Simple Topological Drawings of k-Planar Graphs*

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Abstract

Every graph that admits a topological drawing also admits a simple topological drawing. It is easy to observe that every 1-planar graph admits a 1-plane simple topological drawing. It has been shown that 2-planar graphs and 3-planar graphs also admit 2-plane and 3-plane simple topological drawings respectively. However, nothing has been proved for simple topological drawings of k-planar graphs for $k \geq 4$. In fact, it has been shown that there exist 4-planar graphs which do not admit a 4-plane simple topological drawing, and the idea can be extended to k-planar graphs for $k > 4$. We prove that there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that every k-planar graph admits an $f(k)$-plane simple topological drawing for all $k \in \mathbb{N}$. This answers a question posed by Schaefer.

1 Introduction

1.1 Problem Statement

A *topological drawing* of a graph G in the plane is a representation of G in which the vertices are mapped to distinct points in the plane and edges are mapped to Jordan arcs that do not pass through (the images of) vertices and no three Jordan arcs pass through the same point in the plane. A graph is *k-planar* if it admits a topological drawing in the plane where every edge is crossed by other edges at most k times, and such a drawing is called a *k-plane drawing*. A *simple topological drawing* of a graph refers to a topological drawing where no two edges cross more than once and no two adjacent edges cross. We study simple topological drawings of k-planar graphs.

It is well known that drawings of a graph G that attain the minimum number of crossings (i.e., the *crossing number* $\mathrm{cr}(G)$ of G) are simple topological drawings. However, a drawing that minimizes the total number of crossings need not minimize the maximum number of crossings per edges; and a drawing that minimizes the maximum number of crossings per edge need not be simple. A *k-plane simple topological drawing* is a simple topological drawing where every edge is crossed at most k times. We study the simple topological drawings of k-planar graphs and prove that there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that every k-planar graph admits an $f(k)$-plane simple topological drawing by designing an algorithm to obtain the plane simple topological drawing from a k-plane drawing of a k-planar graph.

In a k-plane drawing of a graph, every edge is crossed at most k times. However, adjacent edges may cross, and a pair of edges may cross multiple times. To obtain a simple topological drawing, we need to eliminate crossings between adjacent edges and ensure that no two edges cross more than once, without introducing self-intersections of edges during the process.

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This is an extended abstract of a presentation given at EuroCG'20. It has been made public for the benefit of the community and should be considered a preprint rather than a formally reviewed paper. Thus, this work is expected to appear eventually in more final form at a conference with formal proceedings and/or in a journal.

1.2 Related Previous Results

It is easy to see that every 1-planar graph admits a 1-plane simple topological drawing [3]. Pach et al. [2, Lemma 1.1] proved that every k-planar graph for $k \leq 3$ admits a k-plane drawing such that any pair of edges have at most one point in common including endpoints. However, these results do not extends to k-planar graphs for $k > 3$. In fact, Schaefer [4, p. 57] constructed k-planar graphs that do not admit a k-plane simple topological drawing for $k = 4$. The construction idea can be extended to all $k > 4$. The *local crossing number* of a graph G, $\text{lcr}(G)$, is the minimum integer k such that G admits a drawing where every edge has at most k crossings. The *simple* local crossing number, $\text{lcr}^*(G)$ minimizes k over simple topological drawings of G. Schaefer [4, p. 59] asked whether the $\text{lcr}^*(G)$ can be bounded by a function of $\text{lcr}(G)$. We answer this question in the affirmative and show that there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that $\text{lcr}^*(G) \leq f(\text{lcr}(G))$.

1.3 Basic Definitions

Given a k-plane drawing D of a graph G, we denote by N the planarization of D, i.e., we introduce a vertex of degree four at every crossing in D. We call this graph a *network*. Since every edge in D has at most k crossings, each edge of G corresponds to a path of length at most $k + 1$ in N. Our algorithm successively modifies the drawing D, and ultimately returns a simple topological drawing D of G. We formulate invariants for our algorithm in terms of the planarization N of the initial drawing. In other words, N remains fixed (in particular, N will not be the planarization of the modified drawings). Specifically, our algorithm maintains the following invariants:

(i) every edge in D' closely follows a path of length at most $k + 1$ in the network N,
(ii) every pair of edges in D' cross only in a small neighborhood of a node of N, and
(iii) any two edges cross at most once in each such neighborhood.

