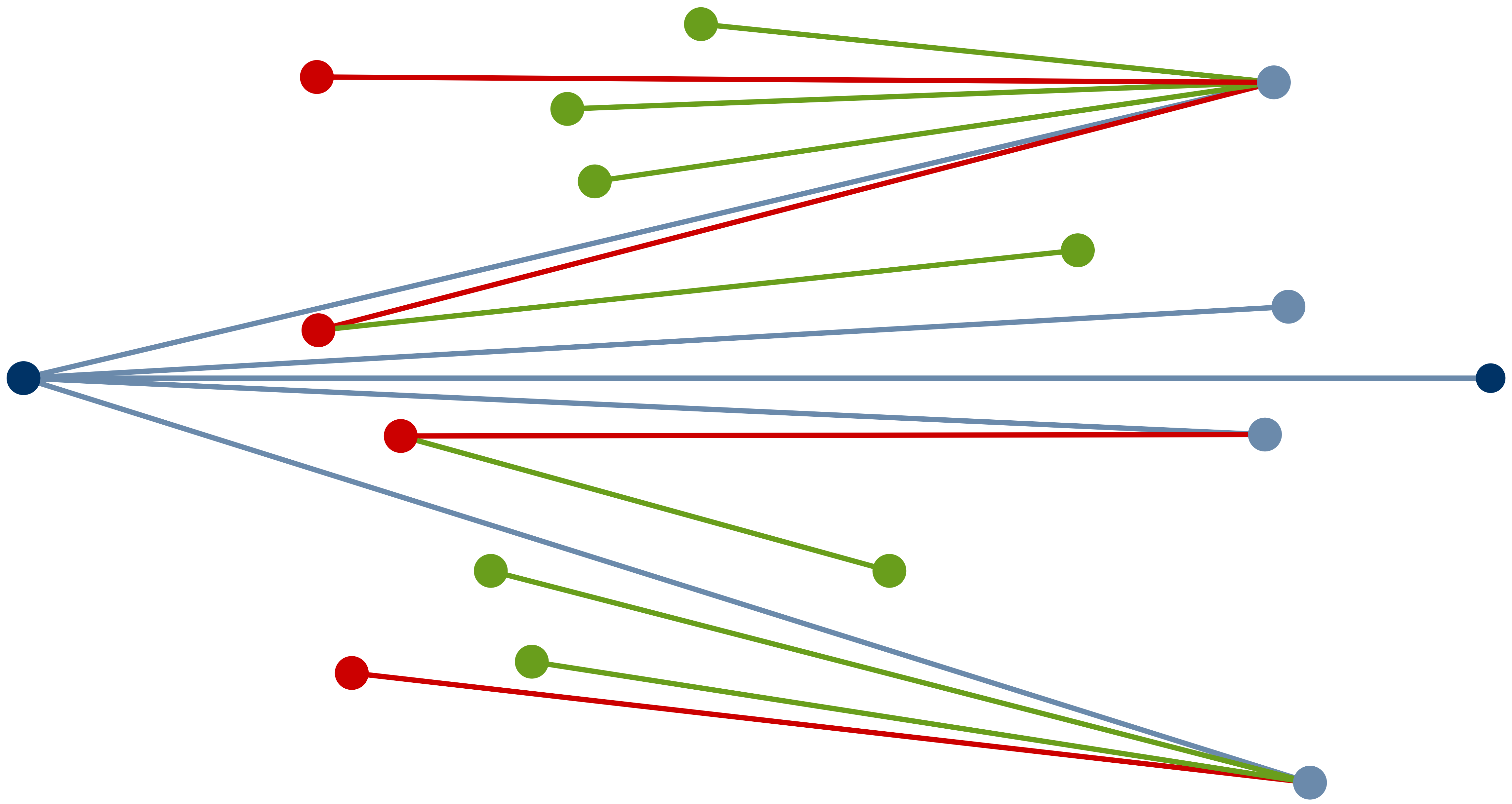
A better approximation for longest noncrossing spanning trees

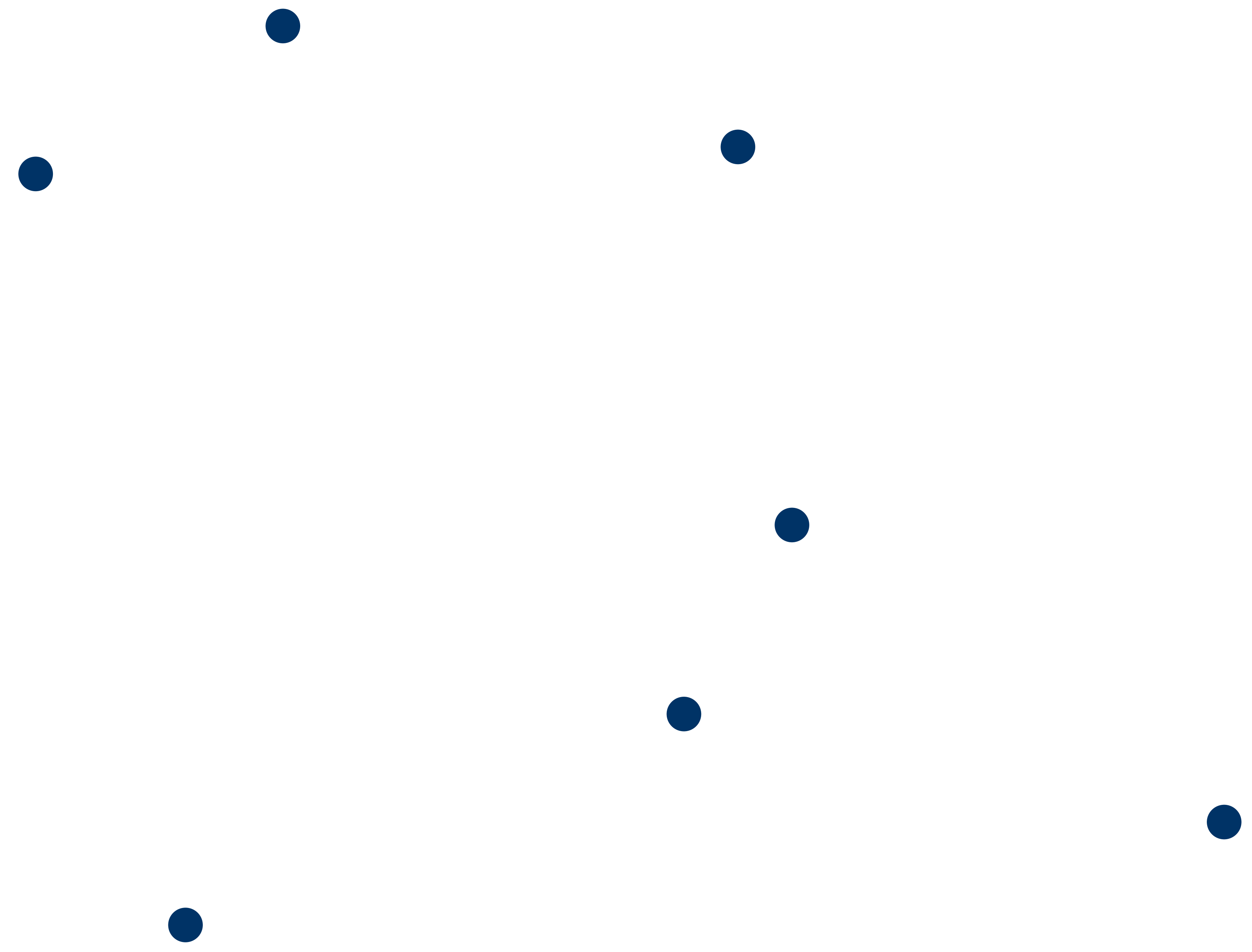


Sergio Cabello, Aruni Choudhary, Michael Hoffmann, **Katharina Klost**, Meghana M. Reddy, Wolfgang Mulzer, Felix Schröder and Josef Tkadlec

EuroCG 2020 Würzburg - Session 2a

Introduction

Motivation: Study *long noncrossing configurations*



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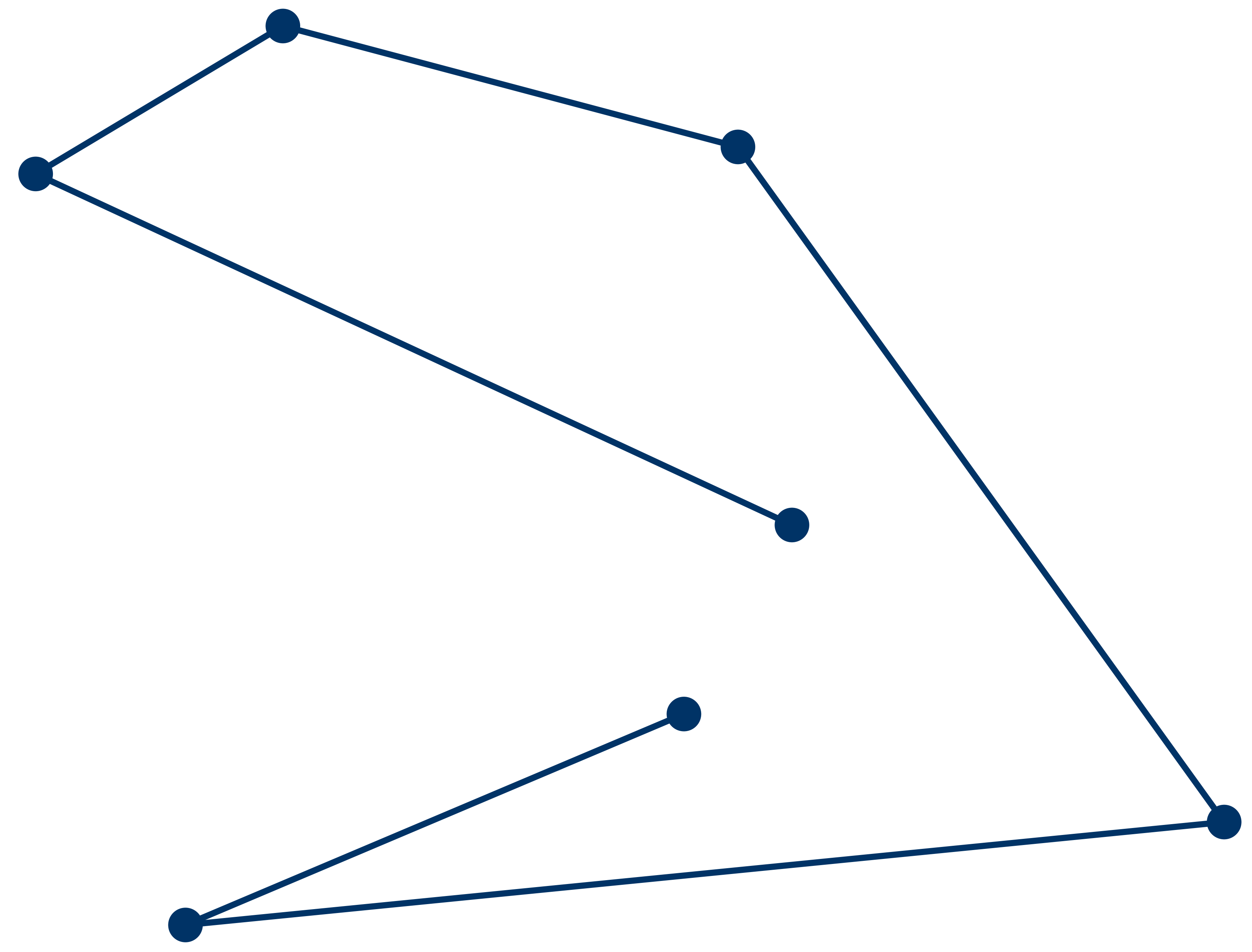

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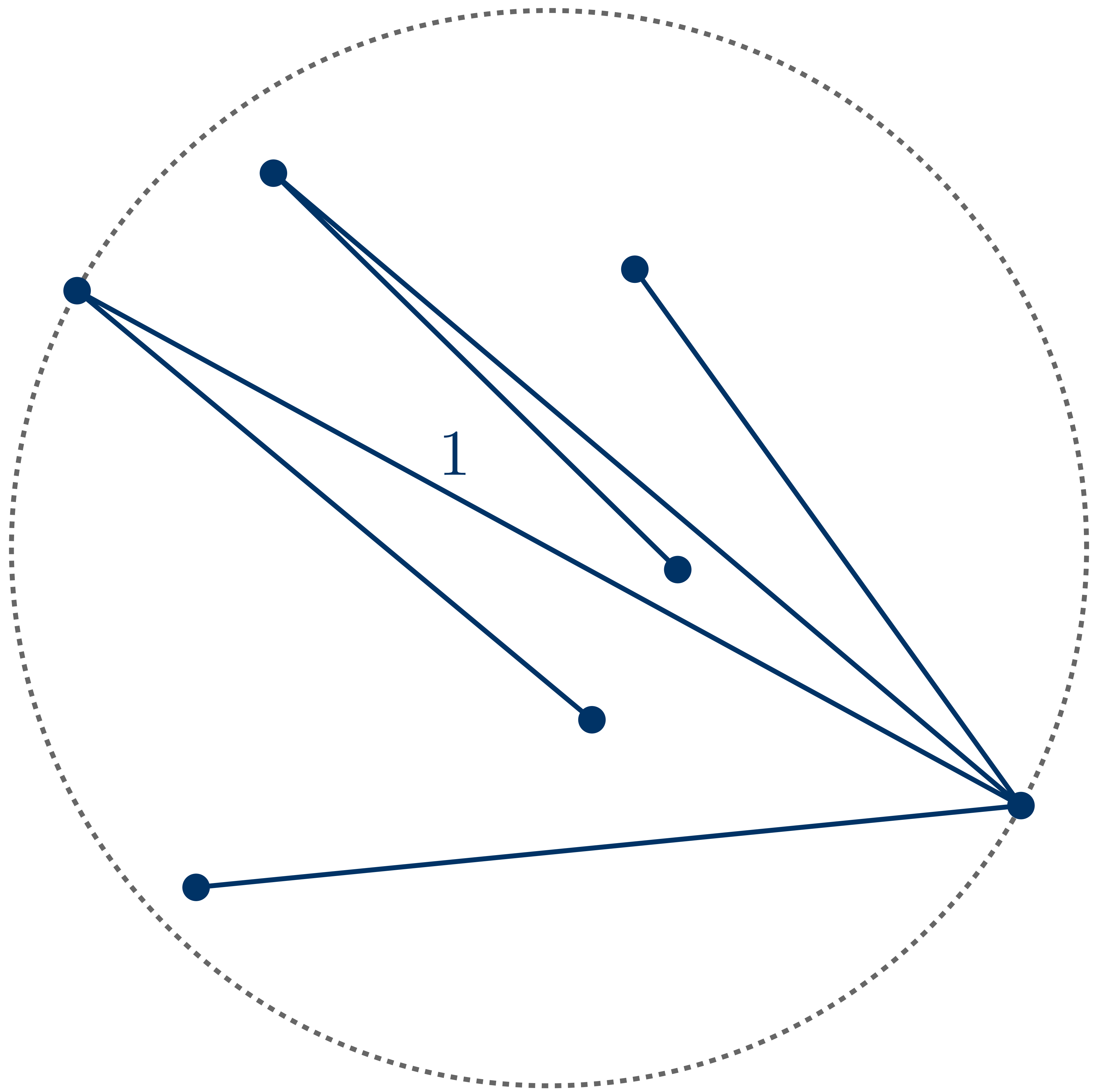
Input: P set of points, $\mathrm{diam}(P) = 1$
Goal: Find *longest* noncrossing spanning tree for P



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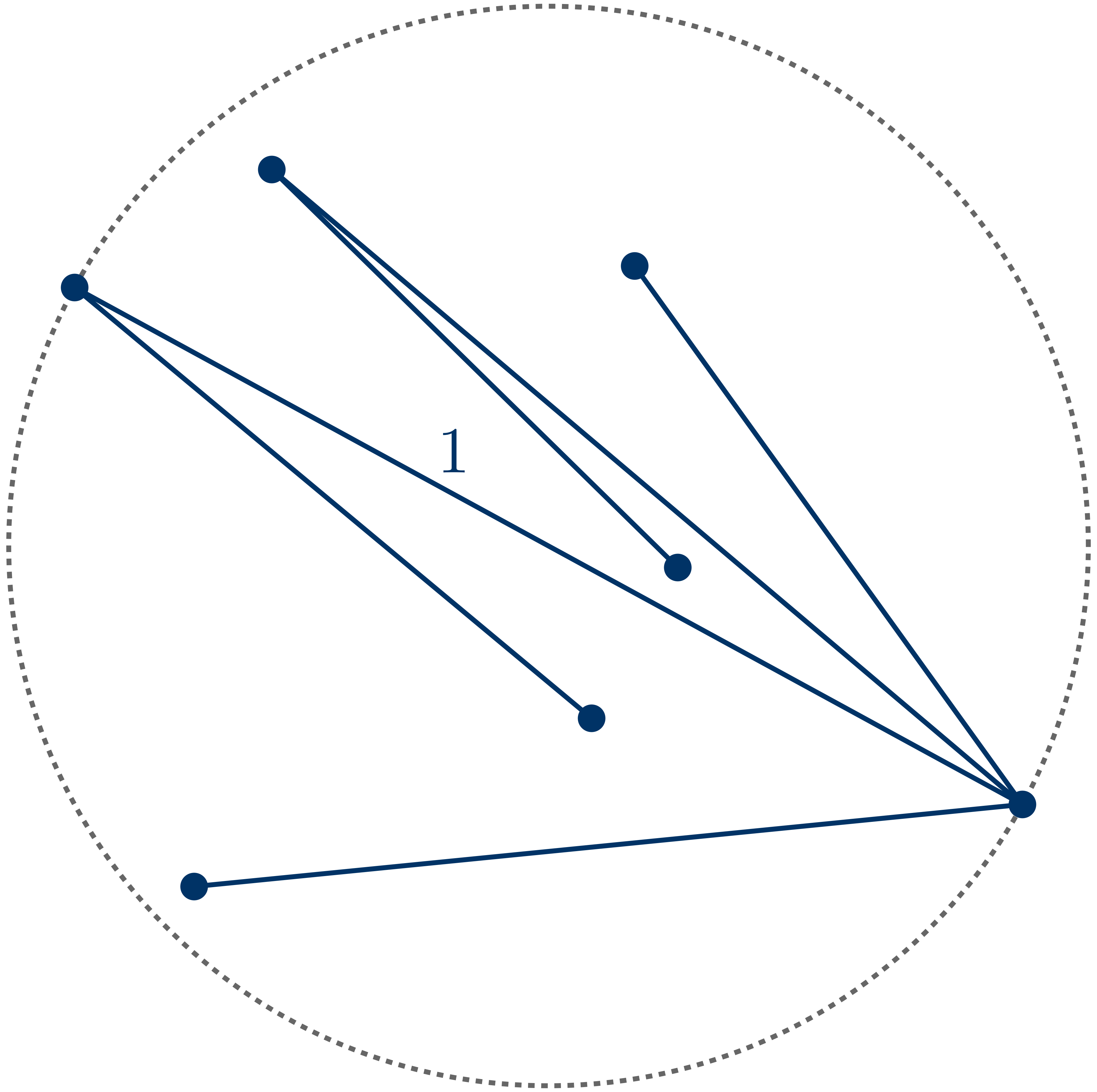

What is known?

Minimum Spanning Tree	**P**	exact
Maximum Spanning Tree		
Maximum noncrossing ST		

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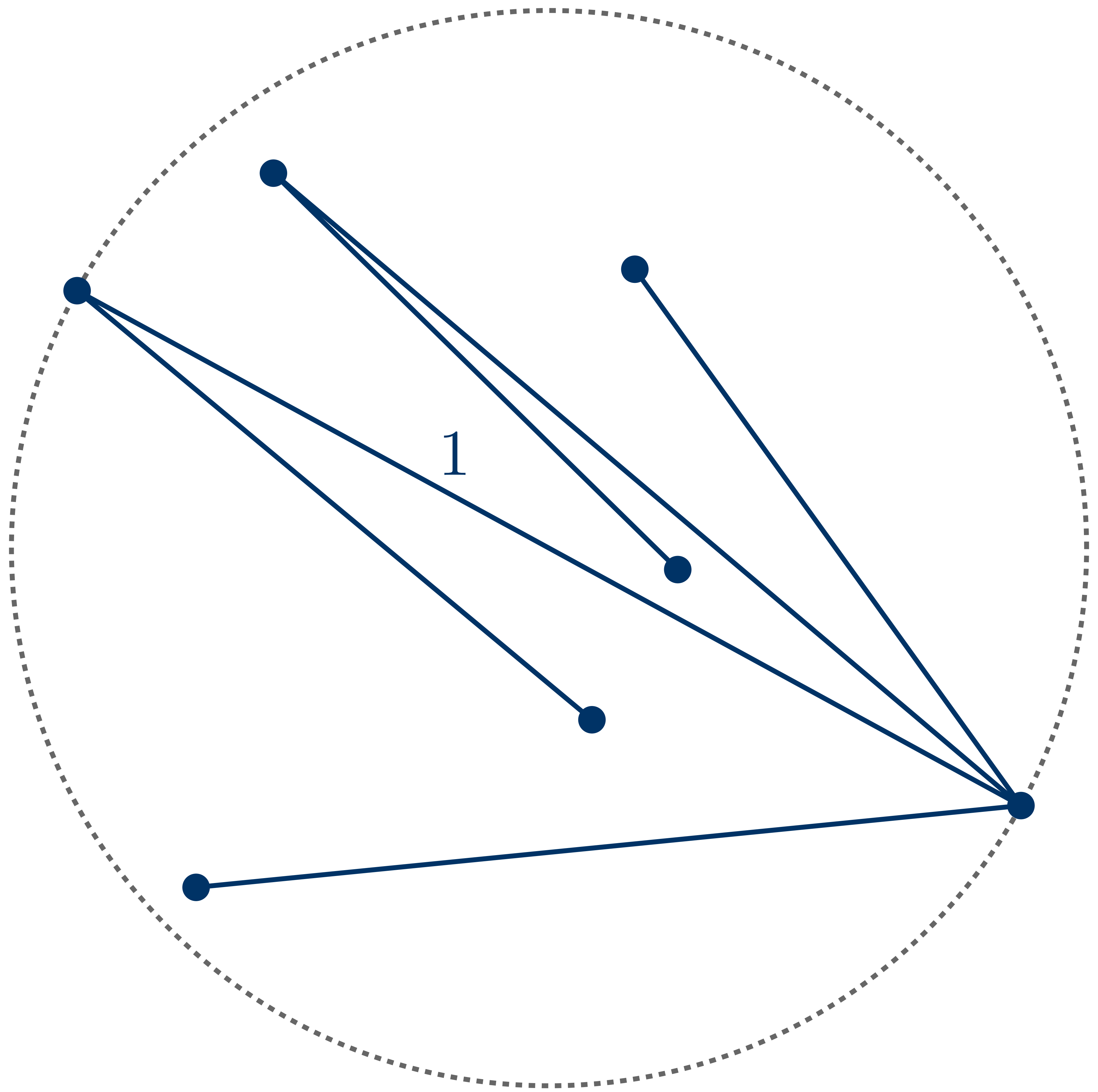

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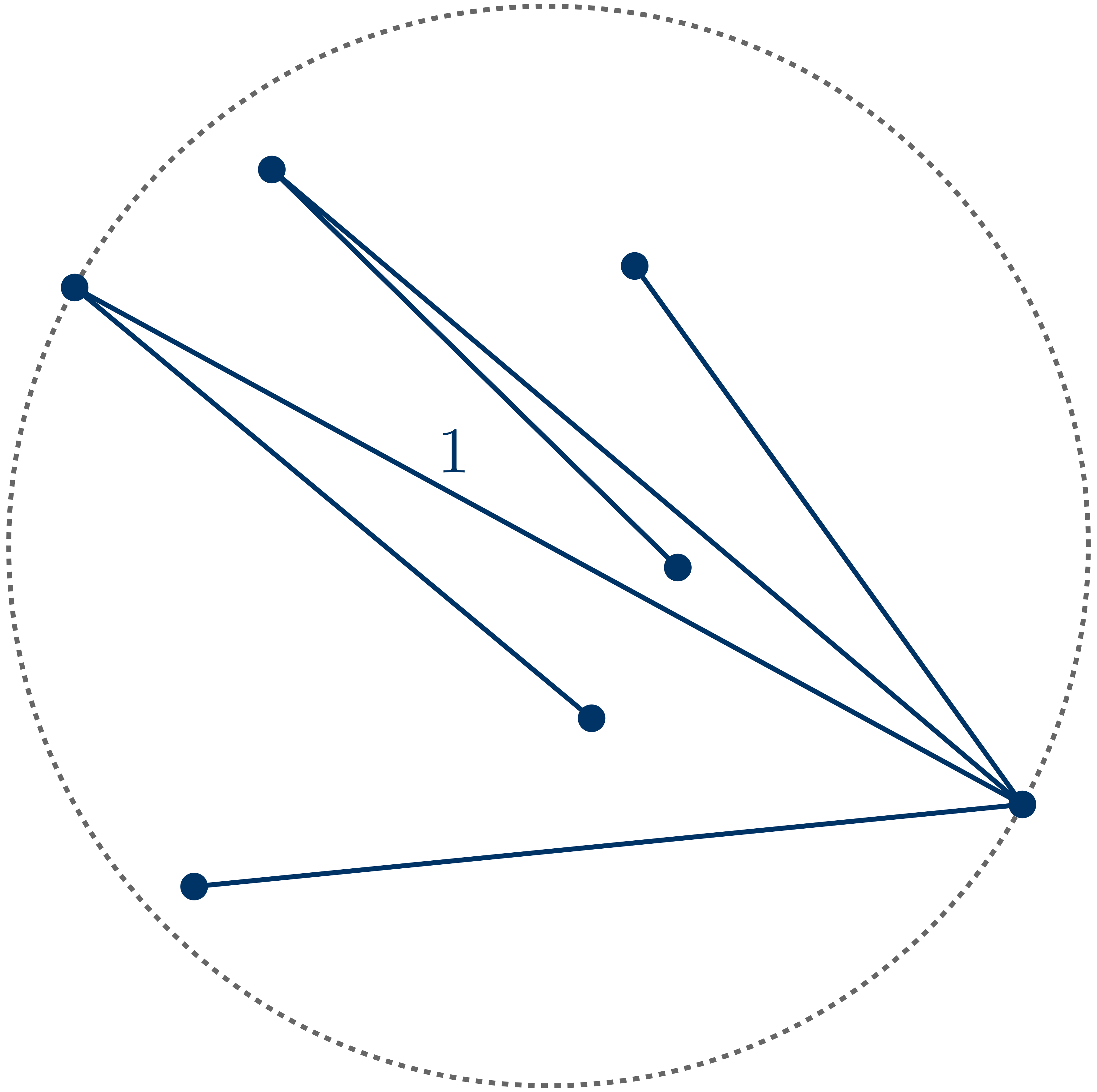

What is known?

