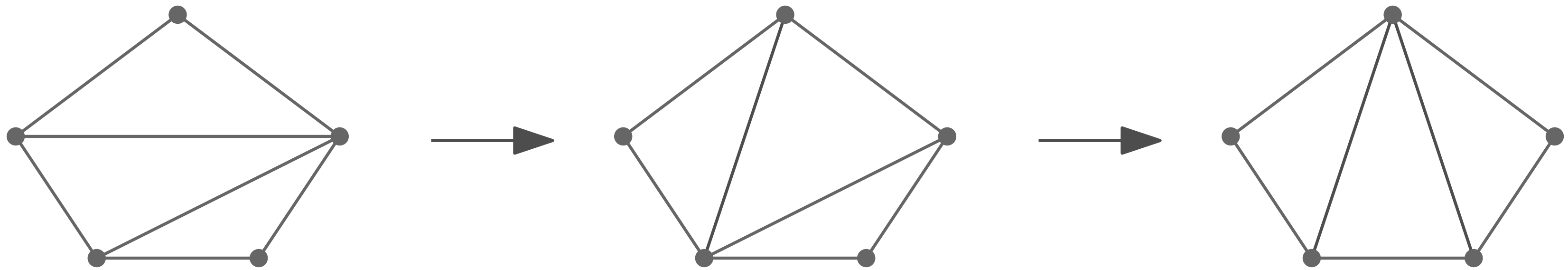
The flip graph and the Delaunay triangulation

The flip graph and the Delaunay triangulation



- Any triangulation can be transformed into the DT by flipping only illegal edges.

The flip graph and the Delaunay triangulation



- Any triangulation can be transformed into the DT by flipping only illegal edges.

$$\Downarrow$$

- The flip graph of any point set S, denoted $\mathcal{T}(S)$, is connected.
- The distance in $\mathcal{T}(S)$ between any two triangulations of S is $O(n^2)$.

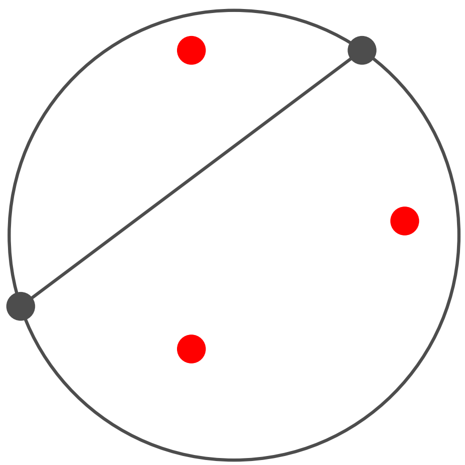
The flip graph and order-k triangulations

The flip graph and order-k triangulations

- Abe and Okamoto observed that the flip graph of order-k triangulations of a point set S, denoted $\mathcal{T}_k(S)$, is connected for $k \leq 2$.

The flip graph and order-k triangulations

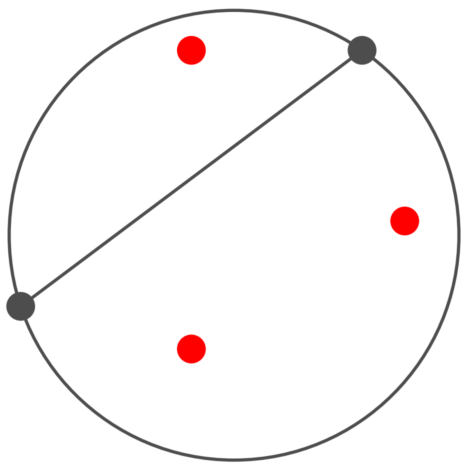
- Abe and Okamoto observed that the flip graph of order-k triangulations of a point set S, denoted $\mathcal{T}_k(S)$, is connected for $k \leq 2$.



order-3 edge

The flip graph and order-k triangulations

- Abe and Okamoto observed that the flip graph of order-k triangulations of a point set S, denoted $\mathcal{T}_k(S)$, is connected for $k \leq 2$.

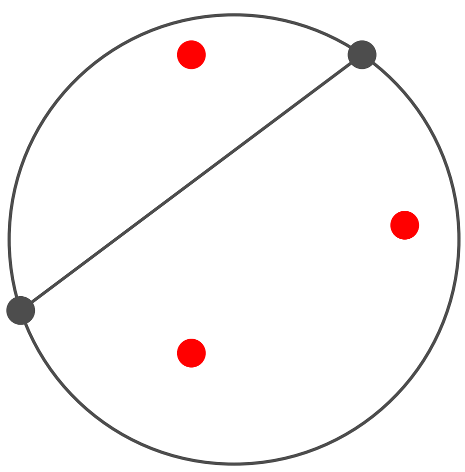


order-3 edge

order-k triangulation $\implies$ order-k edges

The flip graph and order-k triangulations

- Abe and Okamoto observed that the flip graph of order-k triangulations of a point set S, denoted $\mathcal{T}_k(S)$, is connected for $k \leq 2$.



order-3 edge

order-k triangulation $\not\Leftarrow$ order-k edges

The flip graph and order-k triangulations

- Abe and Okamoto observed that the flip graph of order-k triangulations of a point set S, denoted $\mathcal{T}_k(S)$, is connected for $k \leq 2$.

About the flip graph of triangulations with order-k edges, Abellans et al. showed:

The flip graph and order-k triangulations

- Abe and Okamoto observed that the flip graph of order-k triangulations of a point set S, denoted $\mathcal{T}_k(S)$, is connected for $k \leq 2$.

About the flip graph of triangulations with order-k edges, Abellans et al. showed:

- Is connected when $k \leq 1$
- Might be disconnected when $k \geq 2$
- For any k, the flip graph is connected when S is in convex position

The flip graph and order-k triangulations

- Abe and Okamoto observed that the flip graph of order-k triangulations of a point set S, denoted $\mathcal{T}_k(S)$, is connected for $k \leq 2$.

About the flip graph of triangulations with order-k edges, Abellans et al. showed:

- Is connected when $k \leq 1$
- Might be disconnected when $k \geq 2$
- For any k, the flip graph is connected when S is in convex position

exponential upper bound
on the flip distance

Our results

Our results

- For any k there exists a point set S in convex position where $\mathcal{T}_k(S)$ is disconnected. Moreover, $k-1$ flips are sometimes necessary in order to transform an order-k triangulation of S into another.

Our results

- For any k there exists a point set S in convex position where $\mathcal{T}_k(S)$ is disconnected. Moreover, $k-1$ flips are sometimes necessary in order to transform an order-k triangulation of S into another.

- For any order-k triangulation of a point set in convex position there exists an order-k triangulation at distance at most $k-1$ in $\mathcal{T}_{2k-2}(S)$

Our results

- For any k there exists a point set S in convex position where $\mathcal{T}_k(S)$ is disconnected. Moreover, $k-1$ flips are sometimes necessary in order to transform an order-k triangulation of S into another.

- For any order-k triangulation of a point set in convex position there exists an order-k triangulation at distance at most $k-1$ in $\mathcal{T}_{2k-2}(S)$

- Let $k = 2, 3, 4, 5$. For any order-k triangulation of a point set in general position there exists an order-k triangulation at distance at most $k-1$ in $\mathcal{T}_{2k-2}(S)$

Our results

- For any k there exists a point set S in convex position where $\mathcal{T}_k(S)$ is disconnected. Moreover, $k-1$ flips are sometimes necessary in order to transform an order-k triangulation of S into another.

