# On the Number of Delaunay Triangles occurring in all Contiguous Subsequences

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joined work with S. Funke

# Motivation

- Subcomplexes of the Delaunay triangulation useful for representing the shape of objects from discrete samples
  - $\alpha$-shapes, $\beta$-skeleton, the crust

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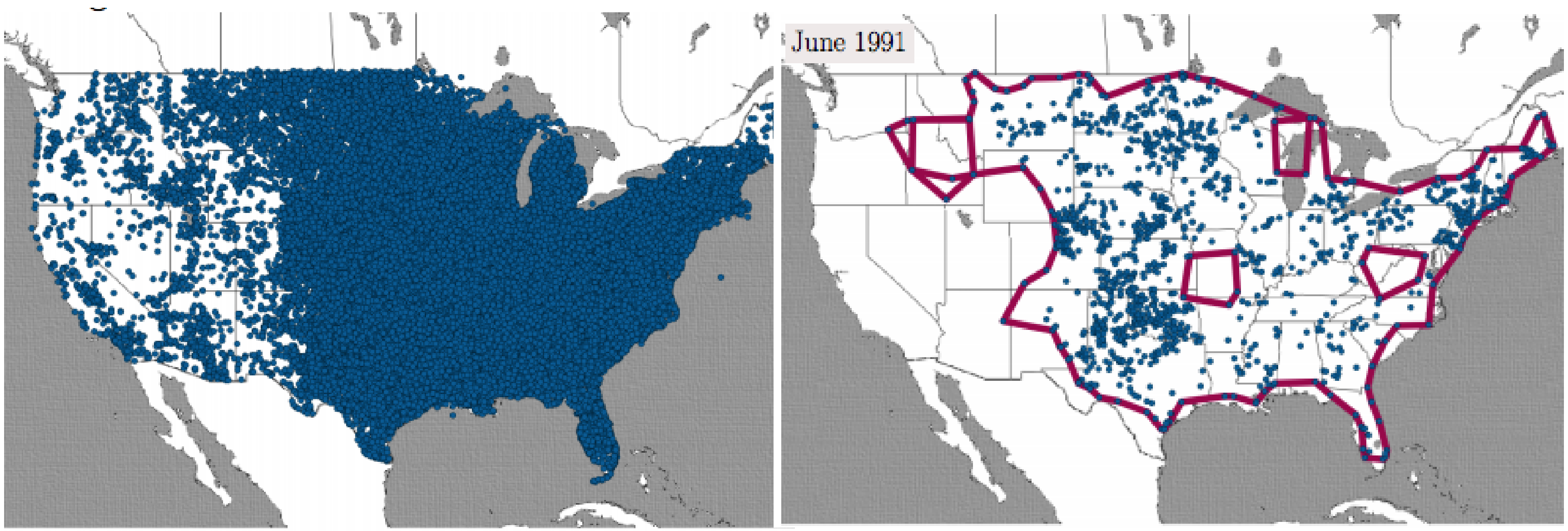
- Restrict temporal samples to shorter time intervals
  - $\alpha$-shapes used to visualize the regions of storm events




[Bonerath et al. '19]

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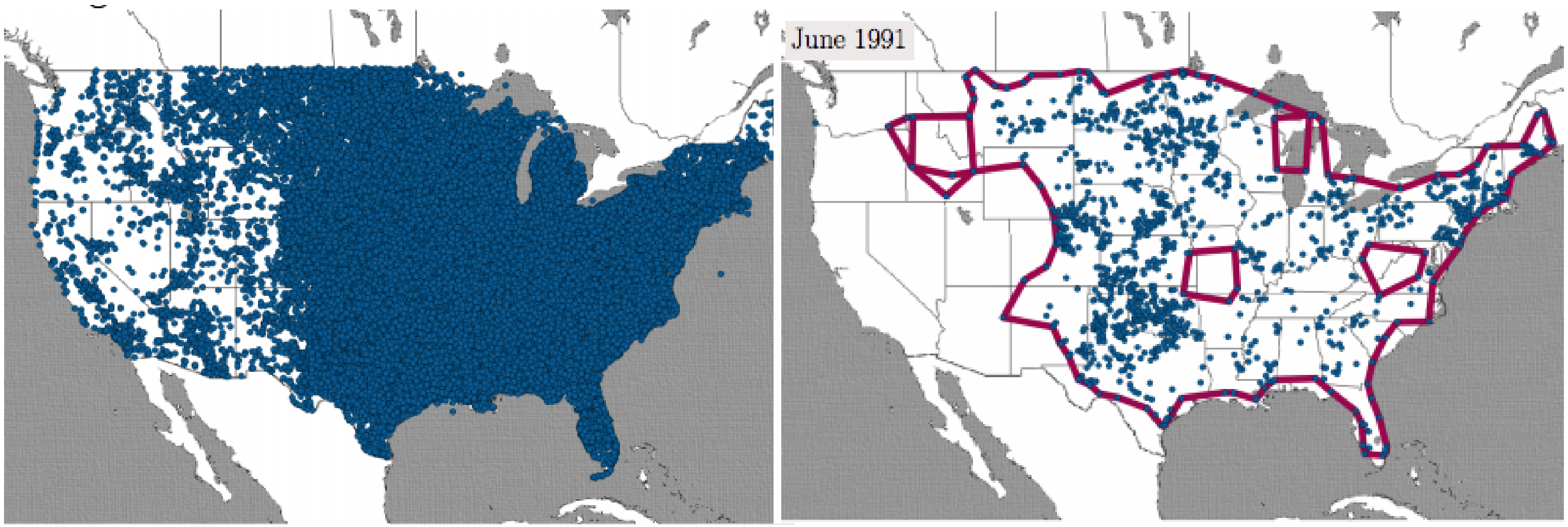


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- Precompute all Delaunay triangles occurring in all contiguous subsequences & index them w.r.t. time, possibly some other parameter ($\alpha$ value, ...) for faster retrieval

# Some Delaunay Triangulations

- $P = \{p_1, p_2, \ldots, p_n\}$, $P_{i,j} := \{p_i, p_{i+1}, \ldots, p_j\}$
- Example: Incremental construction of $DT(P)$ via the sequence $DT(P_{1,3}), DT(P_{1,4}), \ldots DT(P_{1,n})$