Minimum Spanning Tree	$\mathbf{P}$	exact
Maximum Spanning Tree	$\mathbf{P}$	exact
Maximum noncrossing ST	$\mathbf{NP}$-hard??	0.5[1]

[1] Alon et al. (1993)

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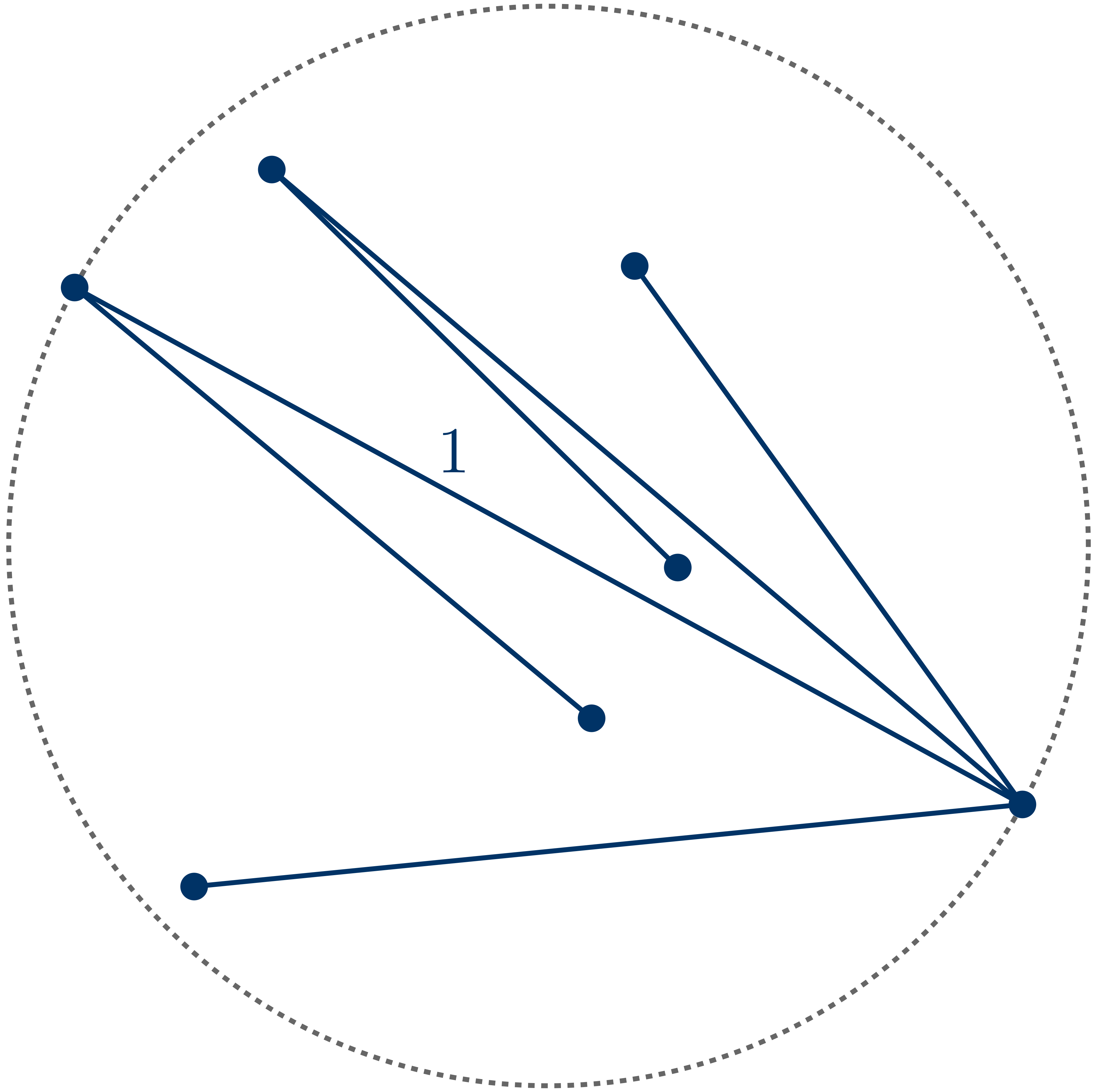

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Minimum Spanning Tree	$\mathbf{P}$	exact
Maximum Spanning Tree	$\mathbf{P}$	exact
Maximum noncrossing ST	$\mathbf{NP}$-hard??	0.502[2]

[2] Dumitrescu and Tóth (2009)

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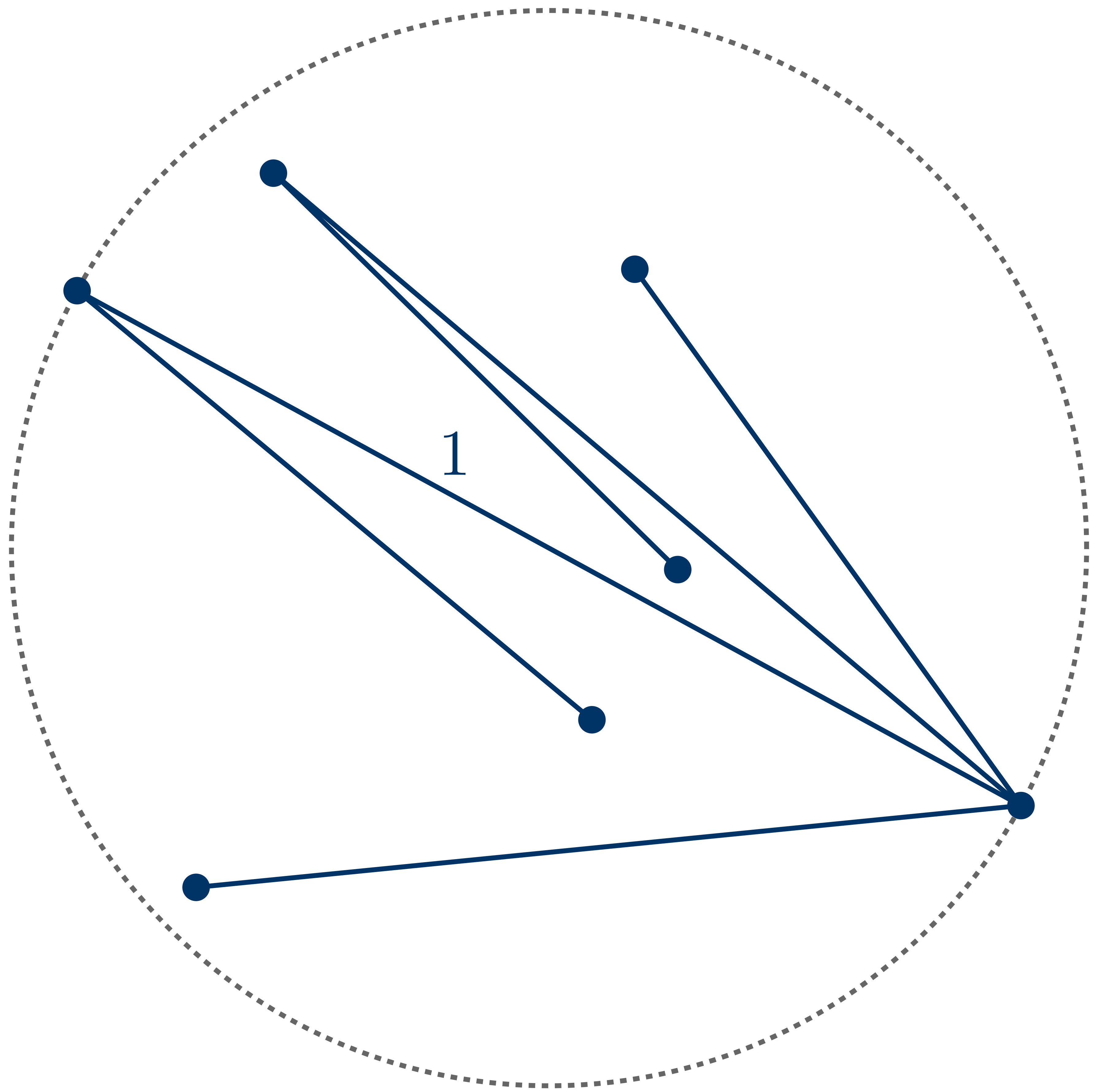

What is known?

Minimum Spanning Tree	$\mathbf{P}$	exact
Maximum Spanning Tree	$\mathbf{P}$	exact
Maximum noncrossing ST	$\mathbf{NP}$-hard??	0.503[3]

[3] Biniaz et al. (2019)

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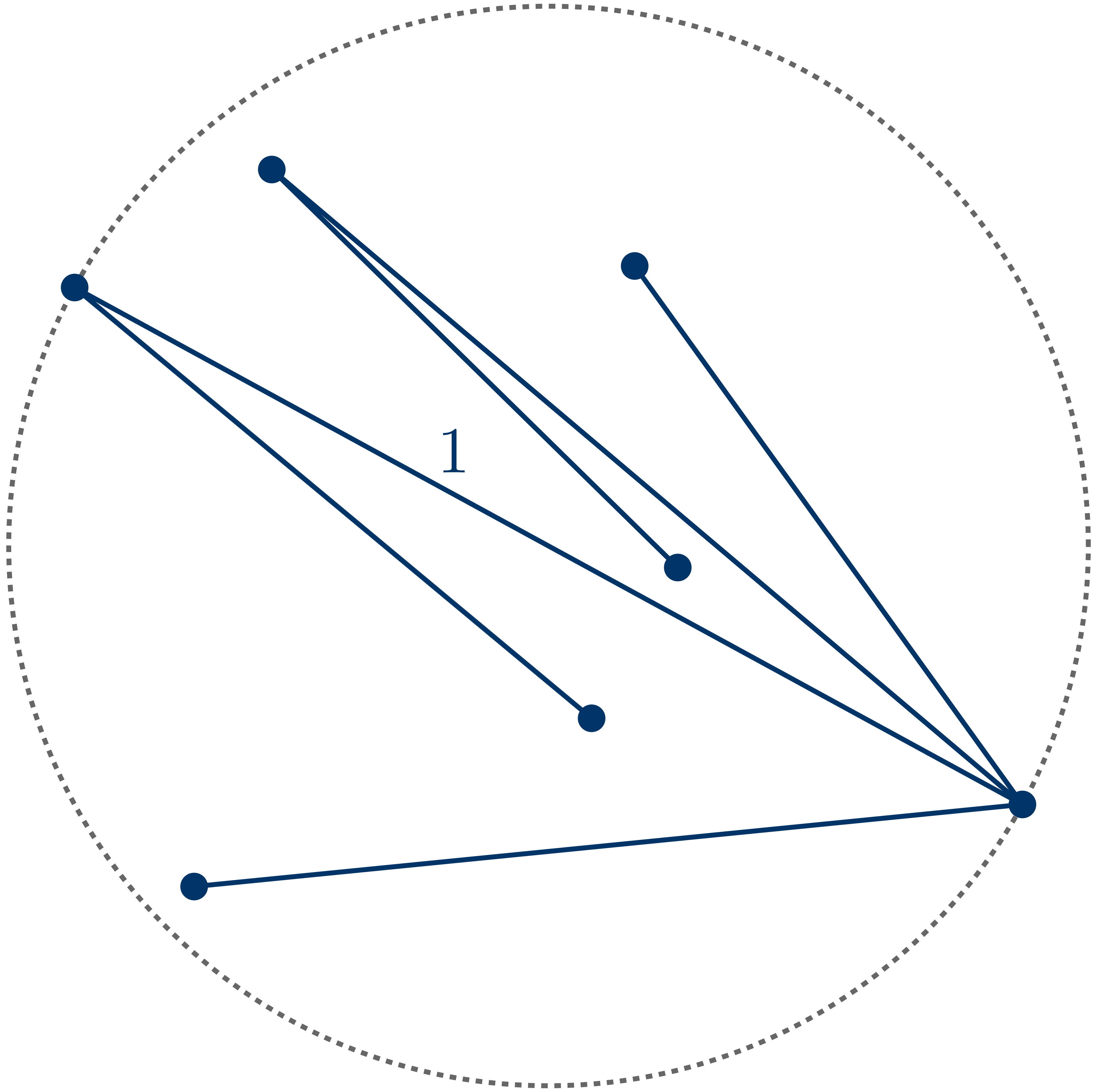
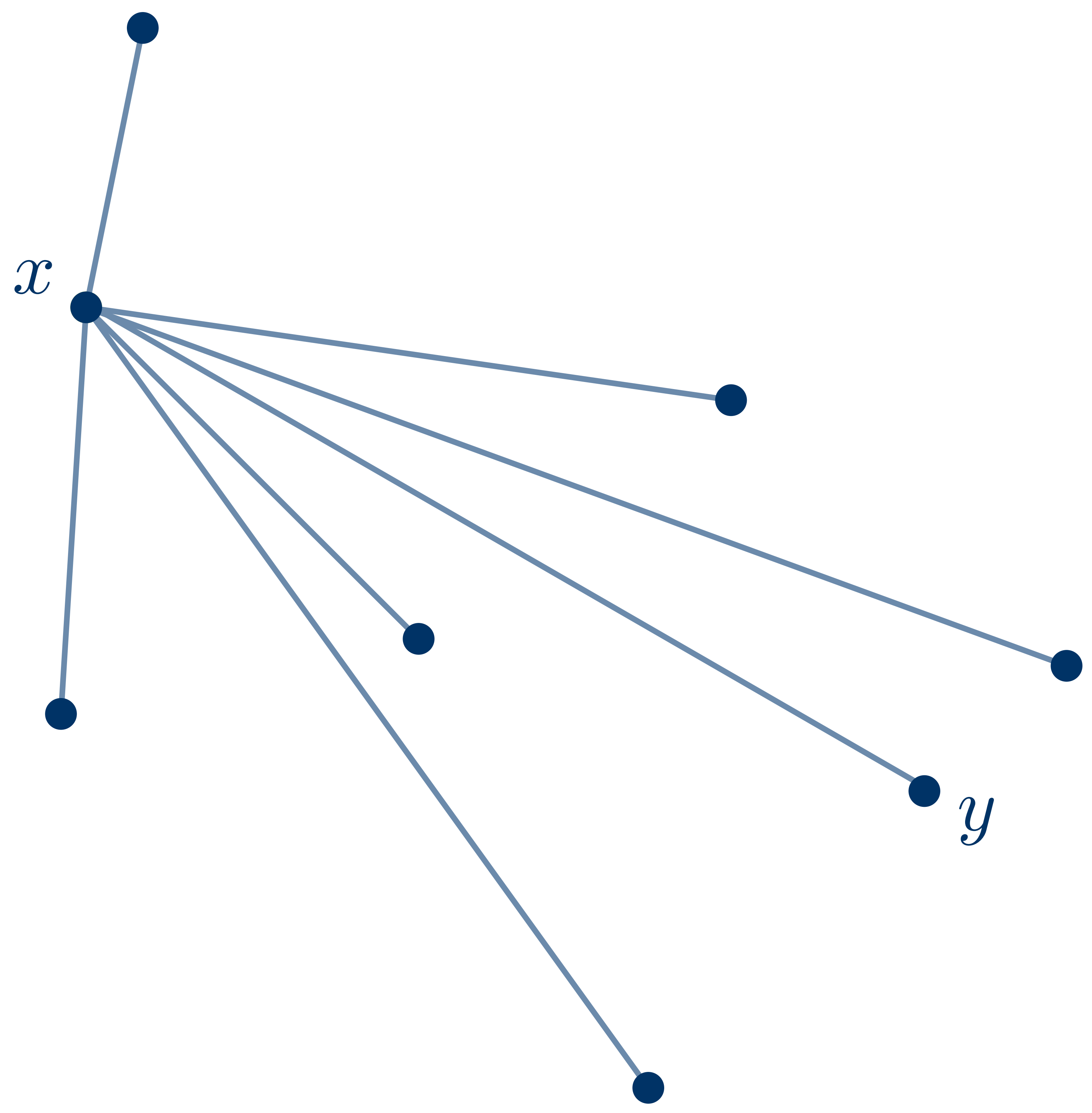
What is known?

Minimum Spanning Tree	**P**	exact
Maximum Spanning Tree	**P**	exact
Maximum noncrossing ST	**NP**-hard??	0.512



General approach



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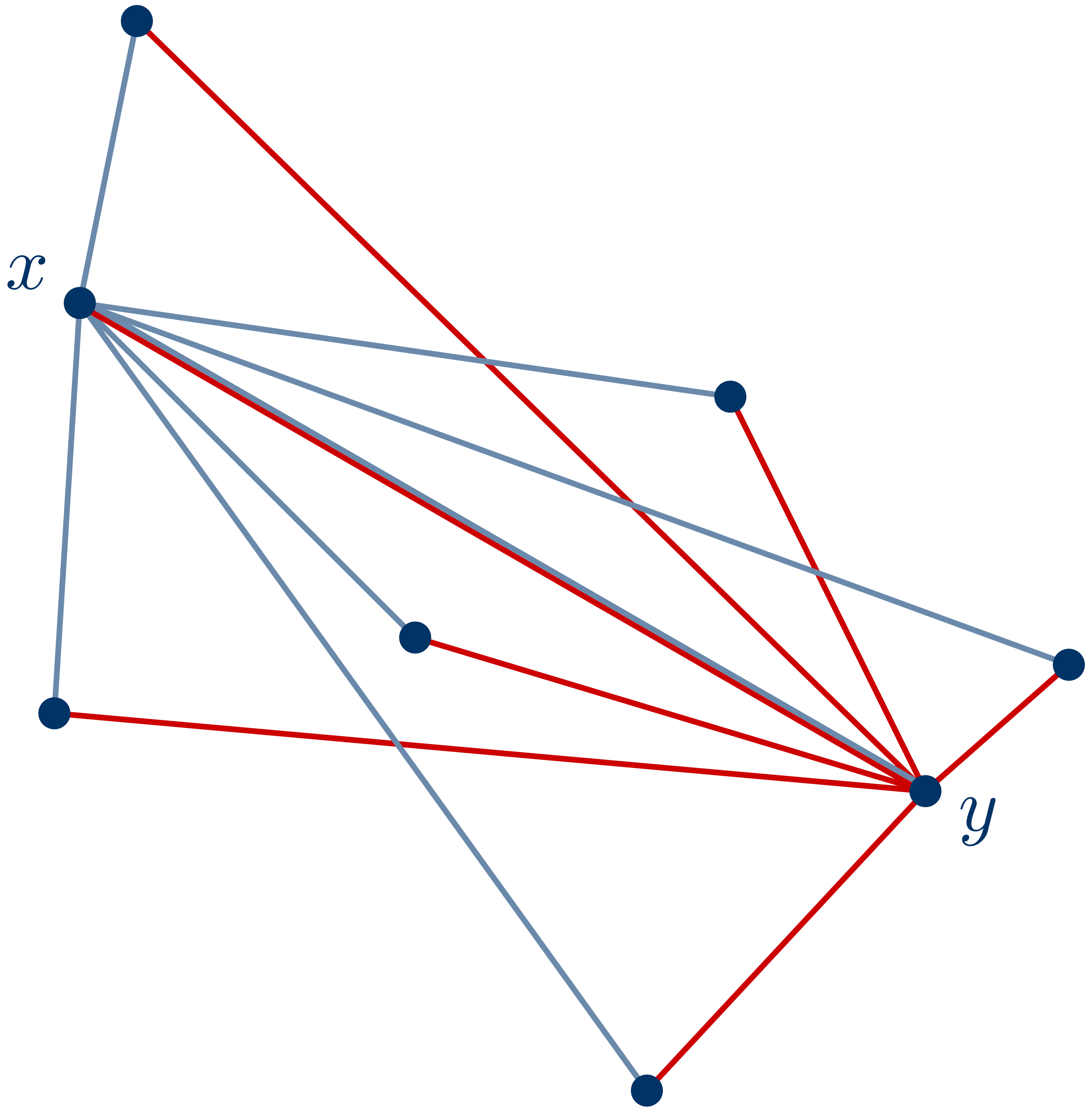

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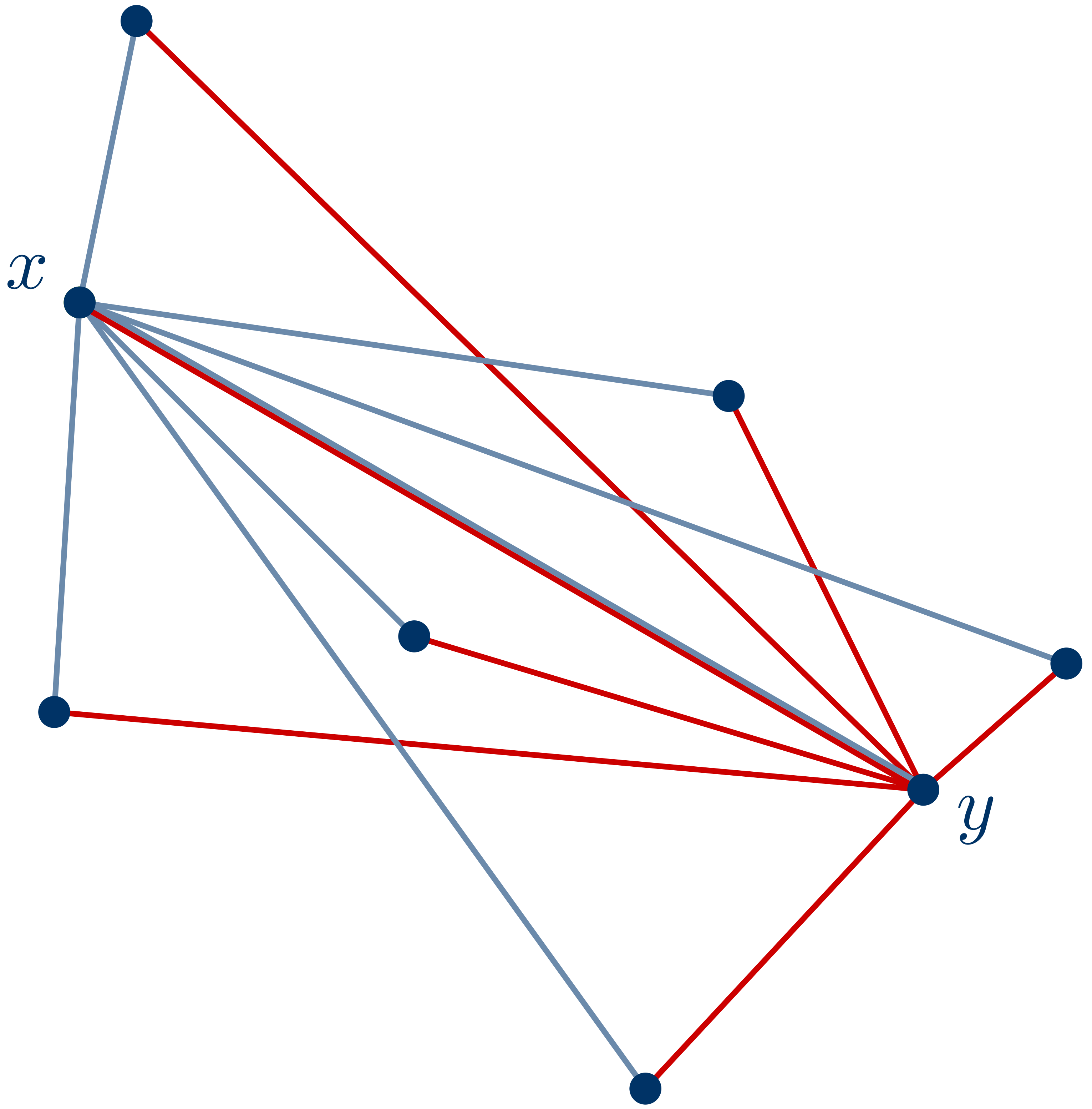
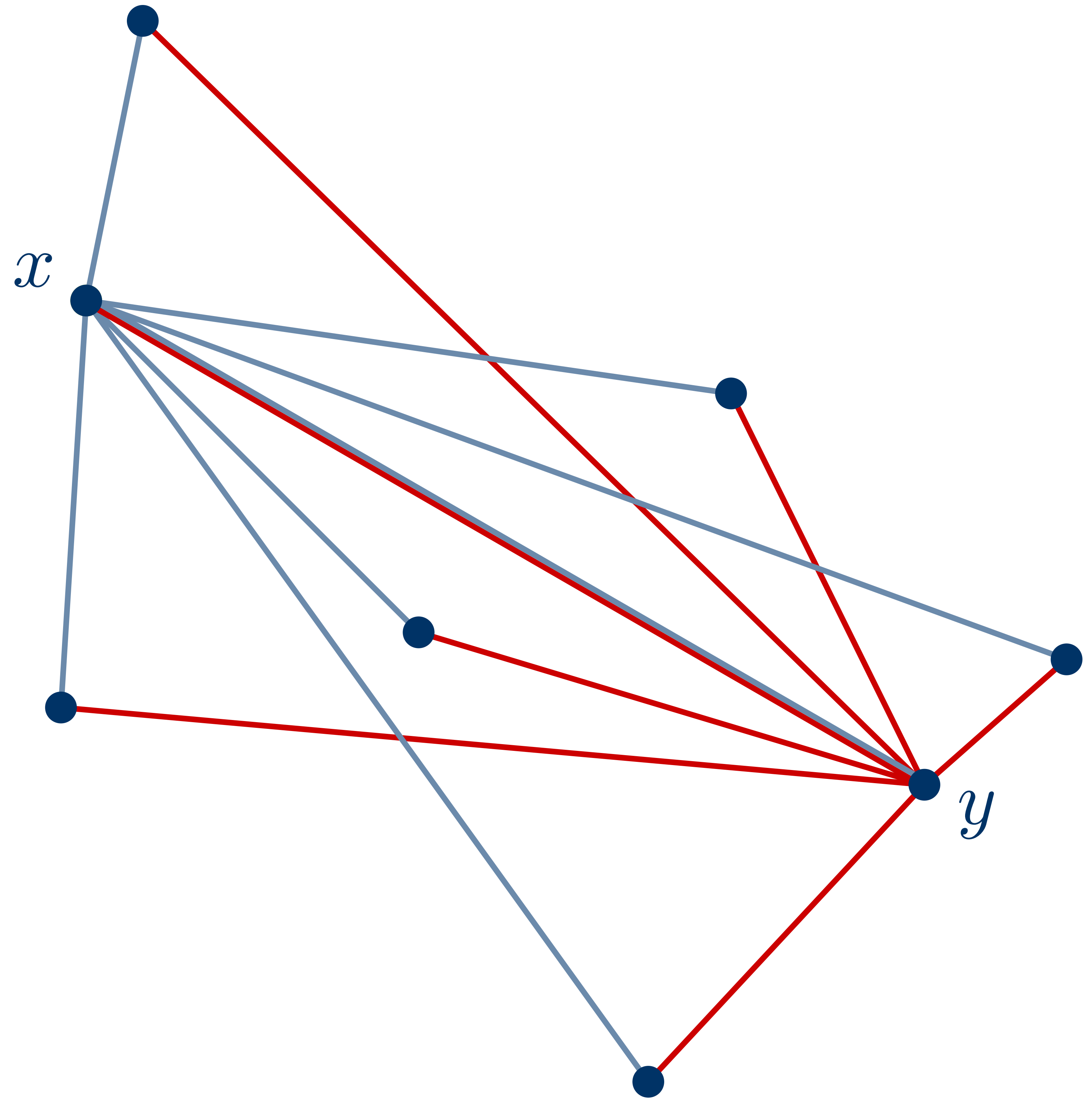


Observation (Dumitrescu, Tóth 2009)**.**
For $x, y \in P$ *we have:*

$$\max\{\text{—}, \text{—}\} \geq \frac{n}{2}\|xy\|$$

General approach



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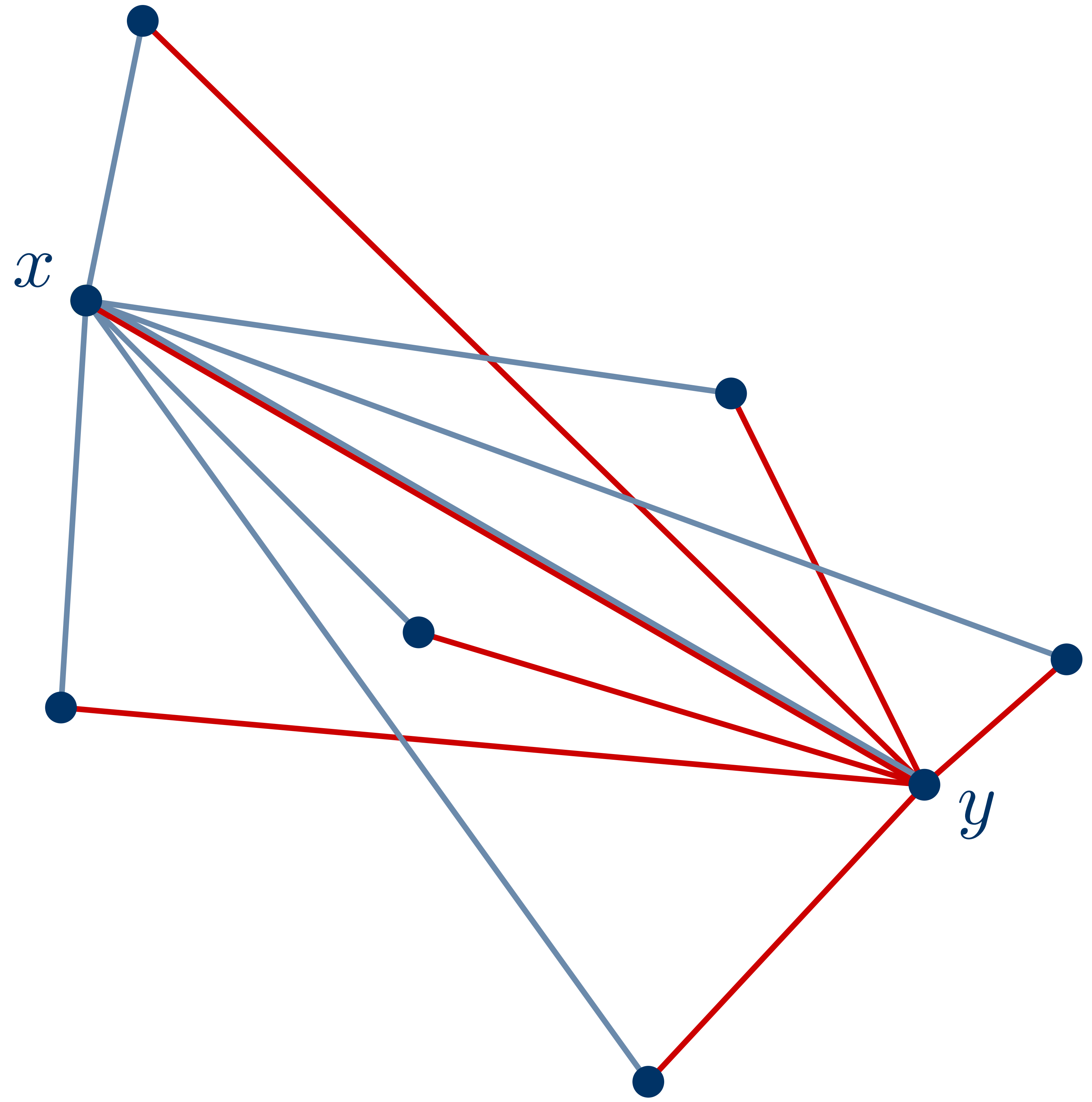

