Balanced Independent and Dominating Sets on Colored Interval Graphs*

Sujoy Bhore[1], Jan-Henrik Haunert[2], Fabian Klute[1], Guangping Li[1], and Martin Nöllenburg[1]

1 **Algorithms and Complexity Group, TU Wien, Vienna, Austria**
`{sujoy, fklute, guangping, noellenburg}@ac.tuwien.ac.at`

2 **Geoinformation Group, University of Bonn, Bonn, Germany**
`haunert@igg.uni-bonn.de`

Abstract

We study two new versions of independent and dominating set problems on vertex-colored interval graphs, namely *f-Balanced Independent Set* (f-BIS) and *f-Balanced Dominating Set* (f-BDS). Let $G = (V, E)$ be a vertex-colored interval graph with a k-coloring $\gamma\colon V \to \{1, \ldots, k\}$ for some $k \in \mathbb{N}$. A subset of vertices $S \subseteq V$ is called *f-balanced* if S contains f vertices from each color class. In the f-BIS and f-BDS problems, the objective is to compute an independent set or a dominating set that is f-balanced. We show that both problems are NP-complete even on proper interval graphs. For the BIS problem on interval graphs, we design two FPT algorithms, one parameterized by (f, k) and the other by the vertex cover number of G. Moreover, we present a 2-approximation algorithm for a slight variation of BIS on proper interval graphs.

1 Introduction

A graph G is an interval graph if it has an intersection model consisting of intervals on the real line. Formally, $G = (V, E)$ is an interval graph if there is an assignment of an interval $I_v \subseteq \mathbb{R}$ for each $v \in V$ such that $I_u \cap I_v$ is nonempty if and only if $(u, v) \in E$. A *proper* interval graph is an interval graph that has an intersection model in which no interval properly contains another [8]. Consider an interval graph $G = (V, E)$ and additionally assume that the vertices of G are k-colored by a mapping $\gamma\colon V \to \{1, \ldots, k\}$. We define and study color-balanced versions of two classical graph problems: maximum independent set and minimum dominating set on vertex-colored (proper) interval graphs. In what follows, we define the problems formally and discuss their underlying motivation.

f-Balanced Independent Set (f-BIS): Let $G = (V, E)$ be an interval graph with vertex coloring $\gamma\colon V \to \{1, \ldots, k\}$. Find an *f-balanced independent set* of G, i.e., an independent set $L \subseteq V$ that contains exactly f elements from each color class.

The classic maximum independent set problem serves as a natural model for many real-life optimization problems and finds applications across fields, e.g., computer vision [2], information retrieval [14], and scheduling [16]. Specifically, it has been used widely in map-labeling problems [1, 4, 17, 18], where an independent set of a given set of label candidates corresponds to a conflict-free and hence legible set of labels. To display as much relevant information as possible, one usually aims at maximizing the size or, in the case of weighted label candidates, the total weight of the independent set. This approach may be appropriate if all labels represent objects of the same category. In the case of multiple categories, however,

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maximizing the size or total weight of the labeling does not reflect the aim of selecting a good mixture of different object types. For example, if the aim was to inform a map user about different possible activities in the user's vicinity, labeling one cinema, one theater, and one museum may be better than labeling four cinemas. In such a setting, the f-BIS problem asks for an independent set that contains f vertices from each object type.

We initiate this study for interval graphs which is a primary step to understand the behavior of this problem on intersection graphs. Moreover, solving the problem for interval graphs gives rise to optimal solutions for certain labeling models, e.g., if every label candidate is a rectangle that is placed at a fixed position on the boundary of the map [9].

While there exists a simple greedy algorithm for the maximum independent set problem on interval graphs, it turns out that f-BIS is much more resilient and NP-complete even for proper interval graphs and $f = 1$ (Section 2.1). Then, in Section 3, we complement this complexity result with two FPT algorithms for interval graphs, one parameterized by (f, k) and the other parameterized by the vertex cover number. We conclude with a 2-approximation algorithm for a slight variation of BIS on proper interval graphs.

The second problem we discuss is

f-Balanced Dominating Set (f-BDS): Let $G = (V, E)$ be an interval graphs with vertex coloring $\gamma \colon V \to \{1, \dots, k\}$. Find an *$f$-balanced dominating set*, i.e., a subset $D \subseteq V$ such that every vertex in $V \setminus D$ is adjacent to at least one vertex in D, and D contains exactly f elements from each color class.

