Improved constant factor for the unit distance problem

Péter Ágoston[1] and Dömötör Pálvölgyi[2]

1 **MTA-ELTE Lendület Combinatorial Geometry Research Group, Institute of Mathematics, Eötvös Loránd University (ELTE), Budapest, Hungary**
agostonp@cs.elte.hu

2 **MTA-ELTE Lendület Combinatorial Geometry Research Group, Institute of Mathematics, Eötvös Loránd University (ELTE), Budapest, Hungary**
dom@cs.elte.hu

Abstract

We prove that the number of unit distances among n planar points is at most $2.09 \cdot n^{4/3}$, improving on the previous best bound of $8n^{4/3}$. We also improve the best known extremal values for $n \geq 22$.

1 Introduction

Call a simple graph a *unit distance graph* (UDG) if its vertices can be represented by distinct points in the plane so that the pairs of vertices connected with an edge correspond to pairs of points at unit distance apart. Denote the maximal number of edges in a unit distance graph with n vertices by $u(n)$. Erdős [4] raised the problem to determine $u(n)$ and this question became known as the *Erdős Unit Distance Problem.* Erdős established the bounds $n^{1+c/\log\log n} \leq u(n) \leq O(n^{3/2})$. The lower bound remained unchanged, but the upper bound has been improved several times, the current best has been $O(n^{4/3})$ [8] for more than 35 years. For a detailed survey, see [10].

It turned out during the Polymath16 project[1] that improved bounds even for small values of n might give better bounds for questions related to the chromatic number of the plane. Our goal is to give an explicit upper bound, thus a constant factor improvement of the $O(n^{4/3})$ bound. Prior to our work, the best explicit constant (we know of) is the one derived from an argument of Székely [9], which gives $u(n) \leq 8n^{4/3}$ for all n. Our main result is the following constant factor improvement.

▸ **Theorem 1.1.** $u(n) \leq \frac{\sqrt[3]{\frac{2}{3}(2+\sqrt{3})\cdot 29}}{2} \cdot n^{4/3} = 2.08\ldots \cdot n^{4/3}$.

Our proof is based on a careful examination of Székely's argument to get rid of a few extra factors, an improved crossing lemma and some simple observations about UDG.

The research was supported by the European Union, co-financed by the European Social Fund (EFOP-3.6.3-VEKOP-16-2017- 00002).

1.1 The crossing lemma

Draw a (not necessarily simple) graph in the plane so that vertices are mapped to points and edges to simple curves that do not go through the images of the vertices other than their endpoints'. The *crossing number of a graph* G, denoted by $cr(G)$, is defined as the minimum number of intersection points among the edges of G in such drawings, counted

[1] https://dustingmixon.wordpress.com/2018/04/14/polymath16

36th European Workshop on Computational Geometry, Würzburg, Germany, March 16–18, 2020.
This is an extended abstract of a presentation given at EuroCG'20. It has been made public for the benefit of the community and should be considered a preprint rather than a formally reviewed paper. Thus, this work is expected to appear eventually in more final form at a conference with formal proceedings and/or in a journal.

with multiplicity. The crossing lemma, which was first proved by Ajtai, Chvátal, Newborn, Szemerédi [2] and, independently, by Leighton [5] is that for any simple graph $cr(G) \geq \Omega(\frac{e^3}{n^2})$ if $e \geq 4n$, where n is the number of vertices and e is the number of edges. The hidden constant has been improved several times; we will need the following variant.

▸ **Lemma 1.2** (Ackerman [1]). *If a simple graph has n vertices and e edges, then $cr(G) \geq \frac{e^3}{29n^2} - \frac{35n}{29}$. Moreover, if $e \geq 6.95n$, then $cr(G) \geq \frac{e^3}{29n^2}$.*

2 Proof of the improved bound

Fix a UDG on n vertices with $u(n)$ edges and the images of the vertices of one of its planar realizations; denote these n points by P and the UDG by G. In the following, we do not differentiate between vertices and their images. Note that due to the maximality of G, any two points at unit distance form an edge.