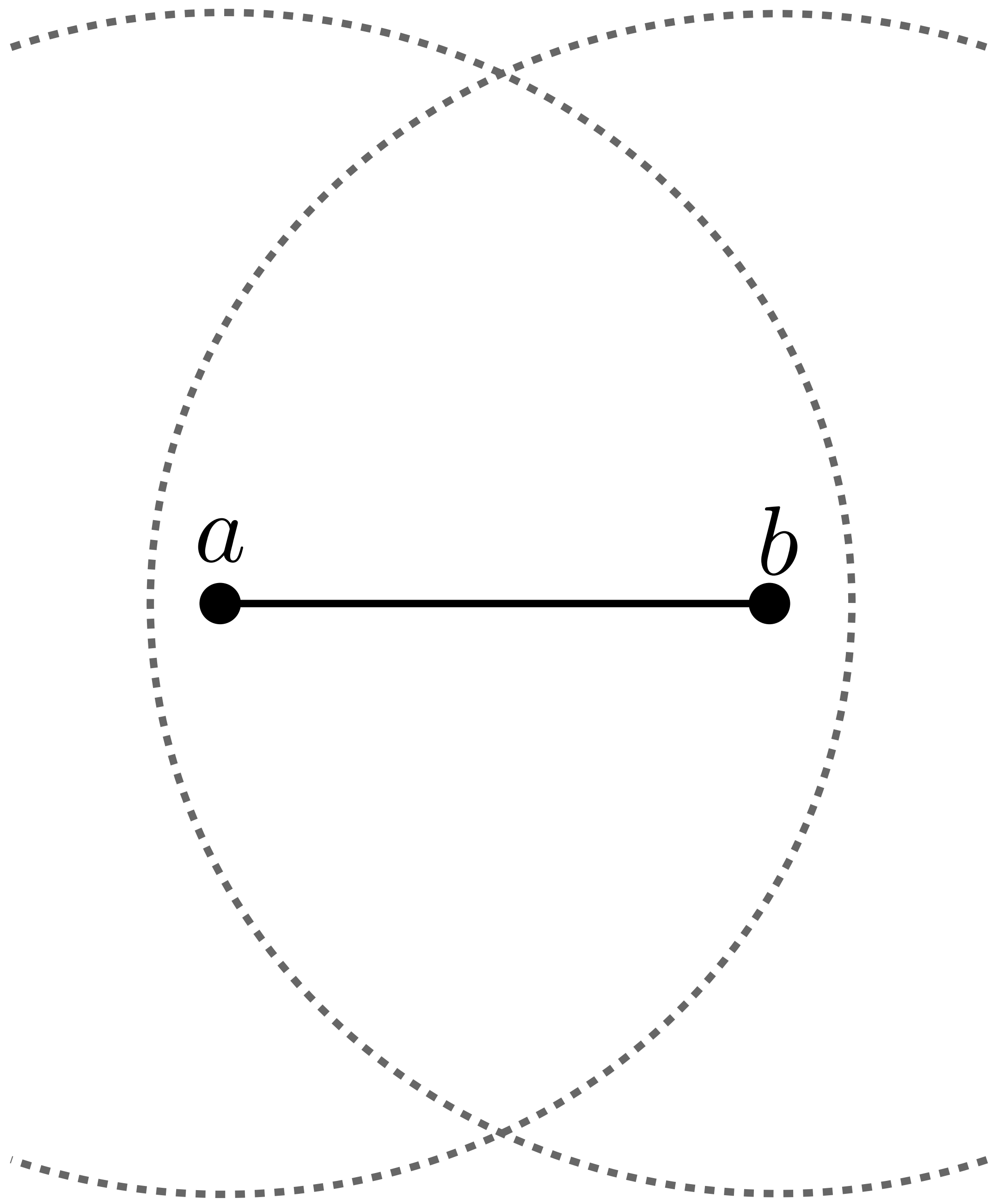

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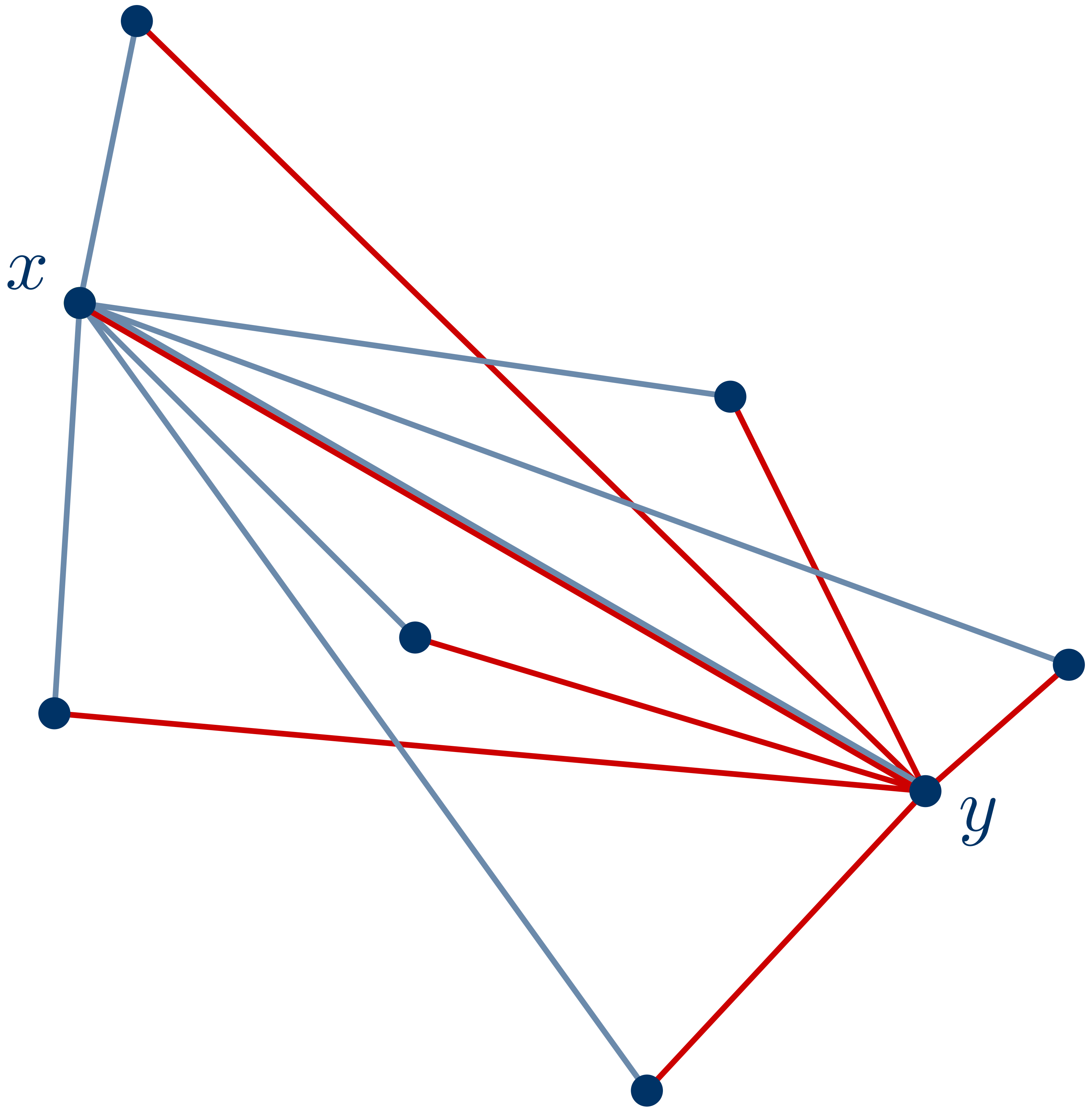

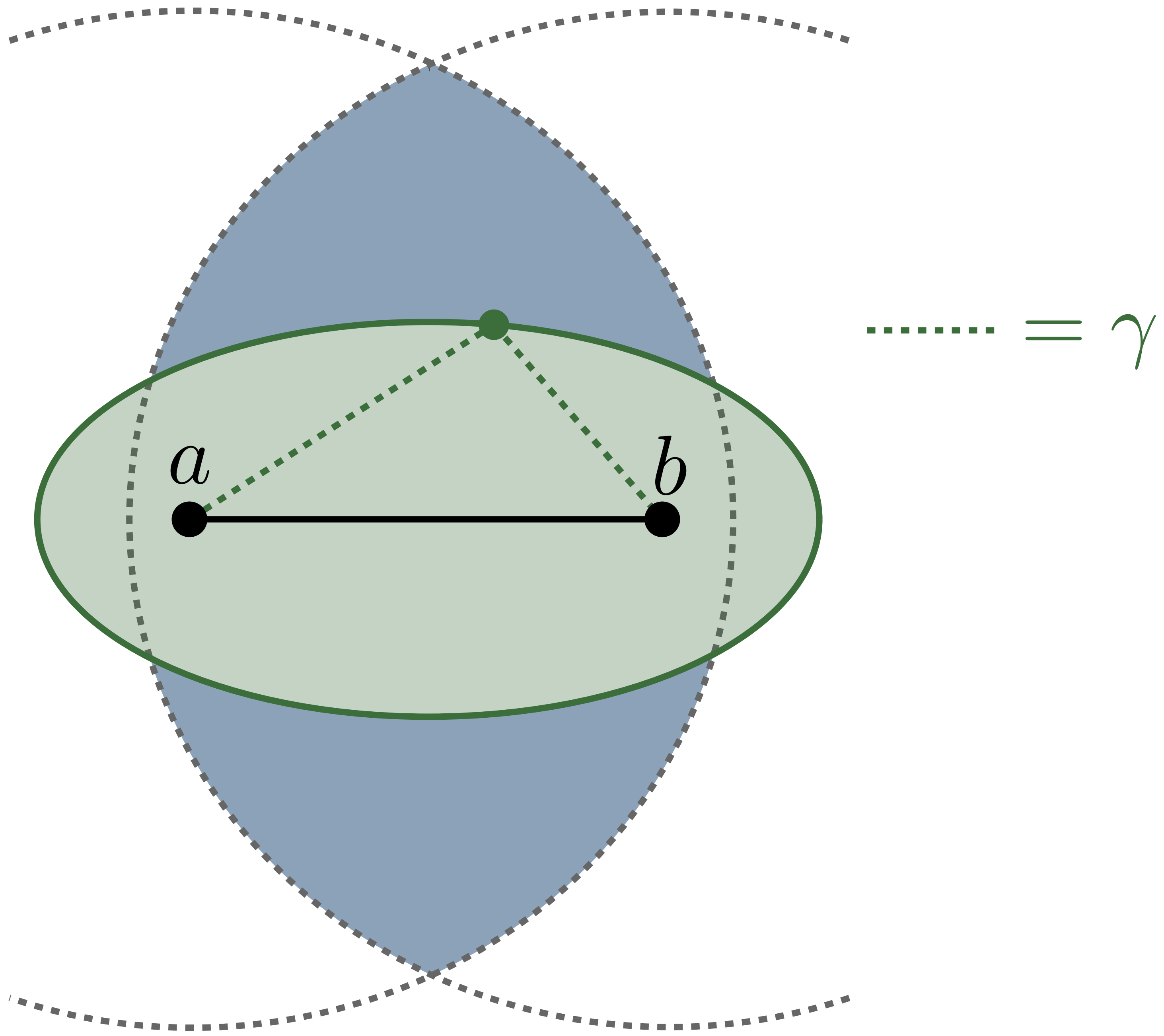

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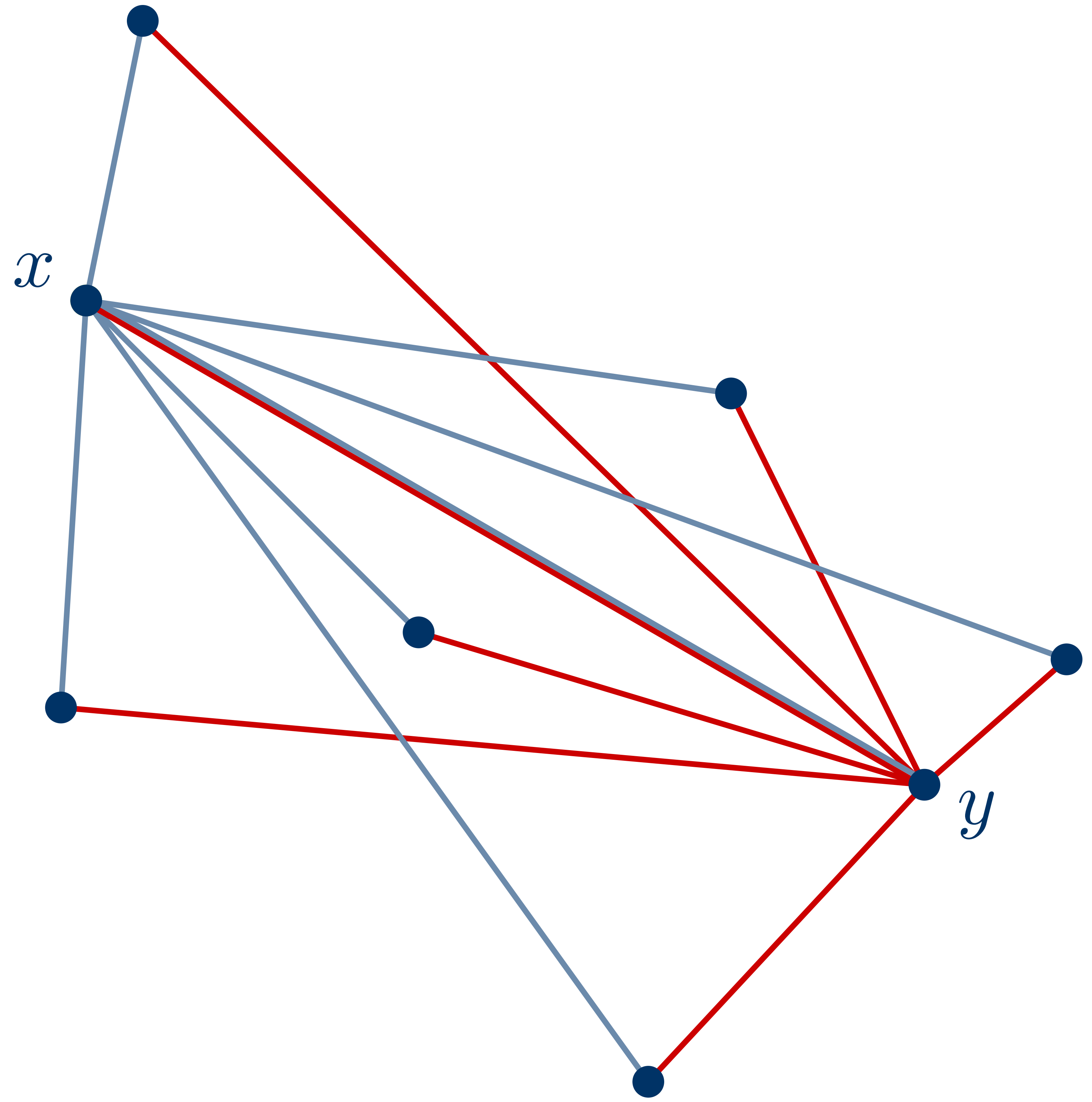

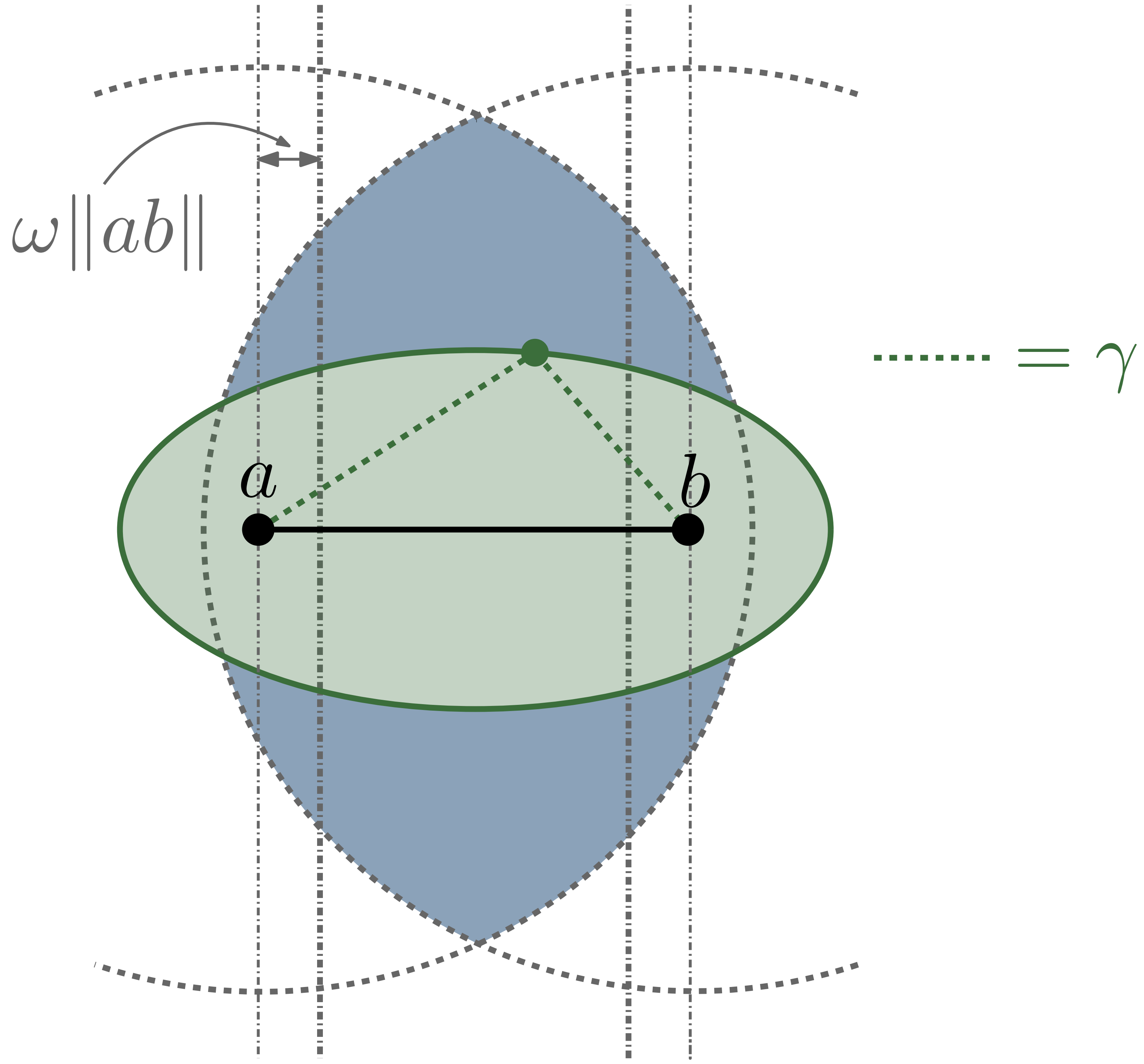

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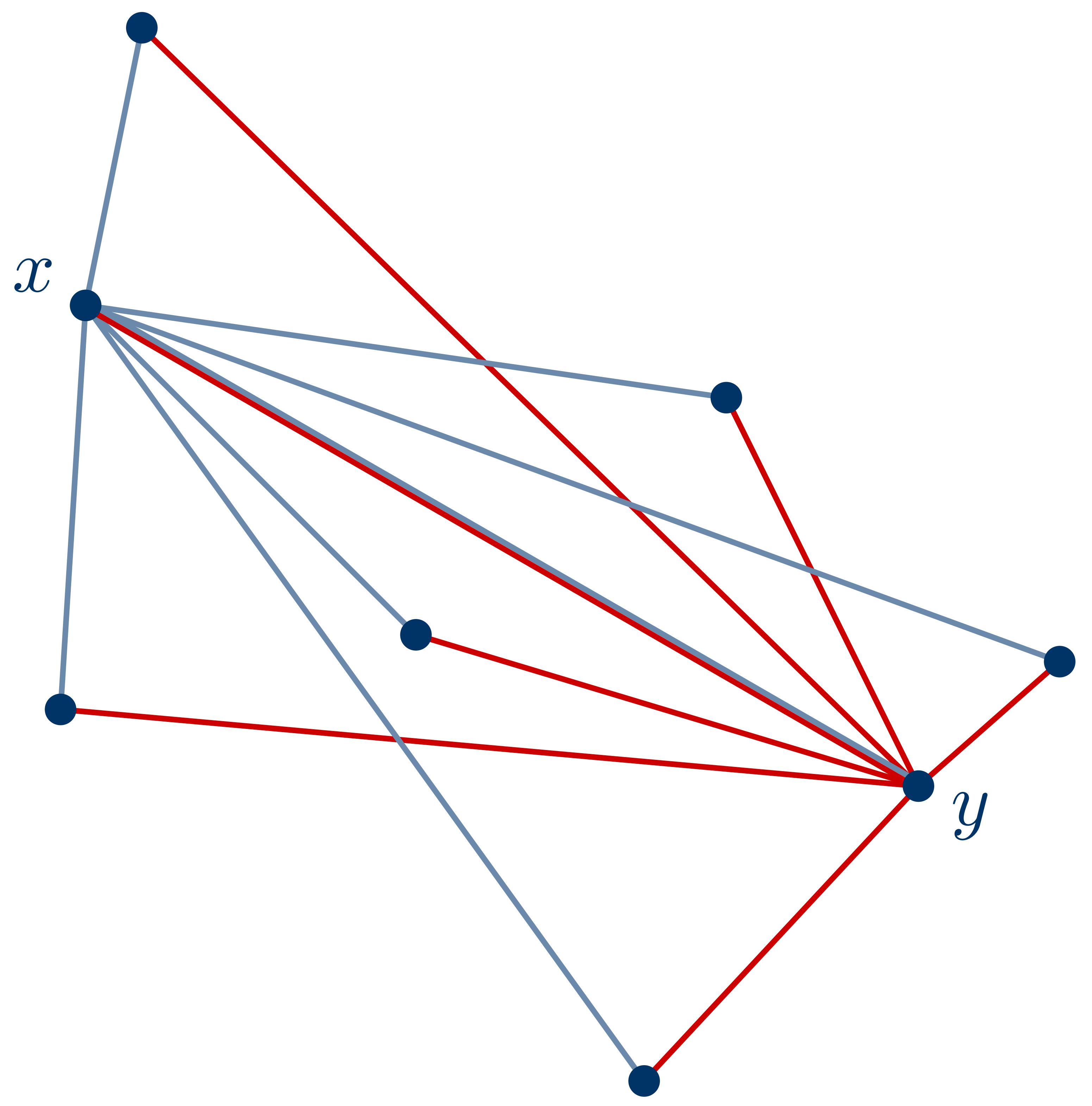

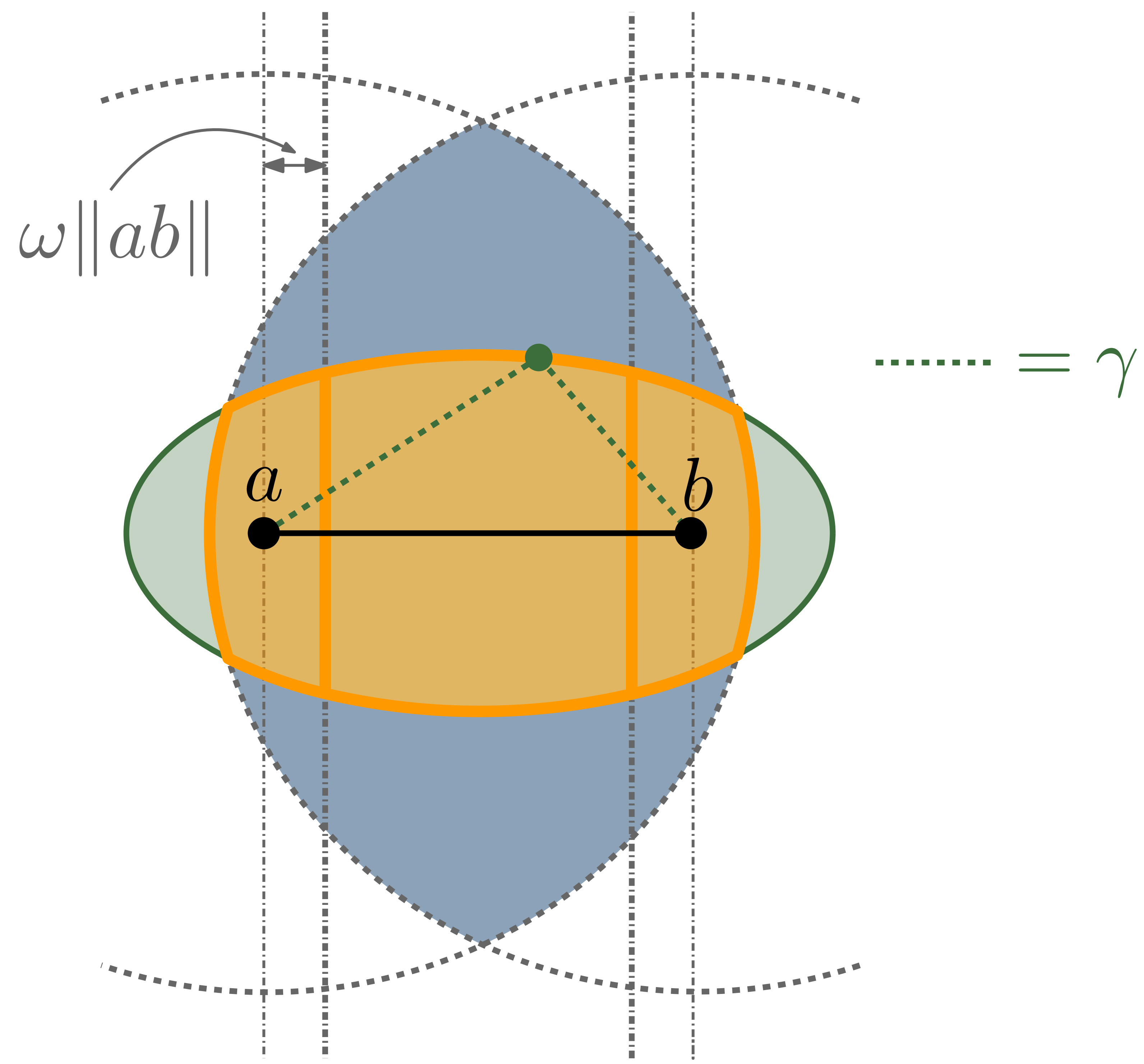

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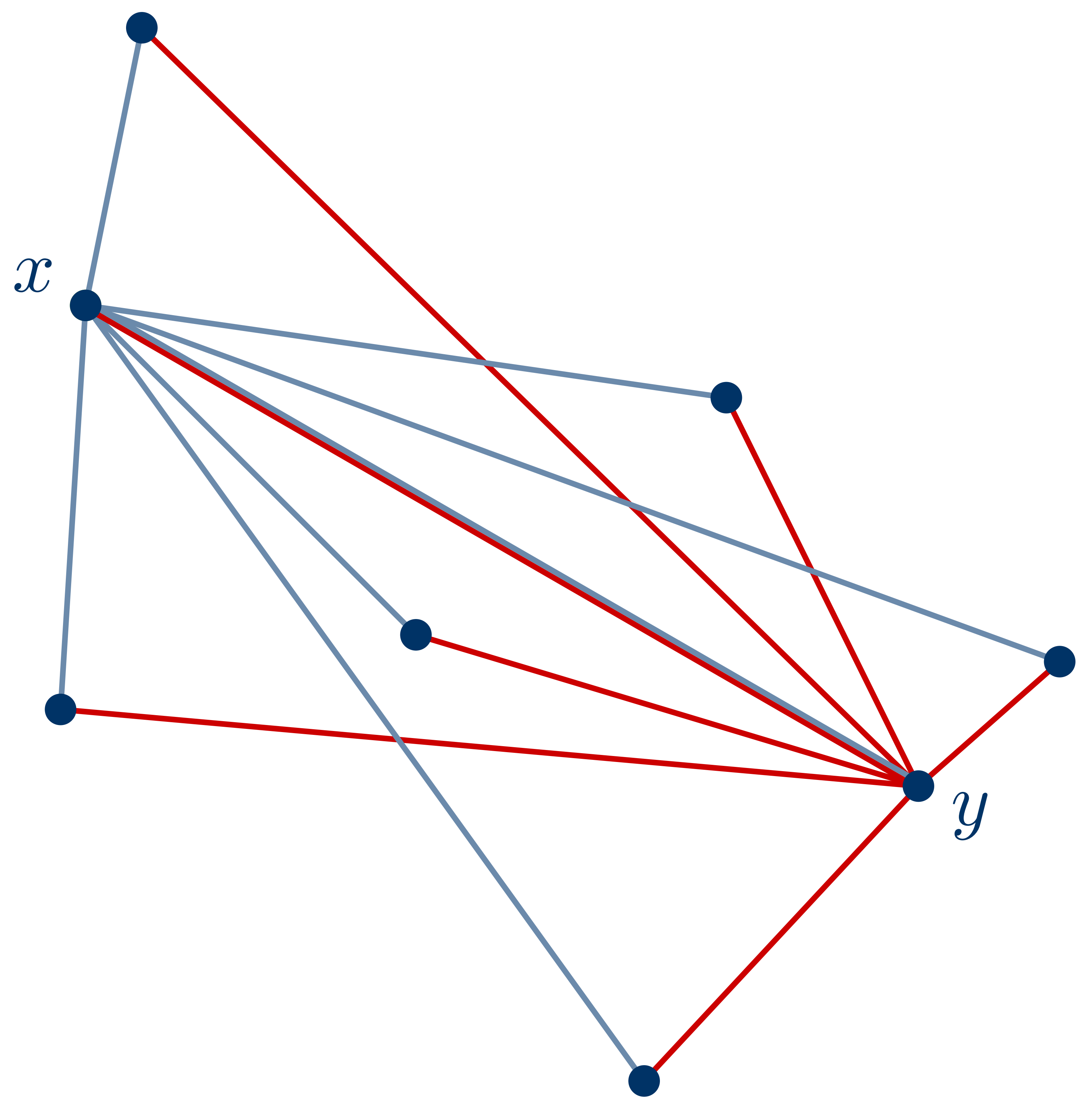