The dominating set problem is another fundamental problem in theoretical computer science which also finds applications in various fields of science and engineering [5, 10]. Several variants of the dominating set problem have been considered over the years: k-tuple dominating set [6], Liar's dominating set [3], independent dominating set [11], and more. The colored variant of the dominating set problem has been considered in parameterized complexity, namely, red-blue dominating set, where the objective is to choose a dominating set from one color class that dominates the other color class [7]. Instead, our f-BDS problem asks for a dominating set of a vertex-colored graph that contains f vertices of each color class. Similar to the independent set problem, we primarily study this problem on vertex-colored interval graphs, which can be of independent interest. In Section 2.2, we prove that f-BDS on vertex-colored proper interval graphs is NP-complete, even for $f = 1$.

2 Complexity Results

In this section we show that f-BIS and f-BDS are NP-complete even if the given graph G is a proper interval graph and $f = 1$. Our reductions are from restricted, but still NP-complete versions of 3SAT, namely 3-bounded 3SAT [15] and 2P2N-3SAT (hardness follows from the result for 2P1N-SAT [19]). In the former 3SAT variant a variable is allowed to appear in at most three clauses and clauses have two or three literals, in the latter each variable appears exactly four times, twice as positive literal and twice as negative literal. Here we give the constructions for both reductions; for detailed proofs we refer to the full version of this paper.

▶ Remark. NP-completeness of 1-balanced independent (dominating) set implies the NP-completeness of f-balanced independent (dominating) set for $f > 1$. Let I_1 be the interval graph in an 1-balanced independent (dominating) set instance. We construct an interval graph I_f consisting of f independent copies of I_1. Clearly I_1 has 1-balanced independent (dominating) set if and only if I_f has an f-balanced independent (dominating) set.

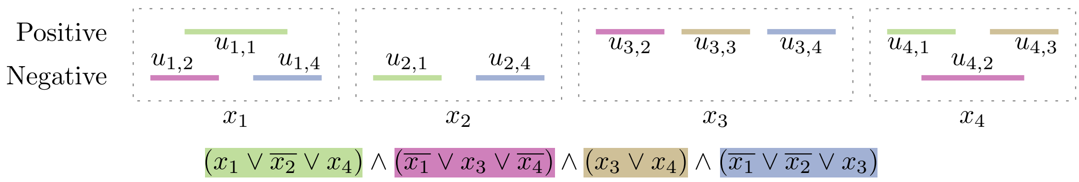

Figure 1 The graph resulting from the reduction for 1-balanced independent set in Theorem 1 depicted as interval representation with the vertex colors being the colors of the intervals.

2.1 f-Balanced Independent Set

Let $\phi(x_1, \dots, x_n)$ be a 3-bounded 3SAT formula with variables $x_1, \dots, x_n$ and clause set $\mathcal{C} = \{C_1, \dots, C_m\}$. From ϕ we construct a proper interval graph $G = (V, E)$ and a vertex coloring γ of V as follows. We choose the set of colors to contain exactly m colors, one for each clause in $\mathcal{C}$ and we number these colors from 1 to m. We add a vertex $u_{i,j} \in V$ for each occurrence of a variable x_i in a clause C_j in ϕ. Furthermore, we insert an edge $u_{i,j}u_{i,j'} \in E$ whenever x_i appears positively in C_j and negatively in $C_{j'}$ (or vice versa). Finally, we color each vertex $u_{i,j} \in V$ with color j. See Figure 1 for an example. The graph G created from ϕ is a proper interval graph as it consists only of disjoint paths of length at most three and can clearly be constructed in polynomial time and space.

▶ **Theorem 1.** *The f-balanced independent set problem on a graph $G = (V, E)$ with vertex coloring $\gamma \colon V \to \{1, \dots, k\}$ is* NP*-complete, even if G is a proper interval graph and $f = 1$.*

2.2 f-Balanced Dominating Set

We reduce from 2P2N-3SAT where each variable appears exactly twice positive and twice negative. Let $\phi(x_1, \dots, x_n)$ be a 2P2N-3SAT formula with variables $x_1, \dots, x_n$ and clause set $\mathcal{C} = \{C_1, \dots, C_m\}$. For variable x_i in ϕ we denote with $\mathcal{C}_{x_i} = \{C_t^1, C_t^2, C_f^1, C_f^2\}$ the four clauses x_i appears in, where C_t^1, C_t^2 are clauses with positive occurrences of x_i and C_f^1, C_f^2 are clauses containing negative occurrences of x_i.