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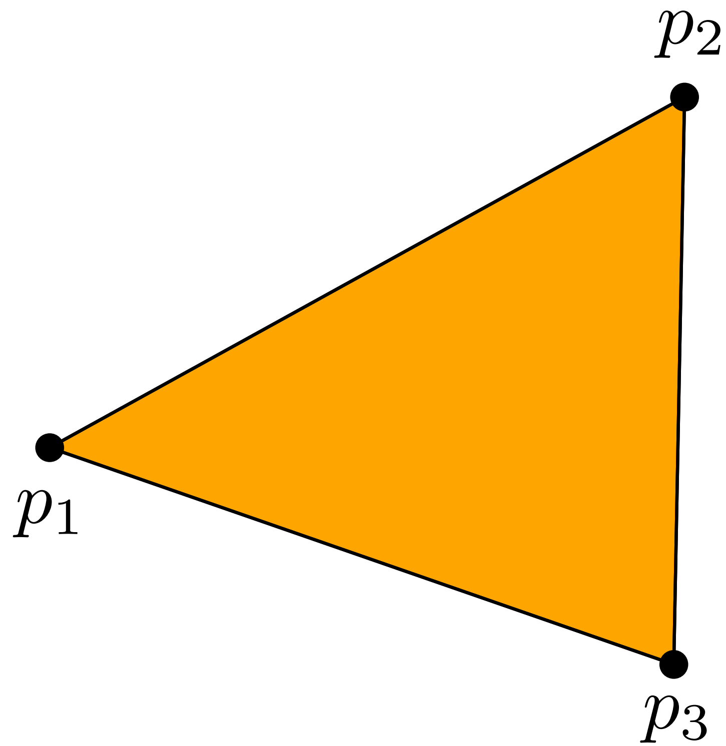

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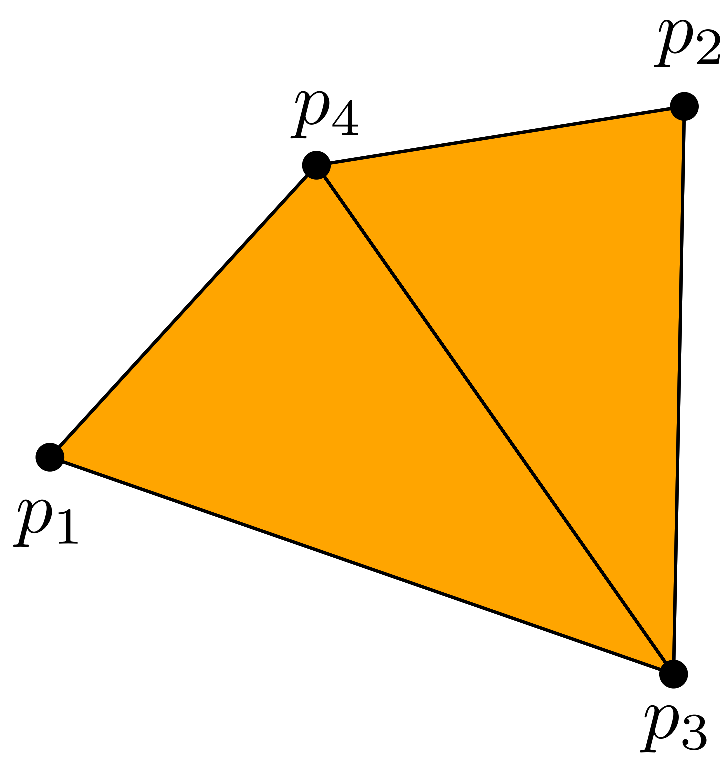

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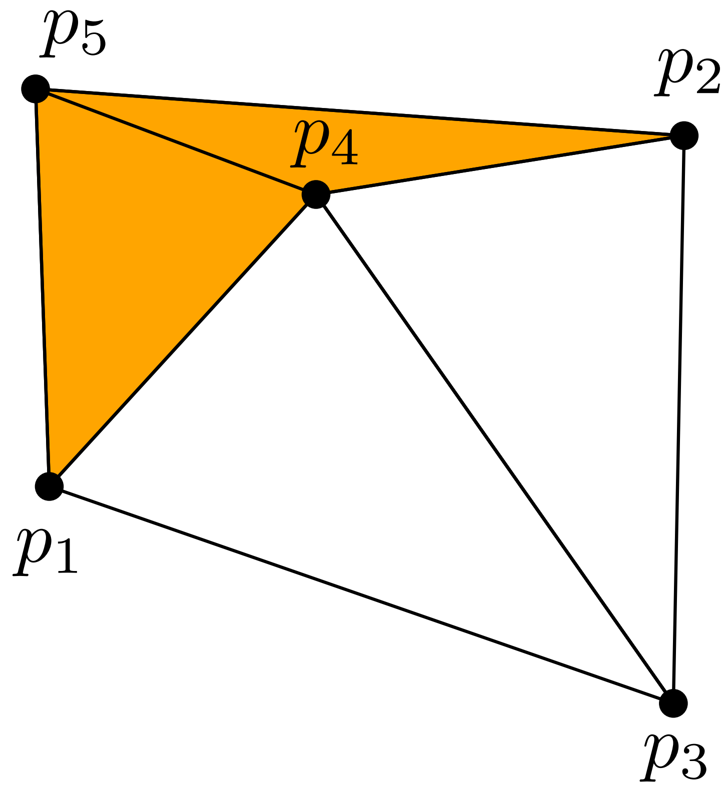

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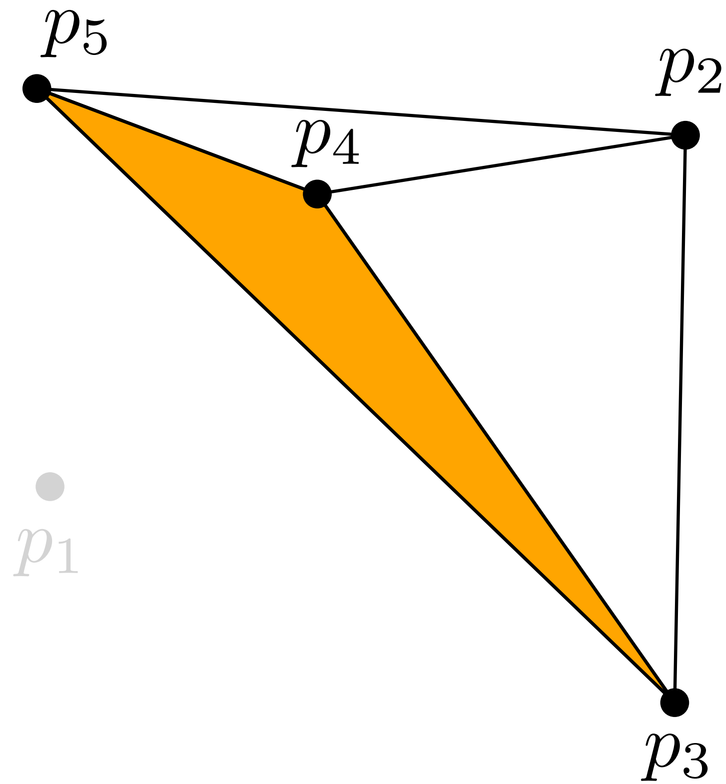