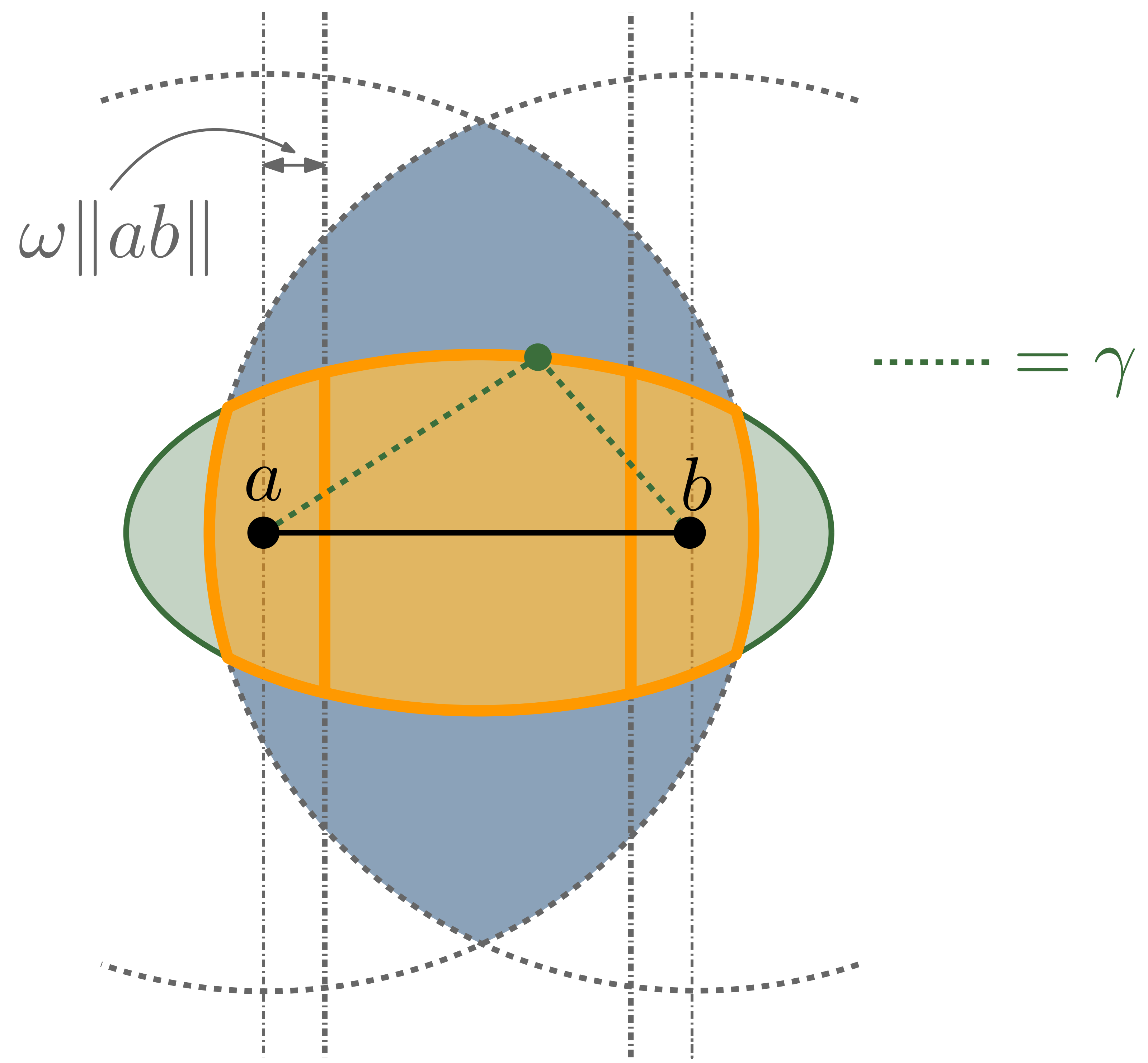

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Observation. *ab longest edge:*

$$|T_{\text{OPT}}| \leq \|ab\|(n-1) \leq \|ab\|n \leq n$$

The 0.512-approximation

Case 1: $\|ab\| \leq d$

$\delta = 0.512$

$d = \frac{1}{2\delta}$

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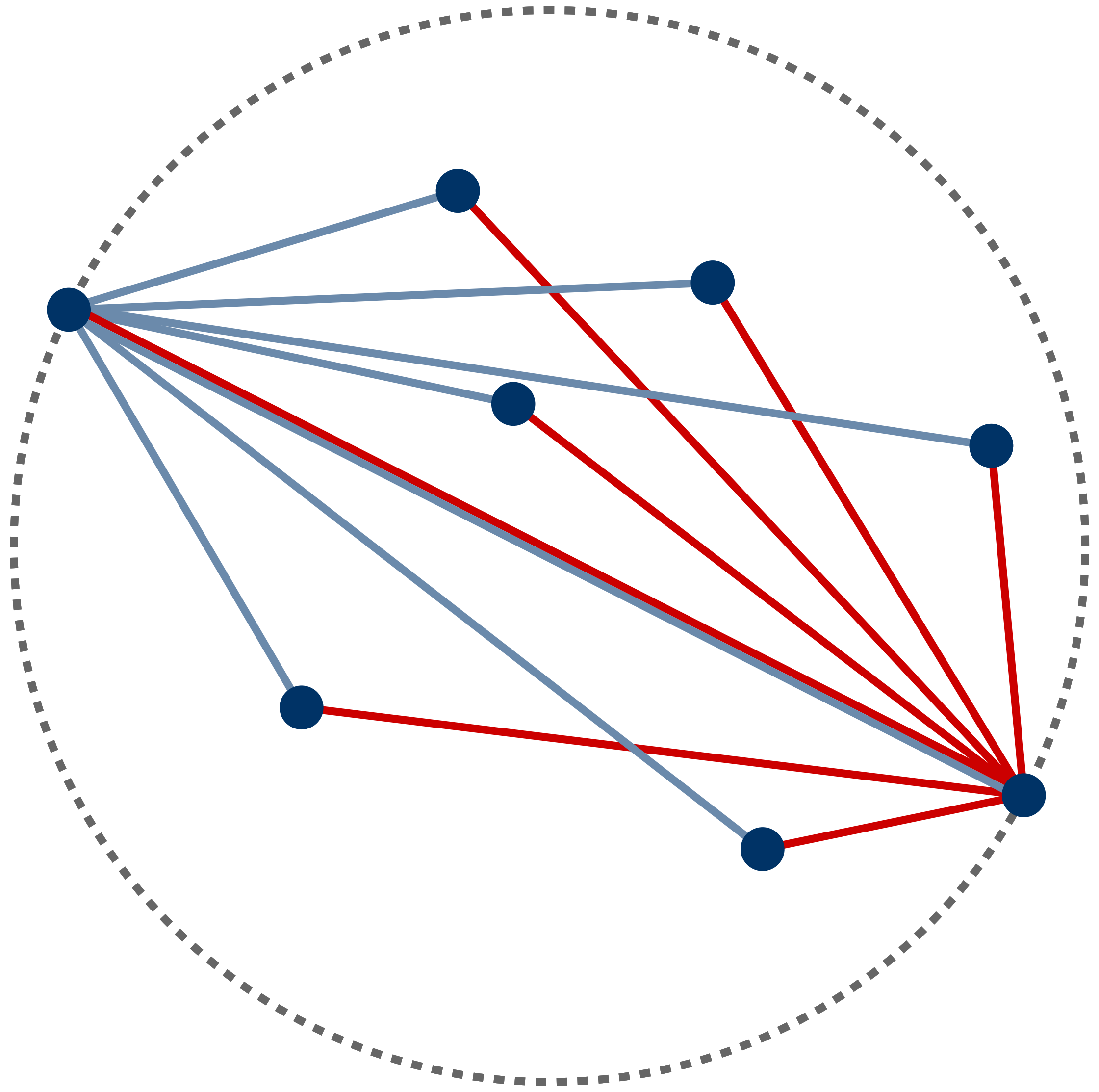

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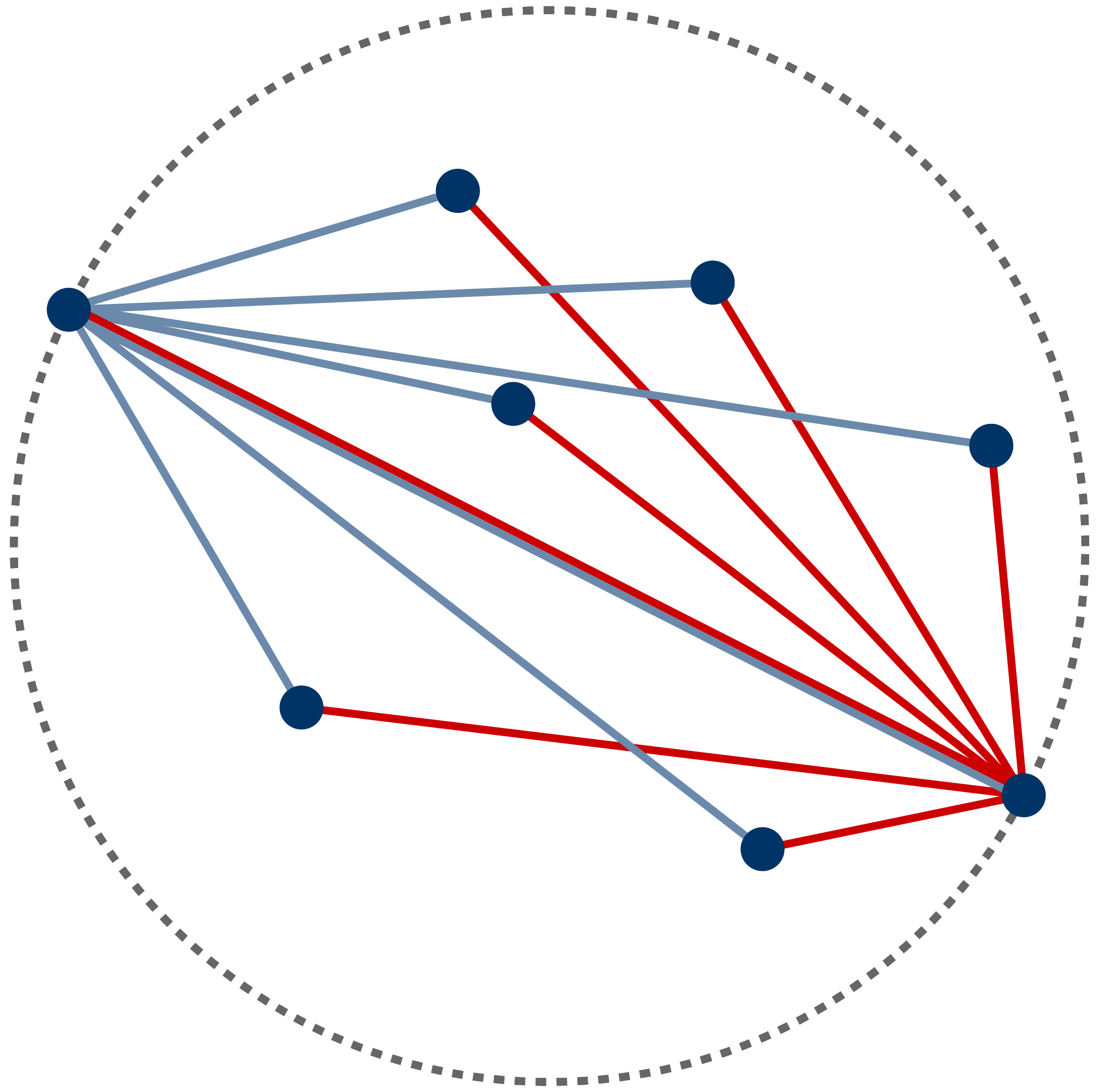

$\max\{$ [blue] $,$ [red] $\} \geq \frac{n}{2}$

$\Longrightarrow$ δ-Approximation

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Case 1: $\|ab\| \leq d$

Case 2: $\|ab\| > d$

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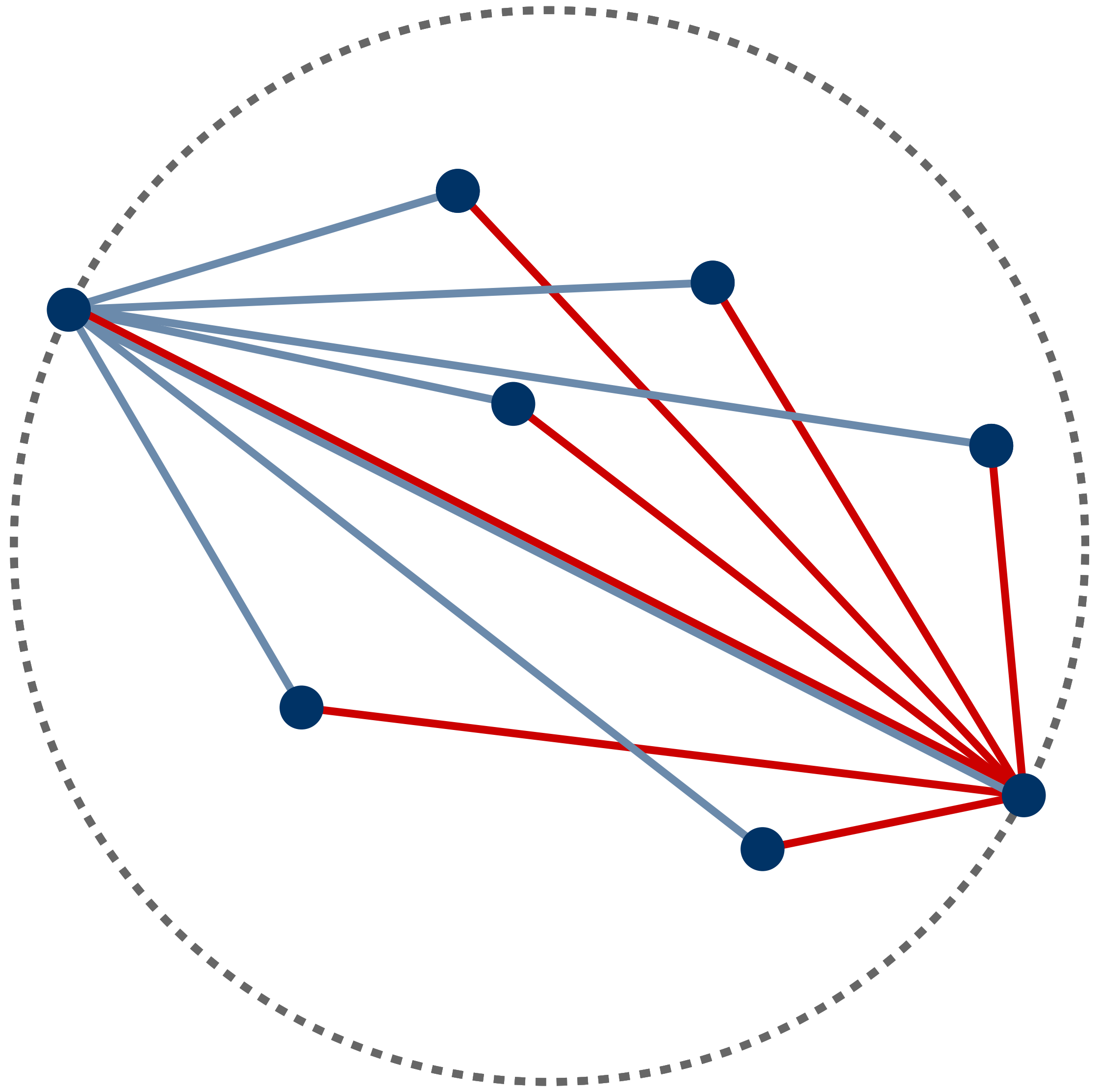

$\max\{$ [blue line] , [red line] $\} \geq \frac{n}{2}$

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The 0.512-approximation

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$\omega = 0.1$

$\hat{\beta} = \frac{\delta - 0.5}{\delta \cdot (1 - \sqrt{1 - d^2(\omega - \omega^2)})}$

Case 1: $\|ab\| \leq d$

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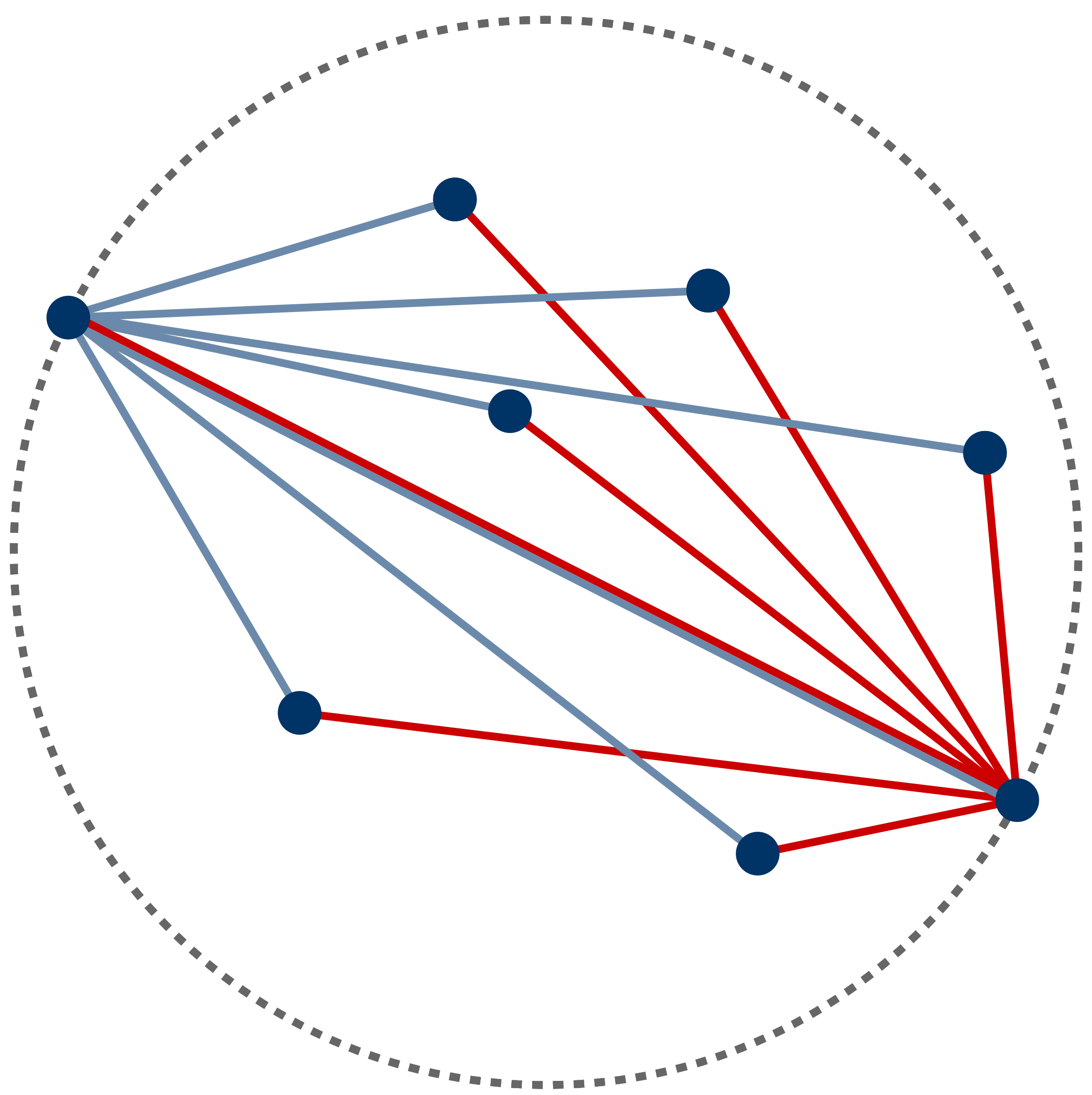

$\max\{$ blue, red $\} \geq \frac{n}{2}$

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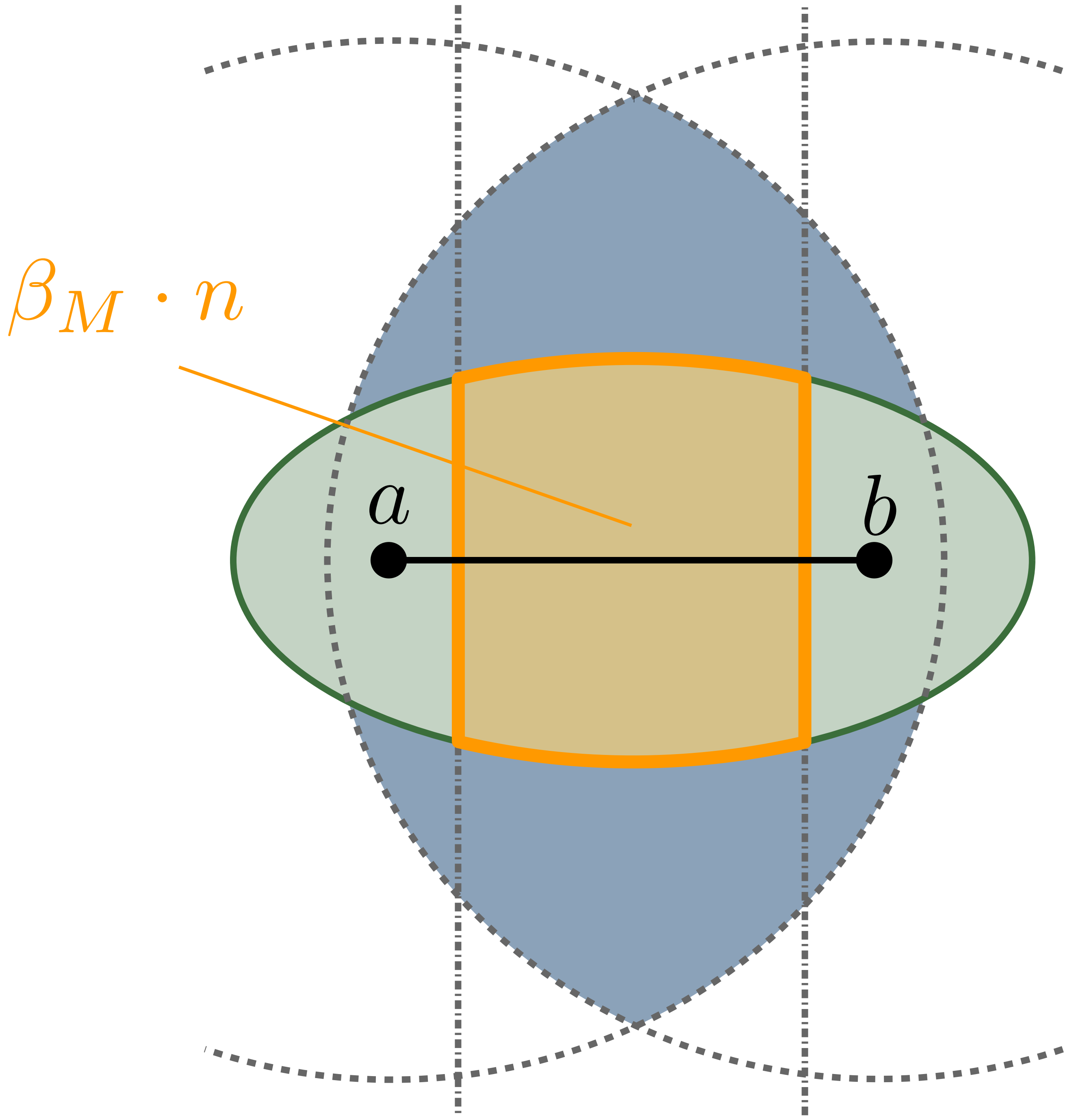
Case 2: $\|ab\| > d$

a) $\beta_M \geq \hat{\beta}$



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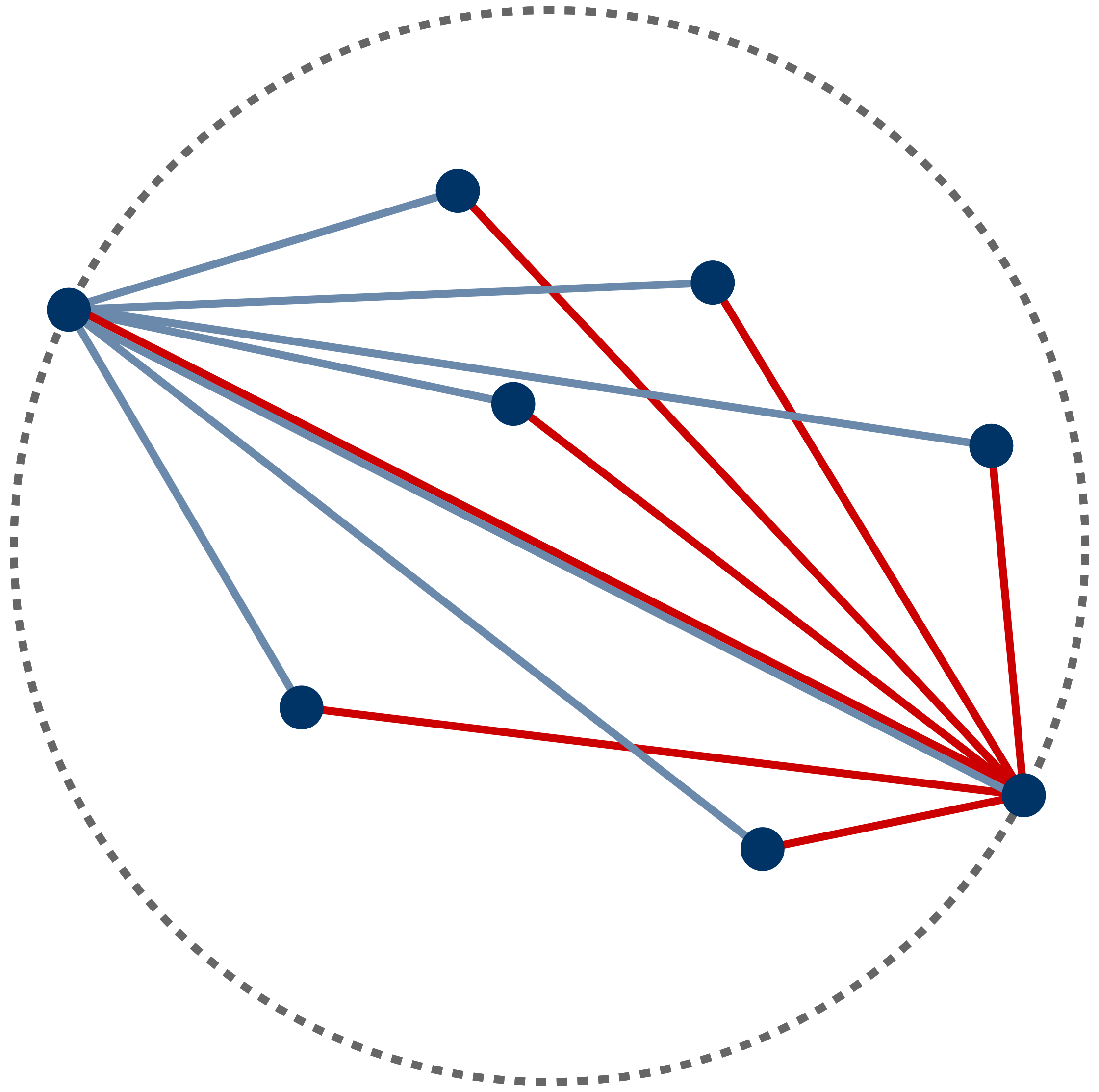

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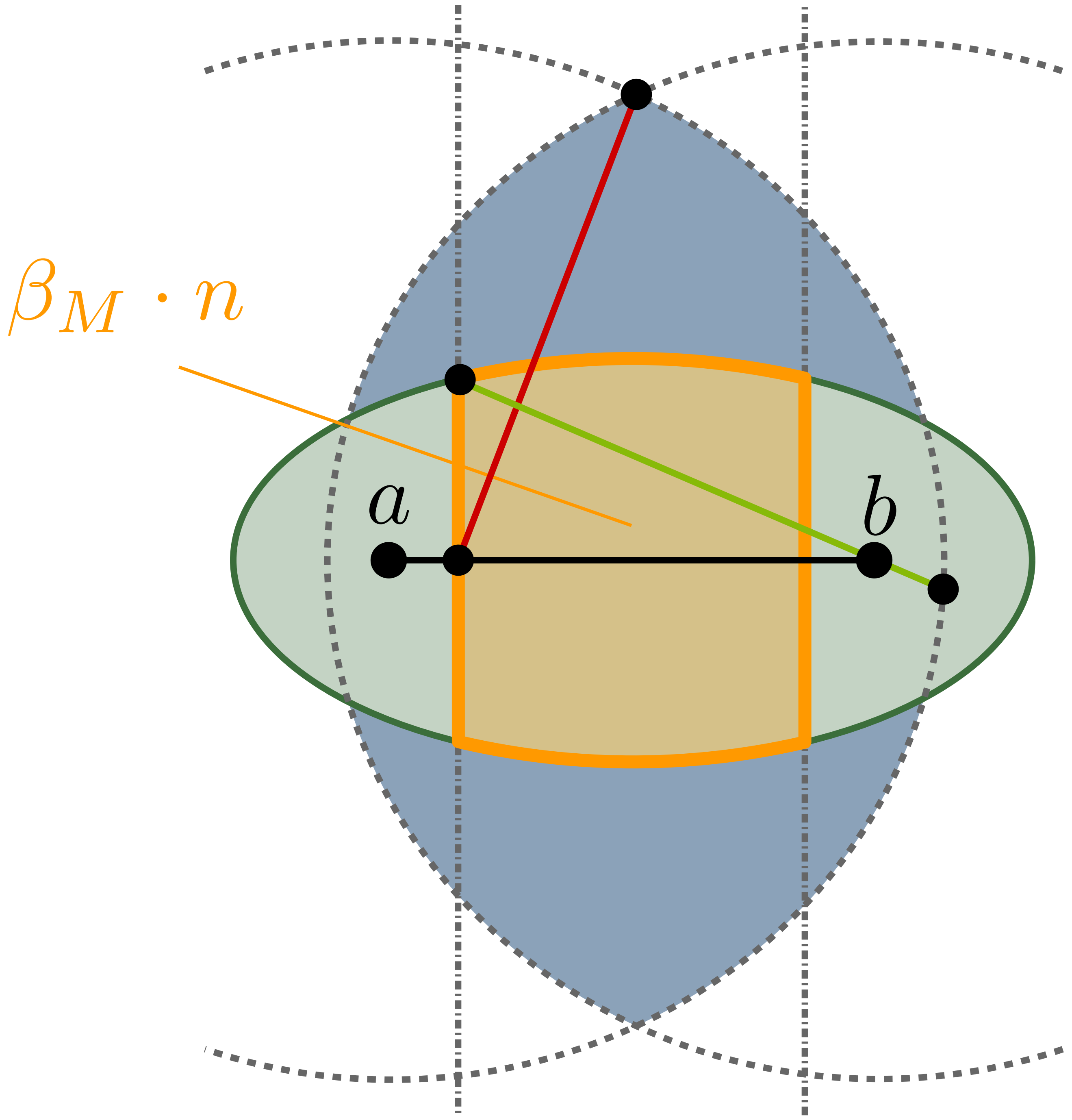