Lenses in a topological drawings. We start with definitions needed to describe the key operations in our algorithm. In a topological drawing, we define a structure called *lens*. Consider two edges, a and b, that cross more than once. Consider two crossings of the edges a and b, denoted by c_1 and c_2. Let a_{12} denote the portion of the edge a between c_1 and c_2, and b_{12} is defined analogously. The arcs a_{12} and b_{12} together are called a *lens* if the arcs do not intersect except at c_1 and c_2. Similarly a lens could also be formed by two adjacent edges cross each other. Let p be the common vertex of two adjacent edges a and b, and let c be a crossing between a and b. The portions of the edges a and b between p and c form a *lens* if the arcs formed by the portions of the edges between p and c do not cross each other.

▶ **Lemma 1.1.** *If two nonadjacent edges a and b cross more than once, then there exist two crossings c_1 and c_2 of the two edges such that a_{12} and b_{12} form a lens, where a_{12} and b_{12} are the portions of the edges a and b between c_1 and c_2.*

Proof. If the edges a and b cross exactly twice, the arcs between the two crossings form a lens. Now, for the edges a and b that cross more than twice, assume there were no two crossings which form a lens. Then the arcs between any two crossings cross each other. Consider two crossings c_1 and c_2 for which the arcs a_{12} and b_{12} have the minimum number of crossings. Let c' be one of these crossings. Then the portions of the arcs a_{12} and b_{12} between c_1 and c' have at least one fewer crossings, contradicting the minimality assumption in the choice of c_1 and c_2. ◀

▶ **Lemma 1.2.** *If two adjacent edges a and b cross each other once, then there exist two crossings c_1 and c_2 of the two edges such that the arcs a_{12} and b_{12} form a lens, where one of the crossings from c_1 and c_2 can be the common endpoint of the two edges, and a_{12} and b_{12} are the portions of the edges a and b between c_1 and c_2.*

Proof. If the edges a and b crossed exactly once, the arcs between the common endpoint and the point of intersection form a lens. If the edges cross more than once, the proof follows from Lemma 1.1. ◀

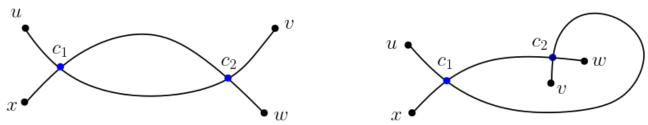


Figure 1 Lenses that can be formed by two edges crossing more than once.

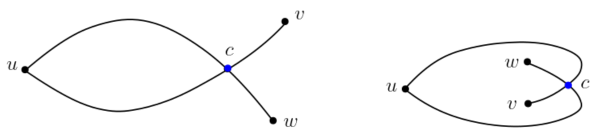


Figure 2 Lenses that can be formed by two adjacent edges crossing each other.

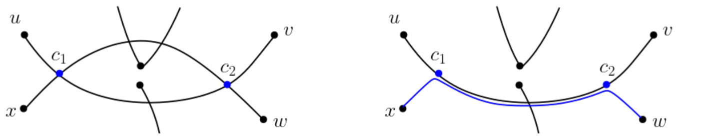
Elimination operation. We further define two operations for our algorithm, each of which modifies one edge in a (current) drawing of G. An *elimination* operation is defined as the redrawing of an edge such that a lens is eliminated.

Let D' be a drawing of G satisfying invariants (i)–(iii). Let $a = (u, v)$ and $b = (x, w)$ be two edges that cross twice, at c_1 and c_2, and let a_{12} and b_{12} be the portions of the edges a and b between c_1 and c_2. Note that other edges may cross the arcs $a_{12}1$ and b_{12}. By (ii) and (iii), the intersection points c_1 and c_2 lie in some small neighborhoods of two distinct nodes of N. By (i), the arcs a_{12} and b_{12} each closely follow some path in N.

We compare a_{12} and b_{12} using two measures: First, the *length* of a_{12} and b_{12}, respectively, is the (graph-theoretic) length of the path in the network N that they follow. Second, we count the number of crossings of a_{12} (resp., b_{12}) with other edges in the current drawing D'. We define a_{12} to be the *efficient arc* (and b_{12} the *inefficient arc*) if a_{12} has smaller length that b_{12}, or if the two arcs have the same length but a_{12} has fewer crossings in D' and b_{12}. If the two arcs have the same length and the same number of crossings, either of the arcs can be considered as the *efficient arc*. The elimination operation reroutes the inefficient arc to follow the efficient arc to eliminate the lens.