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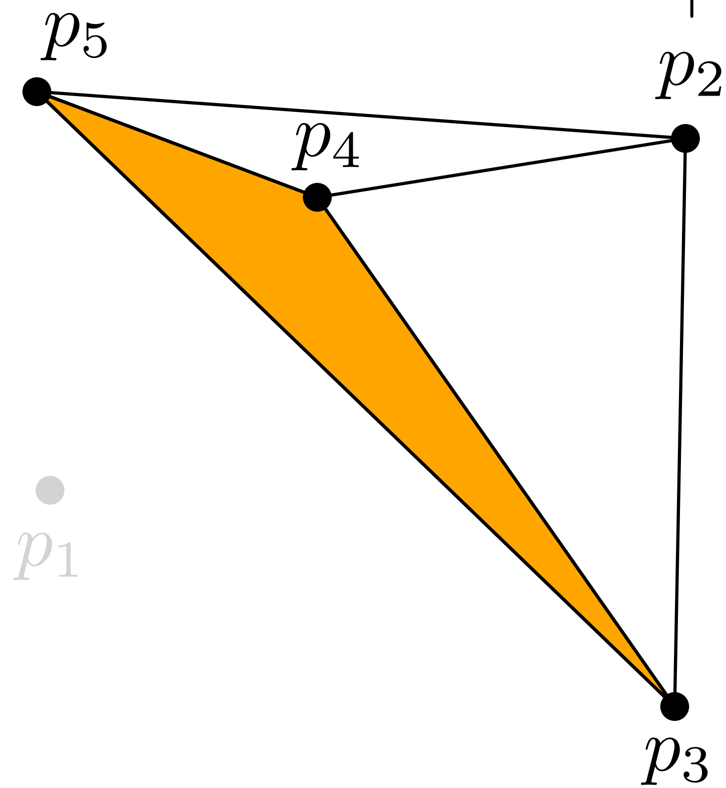
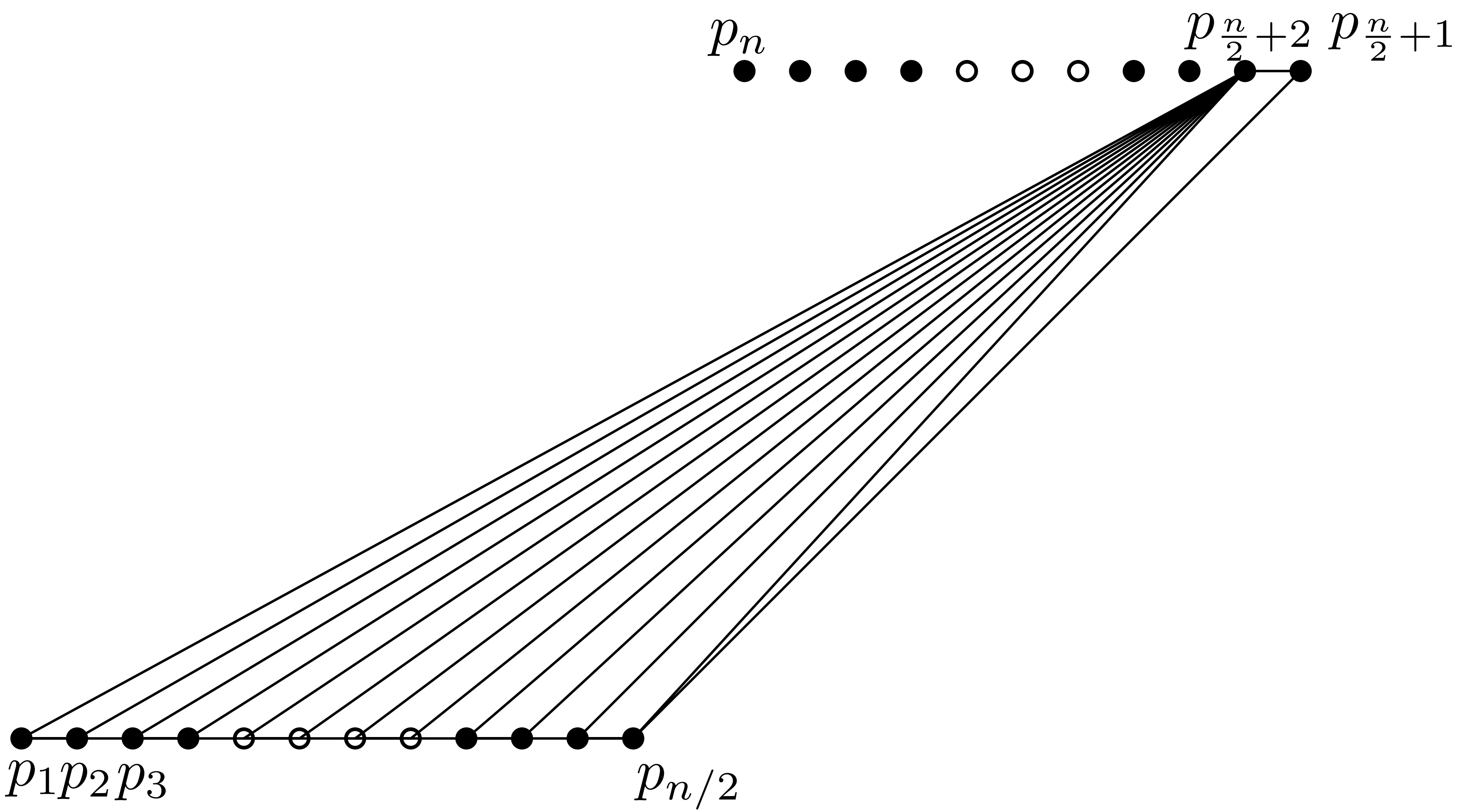
- $T_{i,j}$: triangles of $DT(P_{i,j})$
- $T := \bigcup_{i<j} T_{i,j}$
- $|T| = ?$



# Some Delaunay Triangulations

- $P = \{p_1, p_2, \ldots, p_n\}$, $P_{i,j} := \{p_i, p_{i+1}, \ldots, p_j\}$
- Another example with $|T| \in \Theta(n^2)$