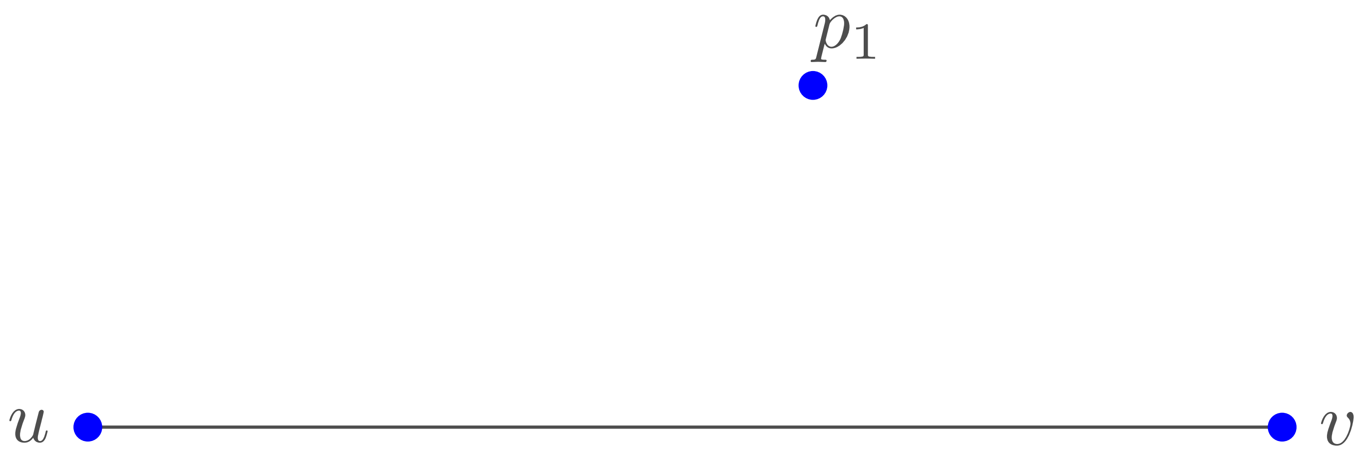
- For any order-k triangulation of a point set in convex position there exists an order-k triangulation at distance at most $k-1$ in $\mathcal{T}_{2k-2}(S)$

- Let $k = 2, 3, 4, 5$. For any order-k triangulation of a point set in general position there exists an order-k triangulation at distance at most $k-1$ in $\mathcal{T}_{2k-2}(S)$