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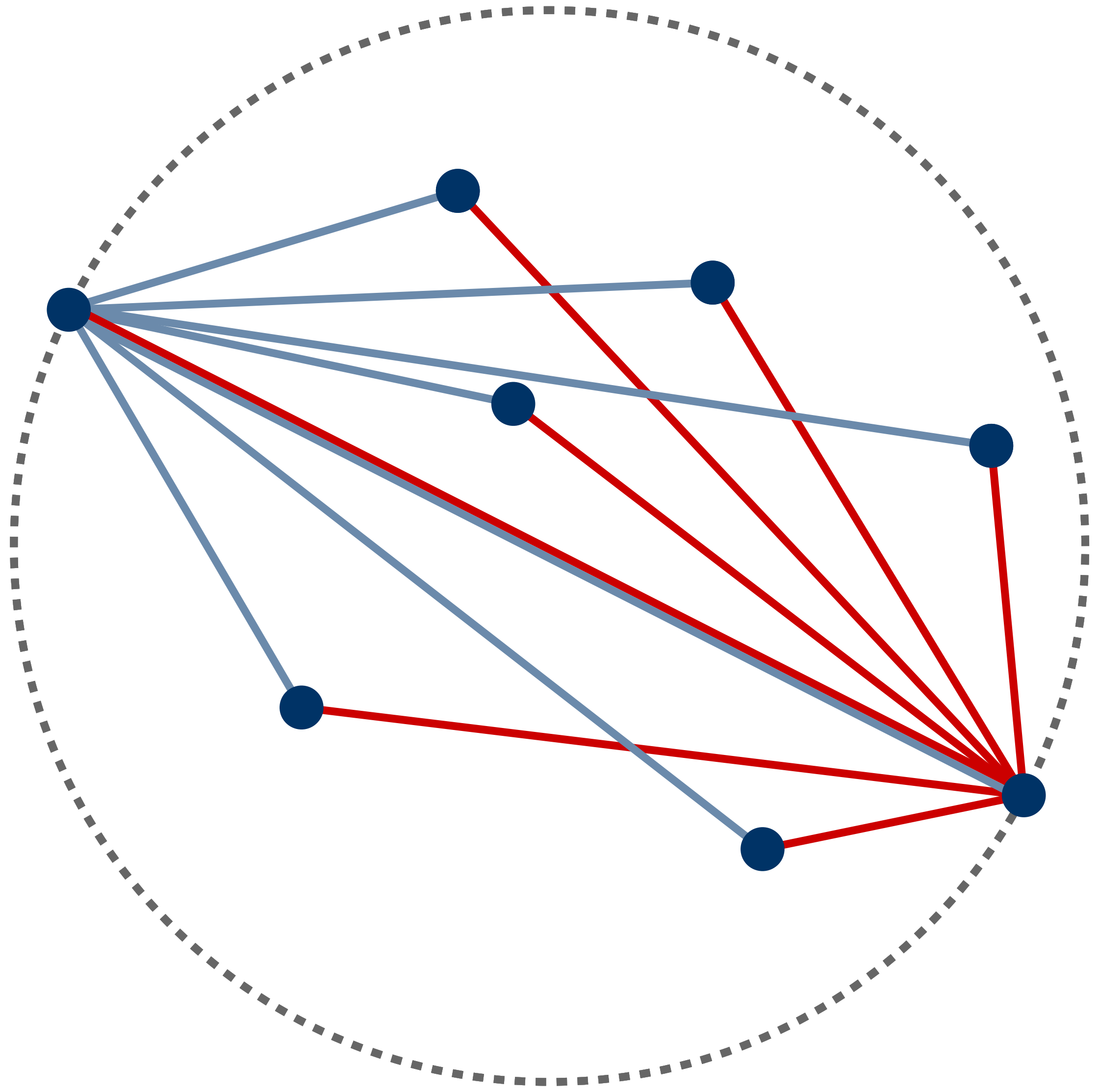

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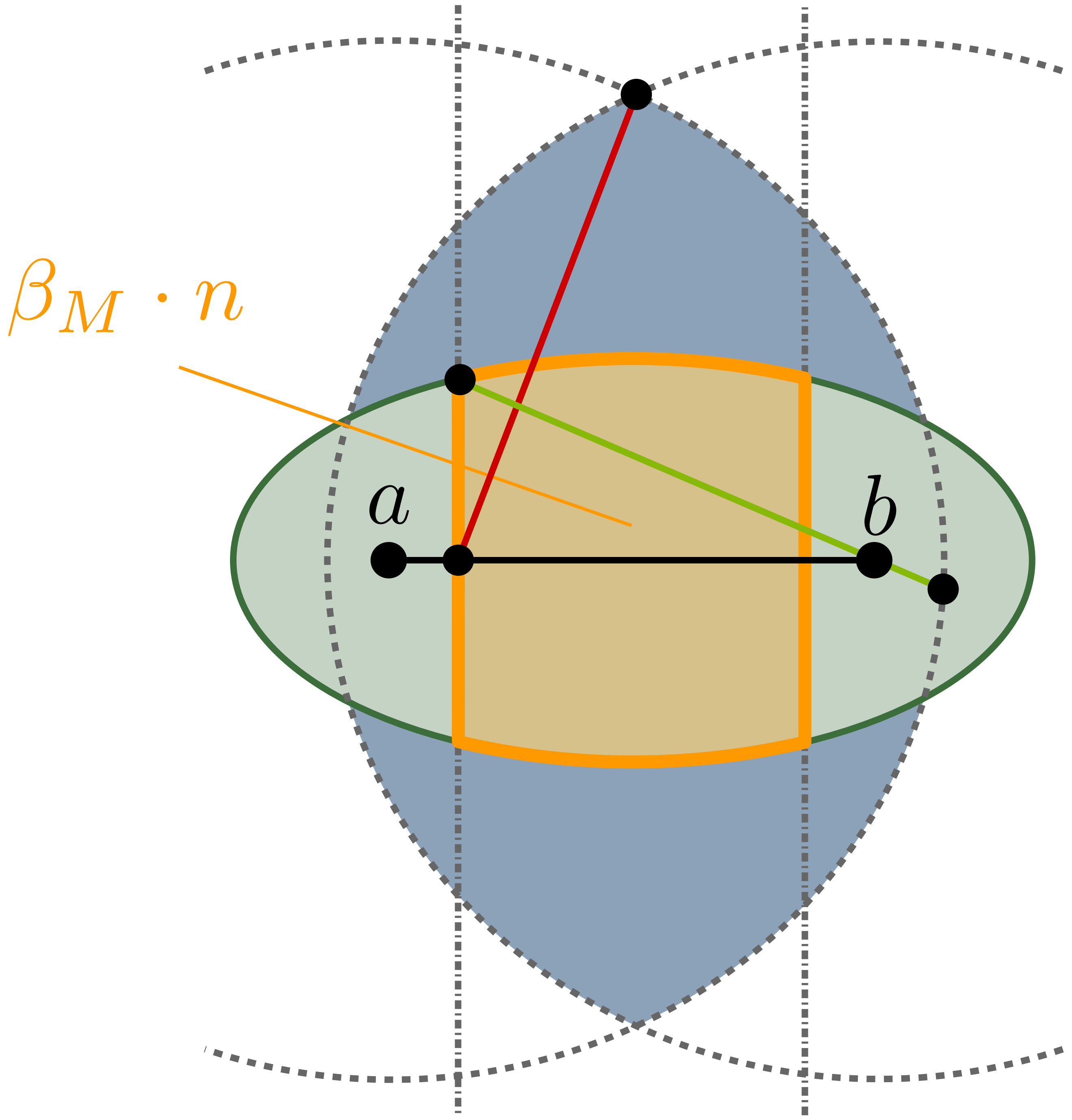

—— $\leq$ ——

Take best star at diameter

The better approximation

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b) $\alpha \geq \hat{\alpha}$

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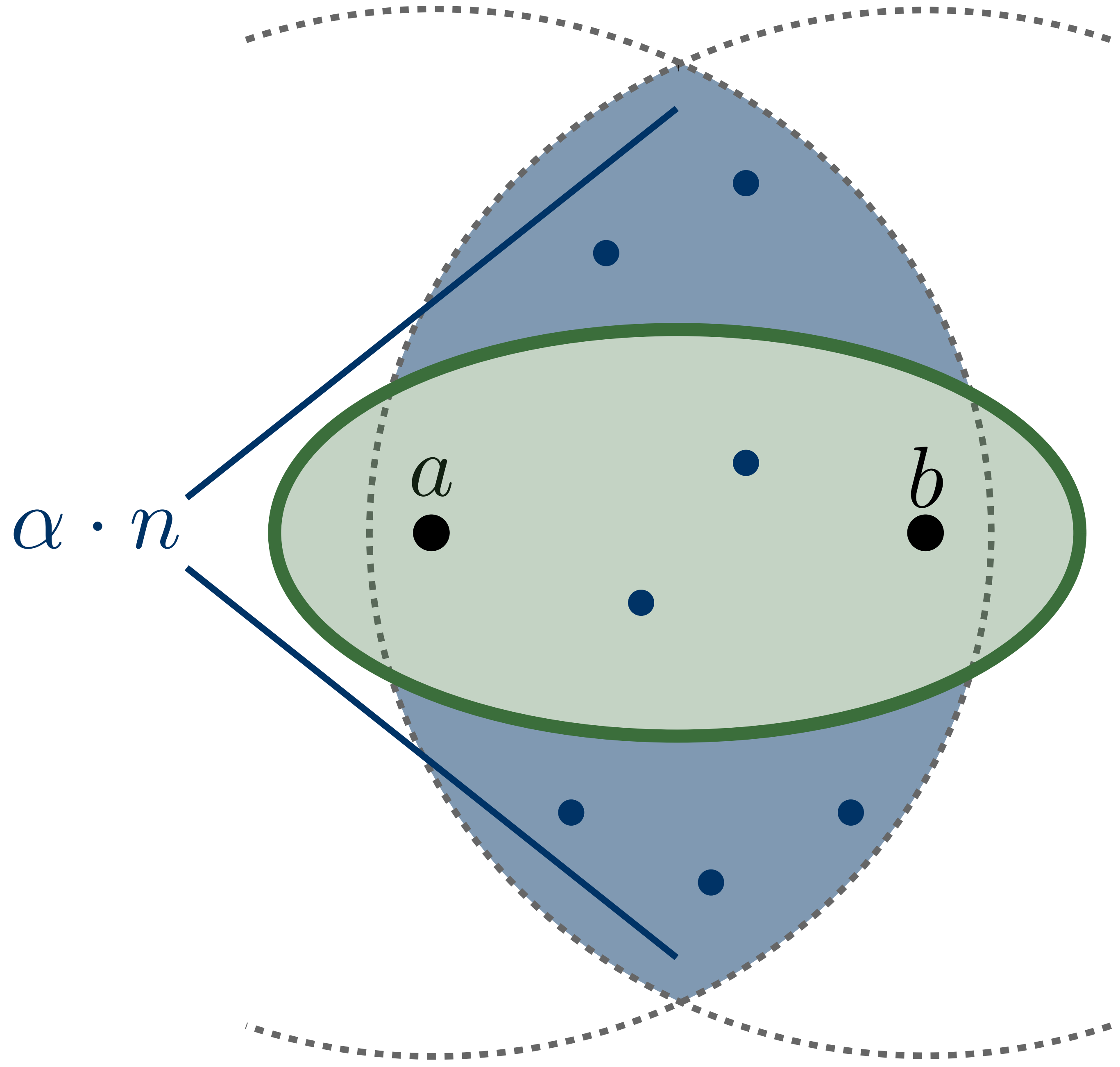

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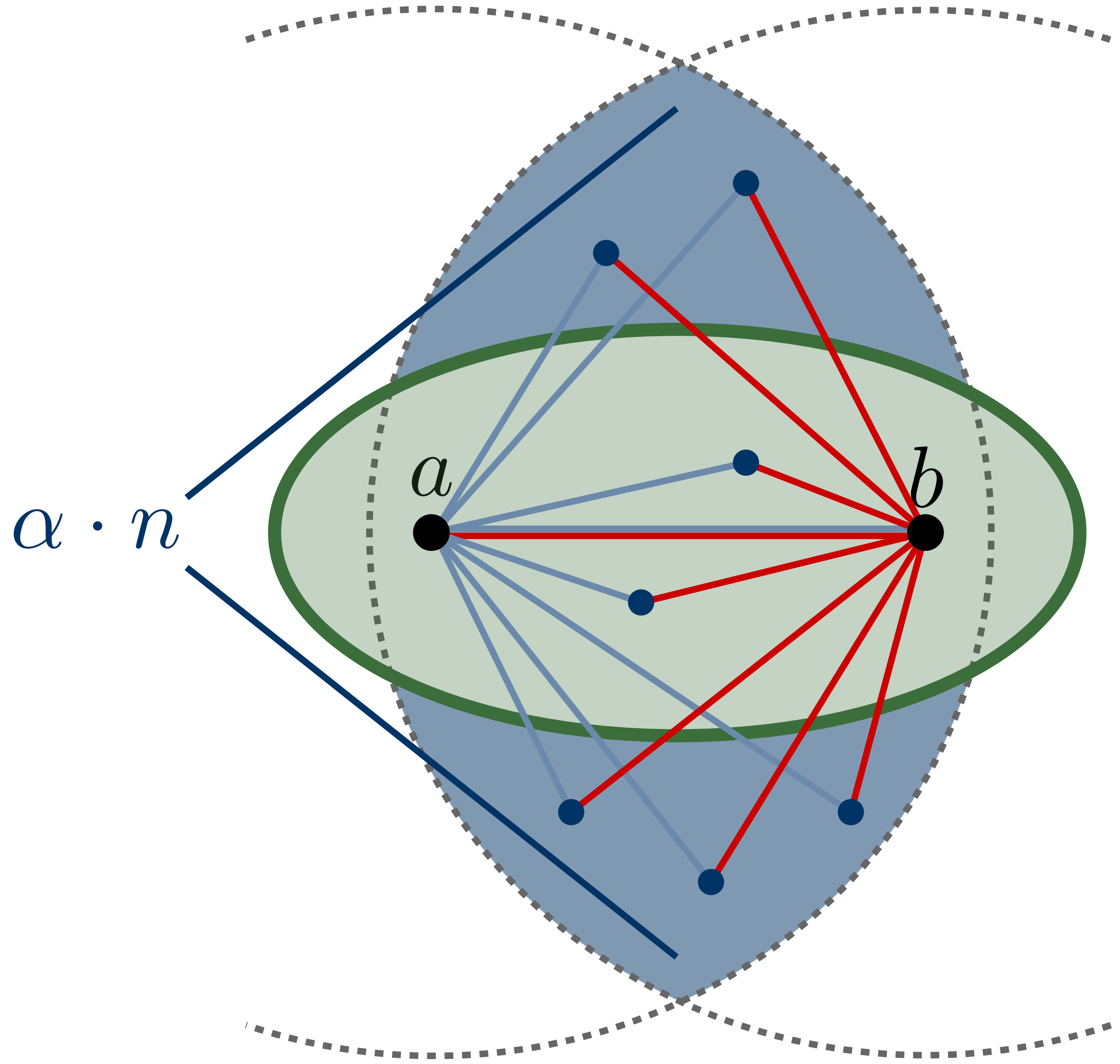

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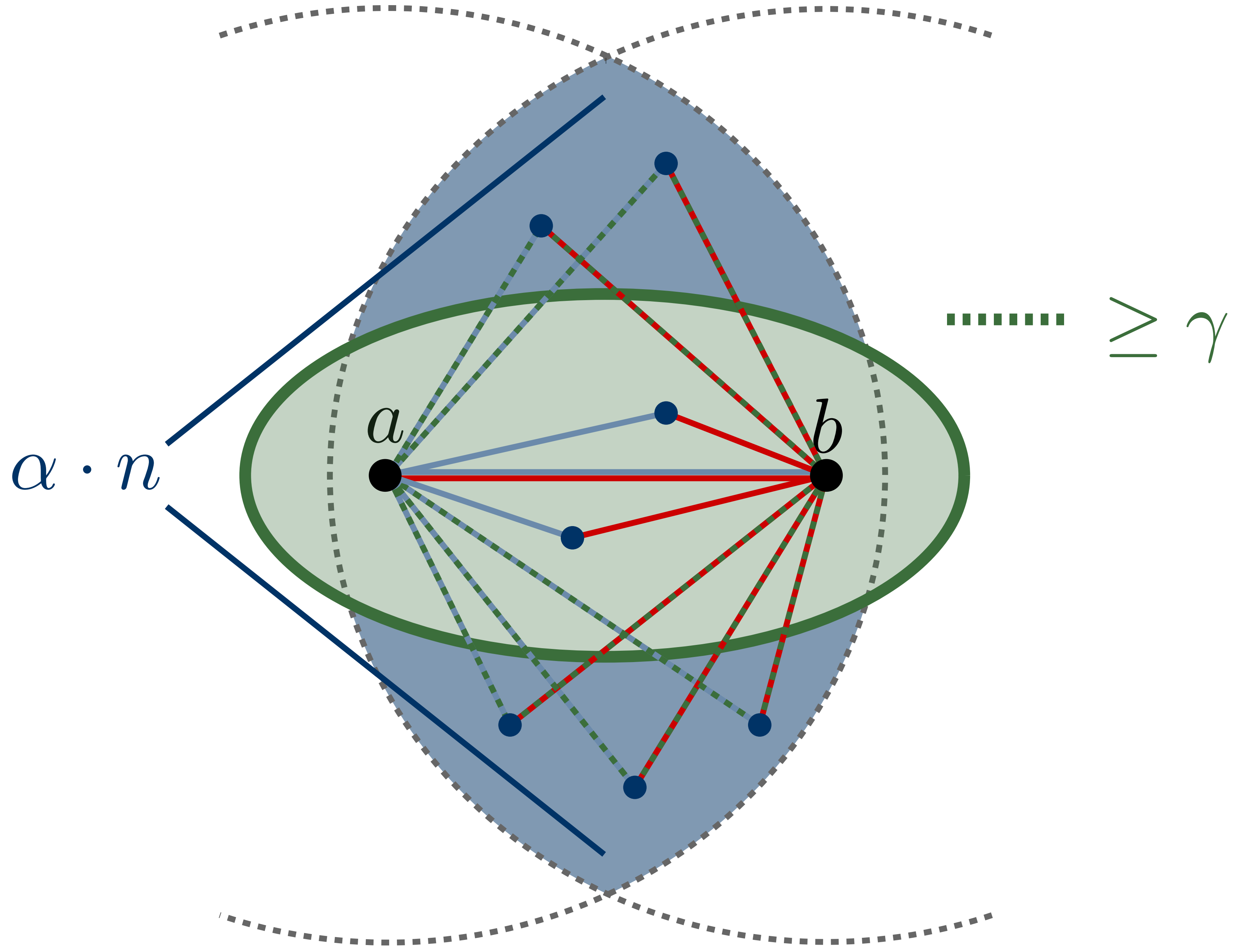

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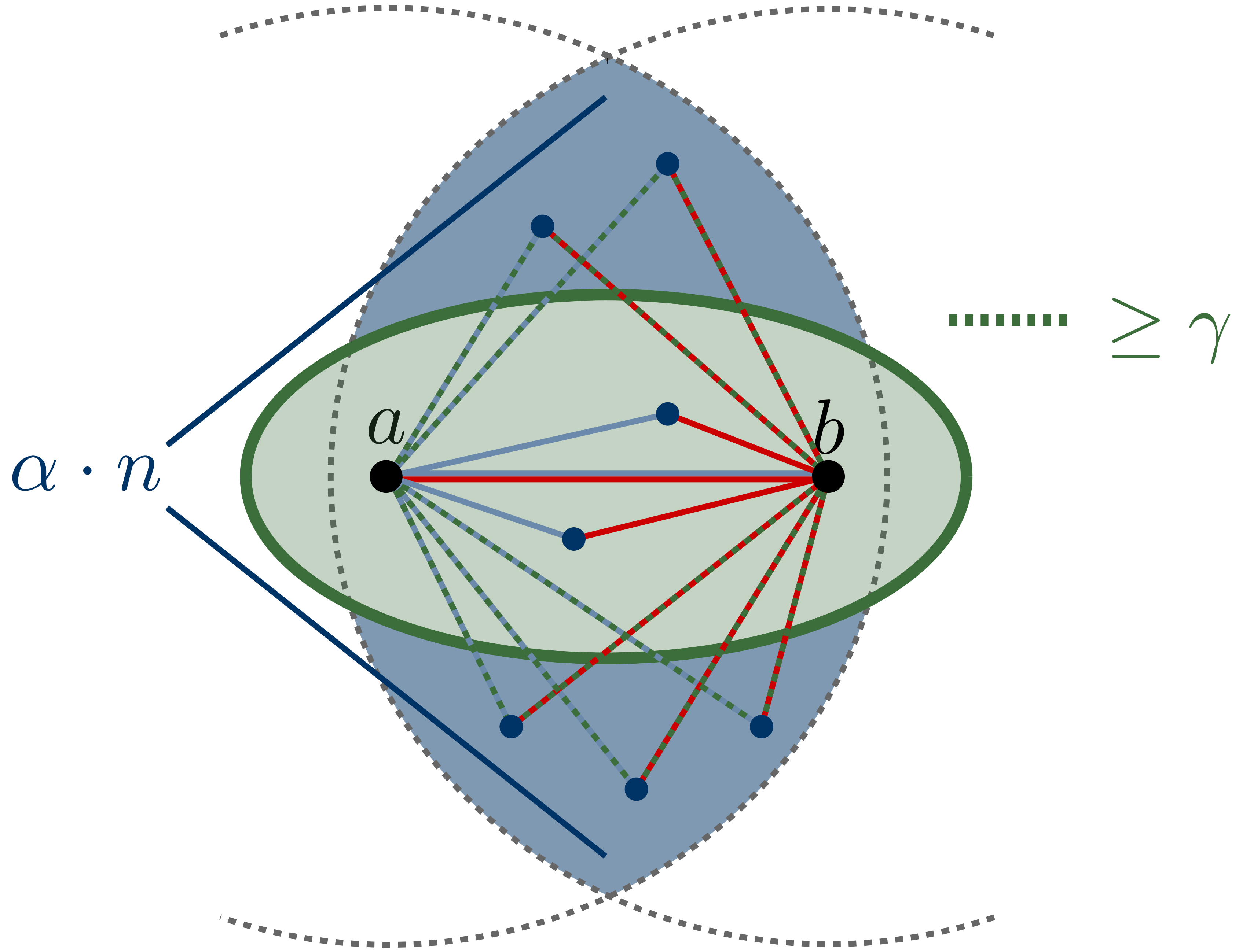

$$\max\{\text{—}, \text{—}\} \geq \frac{n}{2}(\|ab\|(1-\alpha) + \alpha\gamma))$$

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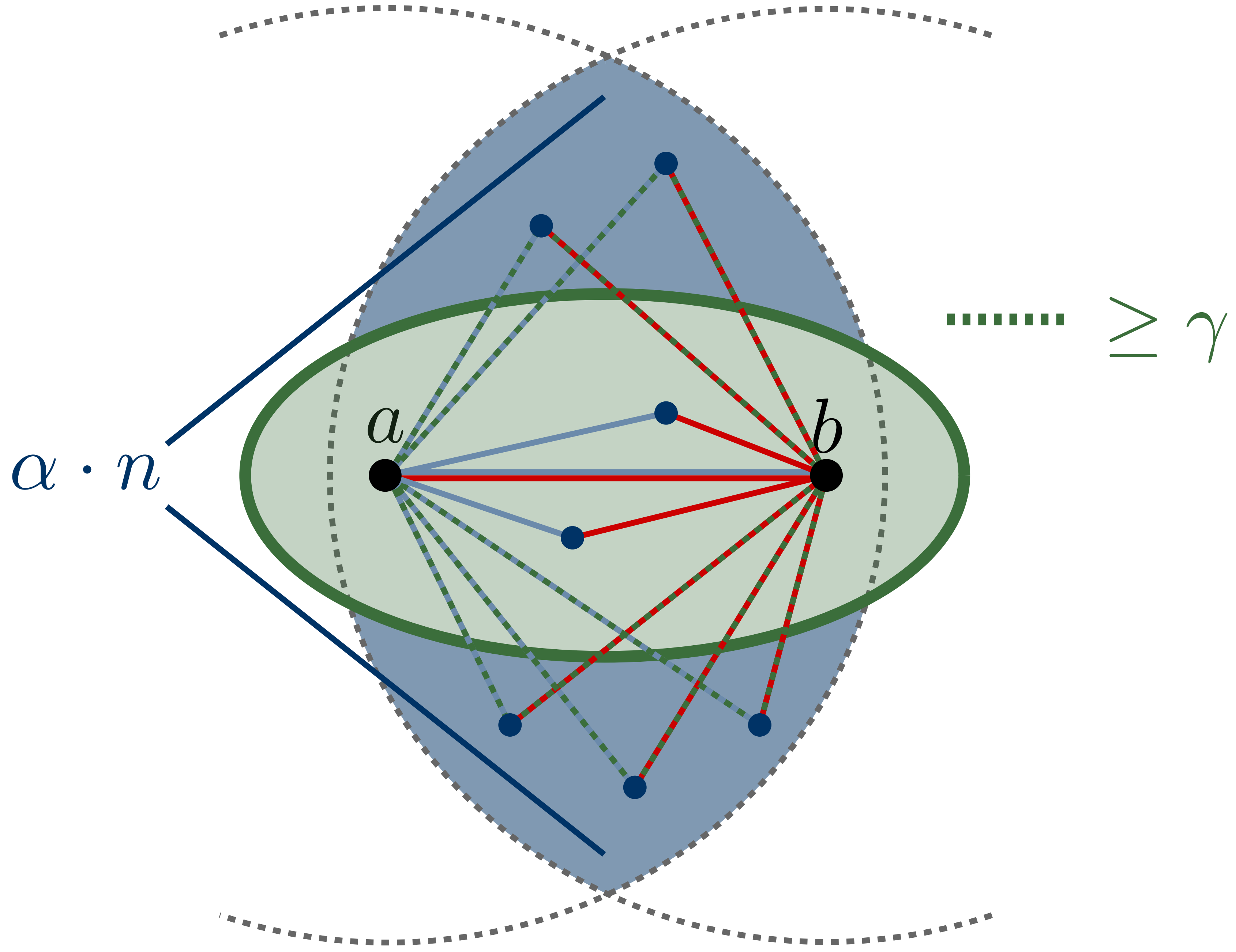
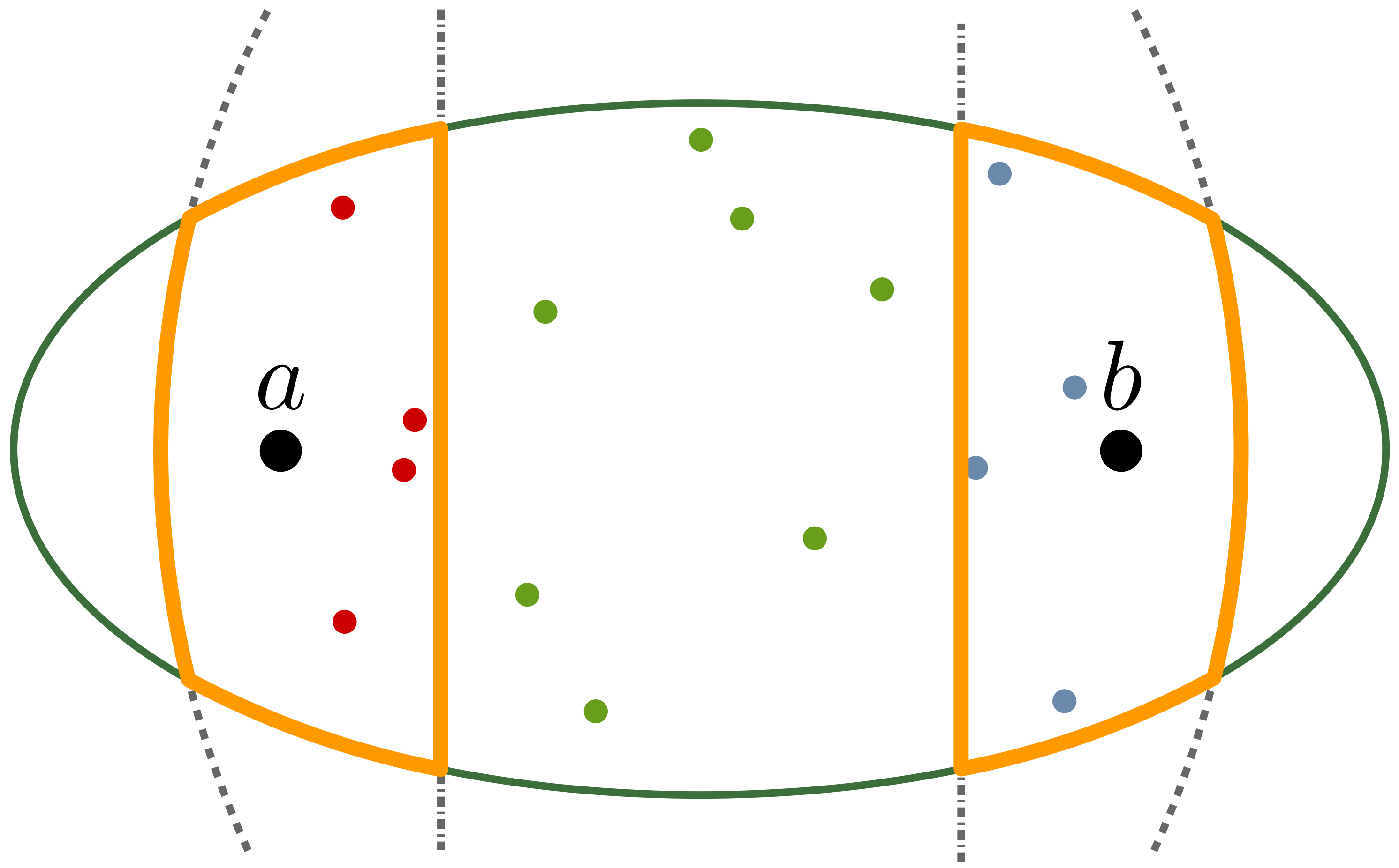