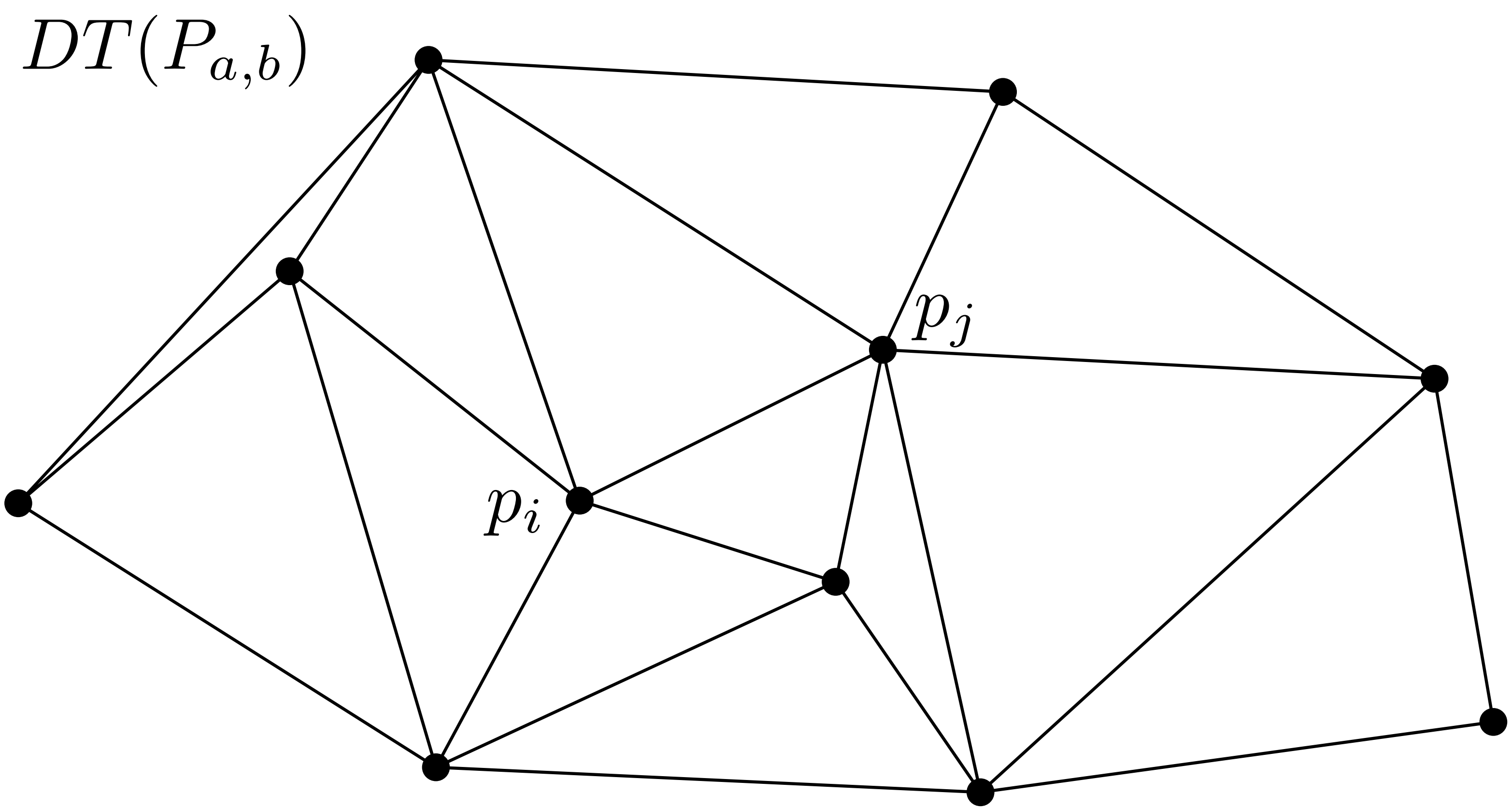
What is the expected number of Delaunay triangles in contiguous subsequences for arbitrary point sets $P$ *ordered uniformly at random*?

# Counting Delaunay Edges and Triangles

- Let $E_T := \{e \mid \exists\, t \in T : e \text{ edge of } t\}$
- Assume non-degeneracy of P
  - No 4 co-circular points, no 3 co-linear points
- Proof:
  1. Bound the expected number of Delaunay edges
  2. Show linear dependence between the number of Delaunay triangles and Delaunay edges

# Lemma 1: Any $e = \{p_i, p_j\} \in E_T$ appears in $DT(P_{i,j})$

There exists some triangle $t \in T$ which uses $e$, so for suitable $a \leq i, b \geq j$, $e$ appears in $DT(P_{a,b})$:



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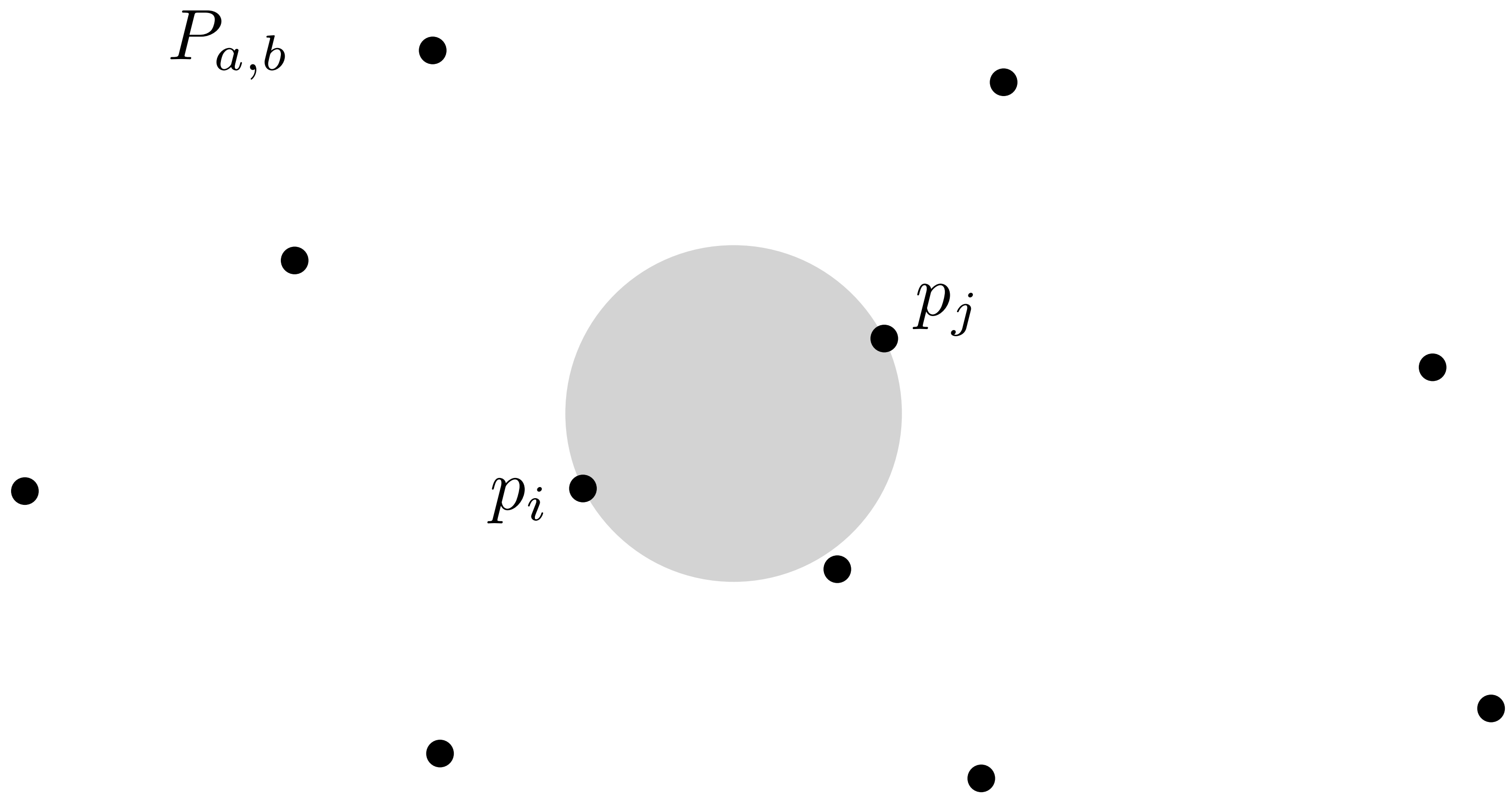