Without loss of generality, assume a_{12} is the efficient arc, $b = xw$, and the points (x, c_1, c_2, w) appear in this order along the drawing of b in D'. To eliminate the lens, we redraw b such that it follows its current arc from x to c_1, and then, without crossing a at c_1, it closely follows the arc a_{12} until c_2, and further follows its current arc from c_2 to w. In this process, the crossing c_1 is eliminated. Figure 3 illustrates this operation where both the crossings are eliminated. Figure 4 illustrates the operation where only one of the crossing is eliminated.

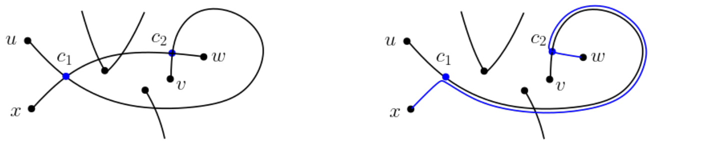
The elimination operation for a lens formed by adjacent edges between the common endpoint and the first crossing is defined similarly. Let $a = (u, v)$ and $b = (u, w)$ be two adjacent edges that cross at c and the arcs a_{12} and b_{12} form a lens. Let a_{12} be the efficient



(a) Lens formed by two edges crossing more than once.

(b) Inefficient curve rerouted and depicted in blue.

Figure 3 Rerouting of edge and eliminating both the crossings.

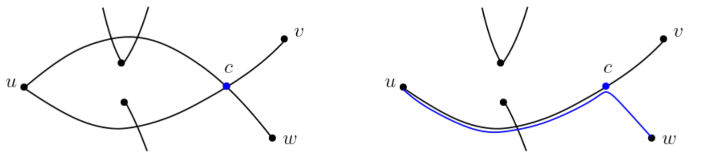


(a) Lens formed by two edges crossing more than once.

(b) Inefficient curve rerouted and depicted in blue.

Figure 4 Rerouting of edge and eliminating crossing c_1.

arc. We redraw the edge b such that it starts at u and follows arc a_{12} until c and then follows its current arc from c until w by avoiding the crossing c. The redrawing of edge b in this method eliminates the crossing c. Figures 5 and 6 illustrate this operation. If the efficient arc a_{12} crosses the arcs $b \setminus b_{12}2$, then we have introduced self-crossing in the modified edge b. We eliminate self-crossings by removing any loops from the modified arc of b. Note that the elimination operation does not increase the length of any arc or any edge.



(a) Lens formed by two adjacent edges.

(b) Inefficient curve rerouted and depicted in blue.

Figure 5 Rerouting of edge to eliminate the lens.

As a result, at least one of the crossings c_1 and c_2 (or crossing c in the case of adjacent edges) is eliminated. However, the elimination of lens may create a new lens if one of the edges crossing the efficient arc crossed the edge corresponding to the inefficient arc as well. This is demonstrated in Figure 7.

In addition, the redrawing of the arcs is done in such a way that the modified arc closely follows an existing path of the network N since the inefficient arc follows closely to the efficient arc. Furthermore, the elimination process can introduce new crossings between the modified inefficient arc and edges crossing the efficient arc. Assume an edge e' crossed the efficient arc at c'. The elimination operation introduces a crossing between e' and the inefficient arc, however, this crossing is very close to the crossing c' and hence is in a small neighborhood of a node of N. Since the new crossing is in a small neighborhood of a node of N, the lengths of the edge e' and the inefficient arc do not increase.

(a) Lens formed by two adjacent edges.

(b) Inefficient curve rerouted and depicted in blue.

Figure 6 Rerouting of edge to eliminate the lens.

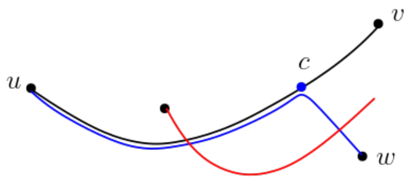


Figure 7 New lens created between red and blue edges after elimination of existing lens.