Case 2: $\|ab\| > d$

c) $\beta_M < \hat{\beta}$ and $\alpha < \hat{\alpha}$



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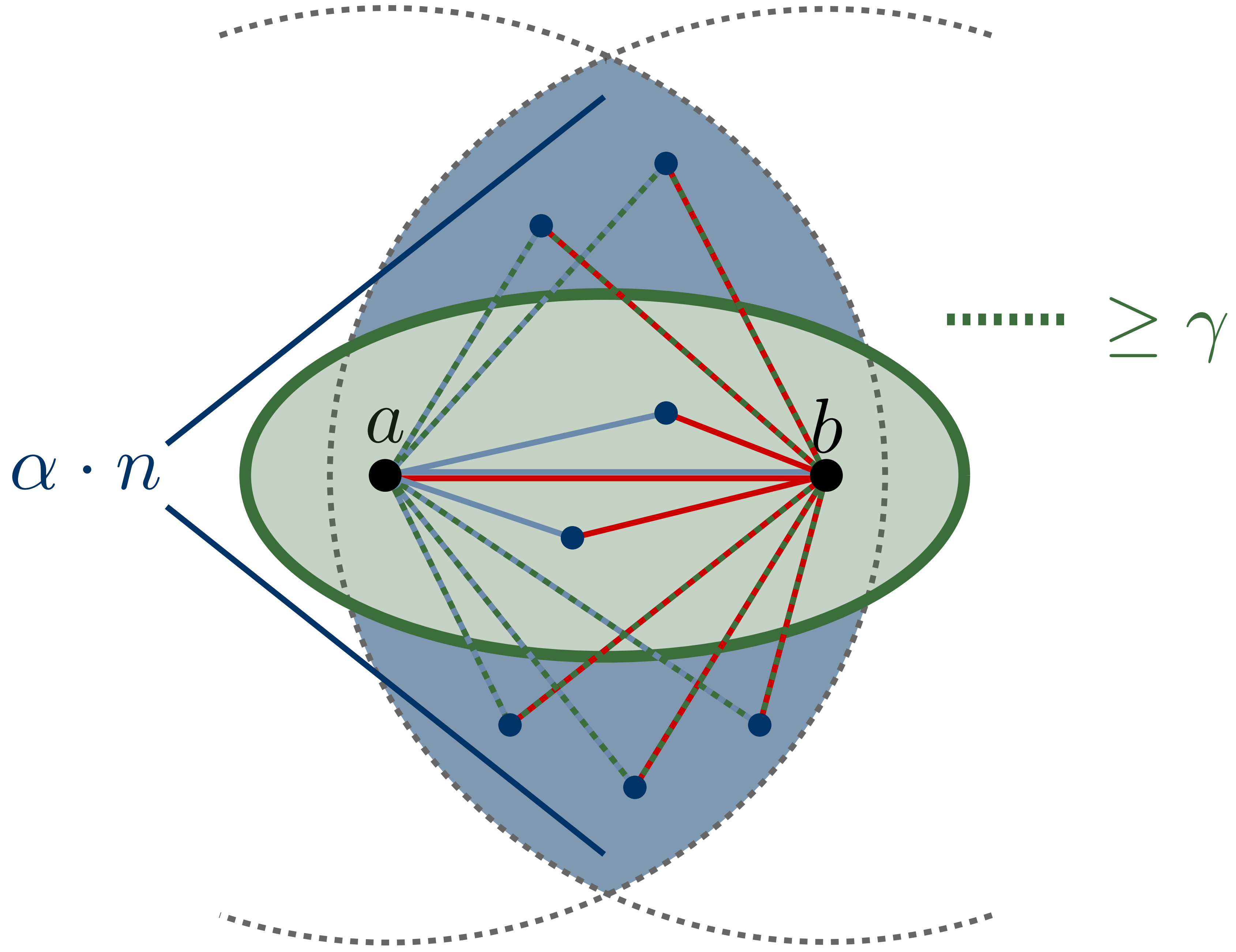

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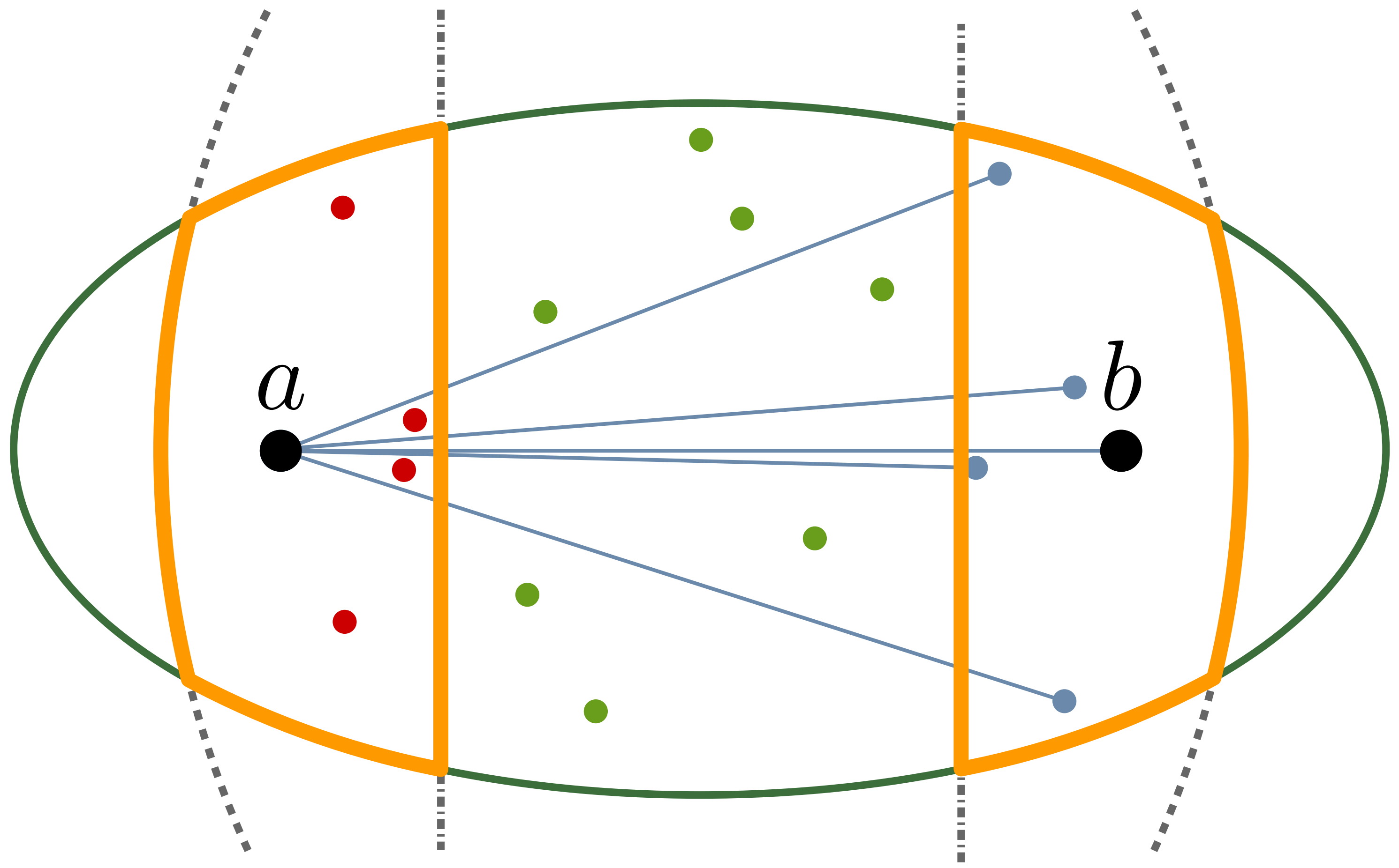

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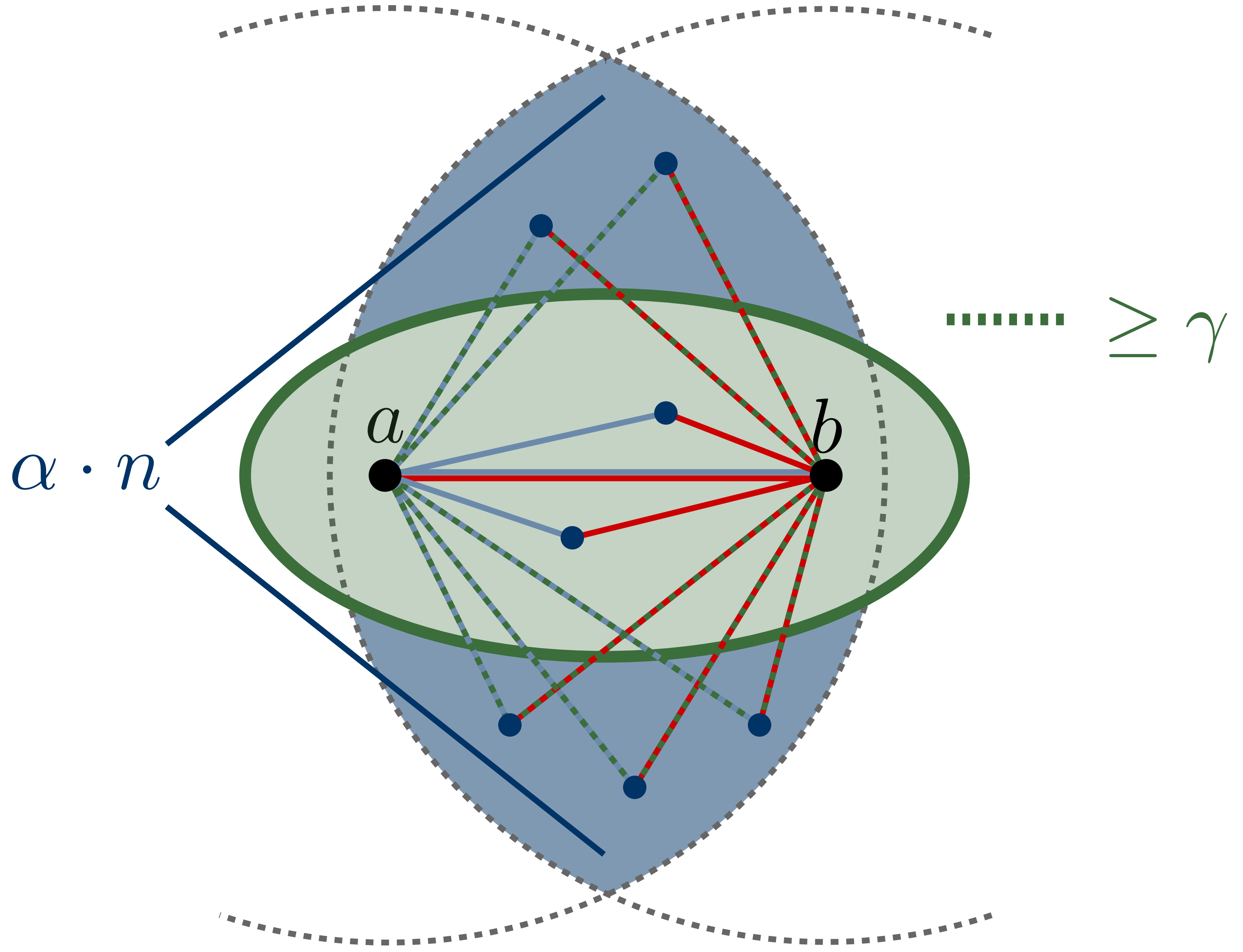

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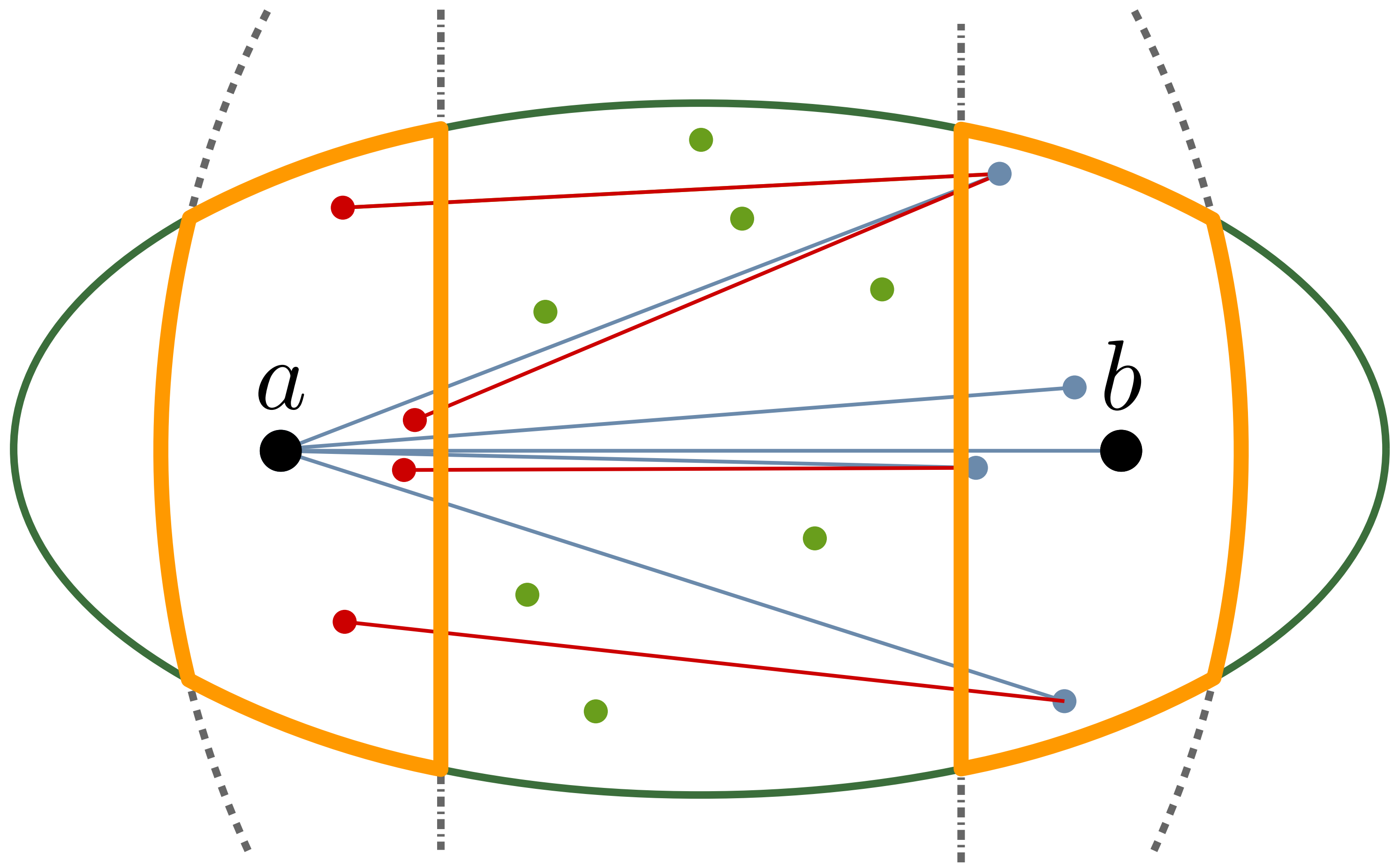

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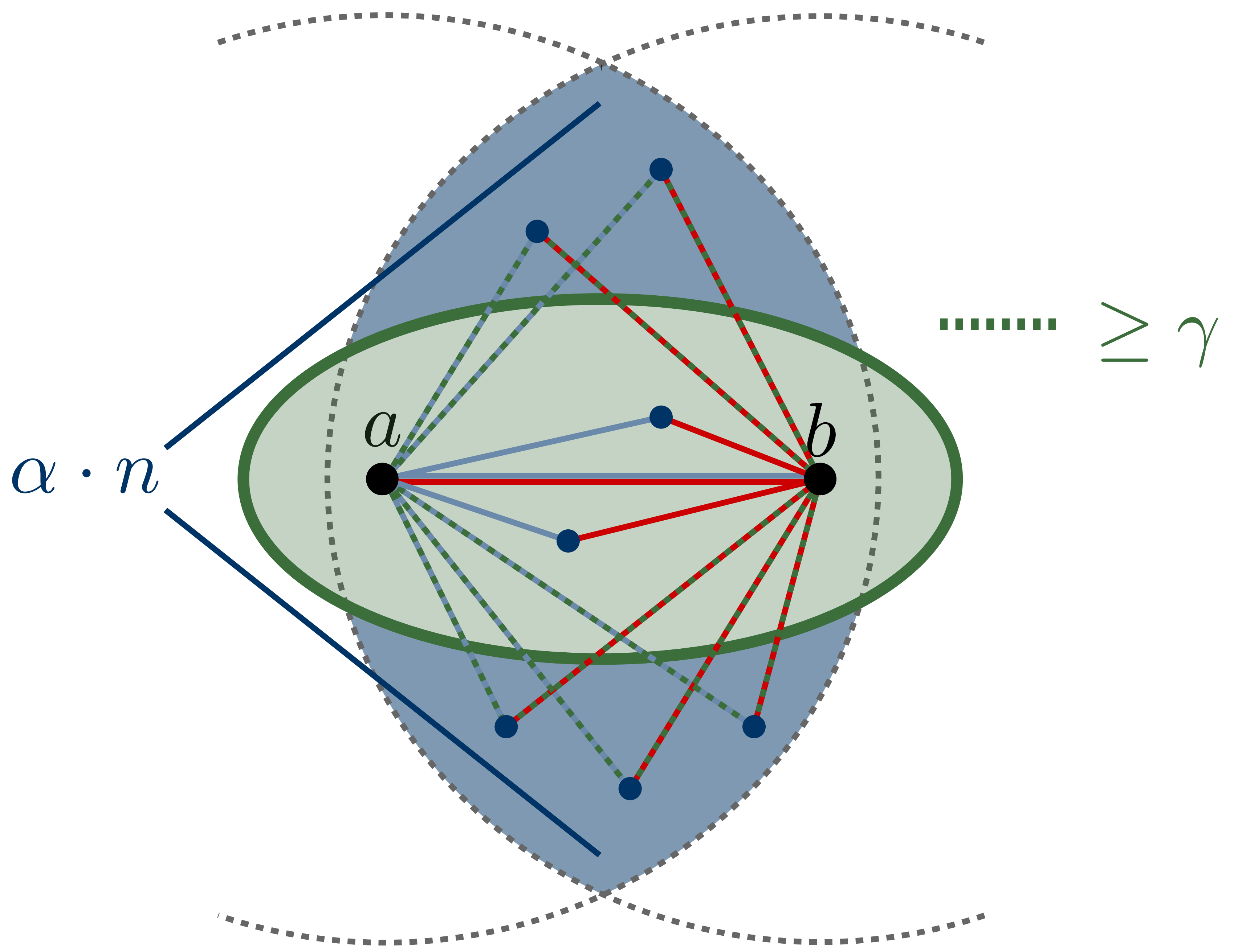
$$\hat{\alpha} = 1 - \frac{2\delta + \hat{\beta}(1-\omega)}{2 - 3\omega}$$
$$\gamma = \frac{2 \cdot \delta - 1 + \hat{\alpha}}{\hat{\alpha}}$$

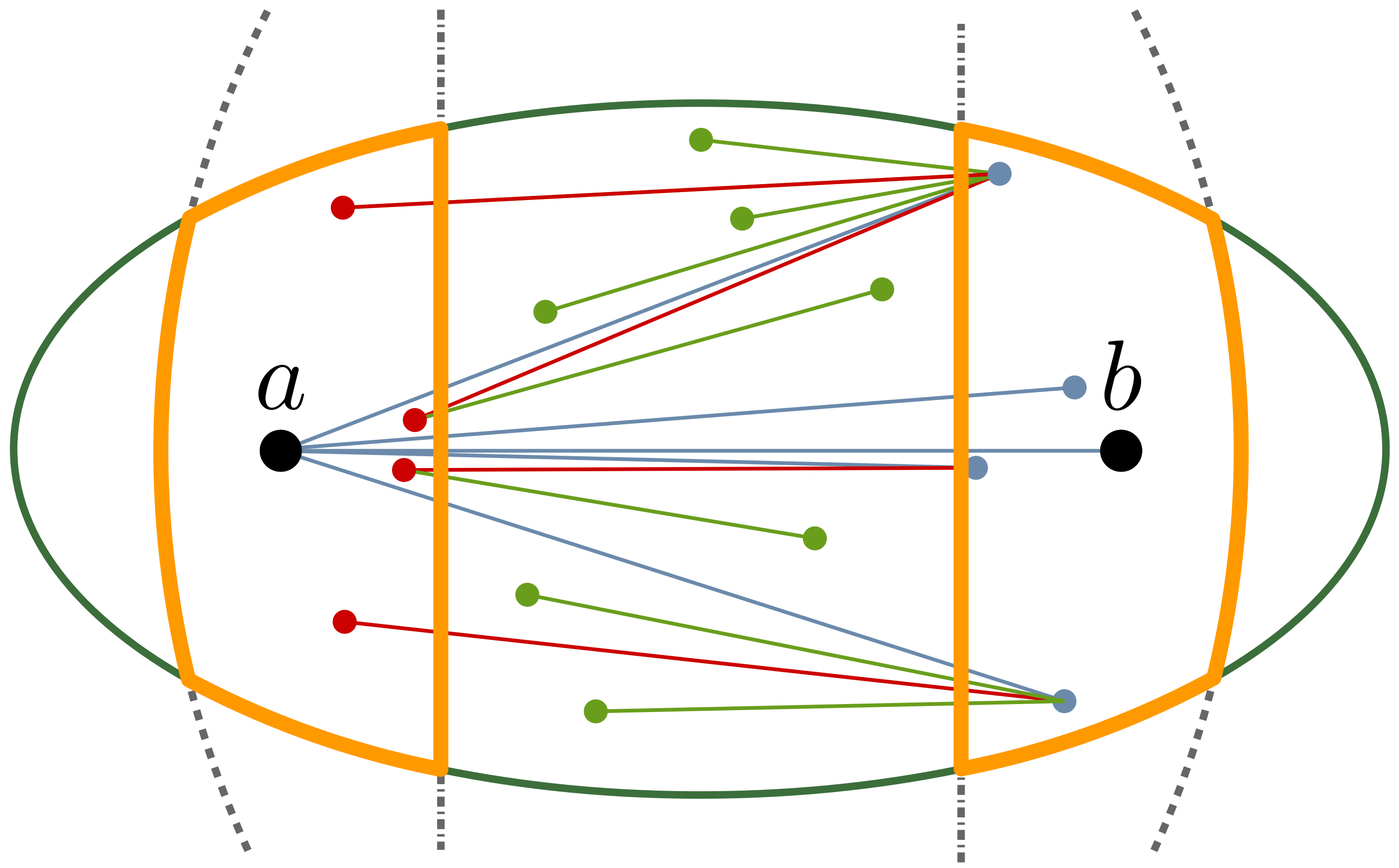
Case 2: $\|ab\| > d$

b) $\alpha \geq \hat{\alpha}$



Case 2: $\|ab\| > d$

c)$\beta_M < \hat{\beta}$ and $\alpha < \hat{\alpha}$



$$\max\{ \text{—}, \text{—} \} \geq \frac{n}{2}(\|ab\|(1-\alpha) + \alpha\gamma))$$

$$\frac{\max\{ \text{—}, \text{—} \}}{|T_{\text{OPT}}|} \geq \delta$$

The better approximation

$$\delta = 0.512$$
$$d = \frac{1}{2\delta}$$
$$\omega = 0.1$$
$$\hat{\beta} = \frac{\delta - 0.5}{\delta \cdot (1 - \sqrt{1 - d^2(\omega - \omega^2)})}$$
$$\hat{\alpha} = 1 - \frac{2\delta + \hat{\beta}(1-\omega)}{2 - 3\omega}$$
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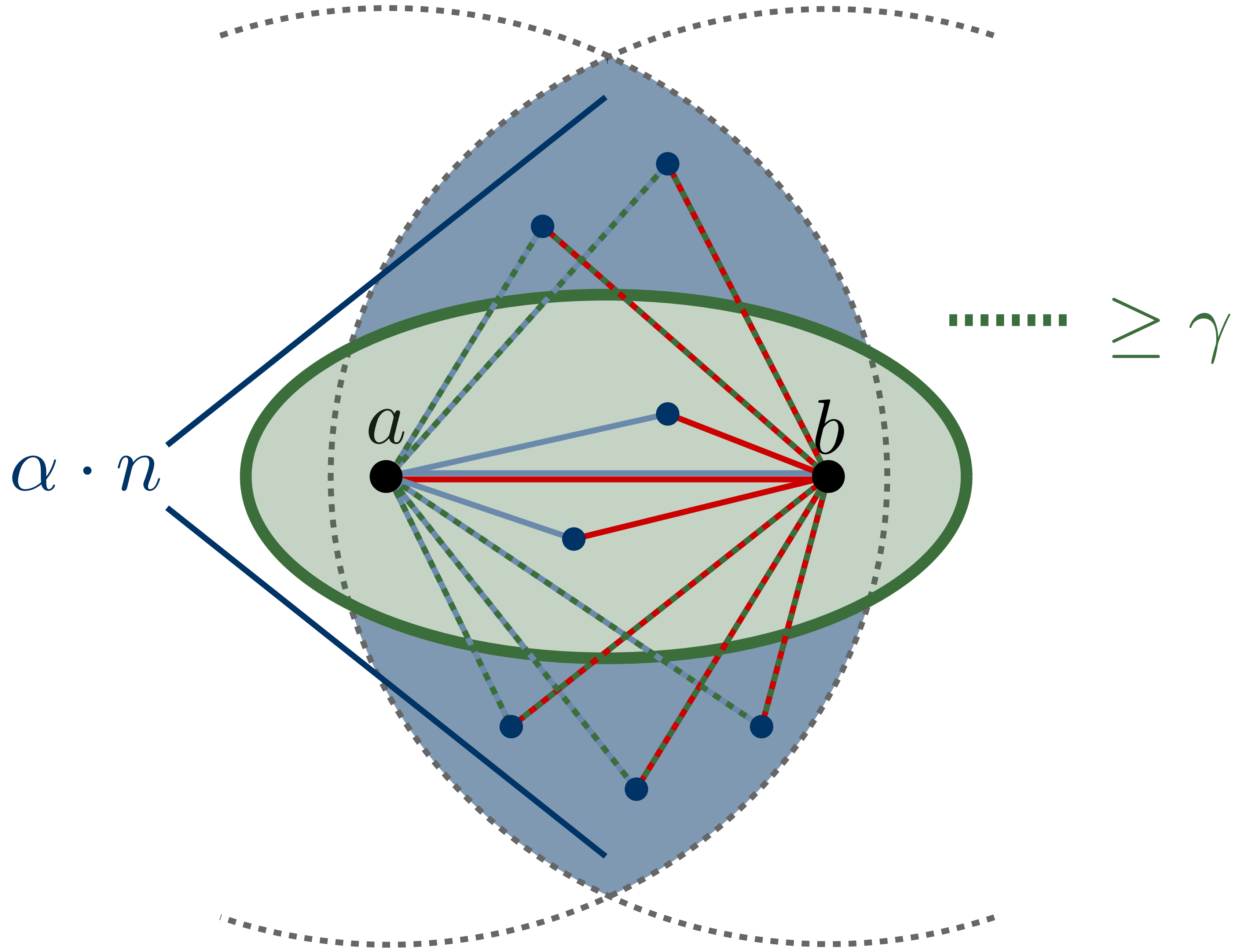
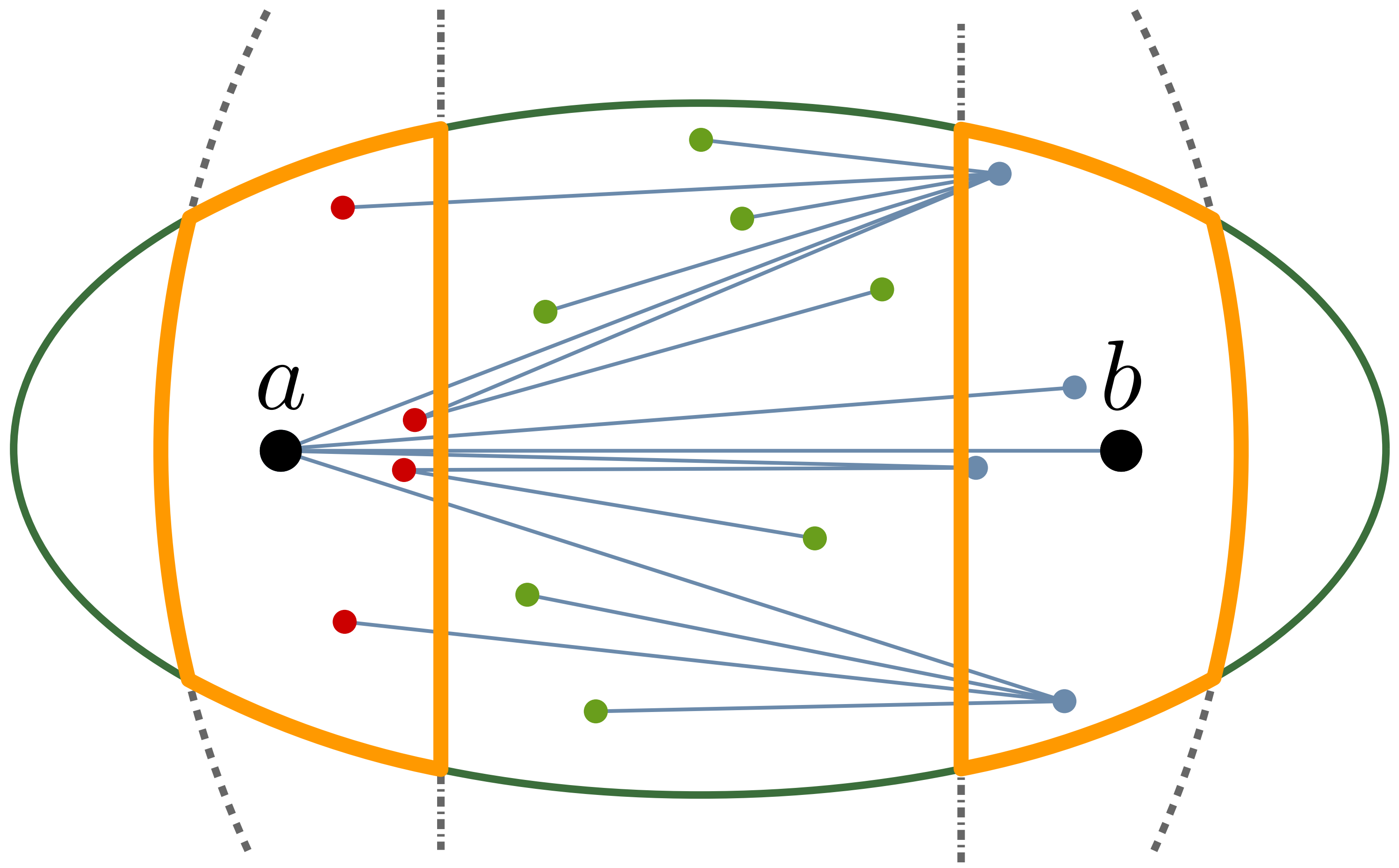