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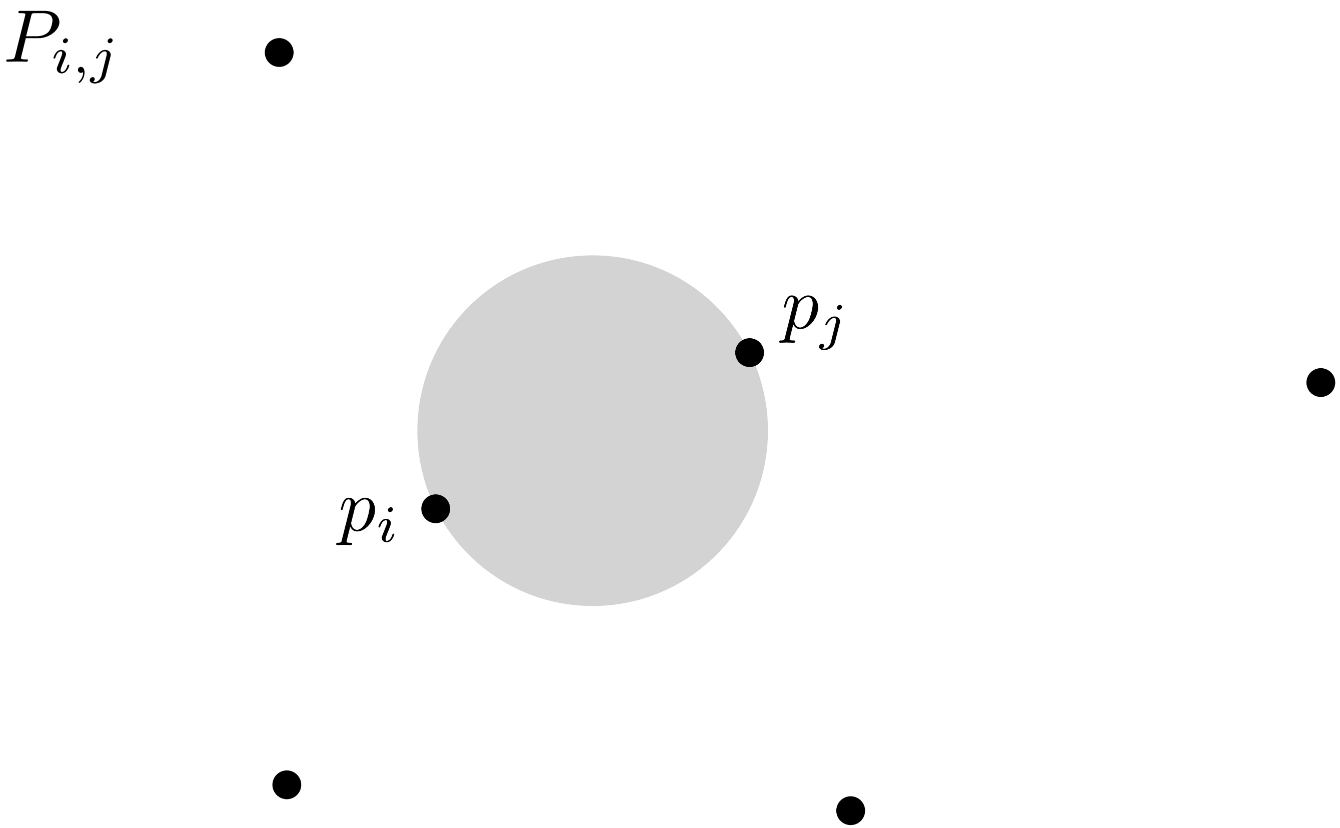

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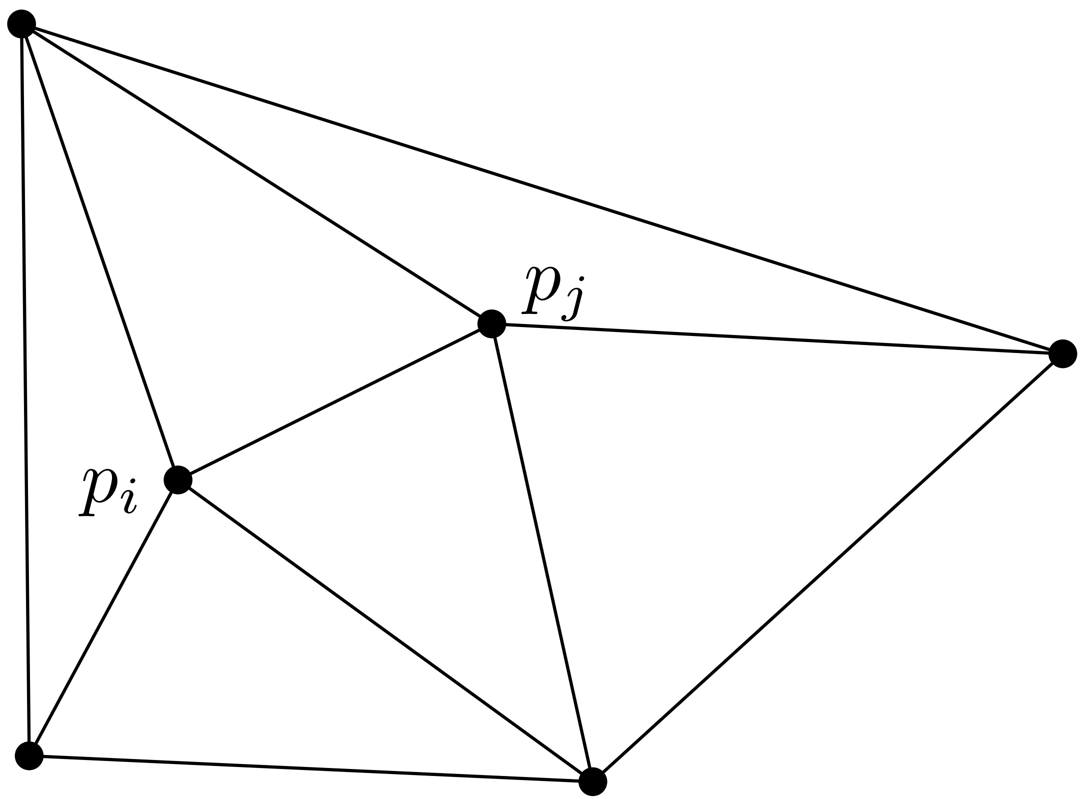


