The Very Best of
Perfect Non-crossing Matchings

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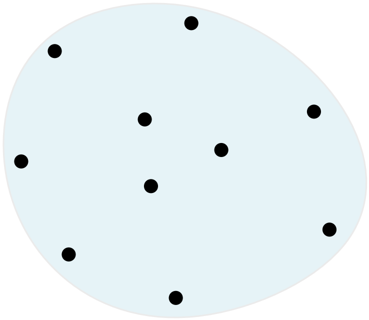
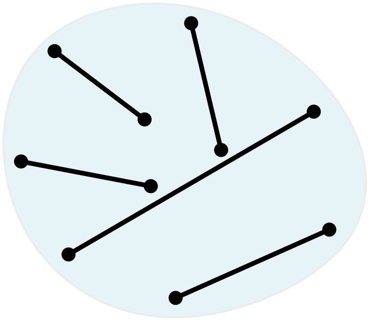
Perfect Non-crossing Matchings

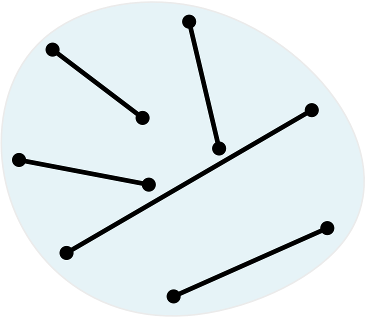


Perfect Non-crossing Matchings



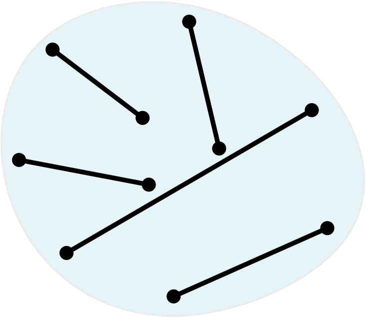
Perfect Non-crossing Matchings

Monochromatic

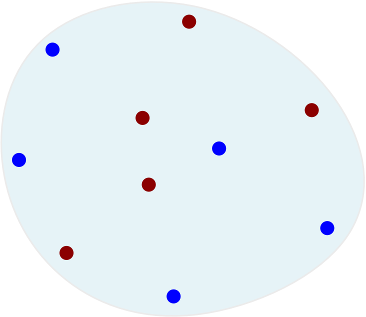


Perfect Non-crossing Matchings

Monochromatic

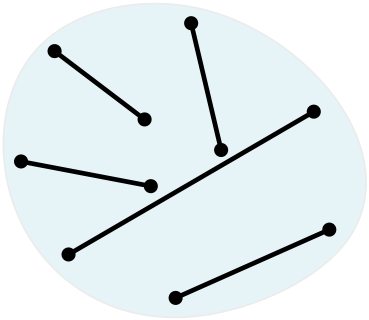


Bichromatic

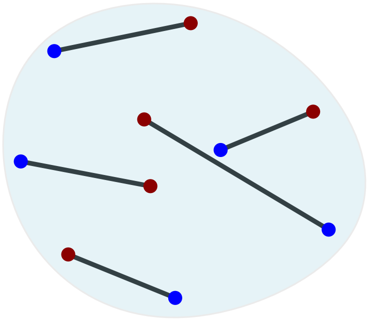


Perfect Non-crossing Matchings

Monochromatic

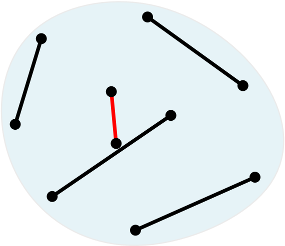


Bichromatic

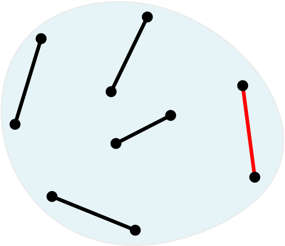


Optimization of Perfect Non-crossing matchings

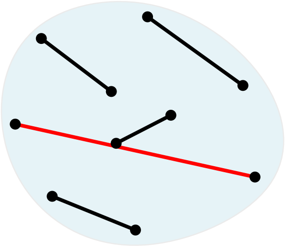
MinMin



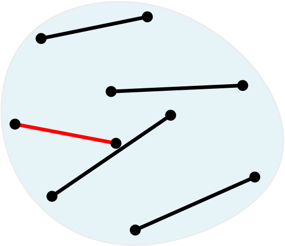
MinMax



MaxMax



MaxMin



Known Results

Monochromatic	**MinMax**
General Position	$\mathcal{NP}$-hard [Abu-Affash et al. 2014]
Convex Position	$O(n^2)$ [Savić, Stojaković 2017]
Points on circle	$O(n)$ [Savić, Stojaković 2017]
Bichromatic	**MinMax**
General Position	$\mathcal{NP}$-hard [Carlsson et al. 2015]
Convex Position	$O(n^2)$ [Savić, Stojaković 2018]
Points on circle	$O(n)$ [Savić, Stojaković 2018]

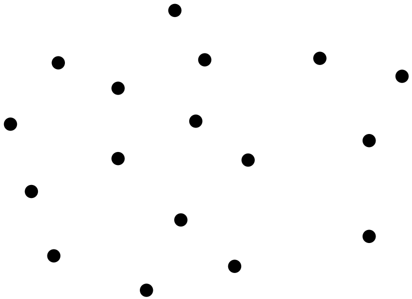
- Only MinMax has been considered before.

Known Results

Monochromatic	**MinMin**	**MaxMax**	**MinMax**	**MaxMin**
General Position			$\mathcal{NP}$-hard	