2 The Algorithm

Given a k-plane topological drawing D of a k-planar graph G, we describe an algorithm using the elimination operations to produce a simple topological drawing D' of graph G from D. Initially, let $D' := D$. While there exists a lens in the drawing D', consider one such lens and eliminate it by modifying the drawing D'. Return the resulting drawing D'.

▶ **Theorem 2.1.** *The algorithm terminates and transforms a k-plane drawing of a k-planar graph into a simple topological drawing.*

Proof. Note that the redrawing of the arcs in the elimination operation was defined with respect to two measures: the length of the arc and number of crossings of the arcs. Let the sum of lengths of all edges in the drawing be defined as the *total length* of the drawing, where the length of an edge is the length of the path in N that the edge closely follows. After each elimination operation, since we reroute the inefficient arc towards the efficient arc, the total length of the drawing monotonically decreases; and if the total length remains the same, then the total number of crossings strictly decreases. Thus, the algorithm terminates, the drawing D' returned by the algorithm does not contain lenses. By Lemmata 1.1 and 1.2, any two edges in the resulting drawing D' cross at most once and adjacent edges do not cross. Additionally, the operations do not introduce any self-crossings. Consequently, the algorithm returns a simple topological drawing. ◀

▶ **Theorem 2.2.** *There exists a function $f(k)$ such that every k-planar graph admits an $f(k)$-plane simple topological drawing, and the $f(k)$-plane simple topological drawing can be obtained from a k-plane drawing of the graph using the above algorithm.*

Proof. Consider the drawing D' returned by our algorithm, and a node v of network N. We analyse the subgraph G_v of G formed by the edges of G that pass through a small neighborhood of v. Let n_v and m_v be the number of vertices and edges of G_v, respectively. Since N is created from a k-plane drawing, and every node corresponding to a crossing has degree 4, there are at most $4 \cdot 3^{k-1}$ vertices of G at distance at most k from v. Hence, $n_v \leq 4 \cdot 3^{k-1}$.

▶ **Lemma 2.3** (Crossing Lemma [1, Theorem 6]). *Let G be a graph with n vertices and m edges and D be a topological drawing of G. Let $cr(D)$ be defined as the total number of crossings in D, and $cr(G)$ be defined as the minimum of $cr(D)$ over all drawings D of G. If $m \geq 6.95n$, then $cr(G) \geq \frac{1}{29}\frac{m^3}{n^2}$.*

We apply Lemma 2.3 to the graph G_v, and obtain two cases:

- Case 1: $m_v < 6.95n$.
- Case 2: $m_v \geq 6.95n$, and thus $\mathrm{cr}(G_v) \geq \frac{1}{29}\frac{m_v^3}{n_v^2}$. Since G_v has m_v edges and each edge has at most k crossings in the drawing D', we obtain $\frac{1}{29}\frac{m_v^3}{n_v^2} \leq \frac{m_v k}{2}$, which implies $m_v \leq \sqrt{\frac{29k}{2}}\, n_v$.

Combining the two cases to obtain an upper bound on m_v, we get $m_v \leq \max\{6.95n_v, \sqrt{\frac{29k}{2}}\, n_v\}$. Further, for $k \geq 4$, $m_v \leq \sqrt{\frac{29k}{2}}\, n_v$.

Since m_v edges pass through a small neighborhood of v, any edge passing through that neighborhood crosses at most $m_v - 1$ edges. Additionally, every edge in G passes through (the neighborhood of) at most k nodes of N. An edge passing through nodes $v_1, \ldots v_k$, crosses at most $\sum_{i=1}^{k}(m_{v_i} - 1)$ edges in D'. Combining the upper bounds on m_v and n_v, we obtain that every edge in the output drawing D' is crossed at most $\sqrt{\frac{29k}{2}} \cdot 4k \cdot 3^{k-1} = \frac{2}{3}\sqrt{58} \cdot k^{3/2} \cdot 3^k$ times for $k \geq 4$. ◀

3 Conclusion

We have proved that every k-planar graph admits a simple topological drawing where every edge is crossed at most $f(k) = \frac{2}{3}\sqrt{58} \cdot k^{3/2} \cdot 3^k$ times. Consequently, $\mathrm{lcr}^* \leq O(\mathrm{lcr}^{3/2} \cdot 3^{\mathrm{lcr}})$. However, we hope that the bound on $f(k)$ can be improved to a polynomial in k by a more careful analysis for the number of crossings per edge created by our algorithm.

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