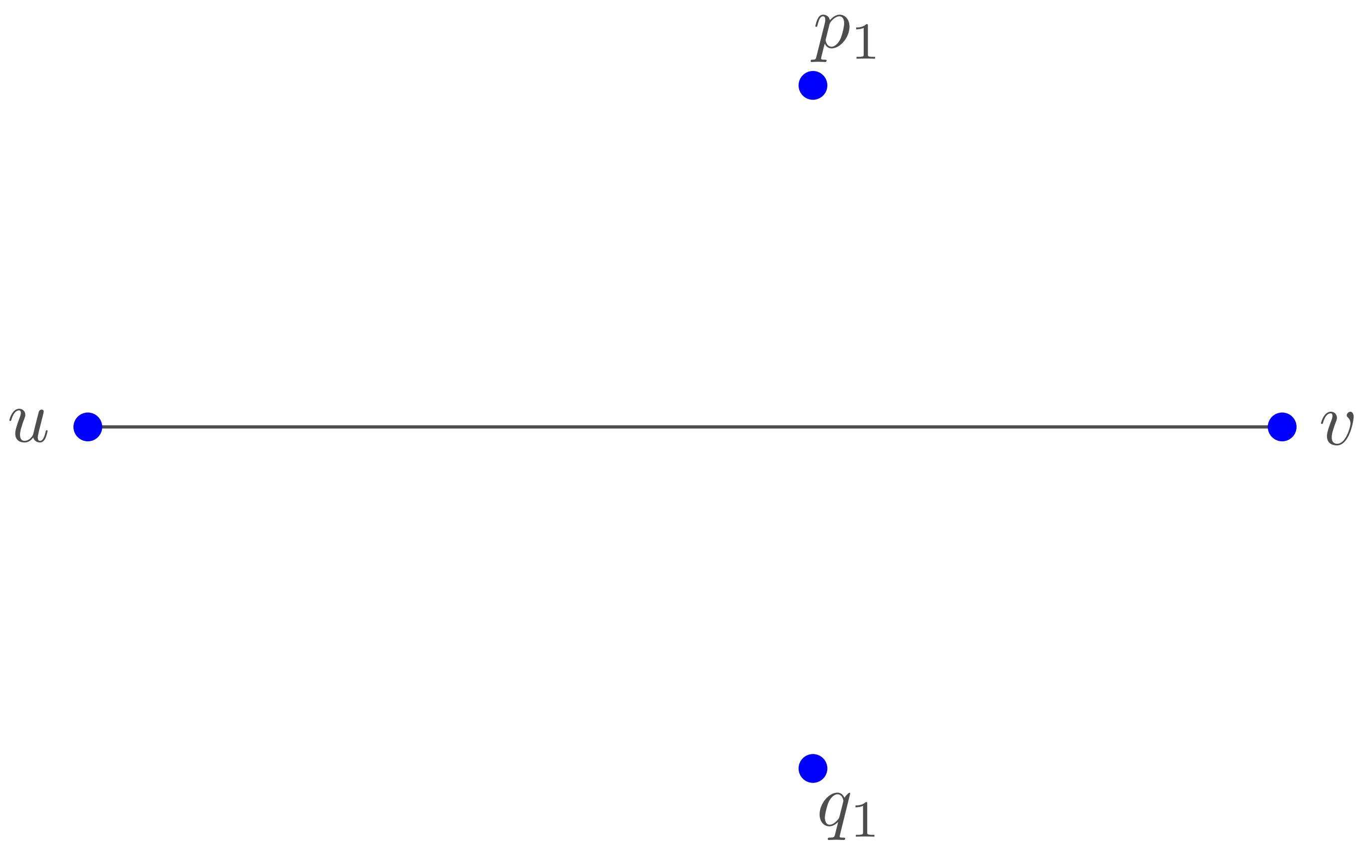
A lower bound

A lower bound

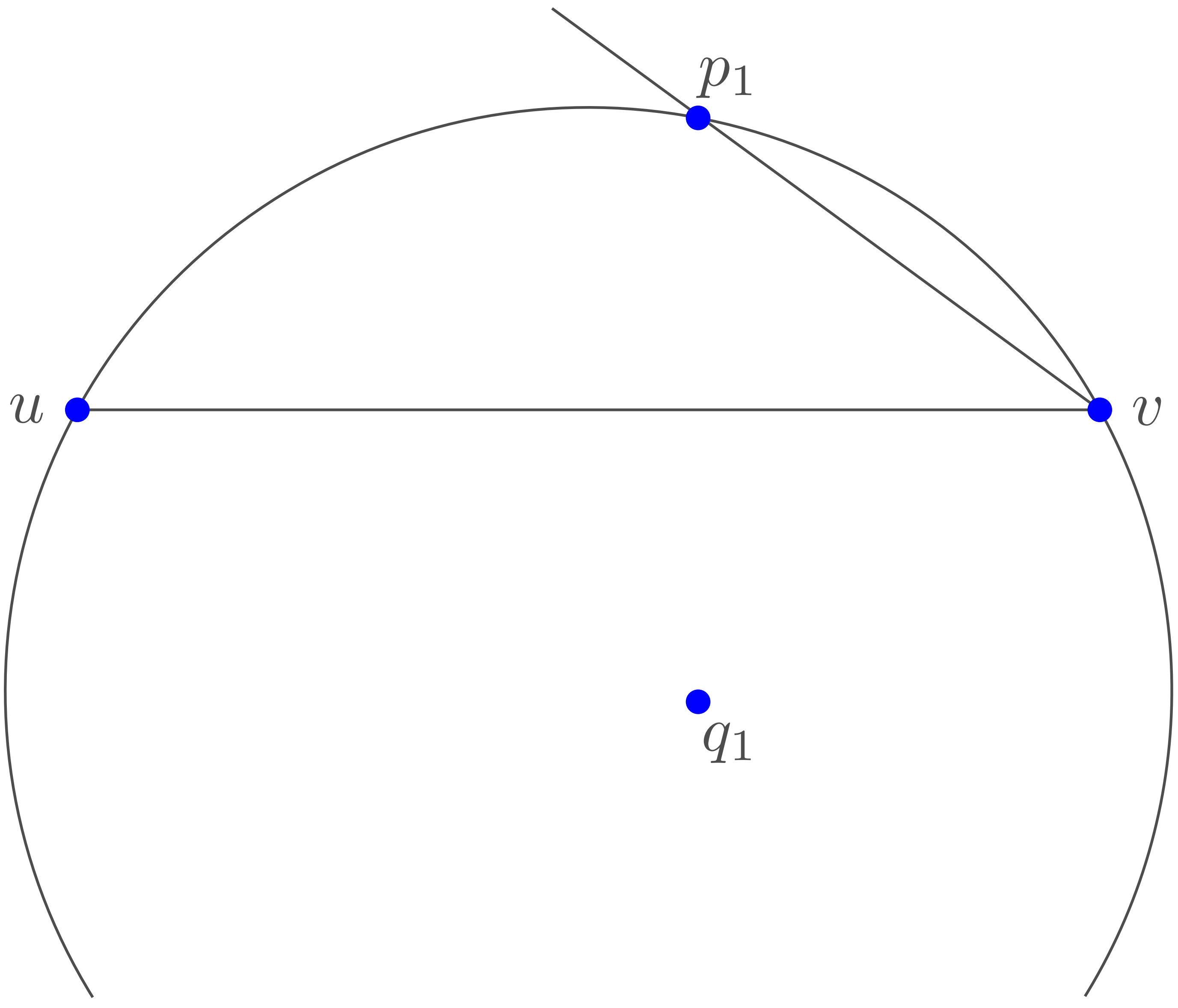
A lower bound



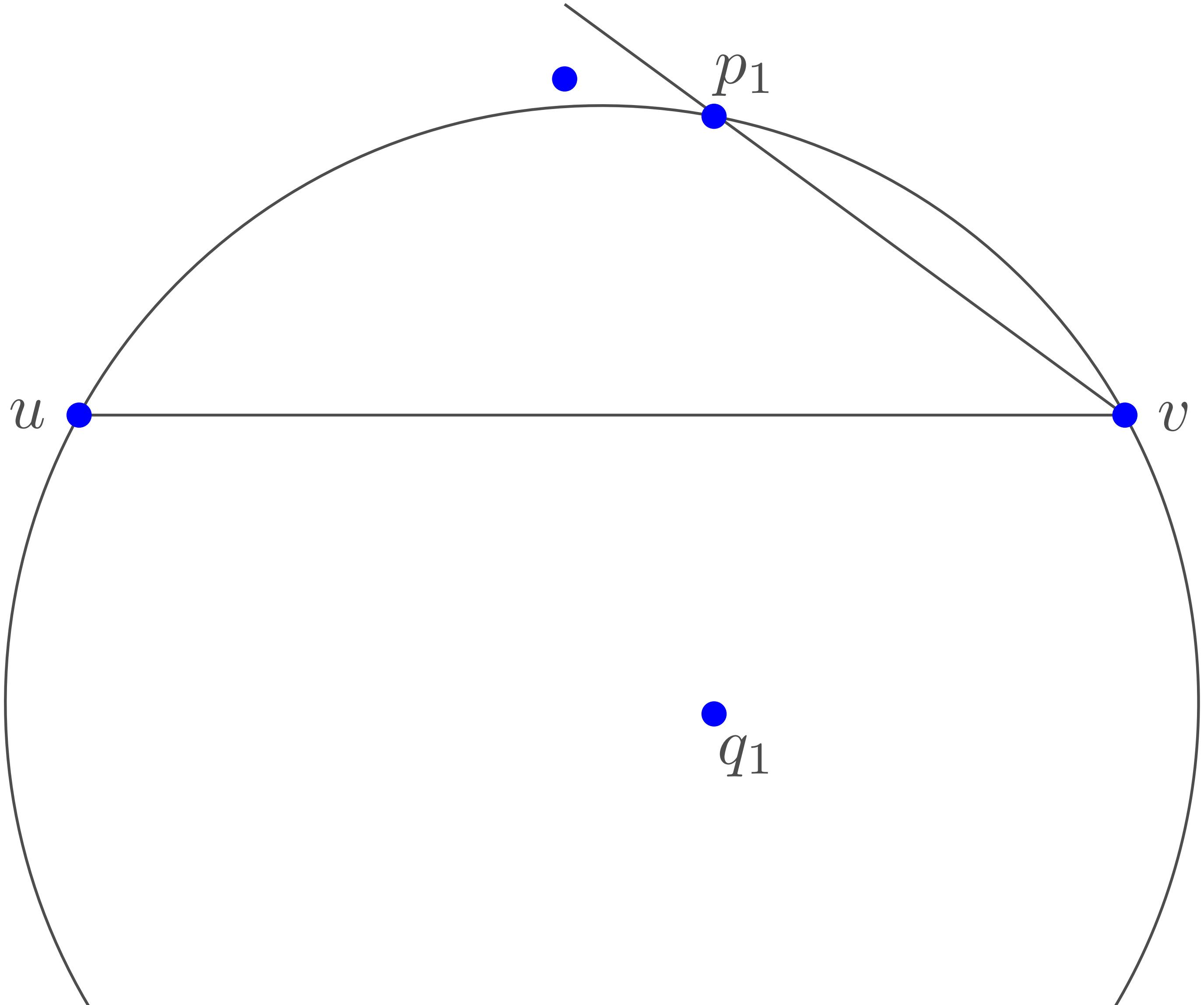
A lower bound



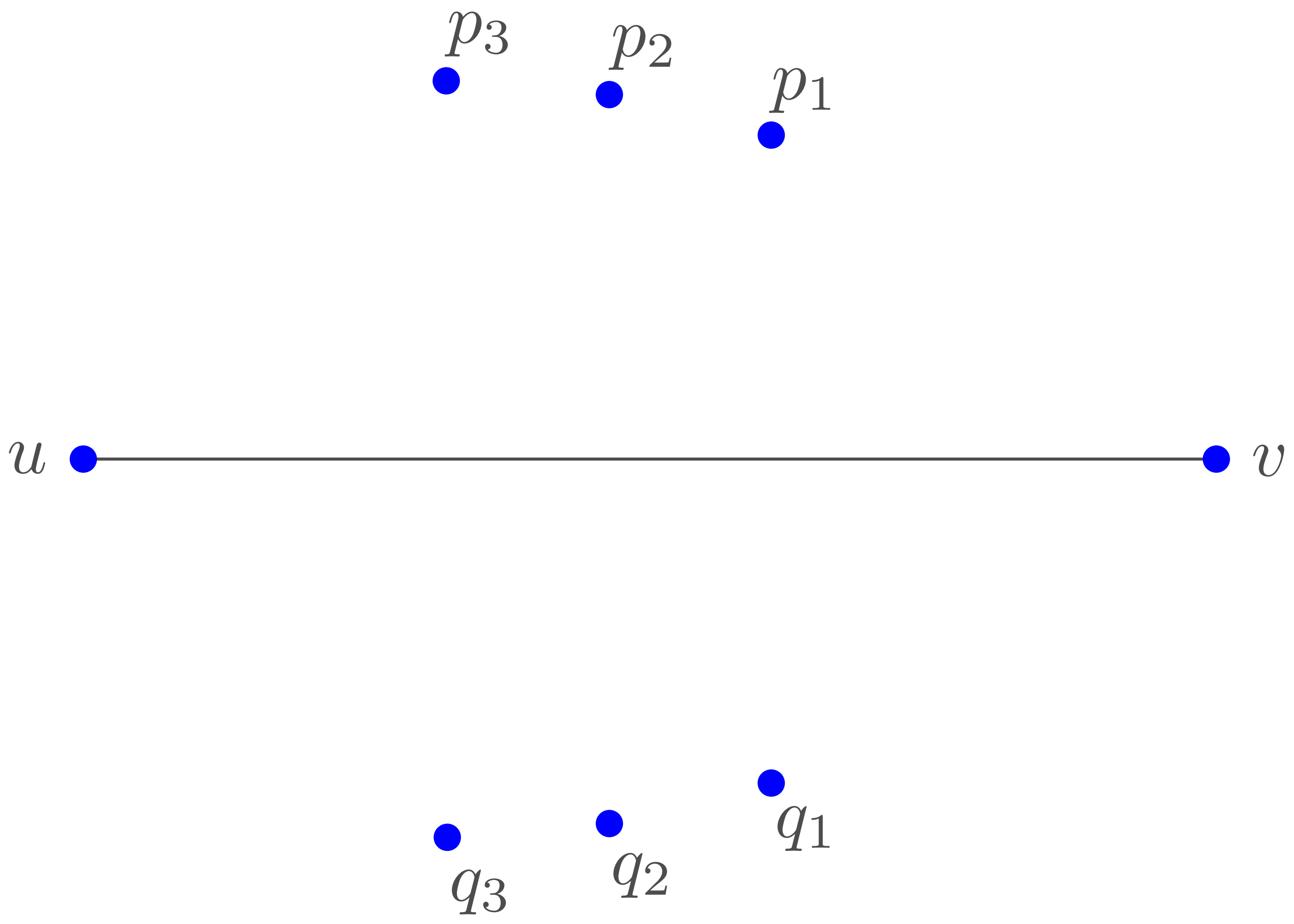
A lower bound



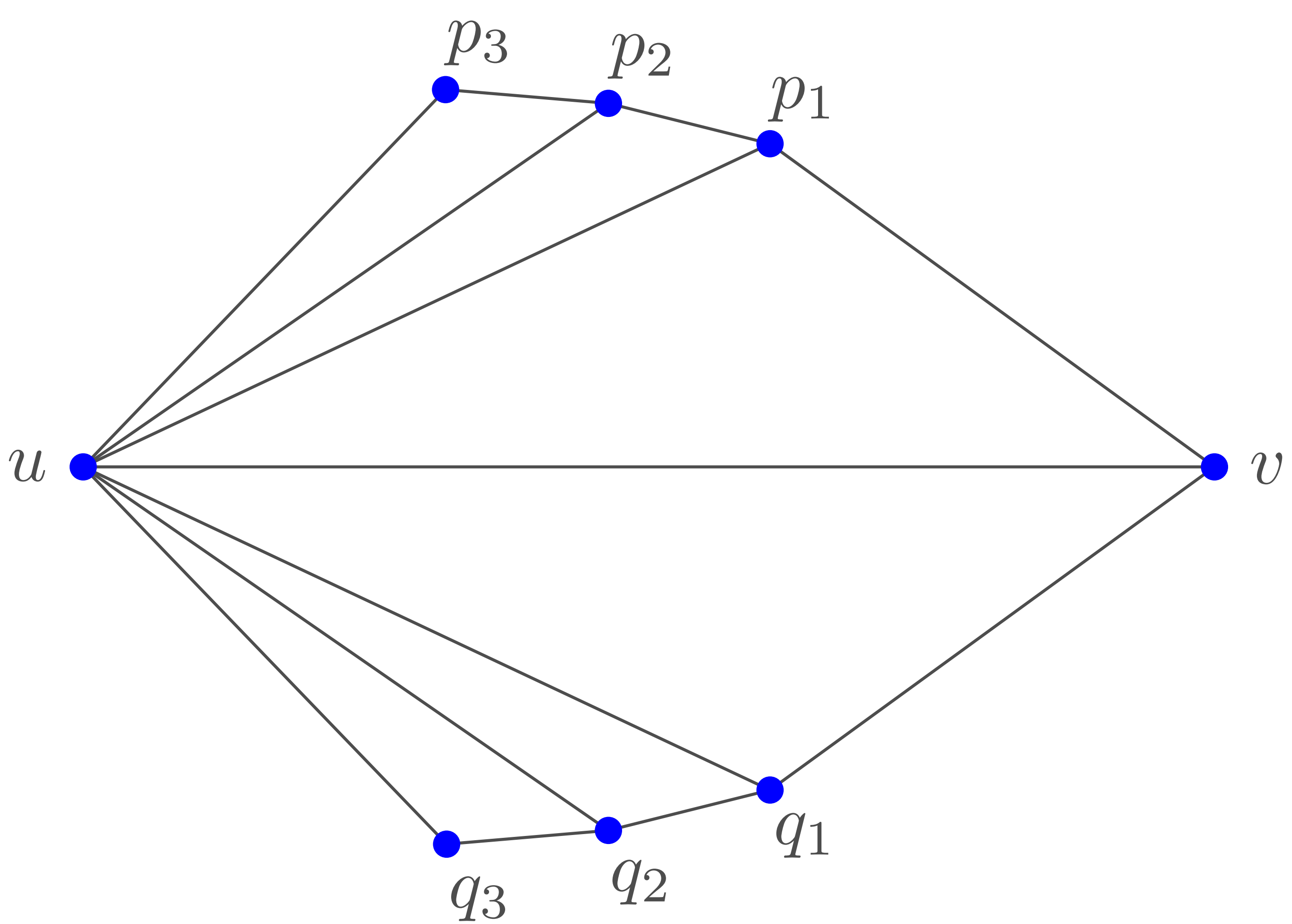
A lower bound



