Quality Measures and Ratios: Assignments

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The lecture on measures and ratios of measures encourages the attention for geometric measures: their design and properties. Well known measures are distance measures for shapes, like Hausdorff distance, Fréchet distance (for curves), and area of overlap (for polygons). Other measures are dilation (detour factor), density, Earth mover's distance, homotopy height, root-mean-square error, ...

In graph drawing, measures are used to capture the quality of a drawing of a graph. All graphs have many different drawings, and some are better than others.

1 Designing measures

We design a new measure if there is no existing measure that captures what we want to represent. Using many small examples, we can test if the measure does what we want: whether it discriminates on the things we care about.

Assignment 1: lake adjacency measure

Design a measure that captures, for a path C, that it often runs parallel to the boundary of big lakes. Assume we have a set of polygons (lakes) given.

Check your measure for being simple, discriminating, scale-invariant, and robust. Note that in many geographic situations, scale-invariance is not an asset; it is just something we can examine about a measure.

To test for being discriminating, draw a reasonable situation and consider the score for a curve. Imagine changing the situation slightly (slight move of the curve, slight enlargement of a lake, local change that makes the curve slightly more parallel to a nearly lake), and see if the measure responds suitably.

Assignment 2: forest density variation measure

Design a measure that captures, for a polygon representing a forest, that its density of trees is varied. Assume we have a set of points (trees) inside a polygon (forest) given as the input.

Check your measure for being simple, discriminating, scale-invariant, and robust. Again, draw examples and examine your measure.

Assignment 3: t-norms and t-conorms

We want to examine possible paths through a nature area for being "good". To this end, imagine a score for being reasonable in its shape (avoiding double-backing, getting far away compared to length of path, no self-intersections, ...), which is a score in $[0, 1]$. Imagine a different score for scenic beauty (different landscapes, sweeping views, ...). For the quality of a path that takes both into account, would you use a t-norm or a t-conorm? Why?

Assuming both separate scores are "reasonable", which t-norm/t-conorm would you choose?

2 Quality ratios

One of the more interesting quality ratio topics is considering the measure angular resolution, and examining how bends can improve this measure. So we will compare free straight-line planar drawings with free 1-bend (or more bend) straight-line planar drawings.

Assignment 4: 1-bend drawings

Suppose we allow one bend in each edge, and we wish to analyze how much better the angular resolution may become when allowing a bend. We choose to let the bend itself also contribute to the angular resolution, so you don't want to make a very sharp turn in an edge.

For specific graphs like $C_3 = K_3$, C_4, and K_4, what are the quality ratio lower bounds we get for these examples?

What are the quality ratios obtained for the Platonic solids?

Can you find any planar graph giving a better lower bound on the quality ratio of 1-bend drawings, versus, free straight line drawings?

Assignment 5: 2-bend drawings

Suppose we allow an extra bend. How much can that help? Answer the questions of the previous assignment by comparing 2-bend drawings with 1-bend drawings, and 2-bend drawings with no-bend drawings.

Assignment 6: many-bend drawings

Suppose you are allowed arbitrarily many bends. What is the quality ratio for many-bend drawings when compared to no-bend drawings?

At what point doesn't it help anymore to get more bends? Is there a value (function of n, the number of vertices in the graph) so that we can always restrict ourselves to at most that many bends? You may wish to distinguish the cases of graphs with no degree-3 or more nodes, and graphs that have at least one node of degree at least 3.

Assignment *

In case your are done and feel like doing more, try other quality measures for ratios of bend drawings.

Another option is to consider any two drawing styles and see what you can get for bounded-face area/diameter ratio as the quality measure.

3 Solutions?

Making these assignments is ideally done in small groups. The assignments were composed to train with the topics. Understanding an assignment and knowing how to think to make them, or solve them, is already enough. It is for now not important to get the best possible solutions.

In many cases I don't know what the best solution is. Below are some thoughts:

Assignment 1

It is somewhat implicit in the statement that the path should be parallel to the nearest big lake, and that lake should be near. Let us assume this.

