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Excercises	
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Excercise 1. Skewness: Cycle vs. Paths

We are given a non-planar biconnected graph $G = (V, E)$ without mulitple edges or self-loops as an input. Recall that in the Skewness ILP we have a binary variable s_e for each $e \in E$ that is 1 if and only if e is deleted to obtain a planar graph. Thus, edges in the computed maximum planar subgraph have $s_e = 0$. Furthermore, there is a constant $D \in \mathbb{N}$ such that we have binary variables c_α for each cycle α with $|\alpha| \leq D$. Variable c_α is 1 if and only if α represents a face in the computed maximum planar subgraph.

Consider a cycle α in G with $|\alpha| \leq D$. Let a_1, b_1, a_2, b_2 denote four of its distinct (but not necessarily neighboring) vertices in this cyclic order around α. Assume that there are two vertex-disjoint paths $A = a_1 \to a_2$ and $B = b_1 \to b_2$, both of which are internally-vertex-disjoint from α.

What can you say about feasible solutions w.r.t. the variables involved in this subgraph? Try to write your insight as a linear constraint.

▶ Sketch of solution.
If the cycle is realized as a face, then both paths A, B need to be outside. But then they would cross. Thus, either the cycle is not chosen as a face, or at least one edge in one of the paths needs to be removed. As a linear constraint we may write:

$$c_\alpha \leq \sum_{e \in A \cup B} s_e.$$

We call these the *cycle-two-paths constraints*; they are part of the mentioned zoo. ◀

Excercise 2. Skewness: Paths vs. Cycles

Consider a vertex $v \in V$ with incident edges $F \subset E$, $|F| \geq 3$. Let $e_1, e_2 \in F$. Assume that G contains two cycles α, β of length at most D, both of which traverse v exactly once. Cycle α enters v via e_1 and leaves via e_2. Inversly, β enters via e_2 and leaves via e_1.

(a) *Assume* that the maximum planar subgraph would be biconnected. What could you say about feasible solutions w.r.t. the variables involved in the subgraph $F \cup \alpha \cup \beta$?

(b) How does the fact that the MPS may be non-biconnected ruin this argument?

(c) Consider a path internally-vertex-disjoint from $\alpha \cup \beta$ that connects v to another vertex on $\alpha \cup \beta$. How and why can you now use the insight of (a) (and (b))? Try to write it as a linear constraint.

▶ Sketch of solution.
ad a) If both α and β constitute faces, the fact that they both use e_1 and e_2 in different orientations tells us that v has to have degree 2 in the MPS; this means that under these circumstances every single edge of $F' := F \setminus \{e_1, e_2\}$ needs to be deleted. We may write this as multiple constraints:

$$c_\alpha + c_\beta - 1 \leq s_e \qquad \forall e \in F'.$$

ad b) In a non-biconnected graph, we may have faces that are not (simple) cycles, but closed walks—i.e., some vertices/edges may be repeated. Since we only want to consider cycles in the ILP, α may be the "representative" of a closed walk β with $\alpha \subset \beta$. In β, the edges F' may be traversed but we simply "forget" this when only looking at α.

ad c) Let P be any path as described in the question. Now, the argment from (a) holds in the following sense: if all of P would be in the MPS, as well as (all of) α and β, then no edge of P can be traversed twice by a single face; this would constitute a contradition. Thus, either α, β or P needs to have one edge removed in the solution. We may write this as

$$c_\alpha + c_\beta - 1 \leq \sum_{e \in P} s_e.$$

We call these the *path-two-cycles constraints*; they are part of the mentioned zoo. ◀

Excercise 3. Skewness: Faces and Kuratowski

Recall the Skewness ILP requires $\sum_{e \in K} s_e \geq 1$ for every Kuratowski subdivision K in G.

(a) Assume there is a cycle $\alpha \in K$ with $|\alpha| \leq D$. Can you rewrite the above Kuratowski constraint to use c_α?

(b) Generalize the constraint to not consider a single cycle, but an (arbitrary) set of cycles C.

▶ Sketch of solution.

ad a) Either some edge of K outside of α needs to be removed, or α needs to be broken. We may thus write

$$\sum_{e \in K \setminus \alpha} s_e \geq c_\alpha.$$

ad b) If all cycles are chosen, we need to remove at least one remaining edge in K. But a single non-chosen cycle *may* invalidate that:

$$\sum_{e \in K \setminus \bigcup_{\alpha \in C} \alpha} s_e \geq \sum_{\alpha \in C} c_\alpha - (|C| - 1).$$

We call these *Kuratowski-cycle constraints*; they are part of the mentioned zoo. ◀

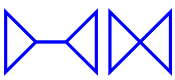
Excercise 4. Genus: Closed Walks with Simplicial Elements

In embeddings, in particular those on a higher genus surface, a face may have *simplicial* edges or vertices, i.e., they appear multiple times along the facial walk around a face (face tracing).

(a) What is the minimum length of a closed walk that contains a simplicial edge? What for a simplicial vertex? Try to argue as concisely as possible.

(b) Answer the above question in dependency on the graph's girth.

▶ Sketch of solution.

a) 8 and 6 edges, respectively: 

b) Let g be the girth of G. The cycles at the left/right need to have length at least g. Thus $2g + 2$ and $2g$, respectively. ◀