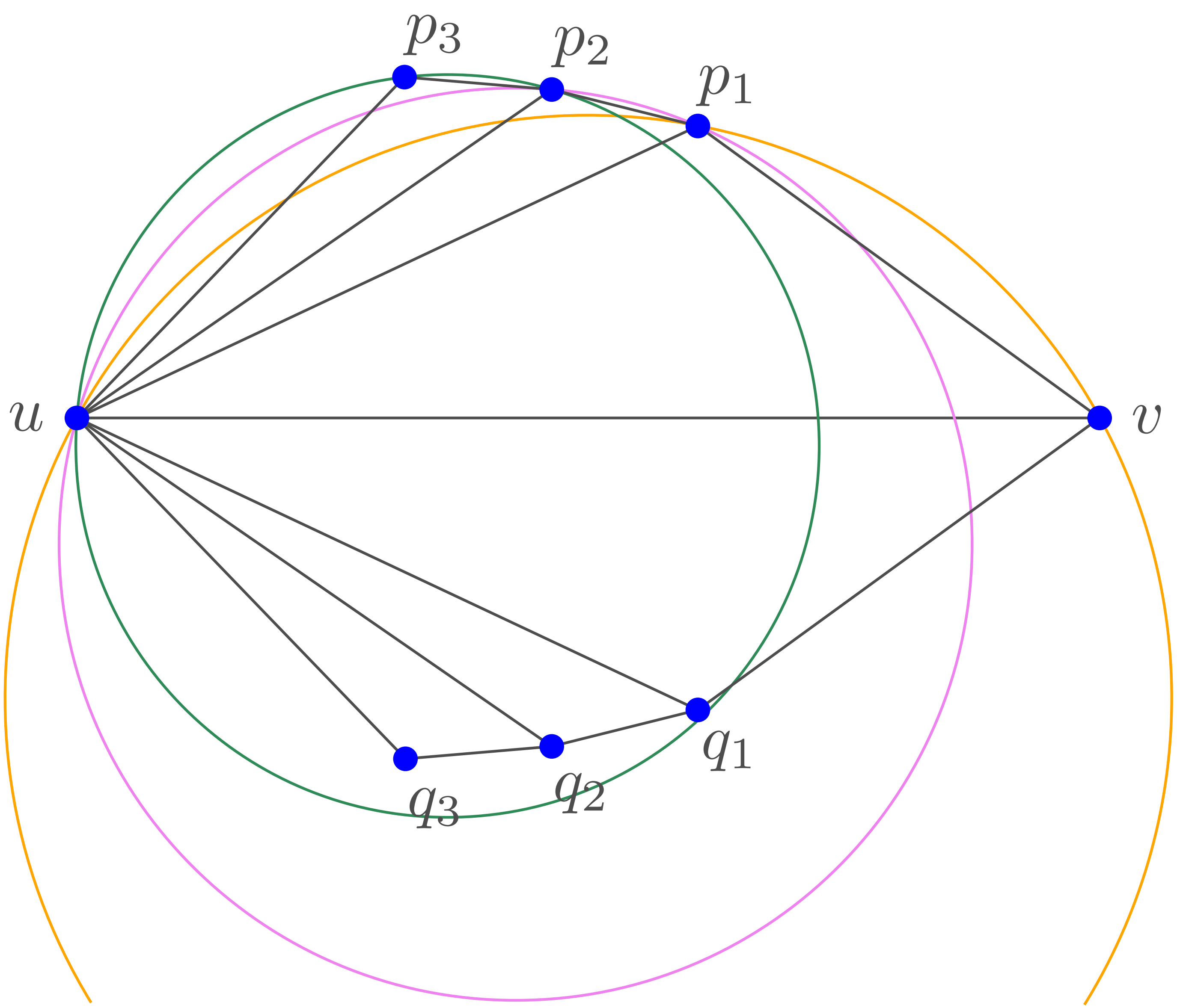
A lower bound



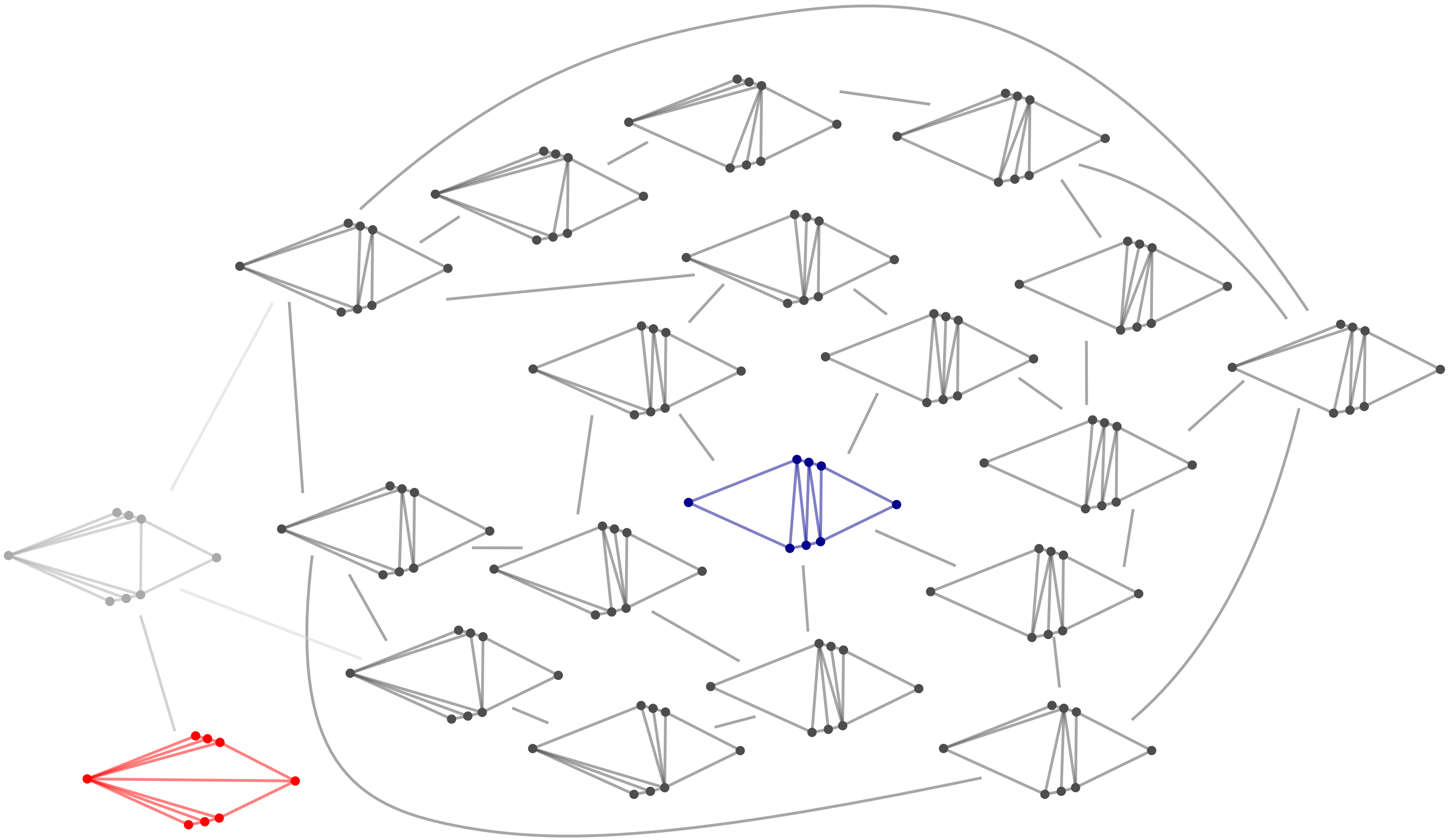
A lower bound



A lower bound



A lower bound



A lower bound

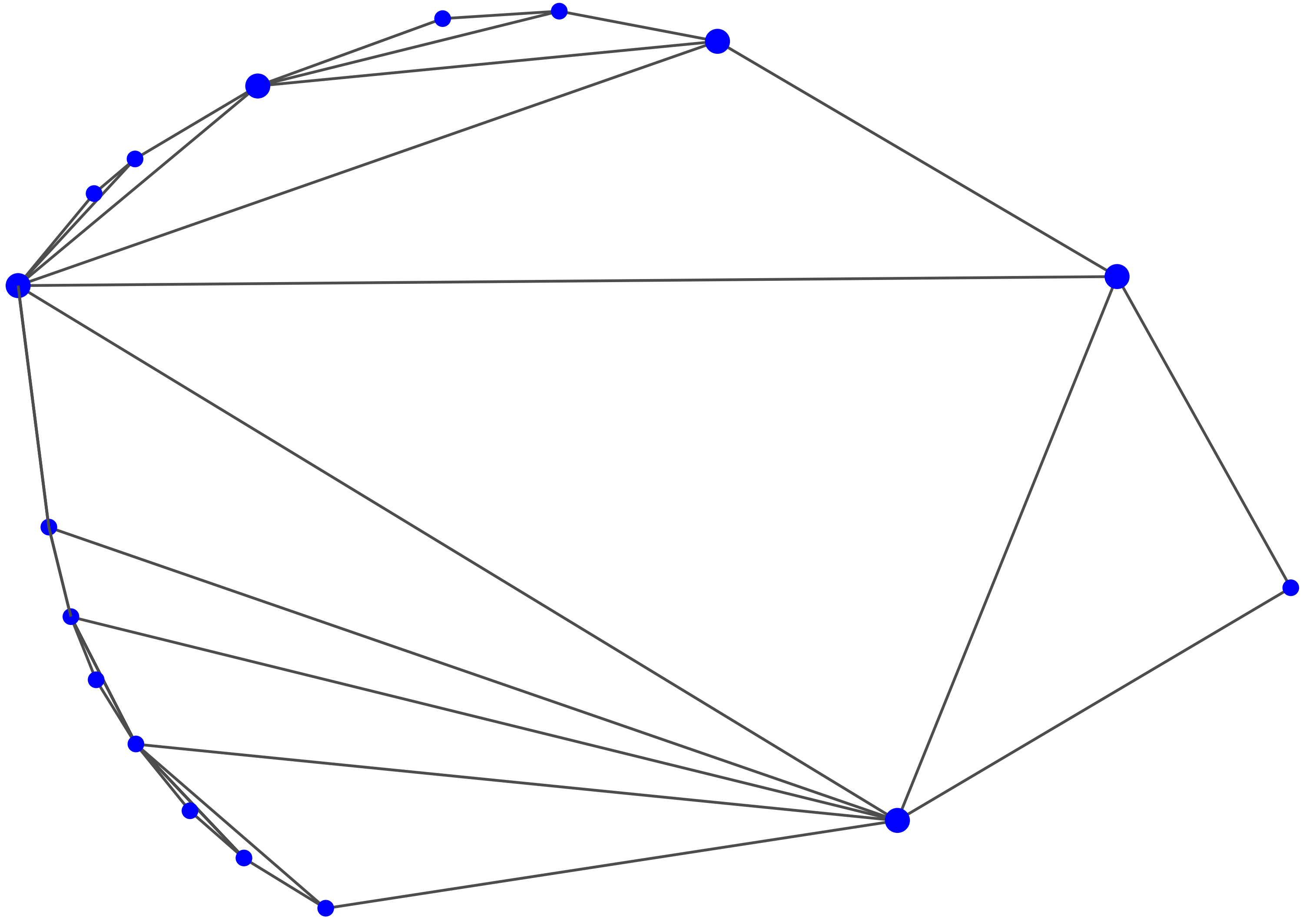
Theorem. For any $k > 2$ there is a set S_k of $2k + 2$ points in convex position such that $G(\mathcal{T}_k(S_k))$ is not