Augmenting Polygons with Matchings

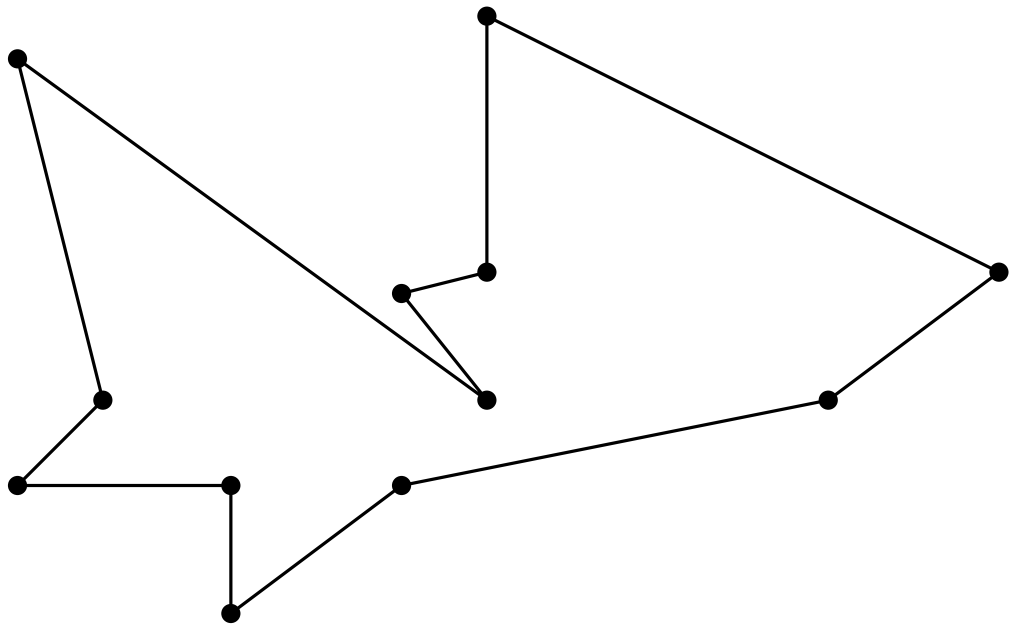
Alexander Pilz, Jonathan Rollin, Lena Schlipf, André Schulz

EuroCG 2020

Problem

Given a simple polygon P (or a geometric graph)

edges drawn with
straight-lines, noncrossing

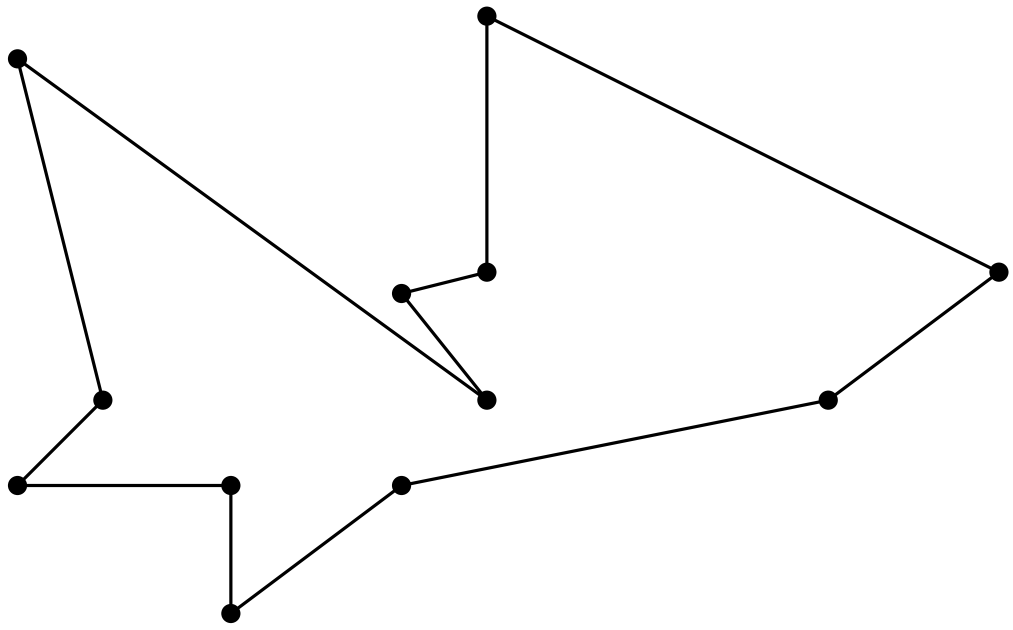


Problem

Given a simple polygon P (or a geometric graph)

find a geometric matching on the vertices of P, such that no edges cross in the augmentation.

$\longrightarrow$ this is called a *compatible* matching

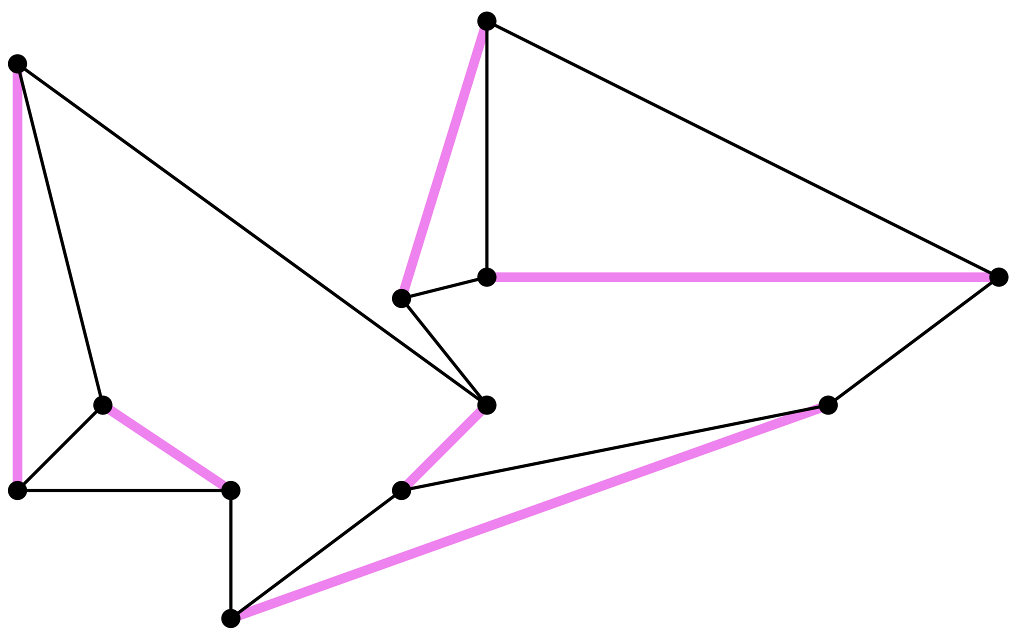


Problem

Given a simple polygon P (or a geometric graph)

find a geometric matching on the vertices of P, such that no edges cross in the augmentation.

$\longrightarrow$ this is called a *compatible* matching

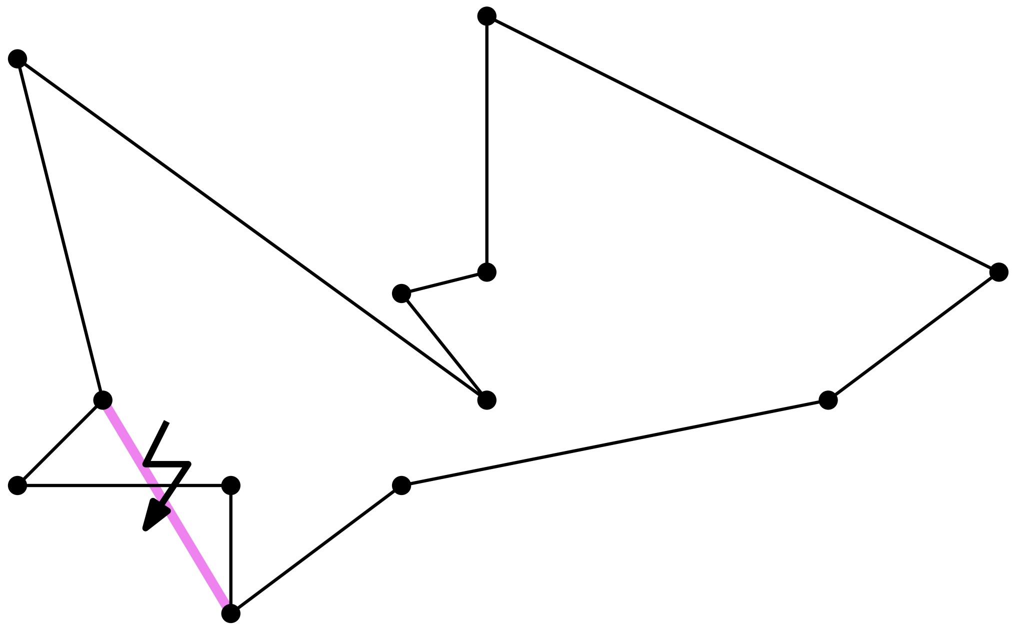


Problem

Given a simple polygon P (or a geometric graph)

find a geometric matching on the vertices of P, such that no edges cross in the augmentation.