▸ **Lemma 2.1.** *If $n \geq 3$, then all vertices of G have degree at least 2, and if $n \geq 7$, then there is at least one UDG with the same vertex and edge number as G, for which there is at most one vertex with degree 2.*

Proof. Suppose that a vertex Z has degree one. Take a Euclidean coordinate system so that no two points have the same x coordinate: if for some coordinate system there were two such points, rotate with a sufficiently small angle. Delete Z and order the unit edges of the remaining graph $G \setminus Z$ such that $AB < CD$ if for the x-coordinates of these four points we have $\min(A_x, B_x) < \min(C_x, D_x)$ or $(\min(A_x, B_x) = \min(C_x, D_x)) \wedge (\max(A_x, B_x) < \max(C_x, D_x))$. The above is a total ordering as for any pair of different edges, either the left or the right endpoints are different from each other and so are their x coordinates. Take the smallest unit edge AB according to this ordering. For (at least) one of the points forming a unit equilateral triangle with AB, say, C, we would have $AC < AB$ or $BC < AB$ if $C \in P \setminus Z$. This would contradict the minimality of AB, thus $C \notin P \setminus Z$. Moreover, $C \neq Z$, as then its degree would have been at least two. But replacing Z with C gives a UDG with more edges, contradicting the maximality of G.

For the second statement, first delete all the vertices with degree at most 2, recursively, until no vertices with degree at most 2 are left. It is not possible that all vertices were deleted as that would mean that there were at most $2n-3$ edges (along with the last two vertices we have deleted at most 1 and 0 edges, respectively, while along with the other ones, we have deleted at most 2), but for $n \geq 7$, there exist UDGs with more than $2n-3$ edges, for example the ones constructed similarly to the graph in Figure 1.

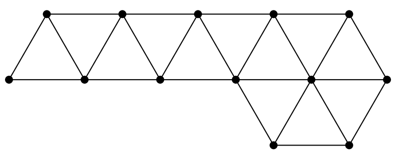


Figure 1

So after the deletions, we have a graph with a positive number of vertices all having degree at least 3. Thus, we can insert a new vertex connected to the two endpoints of a minimal edge in the current graph according to the ordering in the first part. We repeat this for each deleted vertex. In each step at least one of the new edges will be smaller (according to the ordering) than all the previously existing ones, so we always add the new vertex such that it is neighbouring the previously added one, guaranteeing that the previous one now has degree at least 3. So in the end at most one vertex with degree 2 will remain (the one added last) and the graph will have the same number of vertices and edges as G had. ◂

Now we proceed with the proof of Theorem 1.1. The statement is true for $n \le 6$, so from now on we assume $n \ge 7$. From Lemma 2.1, we can assume that P was chosen in a way that G has no vertex with degree at most 1 and has at most one vertex with degree 2.

Following Székely [9], define another graph on P, denoted by H, whose edges are those arcs on the unit circles around the points of P, which end in points from P and do not contain points of P in their interior. As any unit circle around a point of P has at least two points from P on it, H has no loops, but it might have multiple edges. Also, $|E(H)| = 2u(n)$, since for every point of P, there are exactly as many arcs on the unit circle around it as the degree of the corresponding point in G, so each edge is counted twice.

The second part of Lemma 2.1 implies that there is at most one pair of points which are connected on the same circle with two arcs. If this happens, we delete one of these two edges, to get a graph H^- with $|E(H^-)| = 2u(n) - 1$. Otherwise, to simplify presentation, delete an arbitrary edge to obtain H^-. The crossing number of H^- is bounded as $cr(H^-) \le 2 \cdot \binom{n}{2} = n^2 - n$, since all pairs of circles intersect each other in at most 2 points.