connected. Moreover, there is a triangulation T_k in $\mathcal{T}_k(S_k)$ such that in order to transform T_k into any other triangulation in $\mathcal{T}_k(S_k)$ one needs to perform at least $k - 1$ flips.

An upper bound for convex position

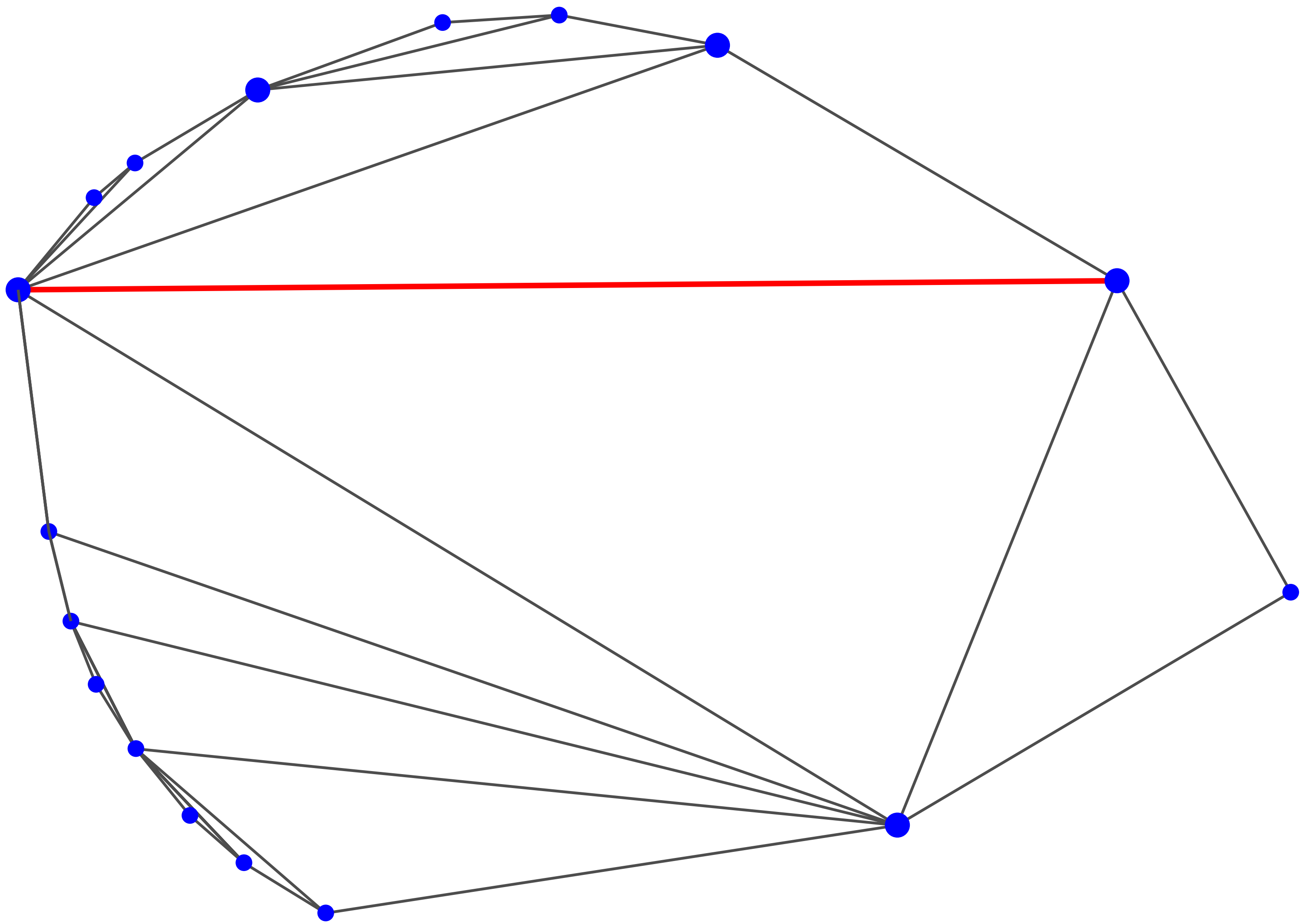
An upper bound for convex position

Theorem. For a point set S in convex position and $k \geq 2$, let $T \neq DT(S)$ be a triangulation in $\mathcal{T}_k(S)$. Then, there exists T' in $\mathcal{T}_k(S)$ such that there is a path between T and T' in $G(\mathcal{T}_{2k-2}(S))$ of length at most $k-1$, where each edge of the path corresponds to flipping an illegal edge.