Convex Position			$O(n^2)$	
Points on circle			$O(n)$	
Bichromatic	**MinMin**	**MaxMax**	**MinMax**	**MaxMin**
General Position			$\mathcal{NP}$-hard	
Convex Position			$O(n^2)$	
Points on circle			$O(n)$	

- Only MinMax has been considered before.
- We consider the remaining cases.

Points in General position



Feasibility criterion

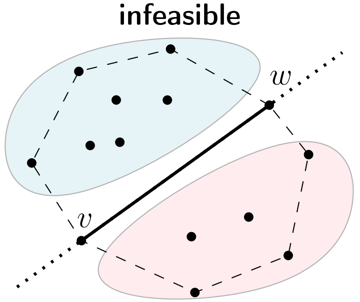
Definition

(v, w) is *feasible* if it is contained in a matching.

Lemma

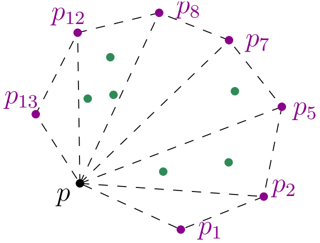
(v, w) *is infeasible iff*

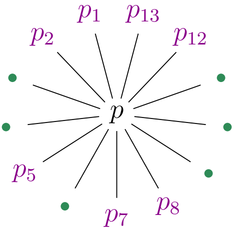
- v, w *on the convex hull* CH,
- *an odd number of points on each side of the connecting line.*



Weak Radial Orderings

- *radial ordering* of p = sorted order of $P \setminus p$ by angle around p
- *weak radial ordering* of p = radial ordering of p with points of $P \setminus \mathrm{CH}$ being indistinguishable





Computation

- Standard (using dual line arrangement): radial orderings of all points in $O(n^2)$
- Modified algorithm: weak radial orderings of all points in CH in $O(nh)$ with $h = |\mathrm{CH}|$

Monochromatic MinMin/MaxMax

Algorithm for MinMin:

- Compute convex hull in $O(n \log h)$
- Compute weak radial orderings in $O(nh)$
- Find shortest edge with at least one point in $P \setminus \mathrm{CH}$ in $O(n \log n)$ (using a Voronoi diagram)
- Find shortest feasible edge with both points in CH in $O(nh)$ using the weak radial orderings