First of all, to deal with the case $n \le 500$, we lower bound the intersections among these unit circles in the following way. For a vertex v with degree $deg(v)$, there are exactly $\binom{deg(v)}{2}$ pairs of circles that intersect in v. Therefore, we have $n^2 - n \ge \sum_v \binom{deg(v)}{2} \ge n \cdot \binom{\sum_v deg(v)/n}{2} = u(n)\left(\frac{2u(n)}{n} - 1\right)$ from Jensen's inequality. This is roughly $u(n) \le \sqrt{n^3/2}$ and the RHS is less than $2n^{4/3}$ if $n < 512$ – a more precise calculation confirms the statement in this range.[2] Moreover, quite surprisingly, this simple bound (combined with a linear lower bound for the number of crossings in H) beats the best previous bound for small n starting from $n = 25$. (These values can be found in Table 1.)

The only possibility left of having multiple edges in H^- is that there can be more unit circles centered at some points of P passing through a pair of points from P. Since there are at most two unit circles through any pair of points, all edges occur with multiplicity at most 2 in H^-. Denote the simple graph formed by the edges of H^- by H_1 and the *simple* graph formed by the edges that have multiplicity 2 by H_2, and the number of their edges by e_1 and e_2, respectively, so $e_1 + e_2 = 2u(n) - 1$ and $e_1 \ge e_2$.

Now we prove $cr(H_1) + 3 \cdot cr(H_2) \le cr(H^-)$. From a drawing of H^- we can obtain a drawing of H_1 or H_2 by picking one of the two embeddings of each edge of H_2 randomly, independently from each other, with probability $\frac{1}{2}$. In this drawing of H_1 any crossing of two edges of $H_1 \setminus H_2$ is preserved, while a crossing of an edge of $H_1 \setminus H_2$ and an edge of H_2 is preserved with probability $\frac{1}{2}$, and a crossing of two edges of H_2 is preserved with probability $\frac{1}{4}$. In the drawing of H_2 only crossings of two edges of H_2 are preserved, each with probability $\frac{1}{4}$. Summing these up, the expected value of crossings in the random drawing of H_1 plus three times the crossings in the random drawing of H_2 obtained this way is at most $cr(H^-)$.

First, suppose that $e_2 \ge 6.95n$. By applying Lemma 1.2 to H_1 and H_2, respectively, we get $e_1 \le \sqrt[3]{29n^2 \cdot cr(H_1)}$ and $e_2 \le \sqrt[3]{29n^2 \cdot cr(H_2)}$. From these, $2u(n) - 1 = e_1 + e_2 \le \sqrt[3]{29n^2 \cdot cr(H_1)} + \sqrt[3]{29n^2 \cdot cr(H_2)}$. If we write $x = \frac{cr(H_1)}{cr(H^-)}$, then $\sqrt[3]{cr(H_1)} + \sqrt[3]{cr(H_2)} \le$

[2] Note that in the later parts of our proof we could also reduce $cr(H^-)$ with $u(n)(\frac{2u(n)}{n} - 1)$ using this argument, but it would not change its order of magnitude or effect the constant we obtain.