$\longrightarrow$ this is called a *compatible* matching

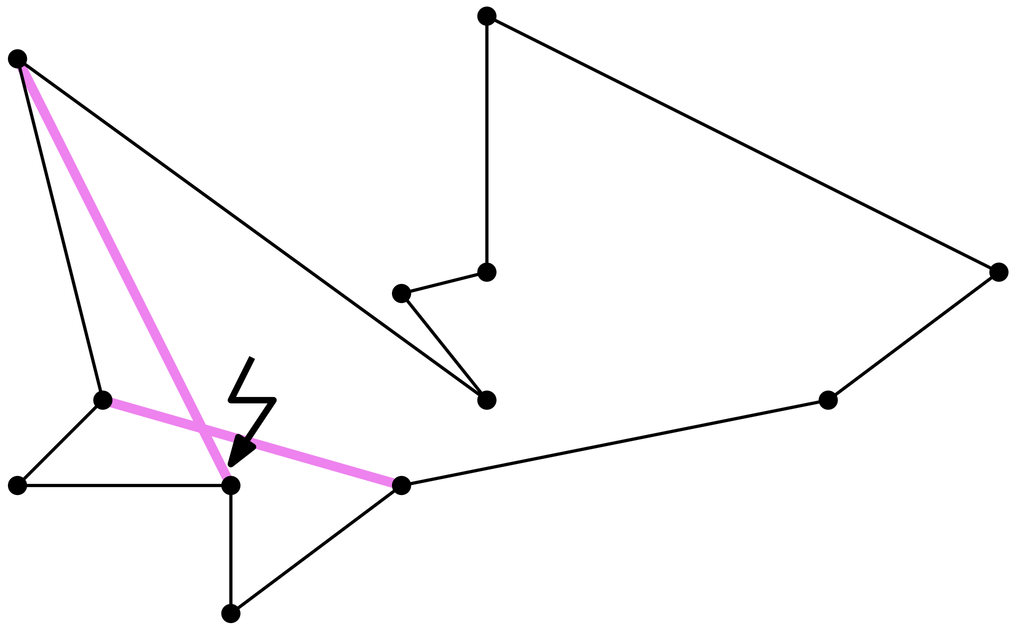


Problem

Given a simple polygon P (or a geometric graph)

find a geometric matching on the vertices of P, such that no edges cross in the augmentation.

$\longrightarrow$ this is called a *compatible* matching

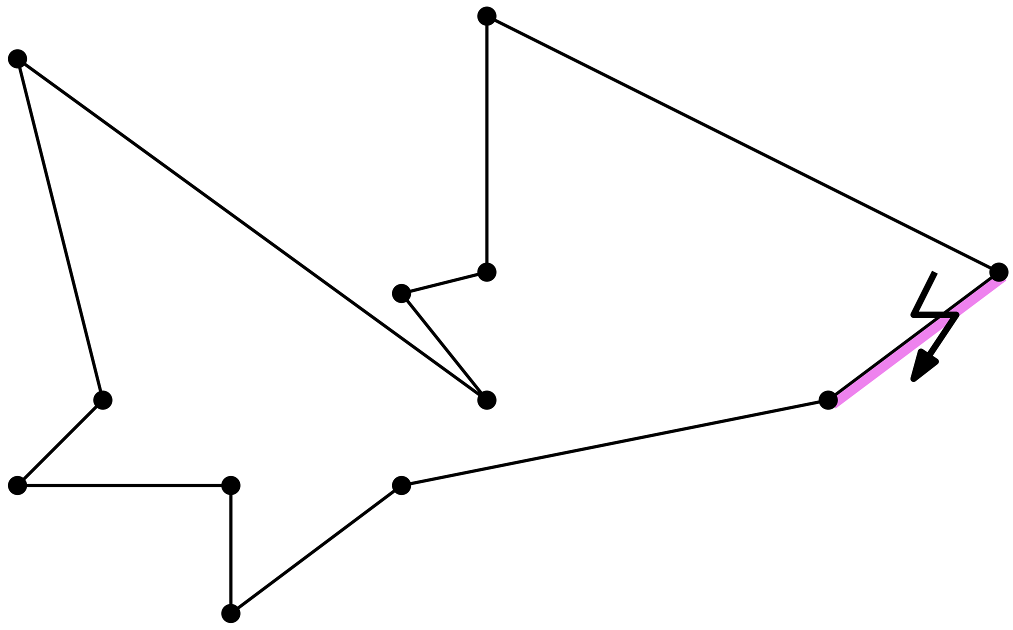


Problem

Given a simple polygon P (or a geometric graph)

find a geometric matching on the vertices of P, such that no edges cross in the augmentation.

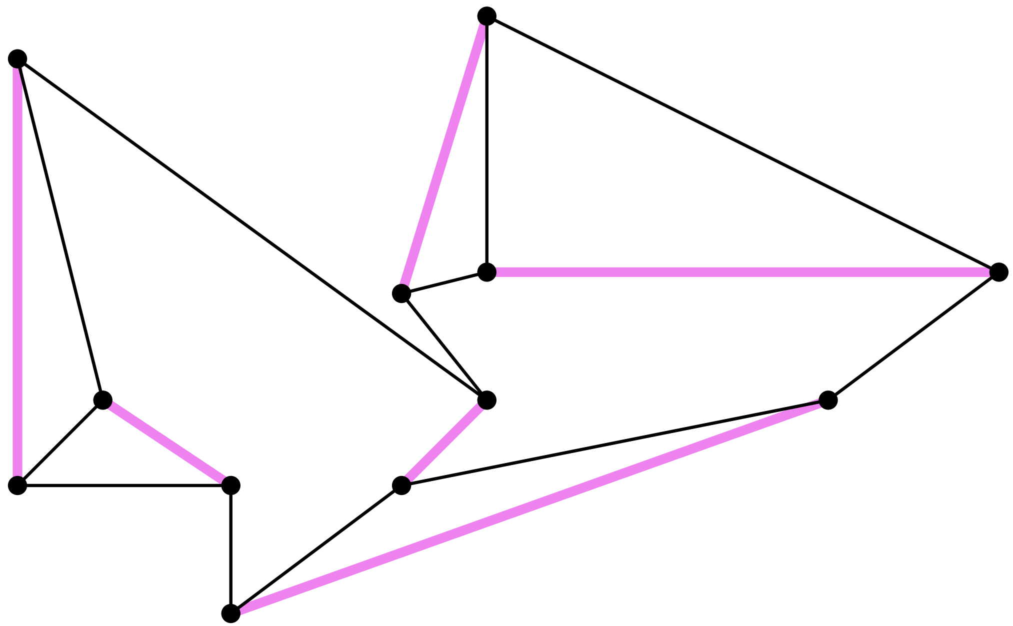
$\longrightarrow$ this is called a *compatible* matching



Problem

Given a simple polygon P (or a geometric graph)

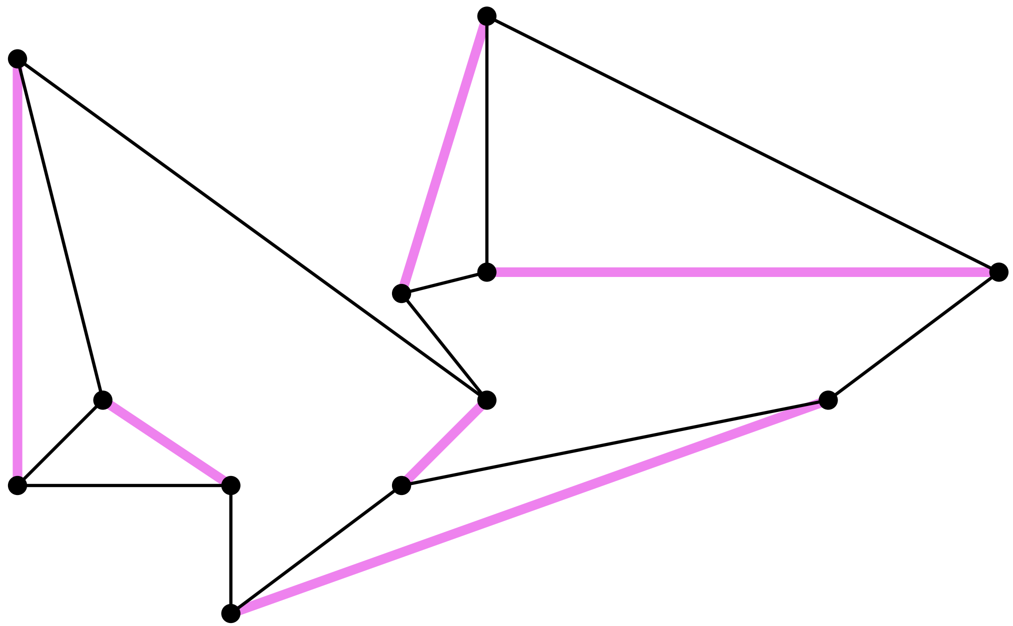
- Is there a compatible perfect matching on the vertices of P?