Case 2: $\|ab\| > d$

c) $\beta_M < \hat{\beta}$ and $\alpha < \hat{\alpha}$



$$\max\{ \text{—}, \text{—} \} \geq \frac{n}{2}(\|ab\|(1-\alpha) + \alpha\gamma))$$

$$\frac{\max\{ \text{—}, \text{—} \}}{|T_{\text{OPT}}|} \geq \delta$$

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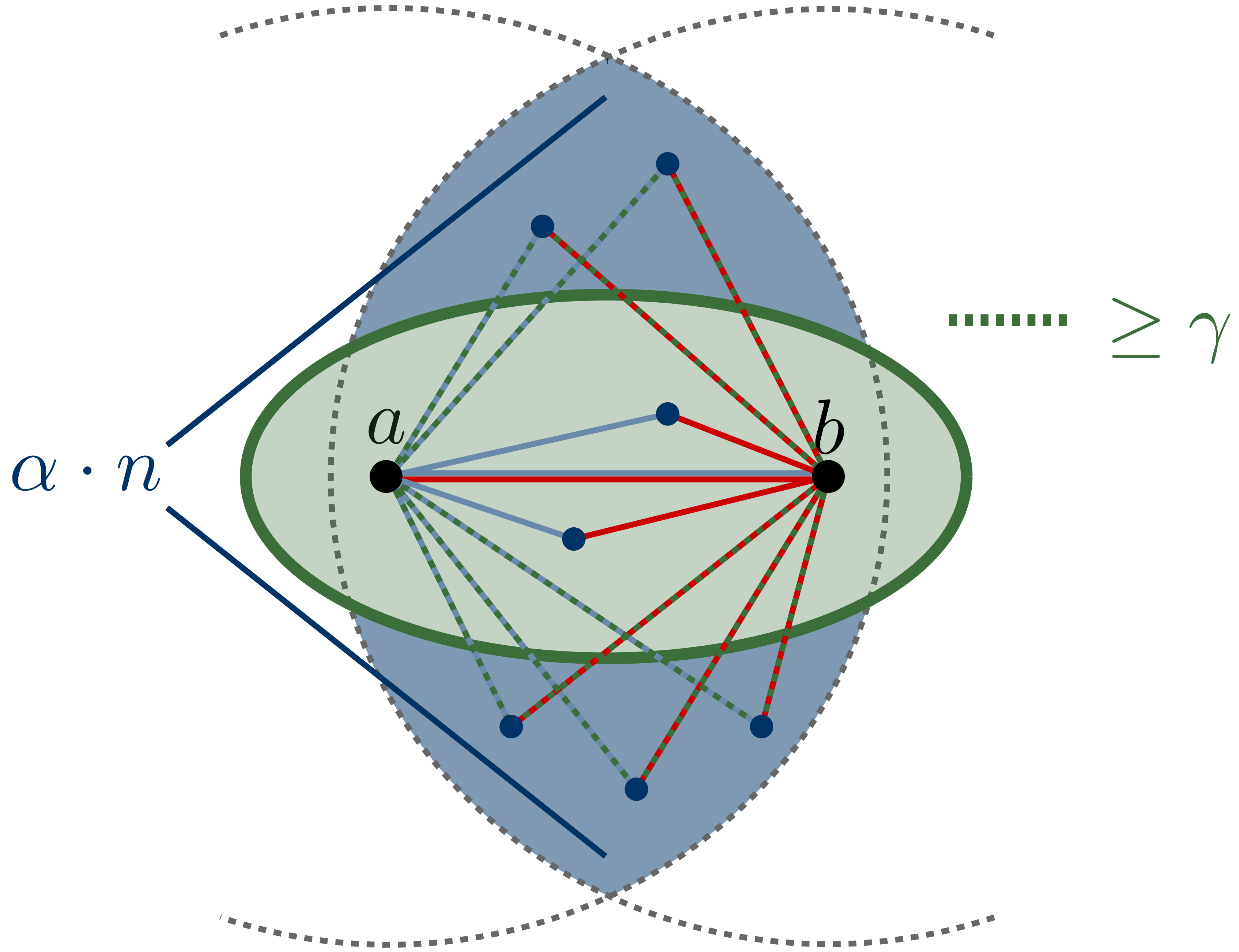
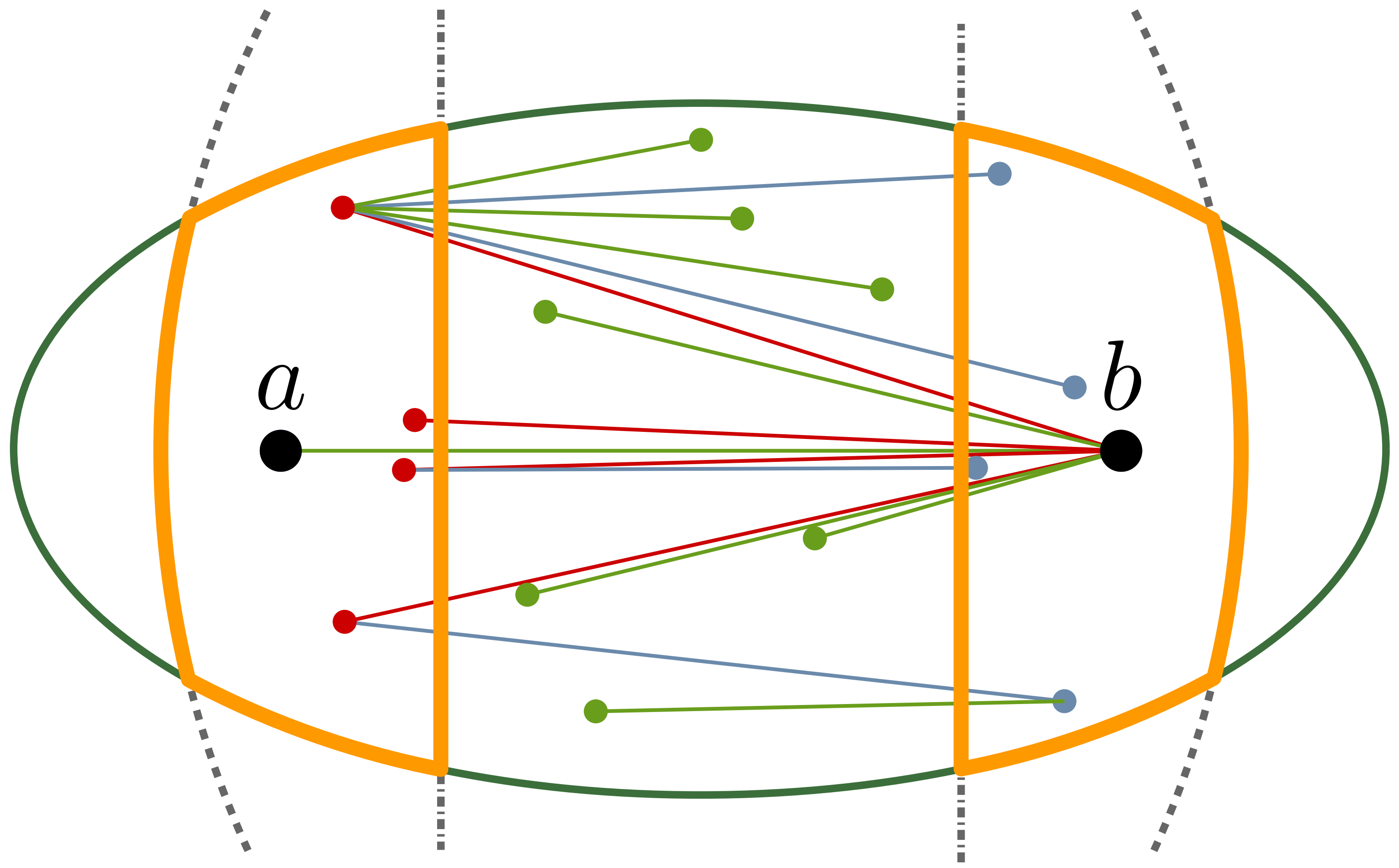


Case 2: $\|ab\| > d$

c)$\beta_M < \hat{\beta}$ and $\alpha < \hat{\alpha}$



$$\max\{\text{blue}, \text{red}\} \geq \frac{n}{2}(\|ab\|(1-\alpha) + \alpha\gamma))$$

$$\frac{\max\{\text{blue}, \text{red}\}}{|T_{\mathrm{OPT}}|} \geq \delta$$

The better approximation

$$\delta = 0.512$$
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$$\omega = 0.1$$
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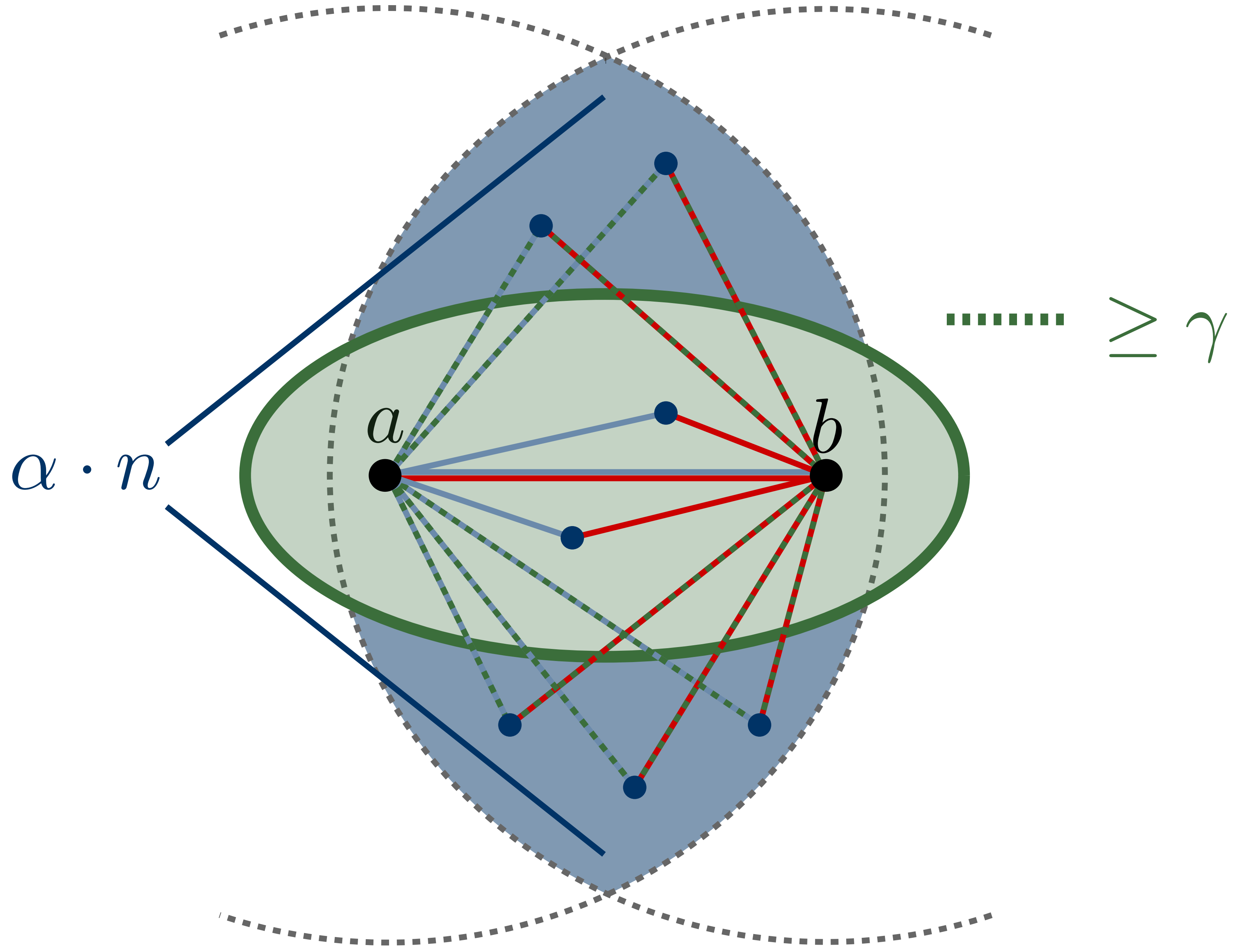
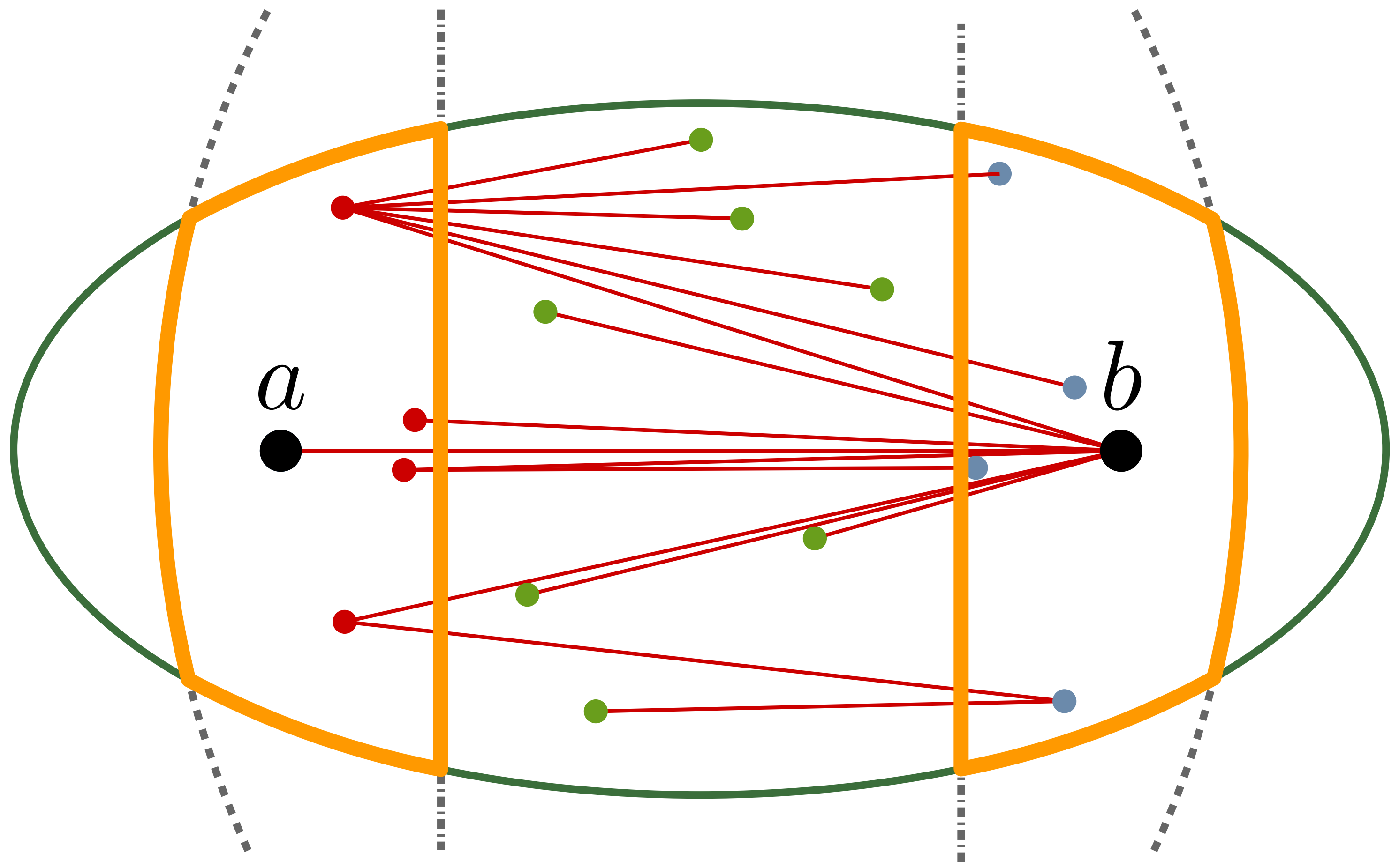
Case 2: $\|ab\| > d$

b) $\alpha \geq \hat{\alpha}$



Case 2: $\|ab\| > d$

c) $\beta_M < \hat{\beta}$ and $\alpha < \hat{\alpha}$



$$\max\{ —, — \} \geq \frac{n}{2}(\|ab\|(1-\alpha) + \alpha\gamma))$$

$$\frac{\max\{ —, — \}}{|T_{\mathrm{OPT}}|} \geq \delta$$

The better approximation

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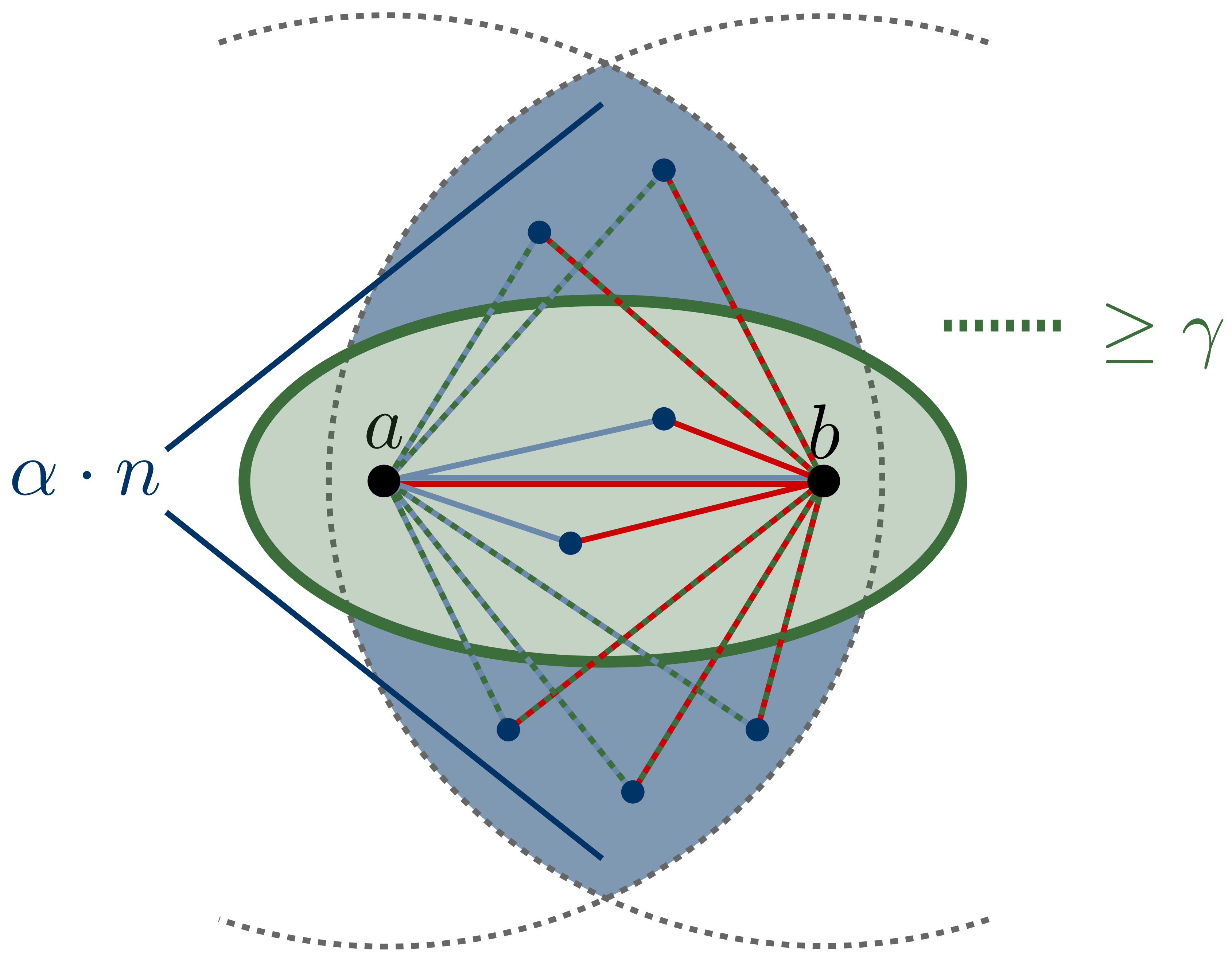
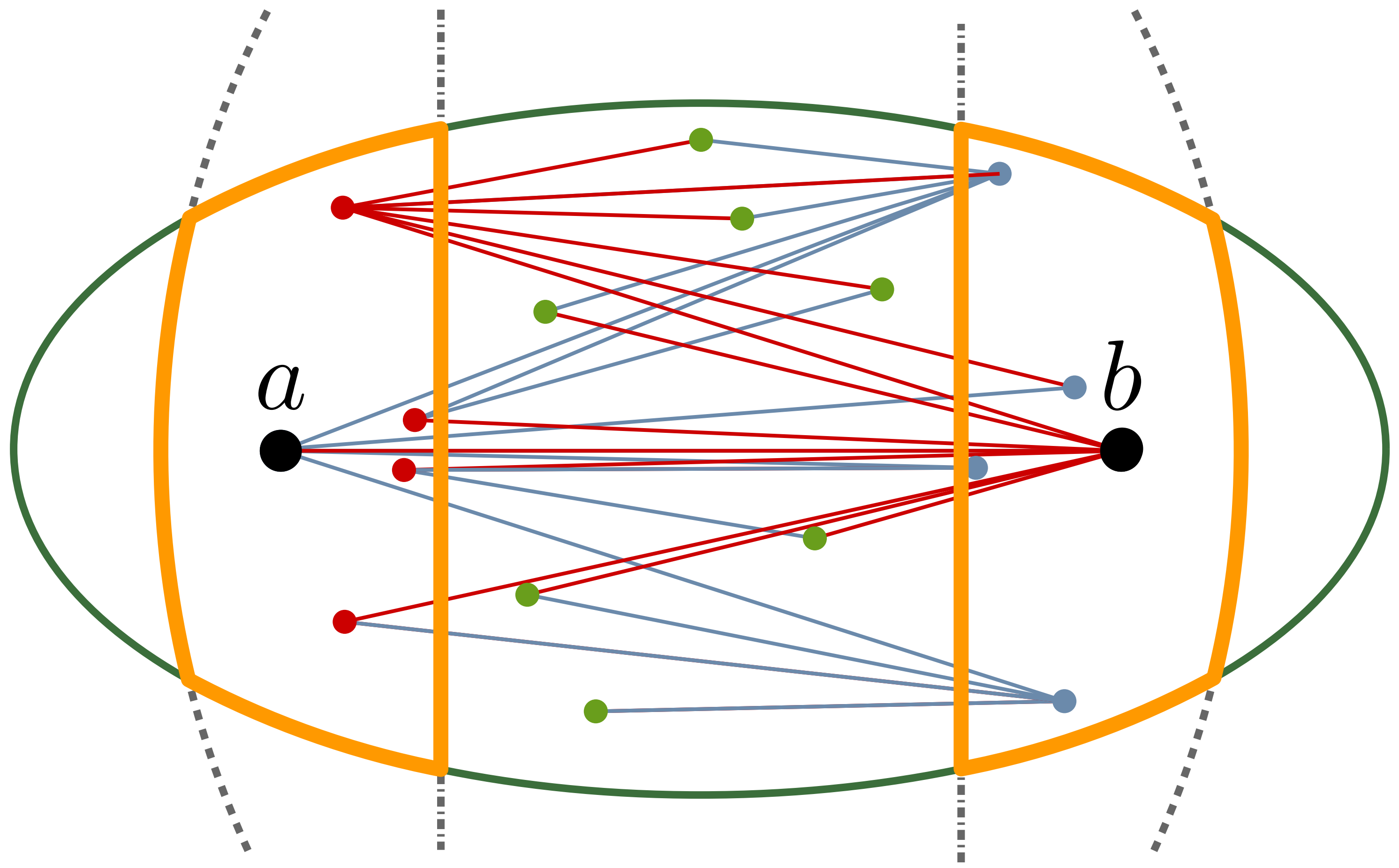


$$\max\{ —, — \} \geq \frac{n}{2}(\|ab\|(1-\alpha) + \alpha\gamma))$$

$$\frac{\max\{ —, — \}}{|T_{\text{OPT}}|} \geq \delta$$

Case 2: $\|ab\| > d$

c)$\beta_M < \hat{\beta}$ and $\alpha < \hat{\alpha}$



$$\max\{ —, — \} \geq$$
$$\frac{n}{2}\|ab\|((1-\hat{\alpha})(2-3\omega) - \hat{\beta}(1-\omega))$$

$$\frac{\max\{ —, — \}}{|T_{\text{OPT}}|} \geq \delta$$

The better approximation

Case 2: $\|ab\| > d$

b) $\alpha \geq \hat{\alpha}$

Case 2: $\|ab\| > d$

c) $\beta_M < \hat{\beta}$ and $\alpha < \hat{\alpha}$

Algorithm:

$$\delta = 0.512$$
$$d = \frac{1}{2\delta}$$
$$\omega = 0.1$$
$$\hat{\beta} = \frac{\delta - 0.5}{\delta \cdot (1 - \sqrt{1 - d^2(\omega - \omega^2)})}$$
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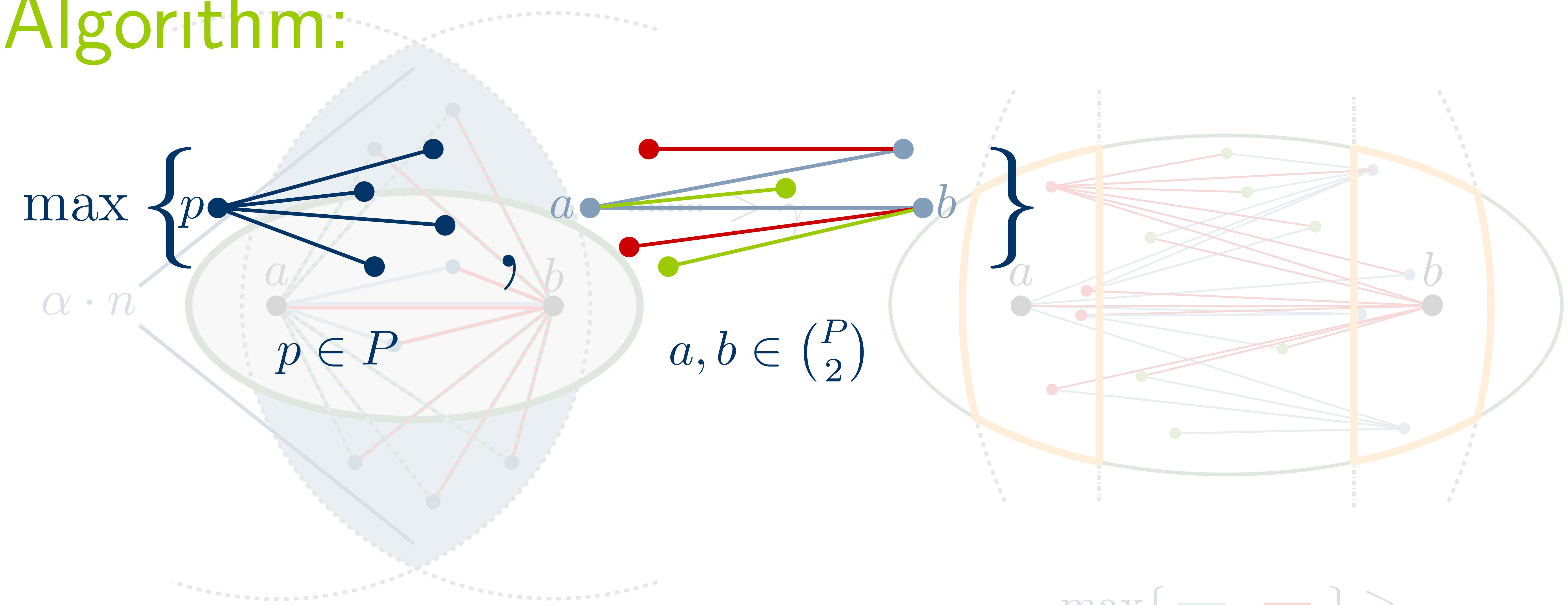


$\max\{\,—\,,\,—\,\} \geq \frac{n}{2}(\|ab\|(1-\alpha) + \alpha\gamma))$

$\frac{\max\{\,—\,,\,—\,\}}{|T_{\text{OPT}}|} \geq \delta$

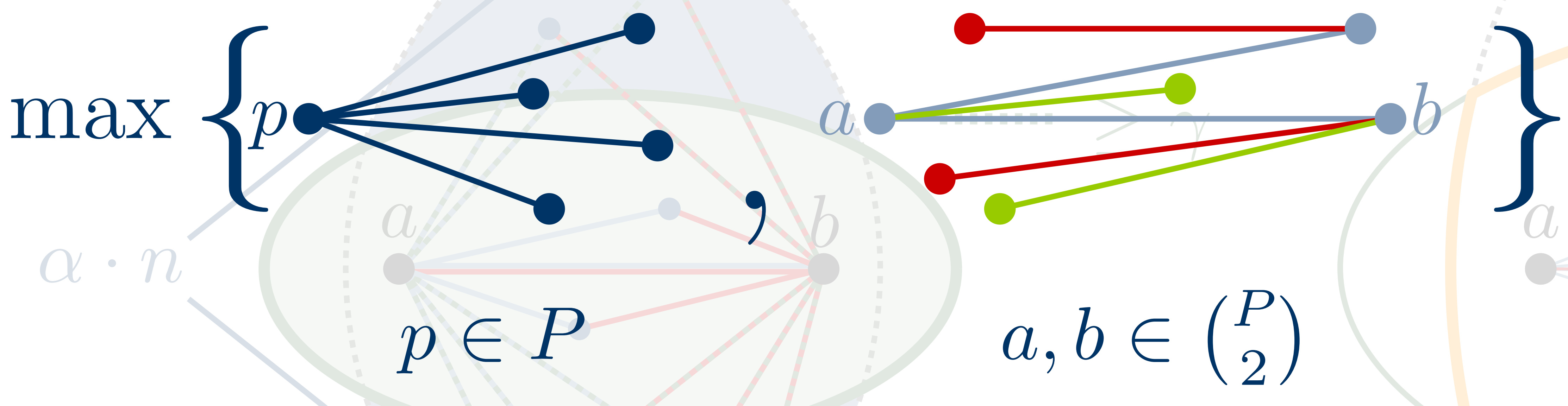
$\max\{\,—\,,\,—\,\} \geq$

$\frac{n}{2}\|ab\|((1-\hat{\alpha})(2-3\omega) - \hat{\beta}(1-\omega))$

$\frac{\max\{\,—\,,\,—\,\}}{|T_{\text{OPT}}|} \geq \delta$

The better approximation

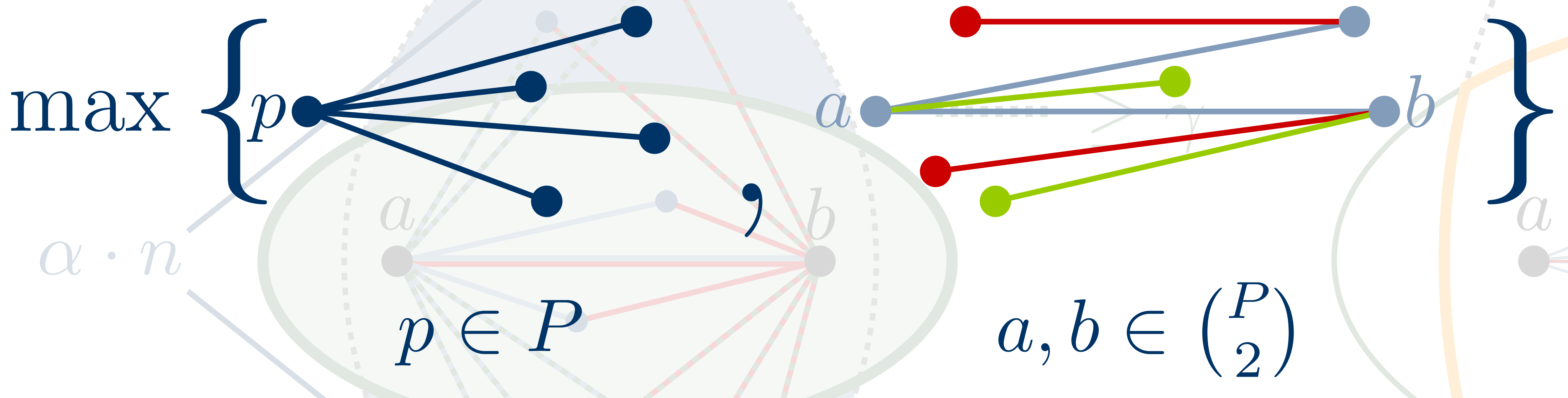
Algorithm:



Theorem. *There is a 0.512-approximation.*

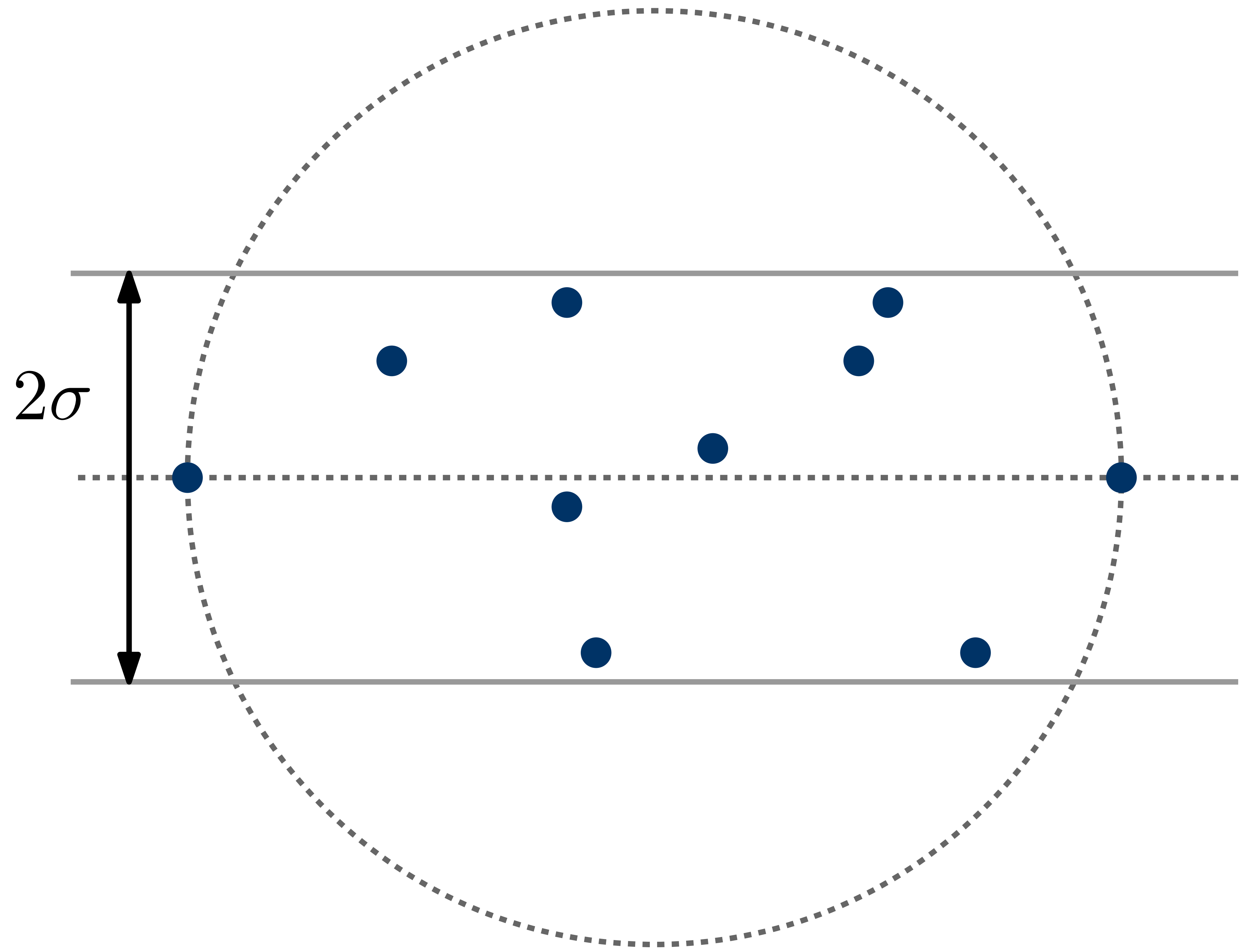
The better approximation

Algorithm:

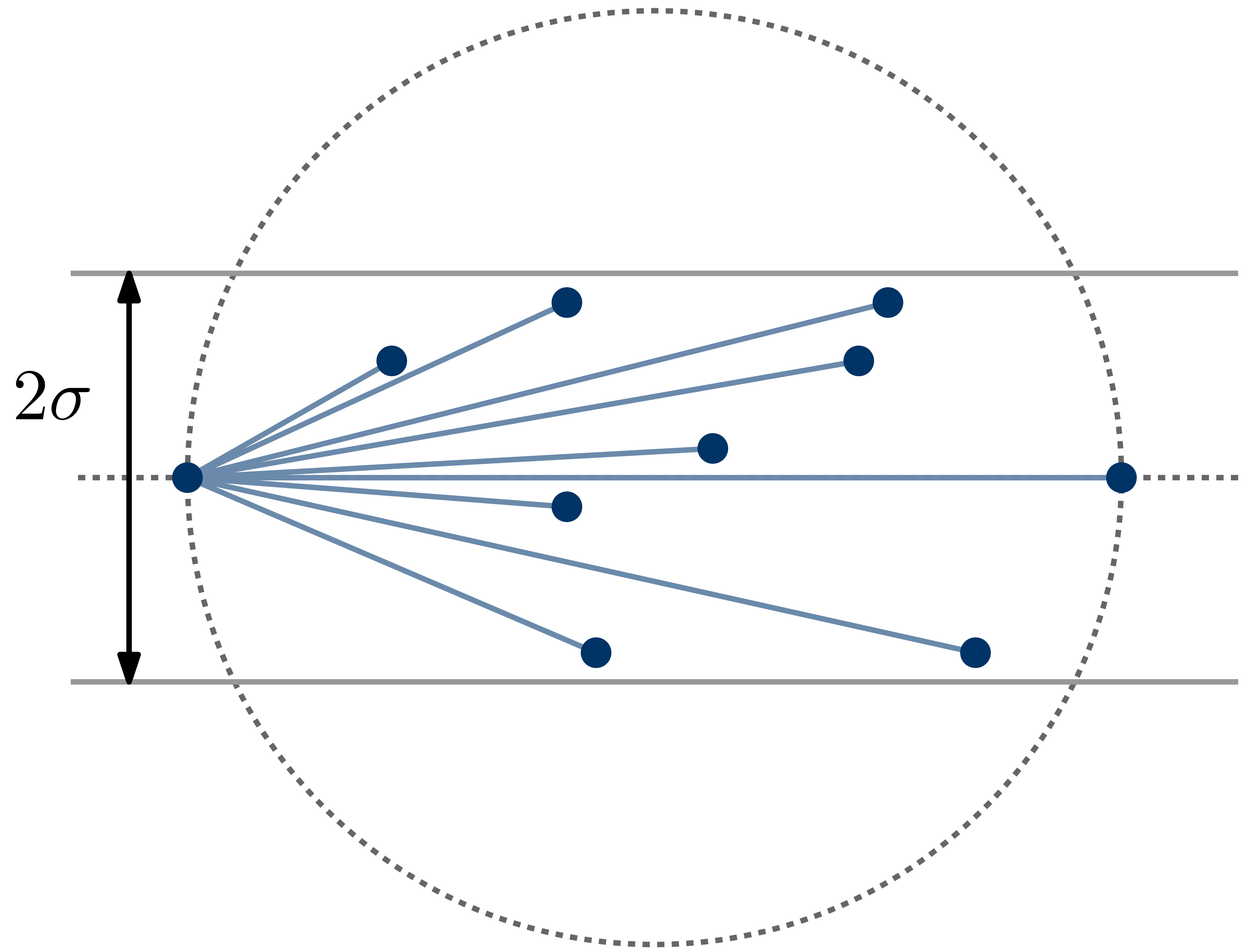


Theorem. *There is a polynomial time* 0.512*-approximation.*

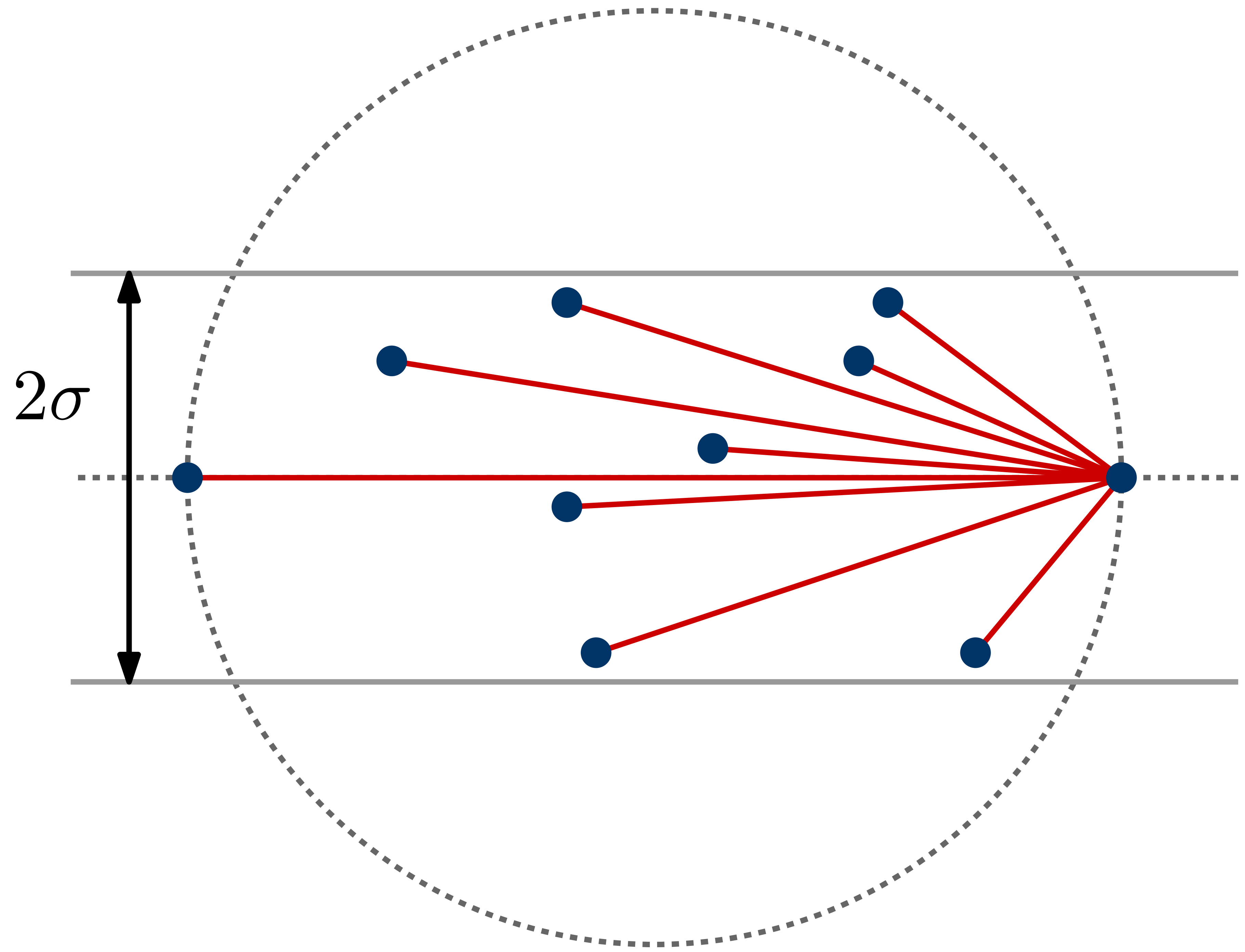
σ-thick Point Sets



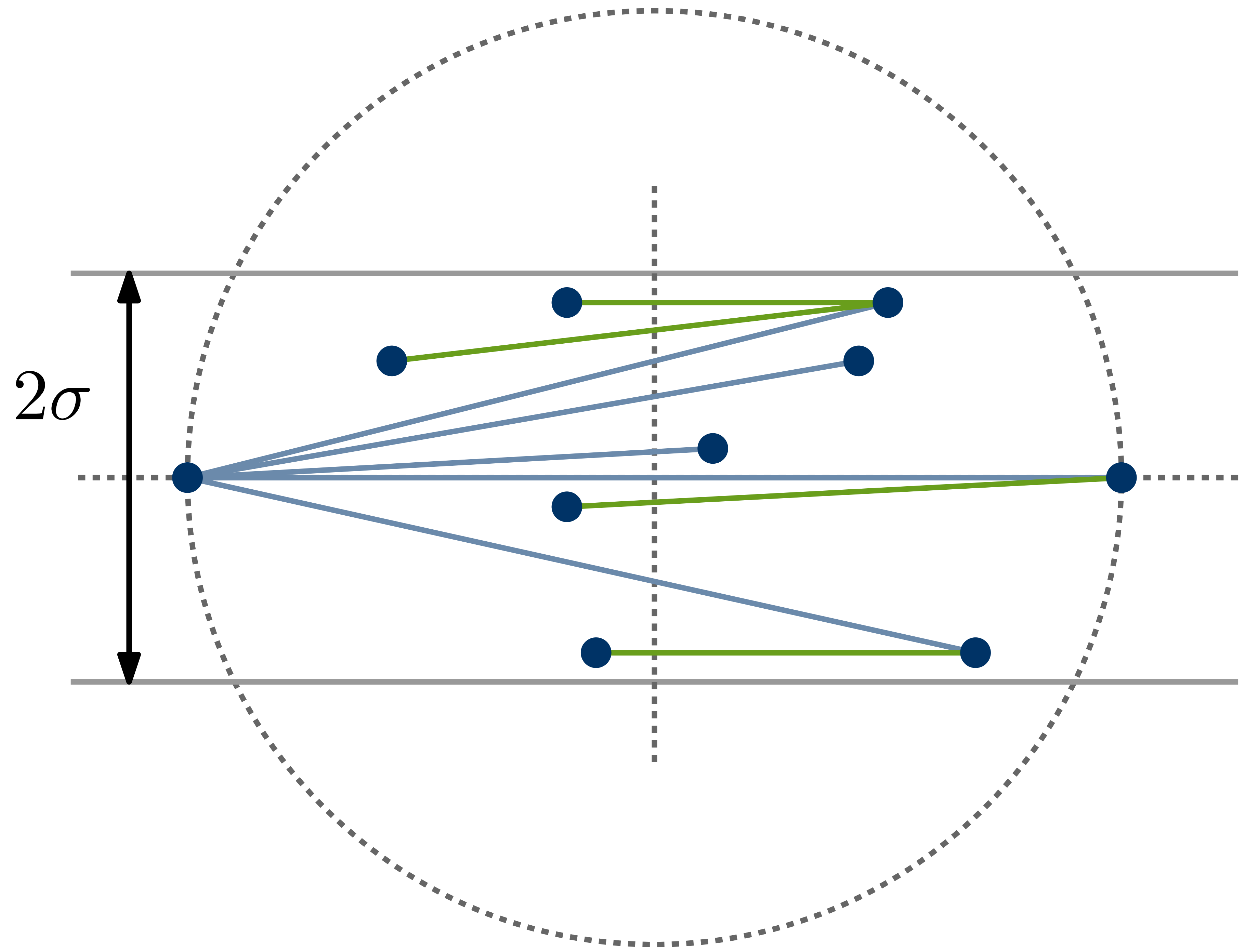
σ-thick Point Sets



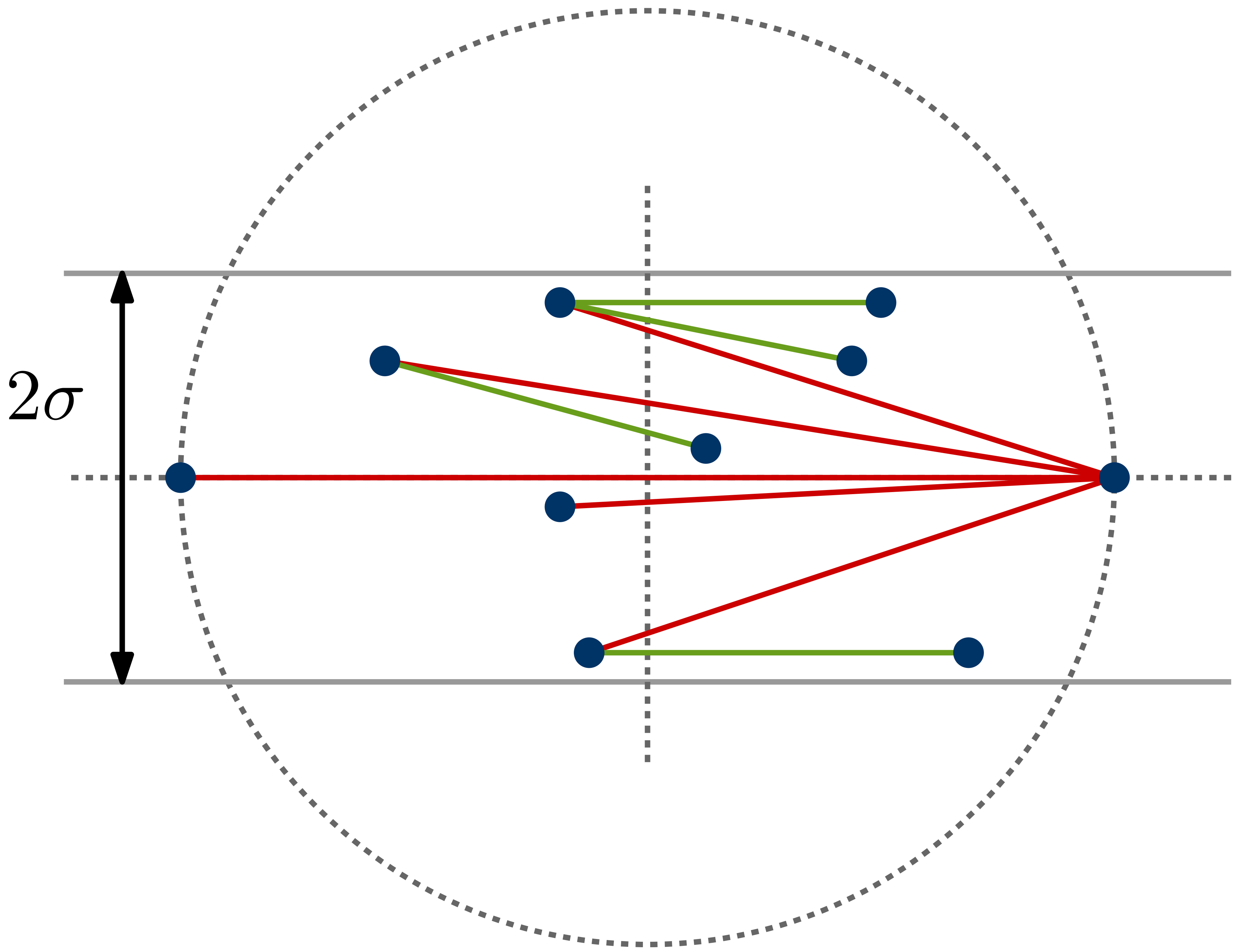
σ-thick Point Sets



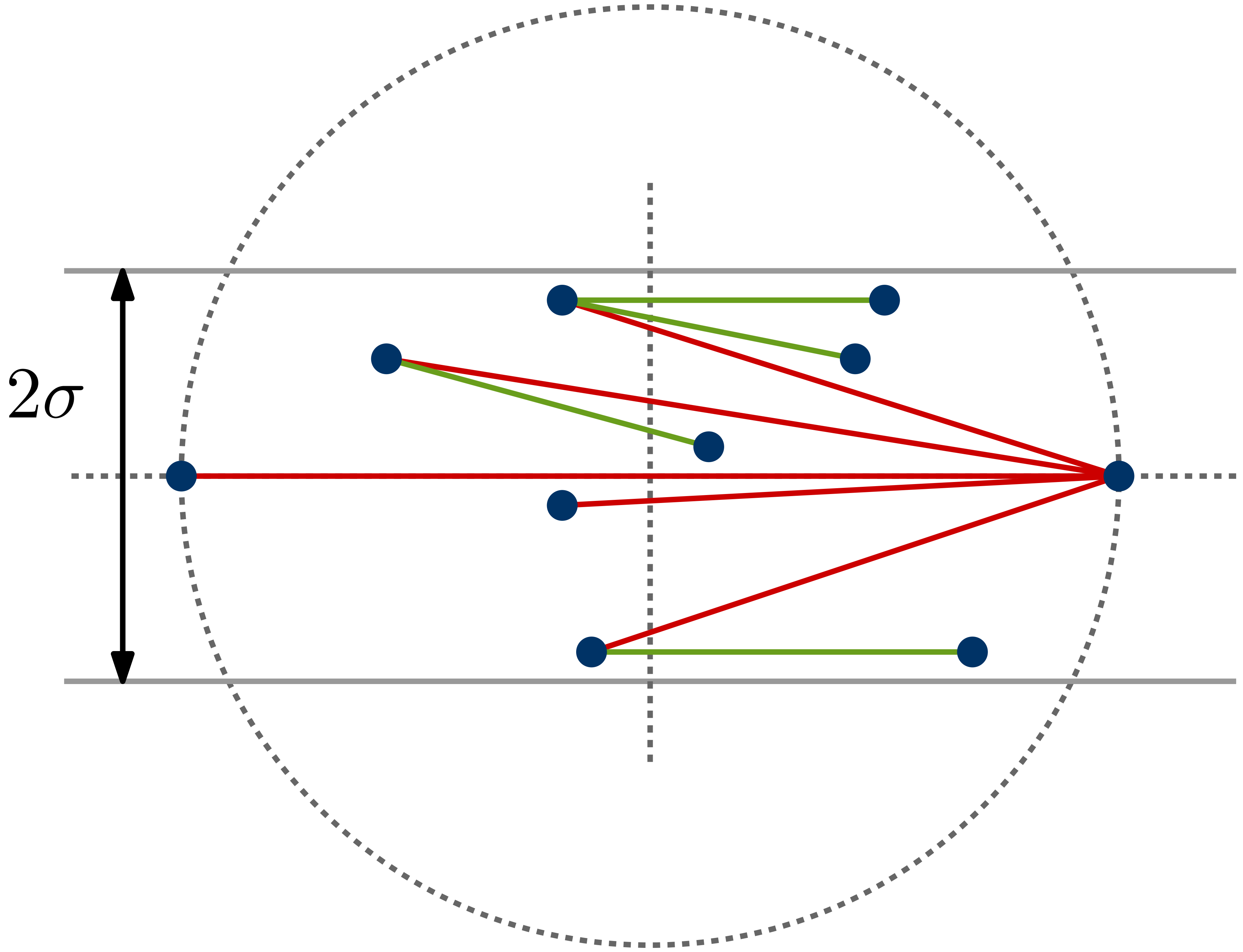
σ-thick Point Sets



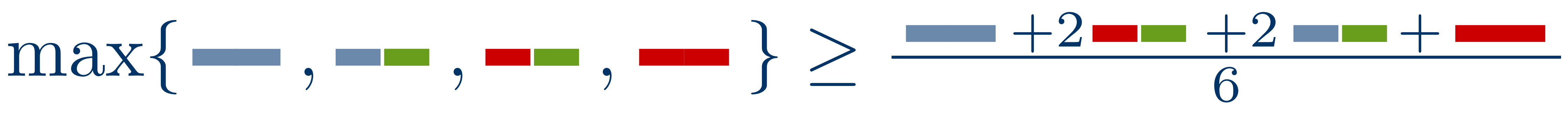
σ-thick Point Sets



σ-thick Point Sets

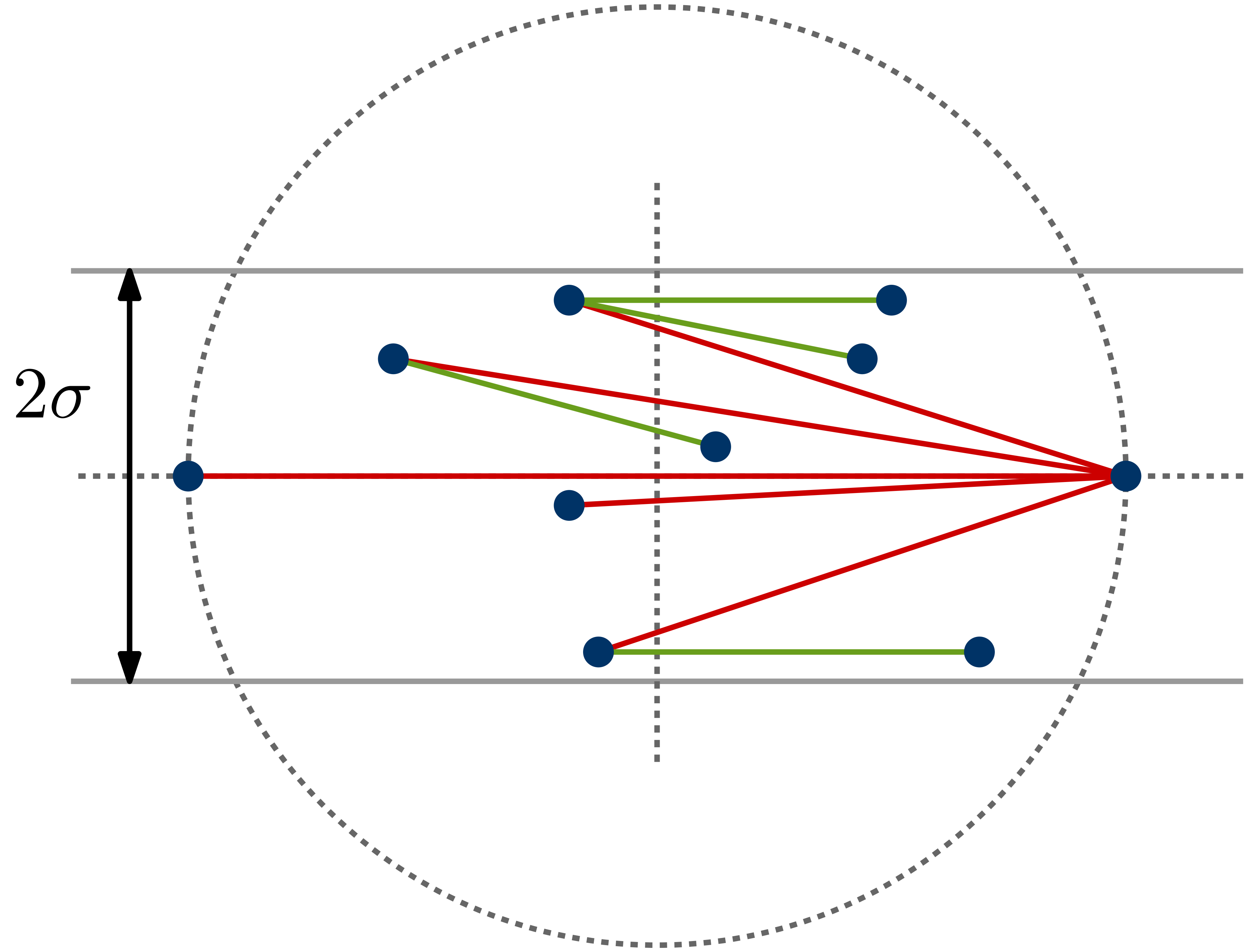


$$f(\sigma) = \frac{2}{3}\sqrt{\frac{1+4\sigma^2}{5-4\sqrt{1-\sigma^2}+4\sigma}}$$



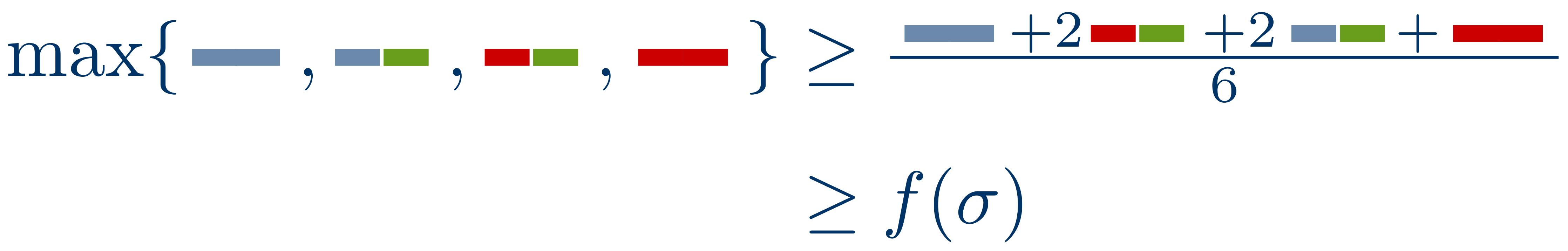
$$\geq f(\sigma)$$

σ-thick Point Sets



Theorem. *For σ-thick: $f(\sigma)$-approximation, with:*

$$f(\sigma) = \frac{2}{3}\sqrt{\frac{1+4\sigma^2}{5-4\sqrt{1-\sigma^2}+4\sigma}}$$



Conclusion

✓

$0.503 \rightarrow 0.512$

even better for σ-thick

Conclusion

✓

$0.503 \to 0.512$

even better for σ-thick

?

improve running time

further improve factor

other special cases

NP-hardness

Conclusion

✓

$0.503 \to 0.512$

even better for σ-thick

?

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Thank you!