$\sqrt[3]{x \cdot cr(H^-)} + \sqrt[3]{\frac{1-x}{3} \cdot cr(H^-)} = \left(\sqrt[3]{x} + \sqrt[3]{\frac{1-x}{3}}\right) \cdot \sqrt[3]{cr(H^-)}$ for $0 \le x \le 1$, so it is enough to maximize $\sqrt[3]{x} + \sqrt[3]{\frac{1-x}{3}}$. And since its derivative is $\frac{1}{3} \cdot (x)^{-\frac{2}{3}} - \frac{1}{9} \cdot \left(\frac{1-x}{3}\right)^{-\frac{2}{3}}$, it has a zero value exactly if $x^{-\frac{2}{3}} = \frac{1}{3} \cdot \left(\frac{1-x}{3}\right)^{-\frac{2}{3}}$, i.e., if $x = \sqrt{3}(1-x)$, which gives $x = \frac{\sqrt{3}}{1+\sqrt{3}}$ (we could do these as both x and $\frac{1-x}{3}$ are positive in the interior of the $[0,1]$ interval). Also, both of the summands in the derivative are monotonously decreasing, thus so is the derivative, so the maximum is attained at the above x, and equals to $\sqrt[3]{\frac{2}{3}\left(2+\sqrt{3}\right)}$. From here the maximum of $2u(n) - 1 = e_1 + e_2$ is $\sqrt[3]{\frac{2}{3}\left(2+\sqrt{3}\right)} \cdot \sqrt[3]{29n^2 \cdot cr(H^-)} \le \sqrt[3]{\frac{2}{3}\left(2+\sqrt{3}\right)} \cdot \sqrt[3]{29 \cdot (n^4 - n^3)}$. This implies $u(n) \le \frac{\sqrt[3]{\frac{2}{3} \cdot (2+\sqrt{3}) \cdot 29}}{2} \cdot n^{\frac{4}{3}} \approx 2.082 \cdot n^{\frac{4}{3}}$.

The second case is when $e_2 < 6.95n$ and $e_1 < 6.95n$. Then $2u(n) - 1 = e_1 + e_2 < 13.9n$, from which $u(n) < 6.95n + 0.5$, which is less than $\frac{\sqrt[3]{\frac{2}{3} \cdot (2+\sqrt{3}) \cdot 29}}{2} \cdot n^{\frac{4}{3}}$ for $n > 37$.

Finally, if $e_2 < 6.95n$ and $e_1 \ge 6.95n$, then $cr(H_1) \le n^2 - n$, which implies $e_1 \le \sqrt[3]{29 \cdot (n^4 - n^3)}$. So $2 \cdot u(n) - 1 = e_1 + e_2 \le \sqrt[3]{29 \cdot (n^4 - n^3)} + 6.95n$ meaning that $u(n) \le \frac{\sqrt[3]{29 \cdot (n^4 - n^3)}}{2} + 3.475n + 0.5$, which is less than $\frac{\sqrt[3]{\frac{2}{3} \cdot (2+\sqrt{3}) \cdot 29}}{2} \cdot n^{\frac{4}{3}}$ for $n > 256$.

This finishes the proof of Theorem 1.1.

3 Best bounds for $n \le 30$

In Table 1 we list the best known bounds, along with constructions.

For $n \le 14$, the exact values of $u(n)$ are known, established in the thesis of Schade [7]. For $n \le 13$, the drawn ones are known to be the only maximal UDGs.[3] In general, the upper bounds can be obtained using the inequality $u(n) \le \lfloor \frac{n}{n-2} \cdot u(n-1) \rfloor$ (for $n \ge 3$), which is true because the edge density of the maximal UDGs with n vertices is monotonously decreasing in n as all subgraphs of a UDG are also UDGs. This was observed by Schade, who sometimes also applied additional tricks – the values where such tricks are needed are denoted by a star.

For $n \ge 15$, the lower bounding graphs are also by Schade, with the exception of the ones for $n = 29$, $n = 30$ and the second graph for $n = 28$, which are our improvements based on the graph for $n = 27$ by Schade. The upper bounding values from $n \ge 22$ are also our improvements, derived from a refinement of the inequality $n^2 - n \ge \sum_v \binom{deg(v)}{2}$. (The improved values are marked in bold.) For the refined inequality, we need the following lemma.