$\Rightarrow e \in DT(P_{i,j})$

# Lemma 2: For $j > i + 1$: $Pr[e \in DT(P_{i,j})] < \frac{6}{j-i}$

- $DT(P_{i,j})$ is a planar graph with $j - i + 1$ nodes
  Euler's formula: $\leq 3(j - i + 1) - 6$ edges

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- So choosing $p_i$ and $p_j$ out of $P_{i,j}$ is the same as choosing one edge (amongst all $\binom{j-i+1}{2}$ possible edges) in a graph with $j - i + 1$ nodes and $\leq 3(j - i + 1) - 6$ edges

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$$\Rightarrow Pr[e \in DT(P_{i,j})] \leq \frac{3(j-i+1)-6}{\binom{j-i+1}{2}} < \frac{6}{j-i}$$

# Lemma 3: The expected size of $E_T$ is $\Theta(n \log n)$

Lower bound:

- Within $P_{1,1}, \ldots, P_{1,n}$, $p_1$'s nearest neighbor changes $\Theta(\log n)$ times in expectation
  - Applies to all $p_i$
- Nearest neighbor graph is a subgraph of the Delaunay triangulation

$$\Rightarrow E[|E_T|] \in \Omega(n \log n)$$

# Lemma 3: The expected size of $E_T$ is $\Theta(n \log n)$

Upper bound: Use linearity of expectation to sum over all potential edges of $E_T$

- Edges $\{p_i, p_{i+1}\}$ always exist
- Other edges $\{p_i, p_j\}$ exist with probability $< \frac{6}{j-i}$

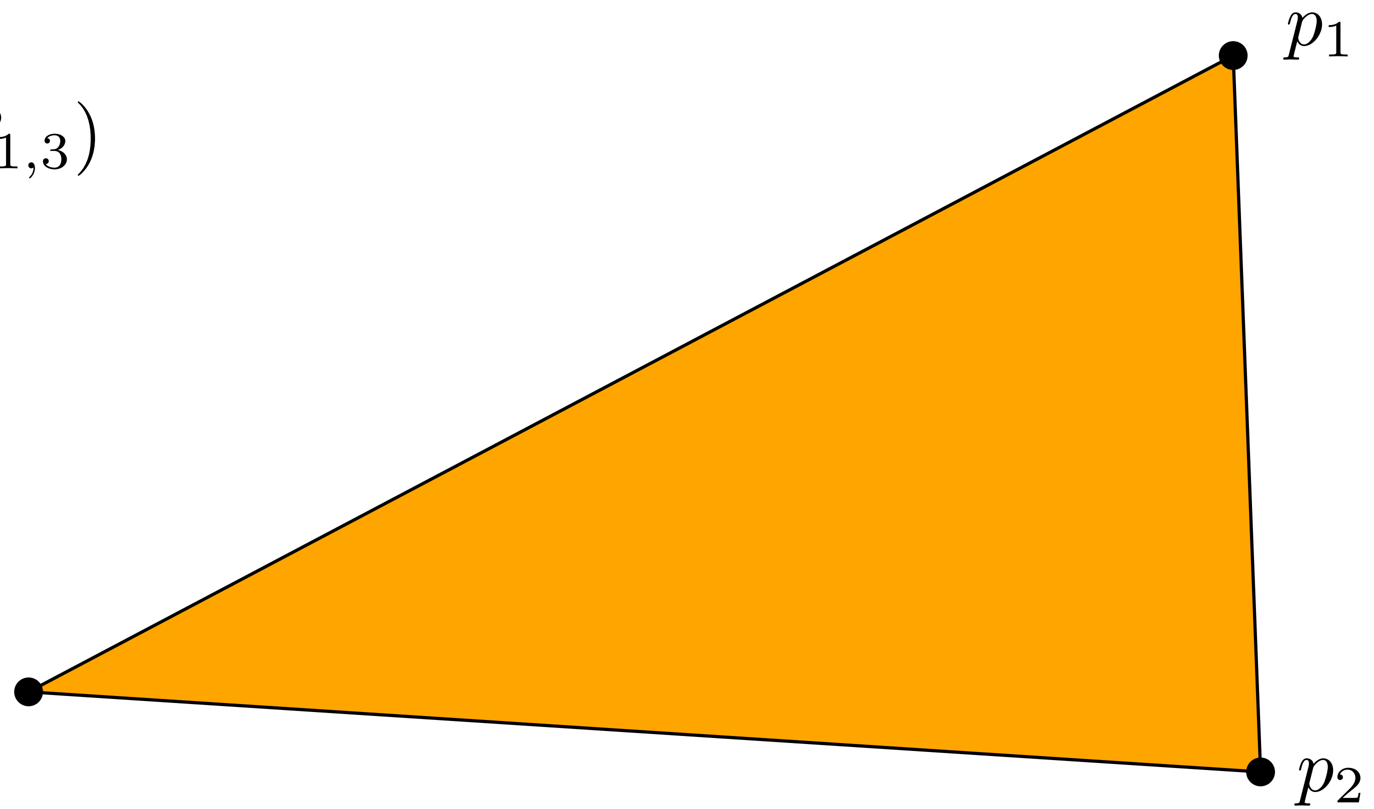
$$E[|E_T|] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} Pr[\{p_i, p_j\} \in E_T]$$

$$\leq \sum_{i=1}^{n-1} \left[1 + \sum_{j=i+2}^{n} \frac{6}{j-i}\right] = (n-1) + 6 \sum_{i=1}^{n-1} \sum_{j=2}^{n-i} \frac{1}{j}$$