Problem

Given a simple polygon P (or a geometric graph)

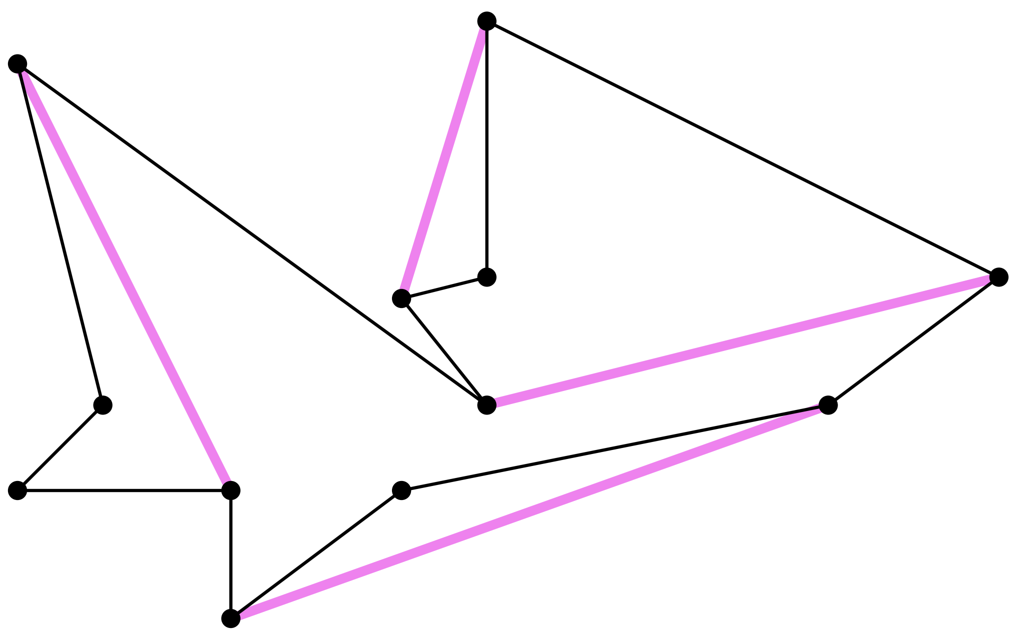
- Is there a compatible perfect matching on the vertices of P?
- What is the smallest size of a compatible maximal matching of the vertices of P?



Problem

Given a simple polygon P (or a geometric graph)

- Is there a compatible perfect matching on the vertices of P?
- What is the smallest size of a compatible maximal matching of the vertices of P?



Known results

- Every polygon with n vertices has a compatible matching of size $\geq \frac{n-3}{4}$ and there are polygons with compatible matchings of size $\leq \frac{n}{3}$.
  [Aichholzer, García, Hurtado, Tejel '11]
- Deciding whether a geometric matching admits a compatible perfect matching such that both matchings together are a cycle is NP-complete.
  [Akitaya, Korman, Rudoy, Souvaine, Tóth '19]
- Each geometric matching of even size admits a compatible perfect matching. [Ishaque, Souvaine, Tóth '13]

Perfect matchings in polygons

Theorem 1 *Given a simple polygon, it is NP-complete to decide whether it admits a compatible perfect matching.*

Perfect matchings in polygons

Theorem 1 *Given a simple polygon, it is NP-complete to decide whether it admits a compatible perfect matching.*

Reduction from POSITIVE PLANAR 1-IN-3SAT

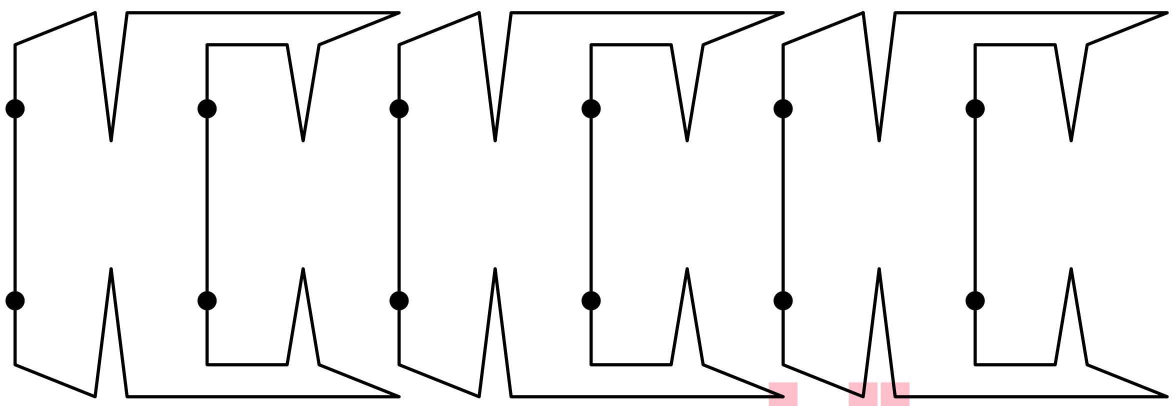
- planar variable-clause incidence graph
- only positive literals
- formula is satisfied if and only if there is exactly one true variable per clause

Perfect matchings in polygons

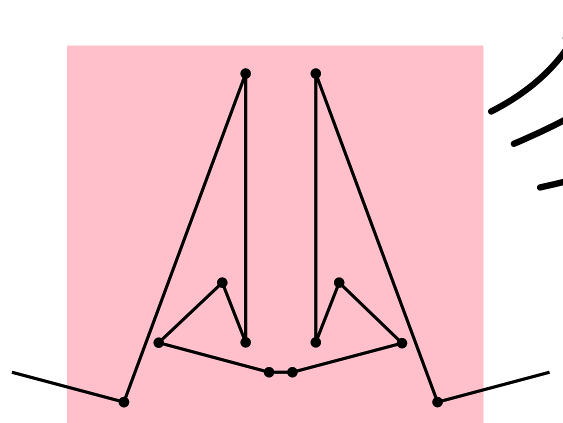
Theorem 1 *Given a simple polygon, it is NP-complete to decide whether it admits a compatible perfect matching.*

Reduction from POSITIVE PLANAR 1-IN-3SAT

- variable gadgets:



- corner gadget:

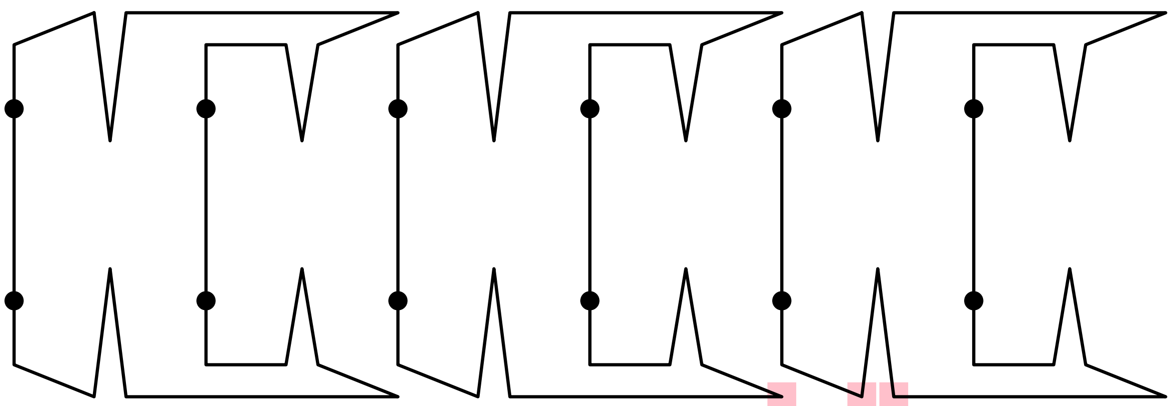


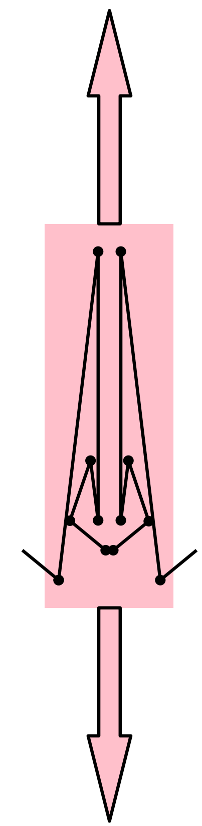
Perfect matchings in polygons

Theorem 1 *Given a simple polygon, it is NP-complete to decide whether it admits a compatible perfect matching.*

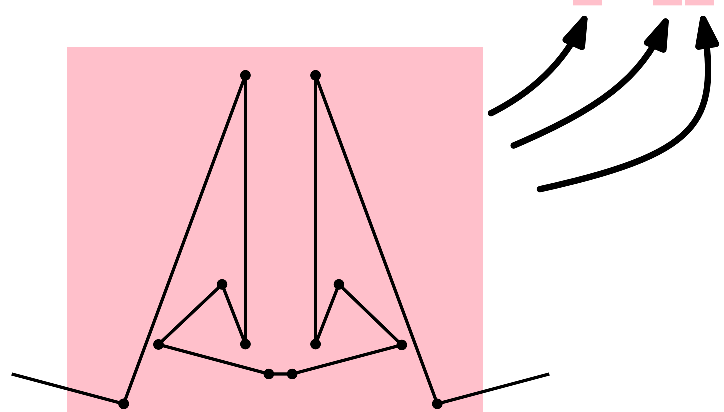
Reduction from POSITIVE PLANAR 1-IN-3SAT

- variable gadgets:





- corner gadget:

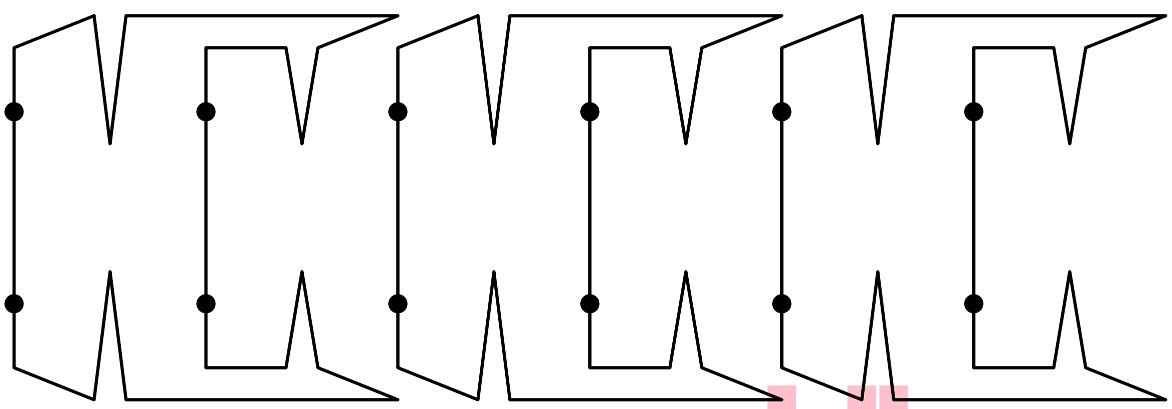


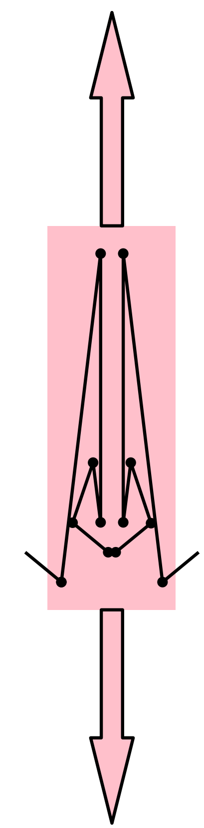
Perfect matchings in polygons

Theorem 1 *Given a simple polygon, it is NP-complete to decide whether it admits a compatible perfect matching.*

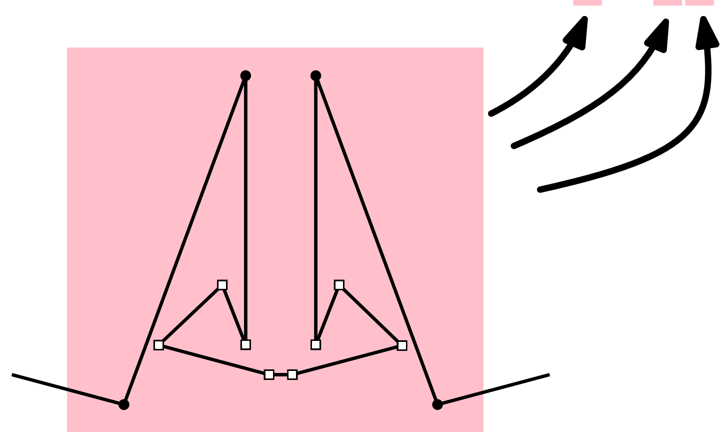
Reduction from POSITIVE PLANAR 1-IN-3SAT

- variable gadgets:





- corner gadget:

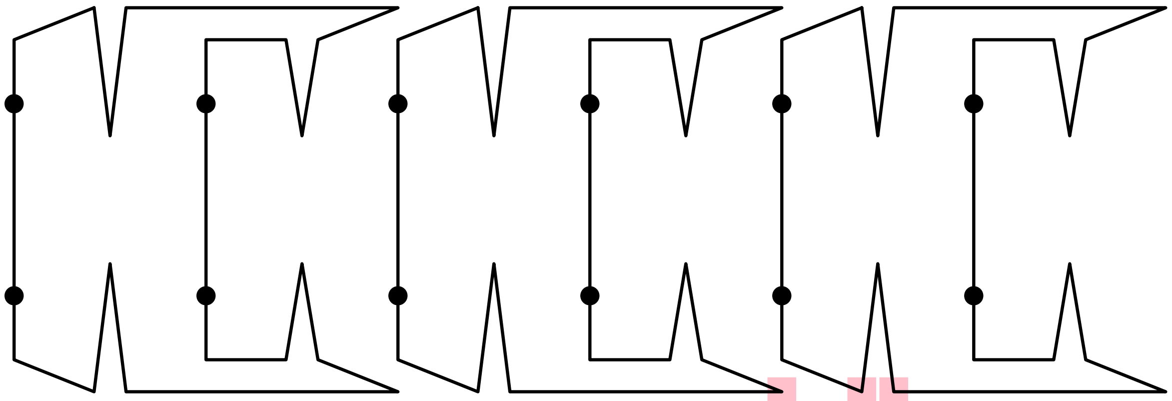


Perfect matchings in polygons

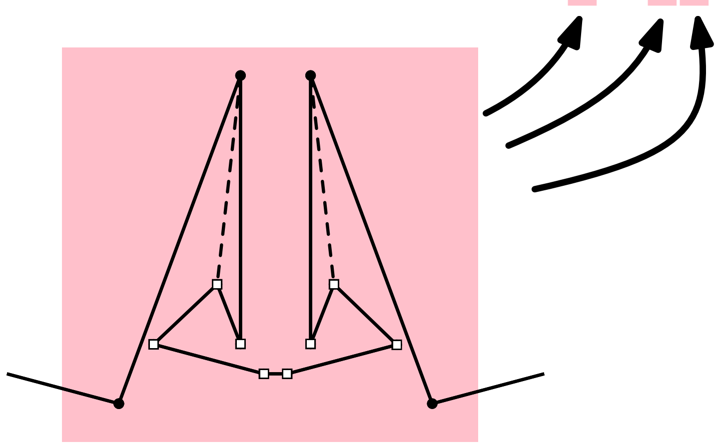
Theorem 1 *Given a simple polygon, it is NP-complete to decide whether it admits a compatible perfect matching.*

Reduction from POSITIVE PLANAR 1-IN-3SAT

- variable gadgets:



- corner gadget:

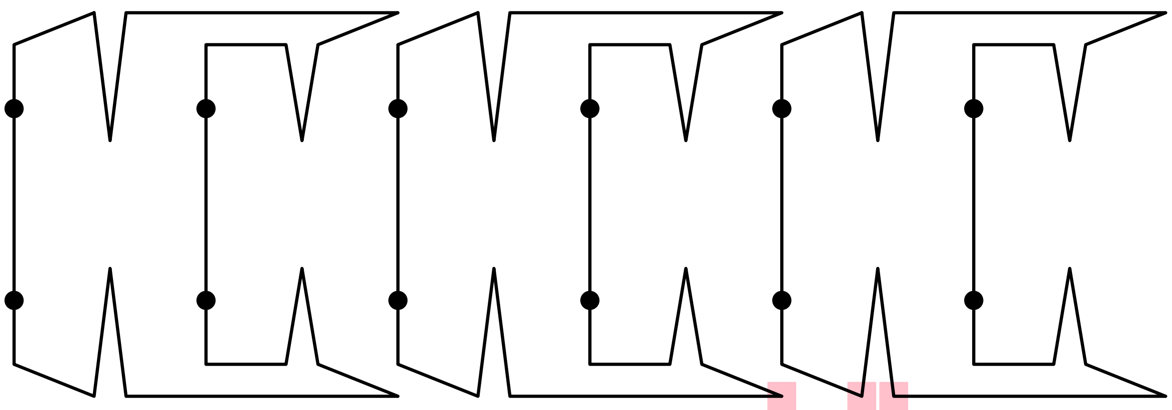


Perfect matchings in polygons

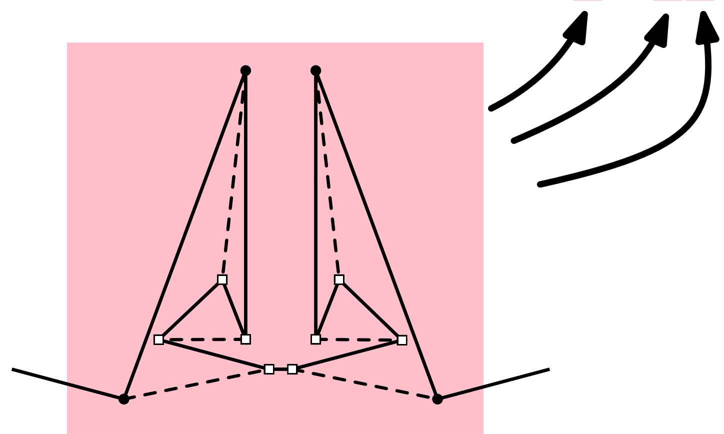
Theorem 1 *Given a simple polygon, it is NP-complete to decide whether it admits a compatible perfect matching.*

Reduction from POSITIVE PLANAR 1-IN-3SAT

- variable gadgets:



- corner gadget:

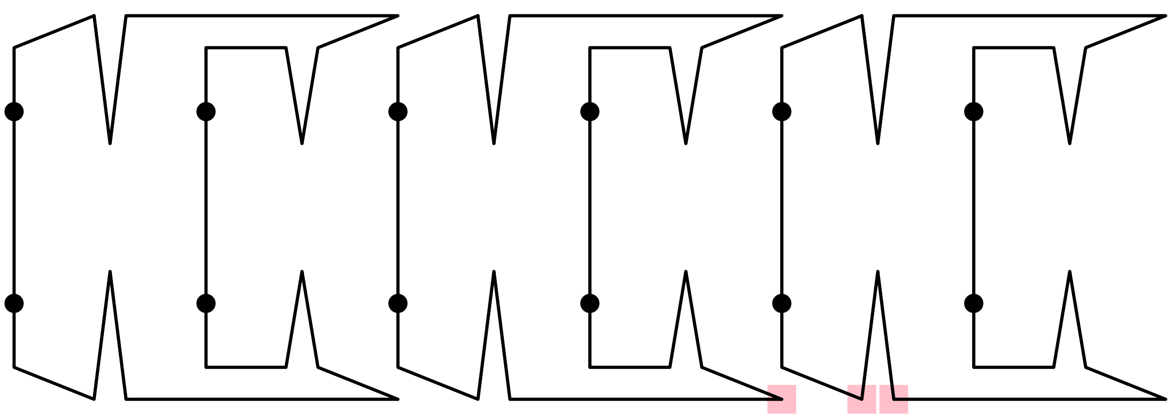


Perfect matchings in polygons

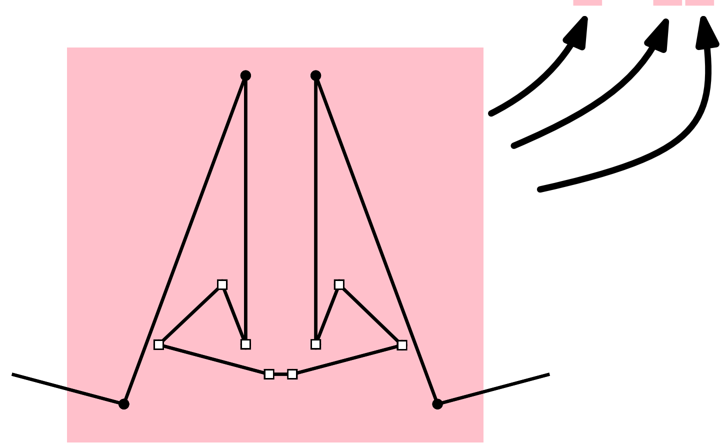
Theorem 1 *Given a simple polygon, it is NP-complete to decide whether it admits a compatible perfect matching.*

Reduction from POSITIVE PLANAR 1-IN-3SAT

- variable gadgets:



- corner gadget:

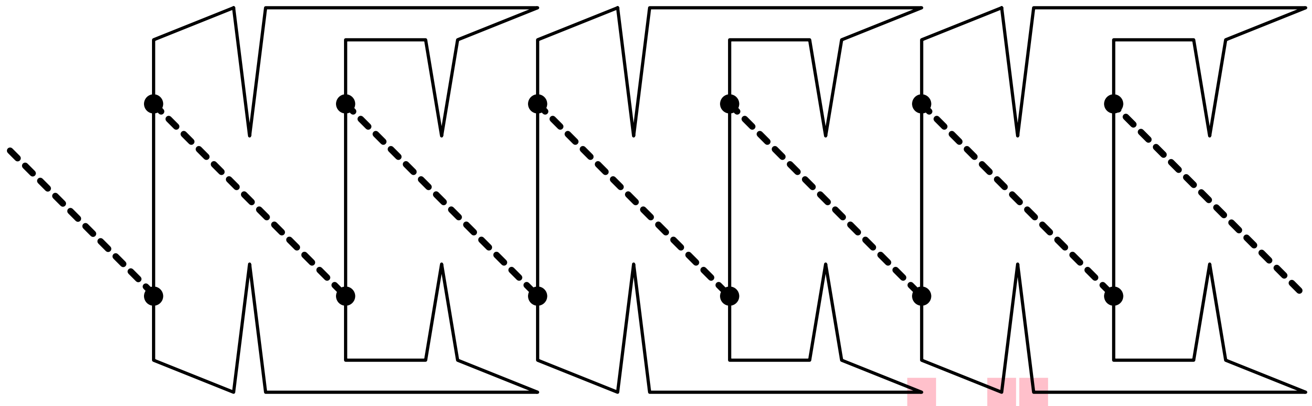


Perfect matchings in polygons

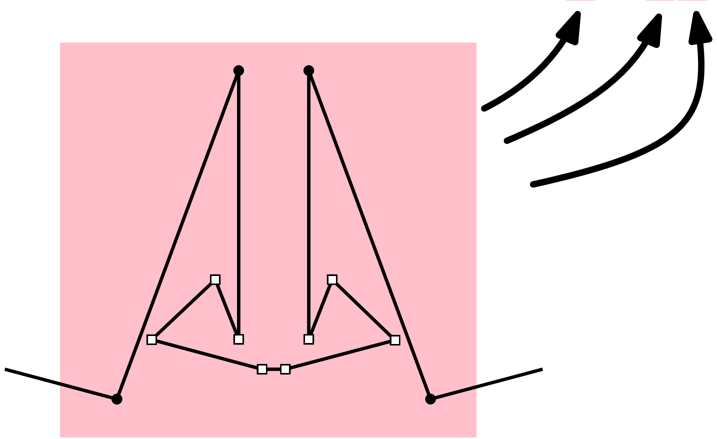
Theorem 1 *Given a simple polygon, it is NP-complete to decide whether it admits a compatible perfect matching.*

Reduction from POSITIVE PLANAR 1-IN-3SAT

- variable gadgets:



- corner gadget:

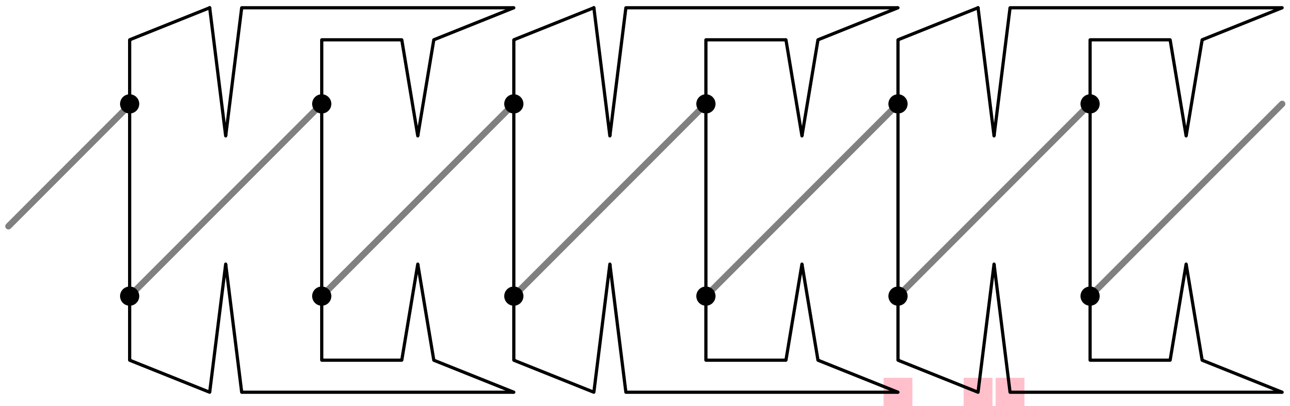


Perfect matchings in polygons

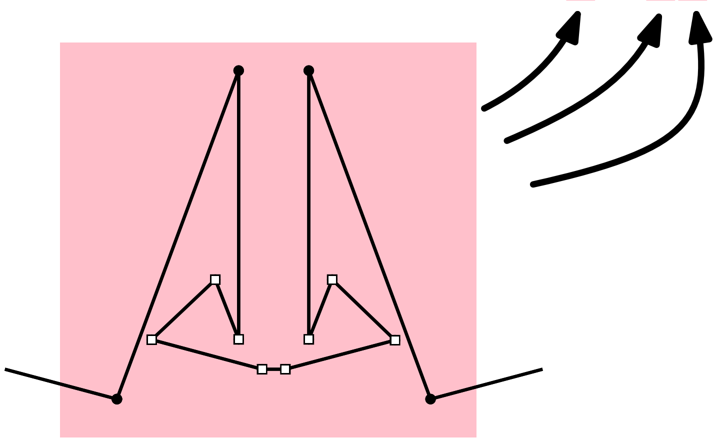
Theorem 2 *Given a simple polygon, it is NP-complete to decide whether it admits a compatible perfect matching.*

Reduction from POSITIVE PLANAR 1-IN-3SAT

- variable gadgets:



- corner gadget:

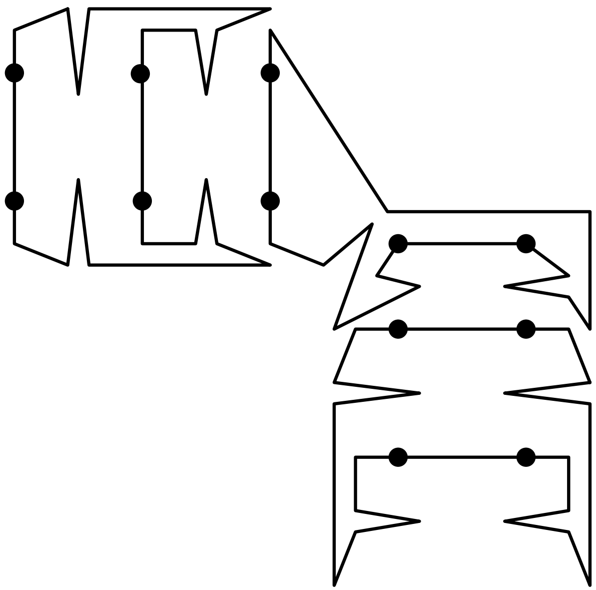


Perfect matchings in polygons

Theorem 2 *Given a simple polygon, it is NP-complete to decide whether it admits a compatible perfect matching.*

Reduction from POSITIVE PLANAR 1-IN-3SAT

- bend in a variable gadget:

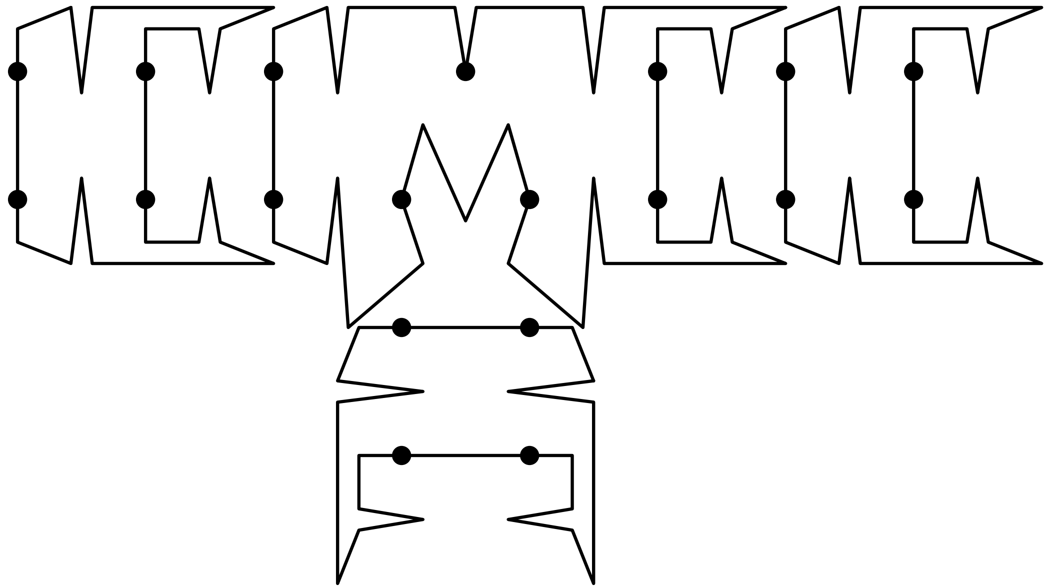


Perfect matchings in polygons

Theorem 2 *Given a simple polygon, it is NP-complete to decide whether it admits a compatible perfect matching.*

Reduction from POSITIVE PLANAR 1-IN-3SAT

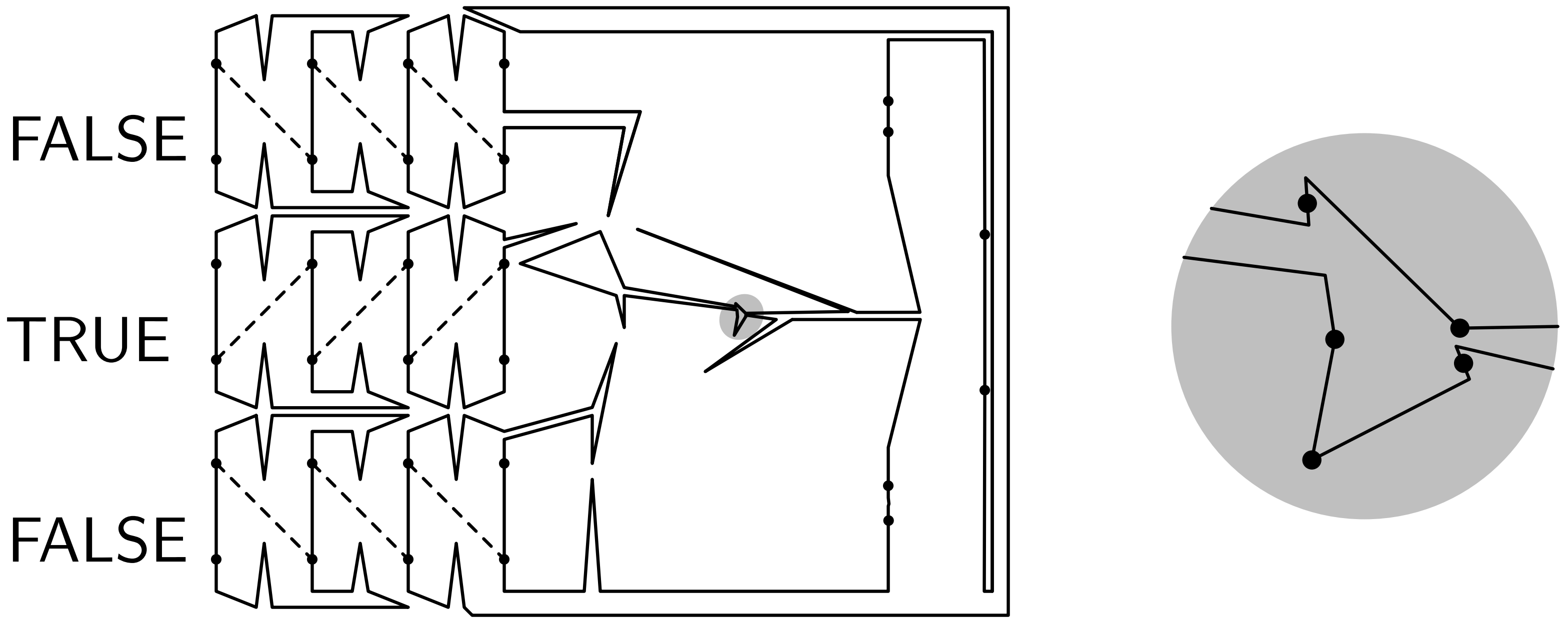
- a split gadget:



Perfect matchings in polygons

Theorem 2 *Given a simple polygon, it is NP-complete to decide whether it admits a compatible perfect matching.*

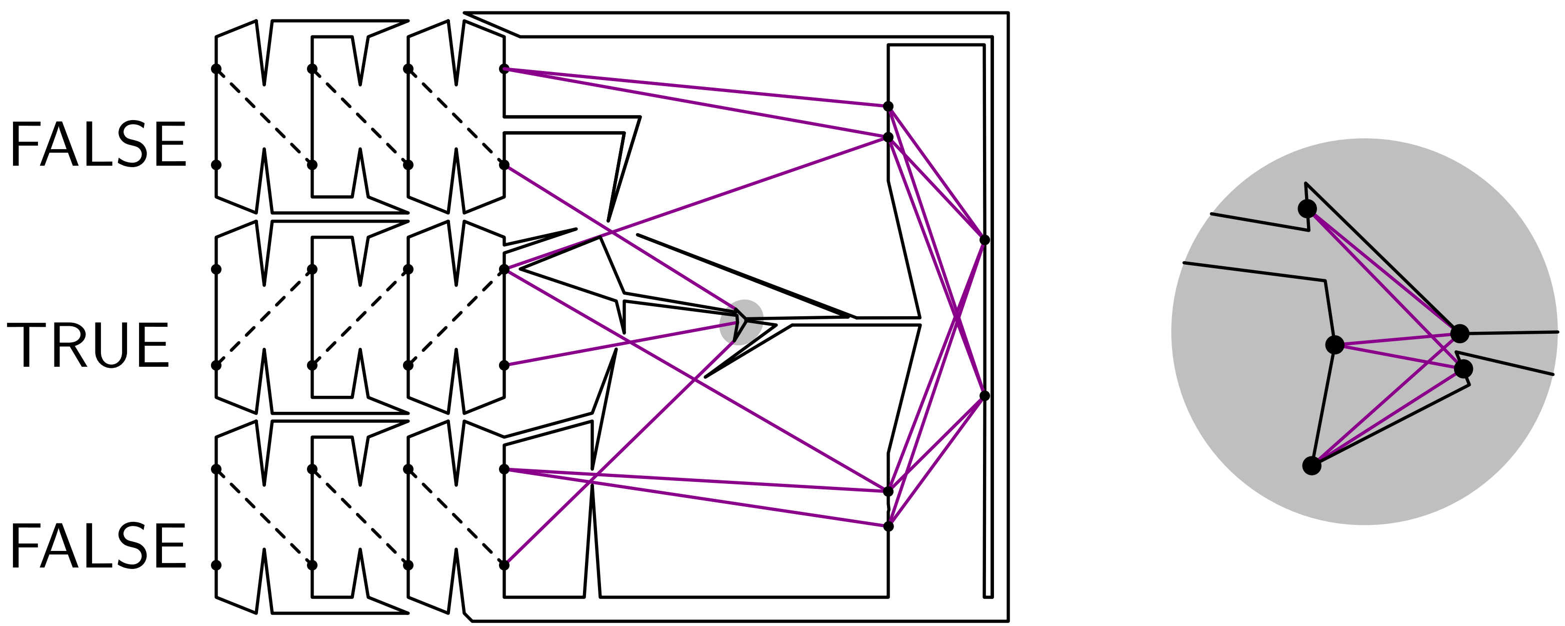
Reduction from POSITIVE PLANAR 1-IN-3SAT