▸ **Lemma 3.1.** *For a graph H with n vertices in which all edges have multiplicity at most 2 and the number of edges (counted with multiplicity) is denoted by e, $cr(H) \ge 2 \cdot (e - 2 \cdot (3n-6)) = 2e - 12n + 24$.*

Proof. Let H be a graph with n vertices and e edges satisfying the conditions for which $cr(H)$ is minimal. Take a drawing of H with the minimal possible number of crossings. We can suppose that the two copies of an edge that occurs with multiplicity run close to each other, otherwise we could redraw any one of them sufficiently close to the other one (to whichever has fewer crossings). We can also suppose that H contains at most one single edge as otherwise we could take any two single edges and replace the one with the more crossings with another one close to (and parallel with) the one with the fewer crossings without increasing the number of crossings (note that this step also changes H, not only the drawing).

[3] Note that in [3] it is incorrectly stated also for $n = 14$ that the constructions were proved to be unique.

Take a maximal plane subgraph H' of H that has the maximal number of edges with multiplicity two in it. This has at most $2 \cdot (3n - 6)$ edges and all the other edges cross at least one of the double edges of H', thus adding at least $2 \cdot (e - 2 \cdot (3n - 6))$ crossings. ◂

Recall that $n^2 - n \geq cr(H) + \sum_v \binom{deg_G(v)}{2}$ where G is a unit distance graph and H is the graph obtained from G, as described previously, with edges of multiplicity at most two, except one, that could have multiplicity three. This, however, could only occur if G had a vertex of degree two, which is impossible if $u(n) \geq u(n-1) + 3$, which we can assume. Thus, applying Lemma 3.1 to H gives $n^2 - n \geq 4 \cdot |E(G)| - 12n + 24 + \sum_{v \in V(G)} \binom{deg(v)}{2} \geq 4 \cdot u(n) - 12n + 24 + n \cdot \left(1 - \left\{\frac{2u(n)}{n}\right\}\right) \cdot \binom{\lfloor \frac{2u(n)}{n} \rfloor}{2} + n \cdot \left\{\frac{2u(n)}{n}\right\} \cdot \binom{\lceil \frac{2u(n)}{n} \rceil}{2}$ (where $\{x\}$ denotes the fractional part of x), which gives our improved upper bounds for $u(n)$.

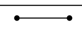
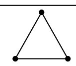
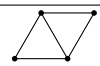
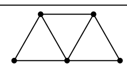
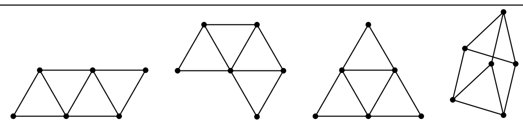
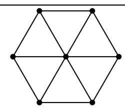
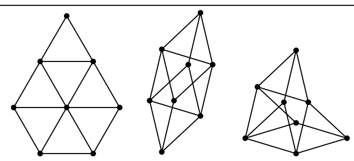
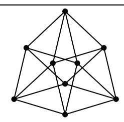
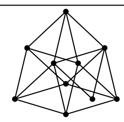
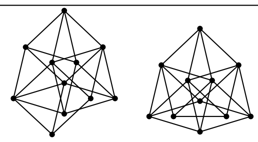
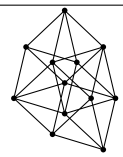
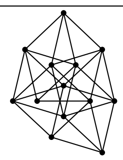
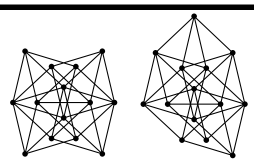
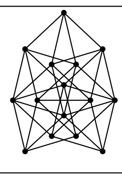
As can be seen, the upper bounds diverge quite fast from the lower bounds. From around $n = 600$, Theorem 1.1 gives the best upper bound.