$$\leq (n-1) + 6 \sum_{i=1}^{n-1} H_n = O(n \log n)$$

# Delaunay edges used by many Delaunay triangles

$DT(P_{1,3})$



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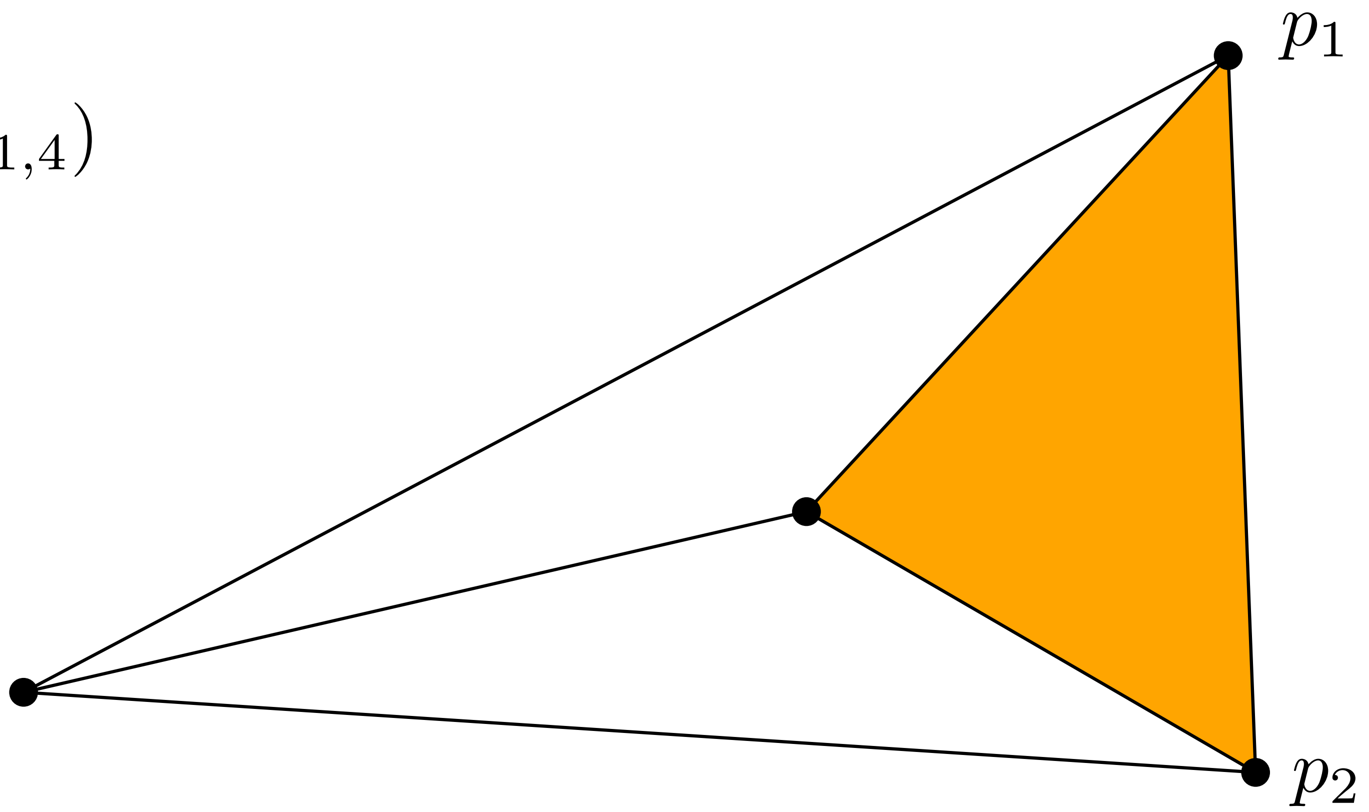

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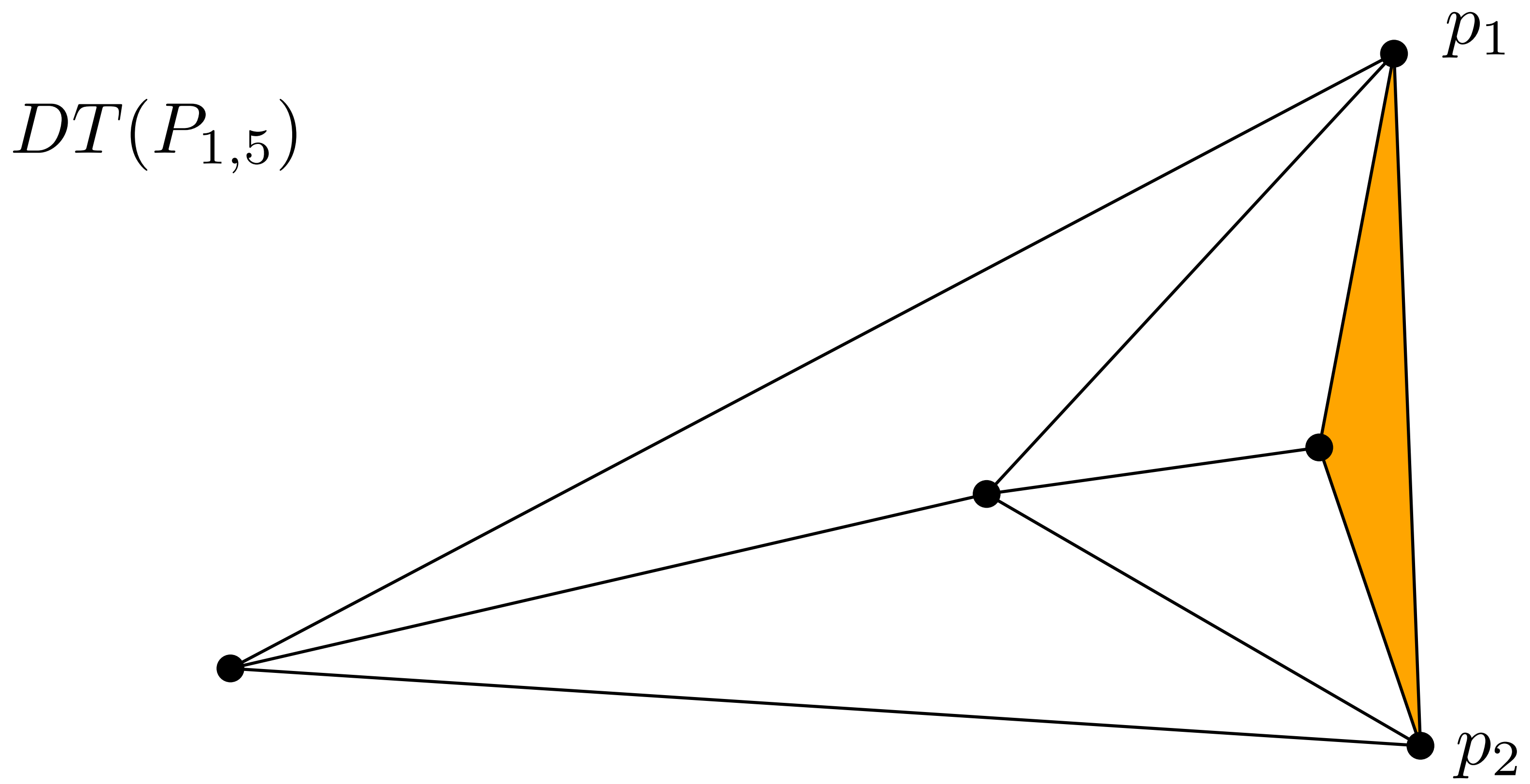