We construct a graph $G = (V, E)$ from $\phi(x_1, \dots, x_n)$ as follows. For each variable x_i we introduce six vertices t_1, t_2, f_1, f_2, h_t, and h_f and for each clause C_j with occurrences of variables x_{j_1}, x_{j_2}, and x_{j_3} we add up to three vertices c_k for each $k \in \{j_1, j_2, j_3\}$ (In case a clause has less than three literals we add only one or two vertices). If the connection to the variable is clear, we also write c_t^1, c_t^2, c_f^1, and c_f^2 for the vertices introduced for this variable's occurrences in the clauses C_t^1, C_t^2, C_f^1, and C_f^2, respectively. Furthermore, we add for each variable x_i the edges (h_t, t_1), (h_t, t_2), (h_f, f_1), and (h_f, f_2), as well as for each clause C_j all possible edges between the three vertices introduced for C_j. For each variable x_i we introduce five colors, namely z_t^1, z_t^2, z_f^1, z_f^2, and z_h. We set $\gamma(h_t) = \gamma(h_f) = z_h$. Finally, we

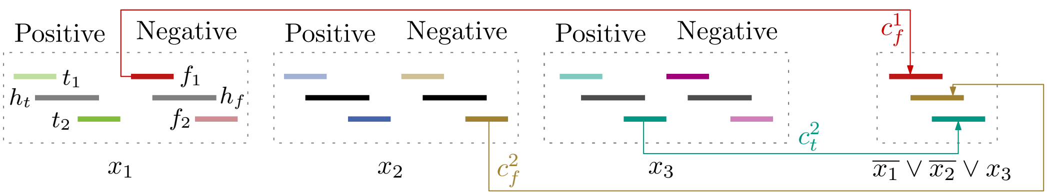

Figure 2 Illustrations of three variable gadgets and a clause gadget from Theorem 2 as interval representations. The vertex colors are given as the colors of the intervals.

set $\gamma(t_1) = \gamma(c_t^1) = z_t^1$. Equivalently for t_2, f_1, and f_2. See Figure 2 for an example.

In total we create $6n + 3m$ many vertices and $4n + 3m$ many edges, thus the reduction is in polynomial time. All variable and clause gadgets are independent components and only consist of paths of length three and triangles, hence G is a proper interval graph. Furthermore, G can clearly be constructed in polynomial time and space.

▶ **Theorem 2.** *The f-balanced dominating set problem on a graph $G = (V, E)$ with vertex coloring $\gamma : V \to \{1, \dots, C\}$ is* NP*-complete, even if G is a proper interval graph and $f = 1$.*

3 Algorithmic Results for the Balanced Independent Set Problem

In this section we first take a parameterized perspective on f-BIS and provide two FPT algorithms[1] with different parameters. Then we give a 2-approximation algorithm for the related problem of maximizing the number of different colors in the independent set. For omitted proofs see the full version of this paper.

3.1 An FPT Algorithm Parameterized by (f, k)

Assume we are given an instance of f-BIS with $G = (V, E)$ being an interval graph with vertex coloring $\gamma\colon V \to \{1, \dots, k\}$. We can construct an interval representation $\mathcal{I} = \{I_1, \dots, I_n\}$, $n = |V|$, from G in linear time [12]. Then our algorithm works as follows. To start we sort the right end-points of the n intervals in $\mathcal{I}$ in ascending order. We define for all intervals I_i with $i > 0$ the index $0 \le p_i < n$ as the index of the interval I_{p_i} whose right endpoint is rightmost before I_i's left endpoint. For each color $\kappa \in \{1, \dots, k\}$, let $\hat{e}_\kappa$ denote the k-dimensional unit vector of the form $(0, \dots, 0, 1, 0, \dots, 0)$, where the element at the κ-th position is 1 and the rest are 0. For a subset $\mathcal{I}' \subseteq \mathcal{I}$ we define a *cardinality vector* as the k-dimensional vector $C_{\mathcal{I}'} = (c_1, \dots, c_k)$, where each element c_i represents the number of intervals of color i in $\mathcal{I}'$. We say $C_{\mathcal{I}'}$ is *valid* if all $c_i \le f$ and the set $\mathcal{I}'$ is independent.