- Total running time is $O(nh + n \log n)$
- Analogue algorithm also for MaxMax

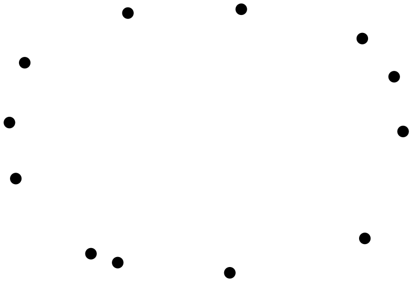
Summary of results

Monochromatic	**MinMin**	**MaxMax**	**MinMax**	**MaxMin**
General Position	$O(nh + n \log n)$	$O(nh + n \log n)$	$\mathcal{NP}$-hard	?
Convex Position			$O(n^2)$	
Points on circle			$O(n)$	
Bichromatic	**MinMin**	**MaxMax**	**MinMax**	**MaxMin**
General Position	?	?	$\mathcal{NP}$-hard	?
Convex Position			$O(n^2)$	
Points on circle			$O(n)$	

Open problems:

- Can feasiblility be decided in polynomial time in the bichromatic case?
- Is MaxMin $\mathcal{NP}$-hard?

Points in Convex position



Points in Convex position

A classical dynamic programming approach works for all optimization criteria:

- A subproblem is a solution to a problem defined by a sequence of consecutive points.
- For each subproblem we test all possible splits into two of its subproblems.
- $O(n^3)$ time and $O(n^2)$ memory in total.

Points in Convex position

Monochromatic	**MinMin**	**MaxMax**	**MinMax**	**MaxMin**
General Position	$O(nh + n \log n)$	$O(nh + n \log n)$	$\mathcal{NP}$-hard	?
Convex Position	$O(n^3)$	$O(n^3)$	$O(n^2)$	$O(n^3)$
Points on circle	$O(n^3)$	$O(n^3)$	$O(n)$	$O(n^3)$
Bichromatic	**MinMin**	**MaxMax**	**MinMax**	**MaxMin**
General Position	?	?	$\mathcal{NP}$-hard	?
Convex Position	$O(n^3)$	$O(n^3)$	$O(n^2)$	$O(n^3)$
Points on circle	$O(n^3)$	$O(n^3)$	$O(n)$	$O(n^3)$

- We can do better for some cases.

MinMin / MaxMax

Monochromatic

- A pair (i, j) is feasible iff $j - i$ is odd.
- P_{even} - even points, P_{odd} - odd points
- Find a minimum / maximum distance pair (i, j)
  $i \in P_{even}, j \in P_{odd}$
  $O(|P_{even}| + |P_{odd}|) = O(n)$ time [Toussaint / Edelsbrunner]

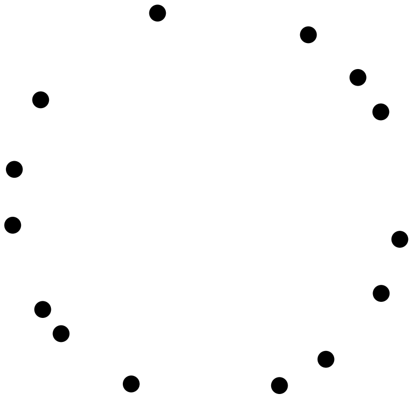
Bichromatic

- Partition P into *orbits* [Savić, Stojaković 2018]
  - A pair (i, j) is feasible iff j and j belong to the same orbit.
  - Colors inside an orbit alternate.
  - The partition into orbits can be computed in $O(n)$.
- Apply the monochromatic approach to each orbit.
- $O(n)$ time total.