Perfect matchings in polygons

Theorem 2 *Given a simple polygon, it is NP-complete to decide whether it admits a compatible perfect matching.*

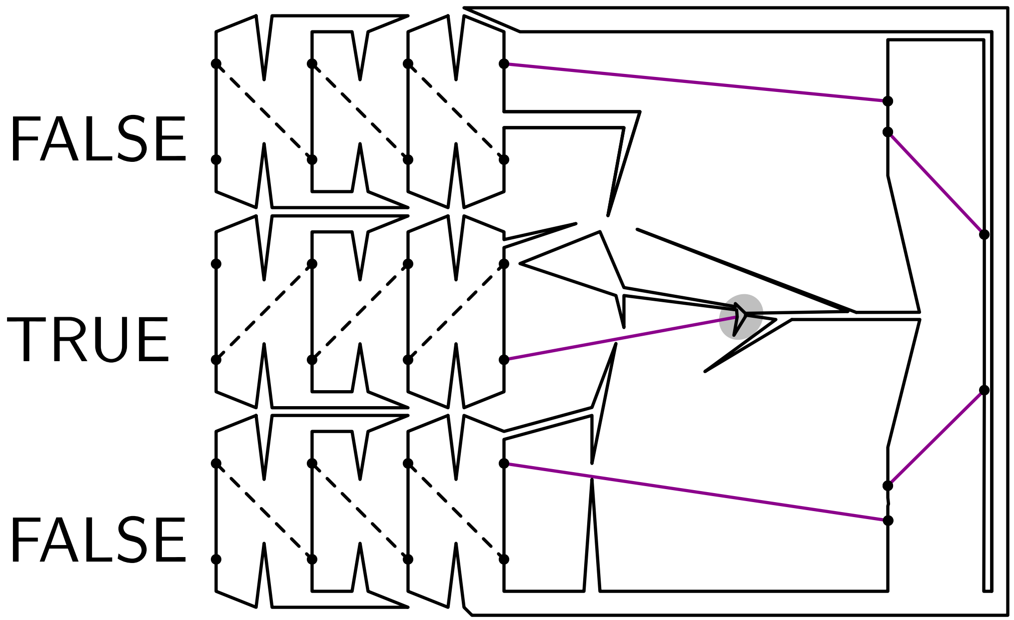
Reduction from POSITIVE PLANAR 1-IN-3SAT

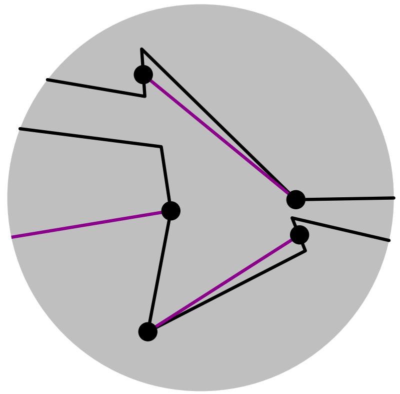


Perfect matchings in polygons

Theorem 2 *Given a simple polygon, it is NP-complete to decide whether it admits a compatible perfect matching.*

Reduction from POSITIVE PLANAR 1-IN-3SAT



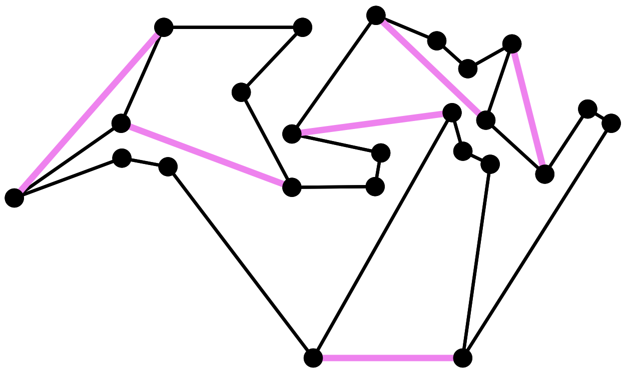


Maximal matchings in polygons

Theorem 3 *Each maximal compatible matching of a polygon with* n *vertices has size* $\geq \frac{n-4}{8}$.
There are polygons with maximal matchings of size $\leq \frac{n}{6}$.

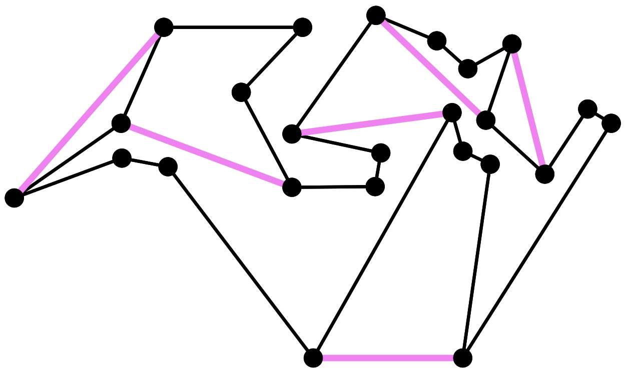
Maximal matchings in polygons

Theorem 3 *Each maximal compatible matching of a polygon with n vertices has size $\geq \frac{n-4}{8}$.*
There are polygons with maximal matchings of size $\leq \frac{n}{6}$.



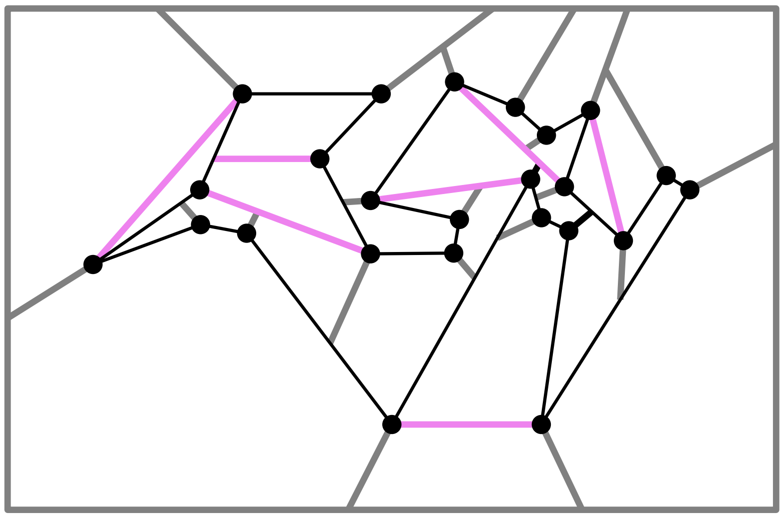
Maximal matchings in polygons

Theorem 3 *Each maximal compatible matching of a polygon with* n *vertices has size* $\geq \frac{n-4}{8}$*.*
There are polygons with maximal matchings of size $\leq \frac{n}{6}$*.*



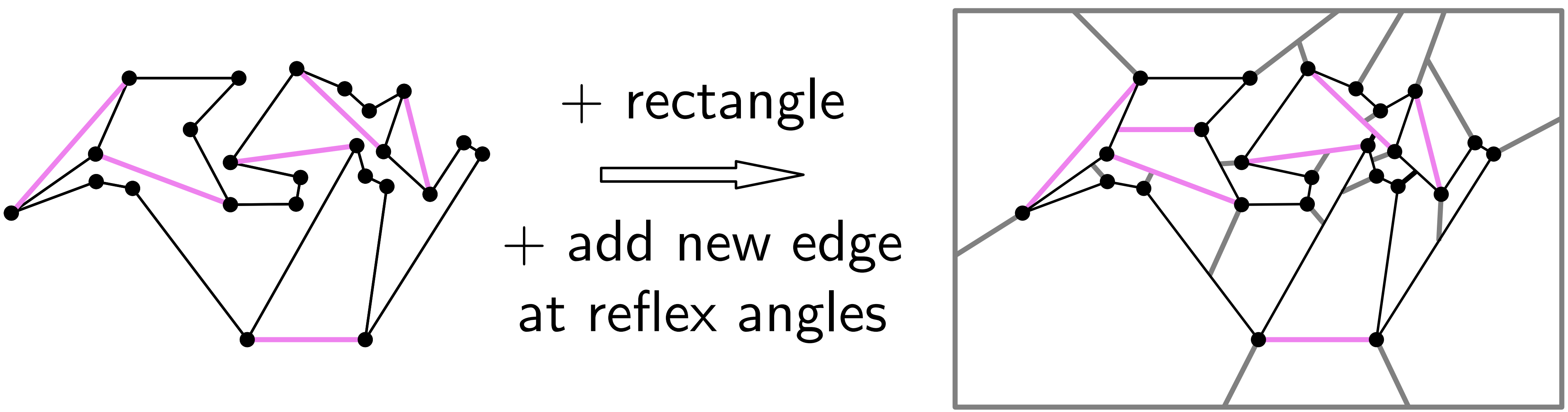
+ rectangle

+ add new edge at reflex angles



Maximal matchings in polygons

Theorem 3 *Each maximal compatible matching of a polygon with n vertices has size $\geq \frac{n-4}{8}$.*
There are polygons with maximal matchings of size $\leq \frac{n}{6}$.