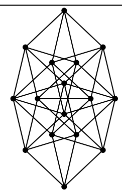
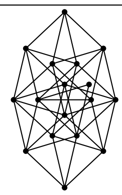
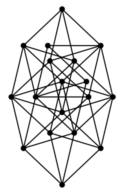
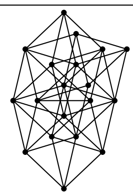
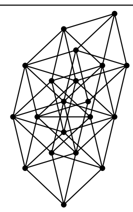
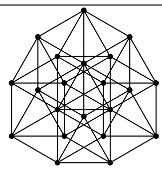
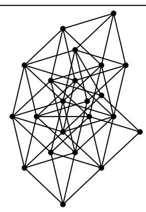
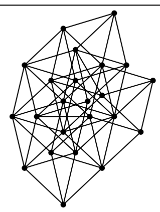
Acknowledgement

The main result was obtained while working on the currently ongoing Polymath16 project about the Hadwiger–Nelson problem and is related, but not directly connected to it.

References

1. E. Ackerman: On topological graphs with at most four crossings per edge *Computational Geometry* **85**, December 2019, 101574, 35 pages.
2. M. Ajtai, V. Chvátal, M. M. Newborn, E. Szemerédi (1982), Crossing-free subgraphs, *Theory and practice of combinatorics*, North-Holland Mathematics Studies, 60, North-Holland, Amsterdam, 9–12.
3. P. Brass, W. O. J. Moser, J. Pach: Research Problems in Discrete Geometry (2005).
4. P. Erdős: On sets of distances of n points, Amer. Math. Monthly 53 (1946), 248-–250.
5. T. Leighton, Complexity Issues in VLSI, Foundations of Computing Series, Cambridge, MA: MIT Press. (1983).
6. J. Pach, R. Radoičić, G. Tardos, G. Tóth: Improving the crossing lemma by finding more crossings in sparse graphs, *Discrete and Computational Geometry* (2006), **36** (4): 527–552.
7. C. Schade: Exakte maximale Anzahlen gleicher Abstände (1993).
8. J. Spencer, E. Szemerédi, W. Trotter Jr.: Unit Distances in the Euclidean Plane, *Graph Theory and Combinatorics* (1984), 293–303.
9. L. Székely: Crossing numbers and hard Erdős problems in discrete geometry, *Combinatorics, Probability and Computing* **6** (1997), 353–358.
10. E. Szemerédi: Erdős's Unit Distance Problem, *Open Problems in Mathematics* (2016), 459–477.

n	$u(n)$	Lower bounding graph(s)
1	0	
2	1	
3	3	
4	5*	
5	7*	
6	9*	
7	12	
8	14*	
9	18	
10	20*	
11	23*	
12	27	
13	30*	
14	33*	
15	37 or 38	

n	$u(n)$	Lower bounding graph(s)
16	41 or 42*	
17	43–47	
18	46–52	
19	50–57*	
20	54–63	
21	57–68*	
22	60–**72**	
23	64–**77**	

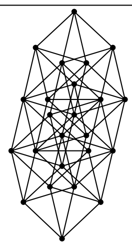
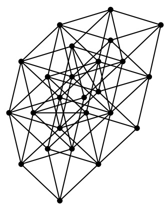
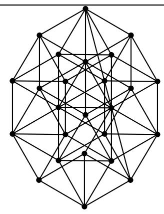
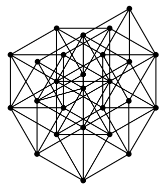
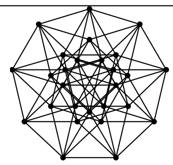
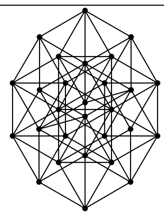
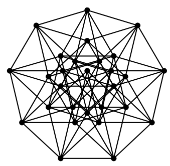
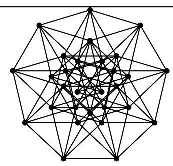
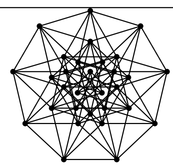
n	$u(n)$	Lower bounding graph(s)
24	68–**82**	
25	72–**87**	 
26	76–**92**	
27	81–**97**	
28	85-**102**	 
29	**89–108**	
30	**93–113**	

Table 1 Best bounds for the maximal number of unit distances $u(n)$ among n planar points.