The key observation here is that there are at most $O(f^k)$ many different valid cardinality vectors as there are only k colors and we are interested in at most f intervals per color. In the following let U_j, $j \in \{1, \dots, n\}$, be the union of all valid cardinality vectors of the first j intervals in $\mathcal{I}$. Let $U_0 = \{(0, \dots, 0)\}$ in the beginning. To compute an f-balanced independent set the algorithm simply iterates over all right endpoints of the intervals in $\mathcal{I}$ and in the i-th step computes U_i as $U_i = \{u + \hat{e}_{\gamma(I_i)} \mid u \in U_{p_i} \text{ and } u + \hat{e}_{\gamma(I_i)} \text{ is valid}\} \cup U_{i-1}$. Finally, we check the cardinality vectors in U_n and return *true* in case there is one with entries being all f and *false* otherwise. An f-balanced independent set can be easily retrieved by backtracking the decisions we made to compute the cardinality vector.

▶ **Theorem 3.** *Let $G = (V, E)$ be an interval graph with a vertex coloring $\gamma : V \to \{1, \dots, k\}$. We can compute an f-balanced independent set of G or determine that no such set exists in $O(n \log n + nf^k \alpha(f^k))$ time, where α is the inverse Ackermann function.*

3.2 An FPT Algorithm Parameterized by the Vertex Cover Number

Here we will give an alternative FPT algorithm for f-BIS, this time parameterized by the *vertex cover number* $\tau(G)$ of G, i.e., the size of a minimum vertex cover of G.

[1] FPT is the class of parameterized problems that can be solved in time $O(g(k)n^{O(1)})$ for input size n, parameter k, and some computable function g.

▶ **Lemma 4.** *Let $G = (V, E)$ be a graph with vertex cover number $\tau(G)$. There are $O(2^{\tau(G)})$ maximal independent sets of G.*

Proof. Consider a minimum vertex cover V_c in G and its complement $V_{\text{ind}} = V \setminus V_c$. Note that since V_c is a (minimum) vertex cover, V_{ind} is a (maximum) independent set. Furthermore, any maximal independent set M of G can be constructed from V_{ind} by adding $M \cap V_c$ and removing its neighborhood in V_{ind}, namely $M = (V_{\text{ind}} \cup (M \cap V_c)) \setminus N(M \cap V_c)$ (see the full version for details). Thus there are $O(2^{\tau(G)})$ maximal independent sets of G. ◀

▶ **Theorem 5.** *Let $G = (V, E)$ be an interval graph with a vertex coloring $\gamma \colon V \to \{1, \ldots, k\}$. We can compute an f-balanced independent set of G or determine that no such set exists in $O(2^{\tau(G)} \cdot n)$ time.*

Proof. According to Lemma 4, there are $O(2^{\tau(G)})$ maximal independent sets of G. The basic idea is to enumerate all the $O(2^{\tau(G)})$ maximal independent sets and compute their maximum balanced subsets. Enumerating all maximal independent sets of an interval graph takes $O(1)$ time per output [13]. Given an arbitrary independent set of G we can compute an f-balanced independent subset in $O(n)$ time or conclude that no such subset exists. Therefore, the running time of the algorithm is $O(2^{\tau(G)} \cdot n)$. ◀

3.3 A 2-Approximation for 1-Max-Colored Independent Sets

Here we study a variation of BIS, which asks for a maximally colorful independent set.

1-Max-Colored Independent Set (1-MCIS): Let $G = (V, E)$ be a proper interval graph with vertex coloring $\gamma \colon V \to \{1, \ldots, k\}$. Find a 1-max-colored independent set of G, i.e., an independent set $L \subseteq V$, whose vertices contain a maximum number of colors.

We note that the NP-completeness of 1-BIS implies that 1-MCIS is an NP-hard optimization problem as well. In the following, we will show a simple sweep algorithm for 1-MCIS with approximation ratio 2.

Our algorithm selects one interval for each color greedily. We maintain an array S of size k to store the selected intervals. After sorting the n right end-points in ascending order, we scan the intervals from left to right. For each interval I_i in this order, we check if the color of I_i is still missing in our solution (by checking if $S[\gamma(I_i)]$ is not yet occupied). If so, we store I_i in $S[\gamma(i)]$ and remove all remaining intervals overlapping I_i. Otherwise, if $S[\gamma(I_i)]$ is not empty, we remove I_i and continue scanning the intervals. This is repeated until all intervals are processed. Using that G is a proper interval graph and a charging argument on the colors in an optimal solution that are missing in the greedy solution, we obtain our approximation result.

▶ **Theorem 6.** *Let $G = (V, E)$ be a proper interval graph with a vertex coloring $\gamma \colon V \to \{1, \ldots, k\}$. In $O(n \log n)$ time, we can compute an independent set with at least $\lceil \frac{c}{2} \rceil$ colors, where c is the number of colors in a 1-max-colored independent set.*

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