One option is to use a threshold for lakes being big. We can also use a threshold for being close to a lake. Thresholds are not good for robustness, though.

It seems reasonable to say that a point on the path and the nearest point on the lake boundary are "parallel" if their tangents are roughly the same. We can use normalized tangent vectors and the dot product to score parallelism.

The measure then becomes an integral over the path length of the dot product of the normalized tangent of a point on the path with the normalized tangent of the nearest point on the boundary of a big lake, provided that big lake is near enough.

We must take special care at vertices of the lake (why not for the path?). Here the tangent is not well-defined; a choice can be some way of taking the tangent of one of the two edges incident to that vertex.

To take lake size and distance to nearest lake into account, we can multiply the outcome of the dot product with the lake size and the inverse of the distance to the nearest lake. This makes the measure more robust, but not fully: the can be an edge of the path that is equidistant to two vertices of lakes of very different size. This edge is then on the bisector between these vertices. A fractional move the one way or the other way can cause a drastic change in score.

Assignment 2

Density is local and not scale-invariant. The density at a point is the number of trees in its neighborhood, divided by the area of that neighborhood. We choose a circular neighborhood.

If we use a small neighborhood of, say, 1 meter, nearly all points will get a tree count of 0 or 1. The best "diversity" would then be an equal area of points with 0 trees and 1 tree. The density variation measure would be how close the situation is to an equal split in area.

If the neighborhood is larger, we can have a range of tree counts. The largest density variation may be equal areas of points with all the counts. We can see this as a histogram with number of trees horizontally and area vertically.

We can use a transportation distance to measure how much energy is needed to convert a given situation to the highest density variation. Distances between densities are often done with the Wasserstein distance, which is a continuous form of the Earth Movers Distance.

Assignment 3

I would use a t-norm. If the path is very non-scenic, then no shape reasonableness should make it a good path. With a t-conorm, a path that has a great shape will have a high score.

Assignment 4

For C_3, a no-bend drawing will always have an angle of ≤ 60, while a 1-bend drawing exists with all angles of 120. So the quality ratio is at least 2.

For C_4, these angles are 90 and 135, so the quality ratio for this graph is 135/90. This is a weaker lower bound for the quality ratio. Longer cycles don't help either.

For K_4, we get a better lower bound: 80/30.

Assignment 5

Assignment 6

The answer is infinity. If we take the graph class of Garg and Tamassia (see the slides), then we know that any planar drawing will have an angle that becomes smaller with $\sqrt{(\log d)/d^3}$.

On the other hand, if we take any straight-line drawing, we can make minor local adaptations very close to the vertices and get an angular resolution of $360/d$, in case $d \geq 3$. We need only constantly many bends for this.

Hence, for this graph class, the ratio goes to ∞. Have more than a constant number of bends does not help anymore, we can already achieve the best possible angular resolution.

In case the graph has no degree-3 vertices (just paths and cycles), this is not quite true. Every extra bend improves the angular resolution (for loops) a tiny bit. The extra benefit becomes smaller and smaller, the more bends we use. For degree at most 2 graphs, the triangle is the critical example, giving 60 degrees without bends, and something approaching 180 degrees for many bends. So the quality ratio tends to 3.

See "Planar Polyline Drawings with Good Angular Resolution" by Gutwenger and Mutzel, Graph Drawing 1998. See also "Universal Slope Sets for 1-Bend Planar Drawings" by Patrizio Angelini, Michael A. Bekos, Giuseppe Liotta, and Fabrizio Montecchiani, Algorithmica 2019. The latter paper proves:

For any $\Delta - 1$, every set S of $\Delta - 1$ slopes is universal for 1-bend planar drawings of planar graphs with maximum degree Δ. Namely, every such graph has a planar drawing with the following properties: (i) each edge has at most one bend; (ii) each edge segment uses one of the slopes in S.

This result solves assignment 4 and assignment 5.