- all faces are convex
- ≤ 2 unmatched vertices per face
- at most $2 + |E(M)| + n$ faces
- unmatched vertices incident to exactly 3 faces

Maximal matchings in polygons

Theorem 3 *Each maximal compatible matching of a polygon with n vertices has size $\geq \frac{n-4}{8}$.*
There are polygons with maximal matchings of size $\leq \frac{n}{6}$.

- all faces are convex
- ≤ 2 unmatched vertices per face
- at most $2 + |E(M)| + n$ faces
- unmatched vertices incident to exactly 3 faces

$$\Rightarrow \quad 3\,(n - 2\,|E(M)|) \leq 2\,(2 + |E(M)| + n)$$

Maximal matchings in polygons

Theorem 3 *Each maximal compatible matching of a polygon with n vertices has size $\geq \frac{n-4}{8}$.*
There are polygons with maximal matchings of size $\leq \frac{n}{6}$.

- all faces are convex
- ≤ 2 unmatched vertices per face
- at most $2 + |E(M)| + n$ faces
- unmatched vertices incident to exactly 3 faces

$$\Rightarrow \quad 3\,(n - 2\,|E(M)|) \leq 2\,(2 + |E(M)| + n)$$

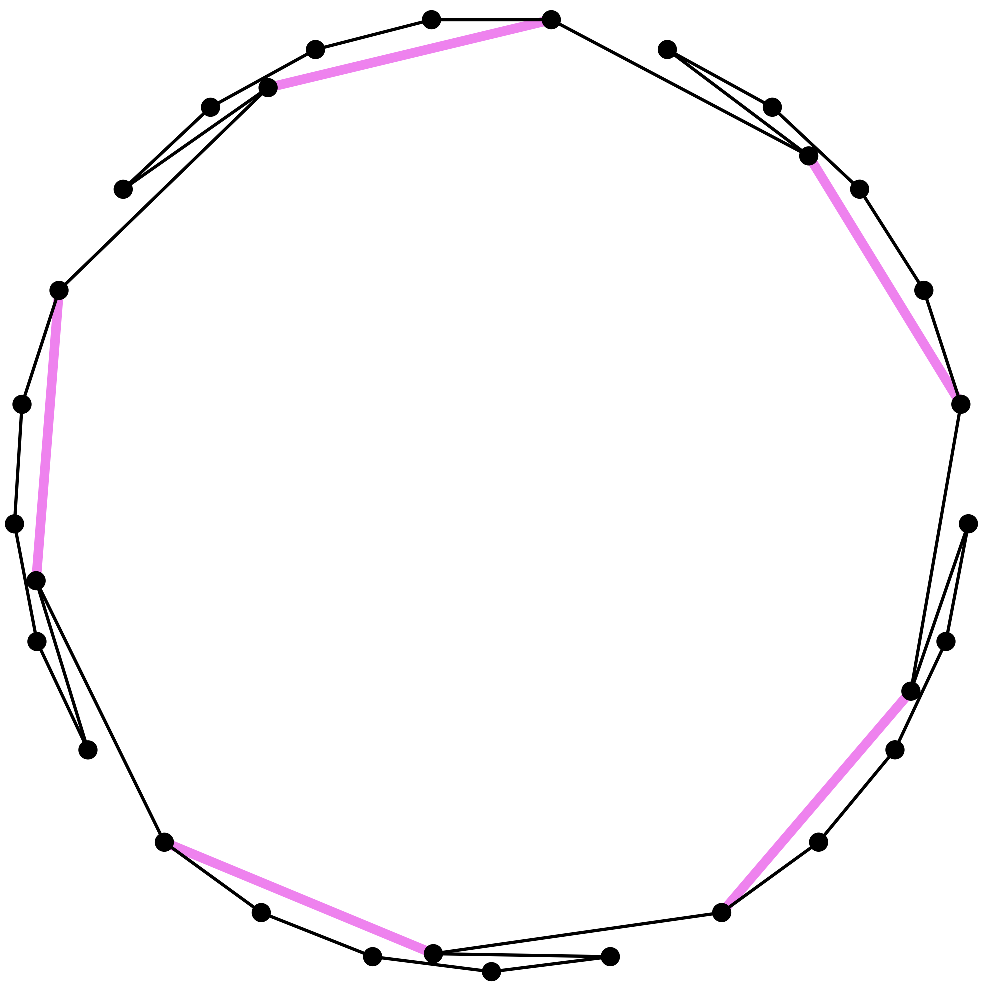
$$\frac{n-4}{8} \leq |E(M)|$$

Maximal matchings in polygons

Theorem 3 *Each maximal compatible matching of a polygon with n vertices has size $\geq \frac{n}{7}$.*
There are polygons with maximal matchings of size $\leq \frac{n}{6}$.

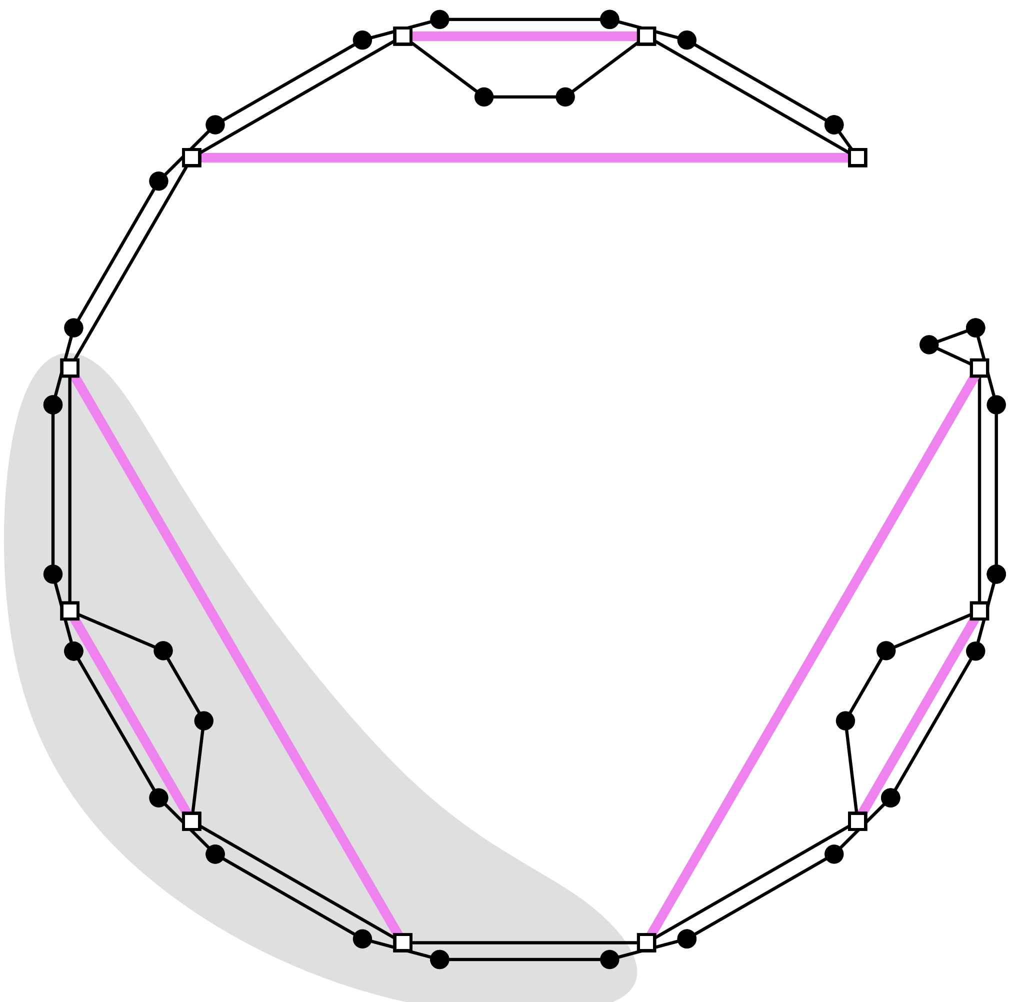
Maximal matchings in polygons

Theorem 3 *Each maximal compatible matching of a polygon with n vertices has size $\geq \frac{n}{7}$.*
There are polygons with maximal matchings of size $\leq \frac{n}{6}$.



Maximal matchings in polygons

Theorem 3 *Each maximal compatible matching of a polygon with n vertices has size $\geq \frac{n}{7}$.*
There are polygons with maximal matchings of size $\leq \frac{n}{7}$.



One more result

Given a geometric graph G, find a set of compatible edges such that the augmented graph has minimum degree 5.

One more result

Given a geometric graph G, find a set of compatible edges such that the augmented graph has minimum degree 5.

Theorem 4 *Given a geometric graph G, it is NP-complete to decide whether there is a set of compatible edges E such that $G + E$ has minimum degree 5.*