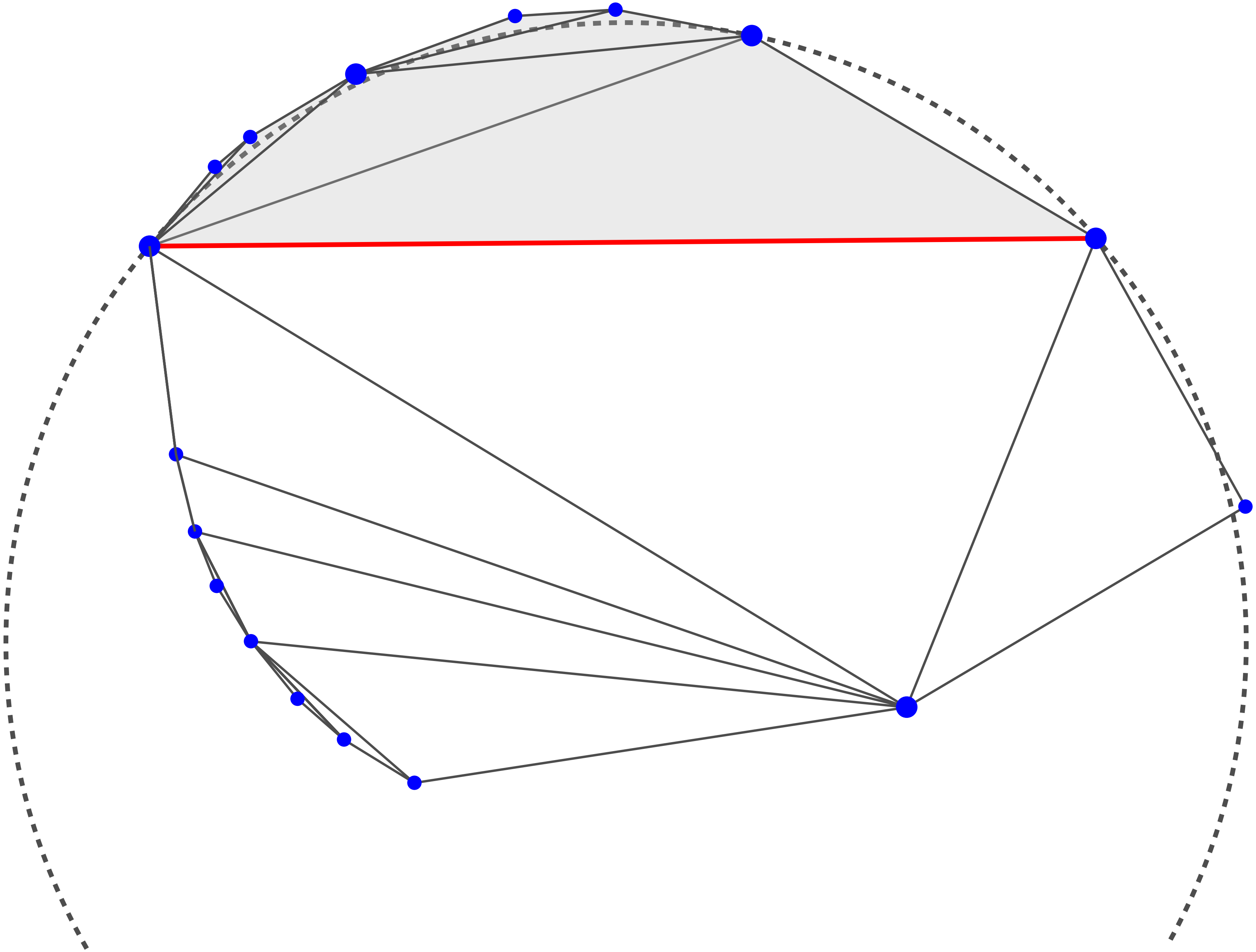
An upper bound for convex position



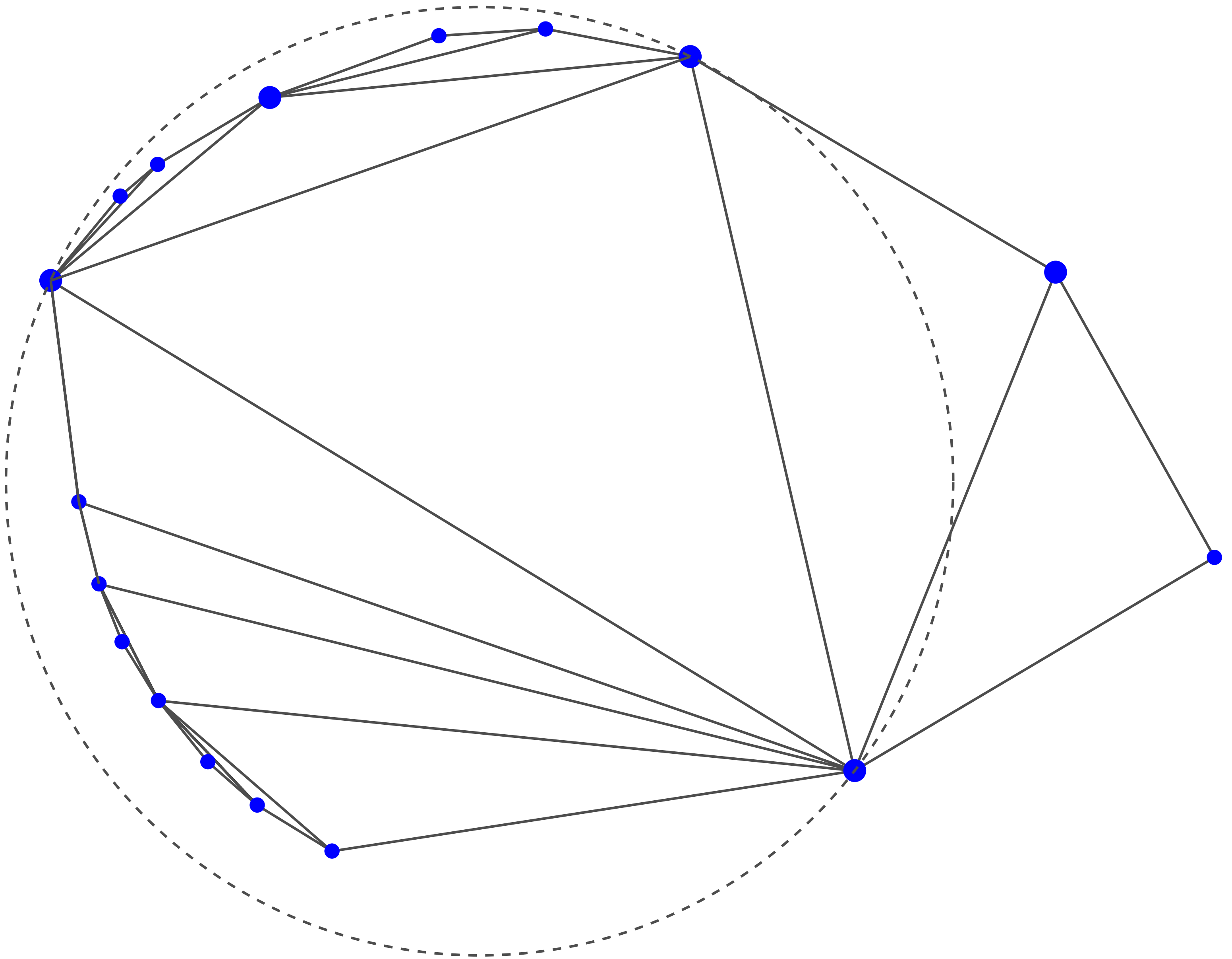
An upper bound for convex position



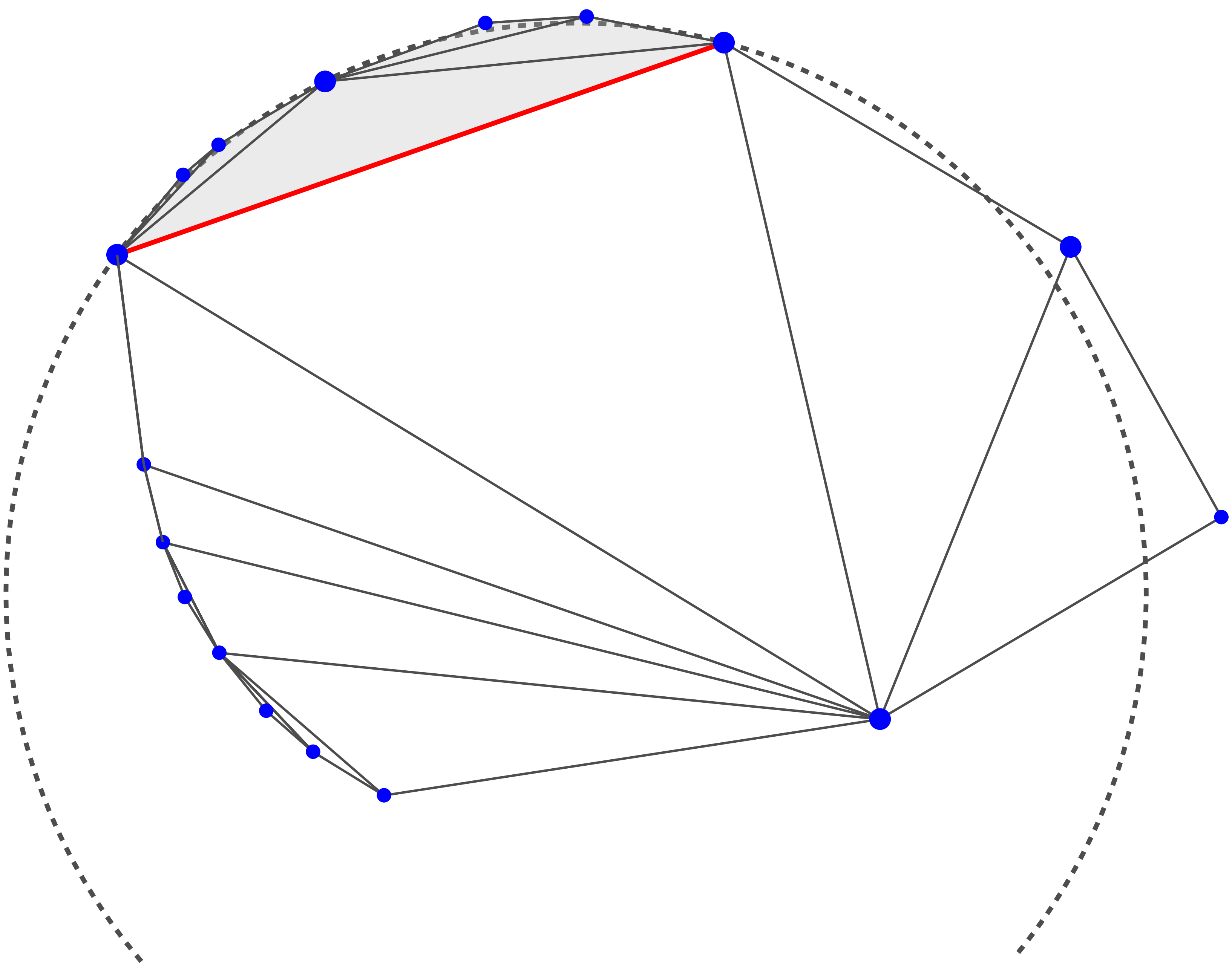
An upper bound for convex position



An upper bound for convex position



An upper bound for convex position



An upper bound for convex position

Theorem. For a point set S in convex position and $k \geq 2$, let $T \neq DT(S)$ be a triangulation in $\mathcal{T}_k(S)$. Then, there exists T' in $\mathcal{T}_k(S)$ such that there is a path between T and T' in $G(\mathcal{T}_{2k-2}(S))$ of length at most $k-1$, where each edge of the path corresponds to flipping an illegal edge.

Summary

Summary

- We showed that $k-1$ flips might be necessary for transforming one k-order triangulation to any other k-order triangulation

- For $k \geq 2$, we showed that any order-k triangulation can be transformed into some other order-k triangulation by at most $k-1$ flips of only illegal edges, such that the intermediate triangulations are of order $2k-2$, in the following settings:

  a) For any $k \geq 2$ and points in convex position
  b) For $k = 2, 3, 4, 5$ and points in general position

Summary

- We showed that $k-1$ flips might be necessary for transforming one k-order triangulation to any other k-order triangulation

- For $k \geq 2$, we showed that any order-k triangulation can be transformed into some other order-k triangulation by at most $k-1$ flips of only illegal edges, such that the intermediate triangulations are of order $2k-2$, in the following settings:

  a) For any $k \geq 2$ and points in convex position
  b) For $k = 2, 3, 4, 5$ and points in general position

Consequences

Summary

- We showed that $k-1$ flips might be necessary for transforming one k-order triangulation to any other k-order triangulation
- For $k \geq 2$, we showed that any order-k triangulation can be transformed into some other order-k triangulation by at most $k-1$ flips of only illegal edges, such that the intermediate triangulations are of order $2k-2$, in the following settings:
  a) For any $k \geq 2$ and points in convex position
  b) For $k = 2, 3, 4, 5$ and points in general position

Consequences

- Any order-k triangulation can be transformed into another by $O(kn)$ flips
- An efficient enumeration algorithm of order-k triangulations

Summary

- We showed that $k-1$ flips might be necessary for transforming one k-order triangulation to any other k-order triangulation

- For $k \geq 2$, we showed that any order-k triangulation can be transformed into some other order-k triangulation by at most $k-1$ flips of only illegal edges, such that the intermediate triangulations are of order $2k-2$, in the following settings:

  a) For any $k \geq 2$ and points in convex position
  b) For $k = 2, 3, 4, 5$ and points in general position

Consequences

- Any order-k triangulation can be transformed into another by $O(kn)$ flips

- An efficient enumeration algorithm of order-k triangulations

Thank you! :-D