Summary of results

Monochromatic	**MinMin**	**MaxMax**	**MinMax**	**MaxMin**
General Position	$O(nh + n \log n)$	$O(nh + n \log n)$	$\mathcal{NP}$-hard	?
Convex Position	$O(n)$	$O(n)$	$O(n^2)$	$O(n^3)$
Points on circle	$O(n)$	$O(n)$	$O(n)$	$O(n^3)$
Bichromatic	**MinMin**	**MaxMax**	**MinMax**	**MaxMin**
General Position	?	?	$\mathcal{NP}$-hard	?
Convex Position	$O(n)$	$O(n)$	$O(n^2)$	$O(n^3)$
Points on circle	$O(n)$	$O(n)$	$O(n)$	$O(n^3)$

Points on a Circle



Points on Circle

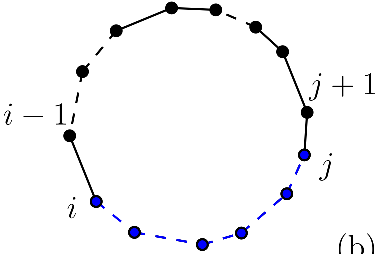
- Any shortest edge of a matching is a boundary edge.

Monochromatic MaxMin

Idea: Try forbidding all boundary edges shorter than μ, and check if there is still a matching.

Lemma

There exist a matching without the forbidden edges iff the length of the longest forbidden chain is less than n.



(b)

Points on Circle - Monochromatic MaxMin

- Problem: Find maximal μ such that there is a matching.
- Idea 1: Binary search ($O(n \log n)$ time)
- Idea 2:
  - Using the lemma:

$$\mu_{max} = \min_{i \in \{0,\dots,2n-1\}} \max_{j \in \{i,\dots,i+n-1\}} |p_j p_{j+1}|$$

  - Use RMQ data structure or sliding window minimum algorithm ($O(n)$ time).
  - It is also possible to construct the optimal solution in $O(n)$ time.

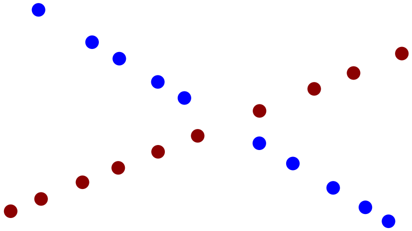
Summary of results

Monochromatic	MinMin	MaxMax	MinMax	MaxMin
General Position	$O(nh + n \log n)$	$O(nh + n \log n)$	$\mathcal{NP}$-hard	?
Convex Position	$O(n)$	$O(n)$	$O(n^2)$	$O(n^3)$
Points on circle	$O(n)$	$O(n)$	$O(n)$	$O(n)$
Bichromatic	**MinMin**	**MaxMax**	**MinMax**	**MaxMin**
General Position	?	?	$\mathcal{NP}$-hard	?
Convex Position	$O(n)$	$O(n)$	$O(n^2)$	$O(n^3)$
Points on circle	$O(n)$	$O(n)$	$O(n)$	$O(n^3)$

Open problem:

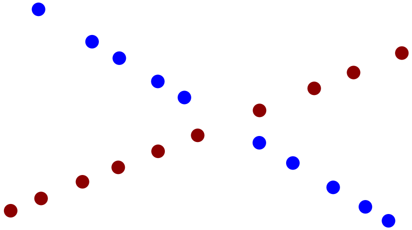
- Can the bichromatic MaxMin problem for circle be solved in $o(n^3)$, possibly employing the theory of orbits?

Doubly collinear points



Doubly collinear points

Bichromatic points with all red points on a line and all blue points on a line



Summary of results

Monochromatic	**MinMin**	**MaxMax**	**MinMax**	**MaxMin**
General Position	$O(nh + n\log n)$	$O(nh + n\log n)$	$\mathcal{NP}$-hard	?
Convex Position	$O(n)$	$O(n)$	$O(n^2)$	$O(n^3)$
Points on circle	$O(n)$	$O(n)$	$O(n)$	$O(n)$
Bichromatic	**MinMin**	**MaxMax**	**MinMax**	**MaxMin**
General Position	?	?	$\mathcal{NP}$-hard	?
Convex Position	$O(n)$	$O(n)$	$O(n^2)$	$O(n^3)$
Points on circle	$O(n)$	$O(n)$	$O(n)$	$O(n^3)$
Doubly collinear	$O(n)$	$O(1)$	$O(n^4 \log n)$	?

Open problem:

- Is the MaxMin doubly collinear problem $\mathcal{NP}$-hard?

Thank you for your attention!