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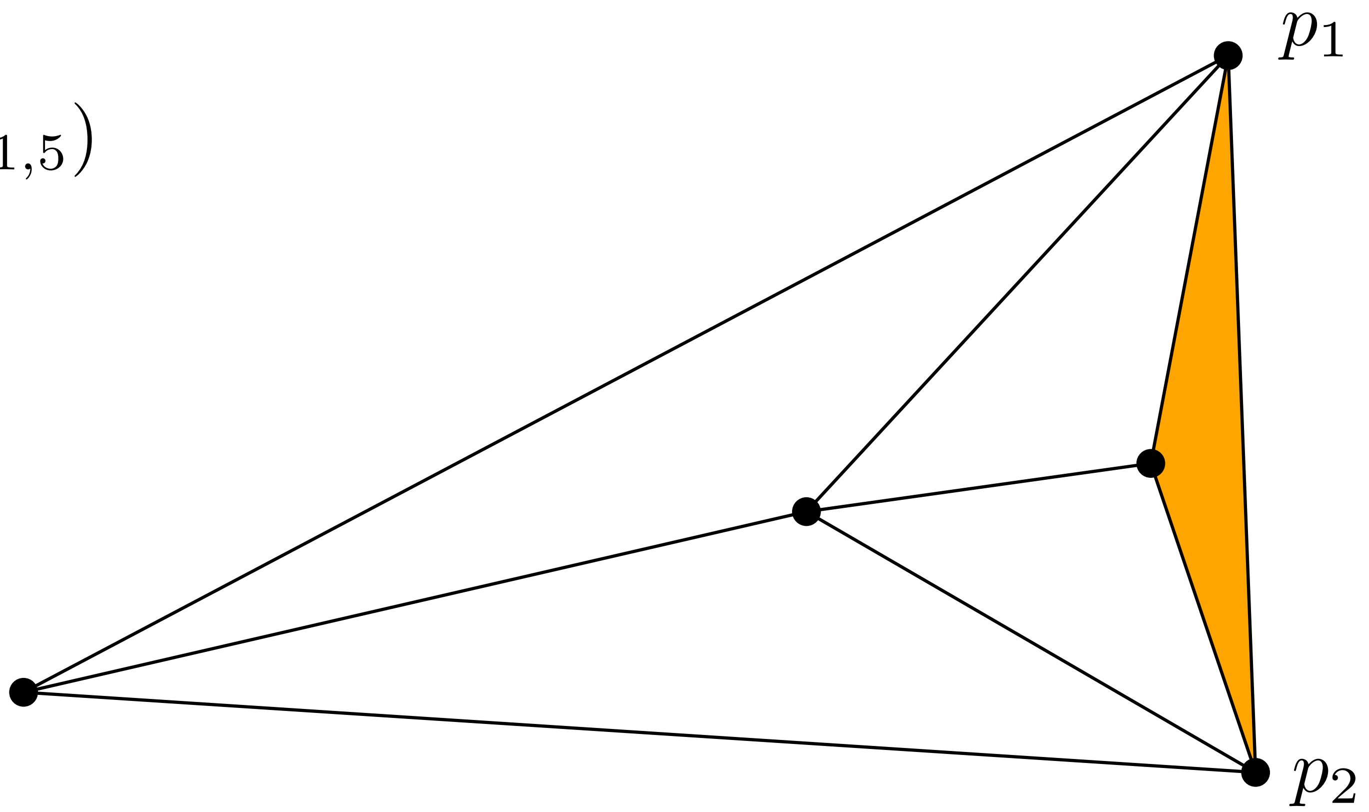


$\rightarrow$ Edge $\{p_1, p_2\}$ used by many Delaunay triangles

# Lemma 4: $|T| \in \Theta(|E_T|)$ for arbitrary orderings of $P$

- For each triangle in $T$, at most 3 edges exist in $E_T$
  $\Rightarrow |E_T| \leq 3|T|$
- For upper bound on edges, charge triangles to edges:
- Delaunay triangle $p_a p_b p_c$ $(a < b < c)$ exists in $DT(P_{a,c})$
- In $DT(P_{a,c})$, at most one other triangle uses edge $\{p_a, p_c\}$
  $\Rightarrow$ Charging $T$'s triangles $p_a p_b p_c$ to $\{p_a, p_c\}$ ensures at most two triangles are charged to each edge in $E_T$
  $\Rightarrow |T| \leq 2|E_T|$

# Putting it all together

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$$E[|E_T|] = \Theta(n \log n) \text{ and } |T| \in \Theta(|E_T|)$$
$$\Rightarrow E[|T|] = \Theta(n \log n)$$

# Experimental results & data

| $n$ | $\lvert\bigcup_{j\leq n} T_{1,j}\rvert$ | $\lvert T\rvert$ | $T$ computation time |
|---|---|---|---|
| $2^{15}$ | 196,168 | 2,860,956 | 6,309 ms |
| $2^{16}$ | 392,592 | 6,267,247 | 14,229 ms |
| $2^{17}$ | 785,879 | 13,622,094 | 32,817 ms |
| $2^{18}$ | 1,572,292 | 29,425,885 | 70,545 ms |
| $2^{19}$ | 3,144,770 | 63,210,634 | 155,370 ms |
| $2^{20}$ | 6,290,562 | 135,134,028 | 347,186 ms |
| $2^{21}$ | 12,581,989 | 287,719,166 | 771,705 ms |

$n$ points sampled from the unit square, averaged over 20 runs

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# More data

| $n$ | $\|T\|$ | $T$ time | $\|\bigcup_{j\leq n} T_{1,j}\|$ | $T_{1,n}$ time |
|---|---|---|---|---|
| $2^{15}$ | 2,860,956 | 6,309 ms | 196,168 | 260 ms |
| $2^{16}$ | 6,267,247 | 14,229 ms | 392,592 | 745 ms |
| $2^{17}$ | 13,622,094 | 32,817 ms | 785,879 | 1,779 ms |
| $2^{18}$ | 29,425,885 | 70,545 ms | 1,572,292 | 4,068 ms |
| $2^{19}$ | 63,210,634 | 155,370 ms | 3,144,770 | 9,008 ms |
| $2^{20}$ | 135,134,028 | 347,186 ms | 6,290,562 | 20,374 ms |
| $2^{21}$ | 287,719,166 | 771,705 ms | 12,581,989 | 44,082 ms |