t-spanners for Transmission Graphs Using the Path-Greedy Algorithm

Stav Ashur[1] and Paz Carmi[1]

1 Ben-Gurion University of the Negev

Abstract

Let $D = \{d_1, \ldots, d_n\}$ be a set of n disks in the plane, and let $C = \{c_1, \ldots, c_n\}$ be their centers. The *transmission graph* $G = (C, E)$ has the centers of D as its vertex set C, and a directed edge (c_i, c_j) if and only if c_j lies in the disks associated with c_i.

A *t-spanner* G' for G is a sparse subgraph of G such that for any two vertices p, q connected by a directed path in G, there is a directed path from p to q in G' of length at most t times the length of the path from p to q in G.

In this paper, we consider the problem of computing a t-spanner with a linear number of edges and bounded in-degree for transmission graphs. We show that the well-known *Path-Greedy* algorithm produces such a t-spanner for transmission graphs, thus, providing a much simpler method than ones that are currently in use.

In addition, we show that the weight of the resulting t-spanner is $O(\log n \cdot wt(MST(D))(1 + \Psi))$, where Ψ is the ratio between the largest and smallest disk radii, and $wt(MST(D))$ is the weight of the MST built over the centers of the disks. To the best of our knowledge, this is the first upper bound on the weight of a t-spanner for transmission graphs.

1 Introduction

Given a directed graph G, let $\delta_G(p, q)$ be the shortest directed path from p to q in G, and let $|\delta_G(p, q)|$ denote its length. A t-spanner for a weighted directed graph $G = (V, E, w)$ is a sparse subgraph $G' \subseteq G$ such that every two vertices $p, q \in V$, connected by a directed path from p to q of weight $|\delta_G(p, q)|$, are connected in G' by a directed path of weight at most $t \cdot |\delta_G(p, q)|$, i.e. $|\delta_{G'}(p, q)| \leq t \cdot |\delta_G(p, q)|$. Algorithms for the construction of t-spanners for geometric graphs have been widely studied, and various results exist for different types of graphs, see [11] for a comprehensive survey.

Transmission graphs

Transmission graphs are a common model for communication networks composed of devices with different transmission ranges. The set of vertices $D = \{d_1, ..., d_n\}$, where every node is tuple $d_i = (c_i, r_i) \in \mathbb{R}^2 \times \mathbb{R}$, represents devices with wireless capabilities that are given as a pair consisting of their location in $\mathbb{R}^2$ (c_i) and their transmission range (r_i). These disks in $\mathbb{R}^2$ induce an intuitive directed graph by connecting two vertices $p = (c_i, r_i)$ and $q = (c_j, r_j)$ if q lies within the transmission range of p, or formally, $||c_j - c_i|| \leq r_i$. See Figure 1 for an example.

Peleg and Roditty [13] presented a method to construct a t-spanner for transmission graphs in metric spaces with constant doubling dimension using $O(\frac{n}{\epsilon^d} \log \Psi)$ edges where $\Psi = \frac{radius_{max}}{radius_{min}}$ is the ratio between the maximum and minimum radii. They later proved [12] that in this setting, it is not possible to guarantee a spanner whose size is independent of Ψ. In later papers, Kaplan et al. [9,10] showed a t-spanner for the Euclidean metric setting with $O(n)$ edges and a lower construction time of $\big(O(n(\log n + \log \Psi)$ or $n \log^5 n$ instead of $n \log n\big)$, thus reducing the running time and removing the dependence on the ratio Ψ.

36th European Workshop on Computational Geometry, Würzburg, Germany, March 16–18, 2020.
This is an extended abstract of a presentation given at EuroCG'20. It has been made public for the benefit of the community and should be considered a preprint rather than a formally reviewed paper. Thus, this work is expected to appear eventually in more final form at a conference with formal proceedings and/or in a journal.

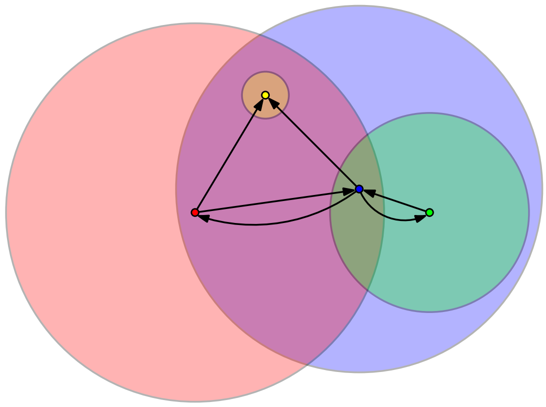

Figure 1 An example of a transmission graph.

The Path-Greedy Algorithm

The simple well-known greedy algorithm for constructing t-spanner was given by Althöfer et al. [1]. They proved that the algorithm can be used to achieve a t-spanner for arbitrary weighted graphs. For euclidean graphs, they prove that the algorithm guaranties a linear number of edges and a bounded degree, both depending on t. A weight bound of $O(wt(MST(P)))$, where P is a set of points in $\mathbb{R}^d, d \leq 3$ was given by Das et al. [4], and generalized for any dimension by Das and Narasimhan [5]. In their paper, Althöfer et al. also mentioned that the algorithm was independently proposed by Bern.

The algorithm itself is very simple, and it is a natural generalization of Kruskal's algorithm for finding an MST of a given graph, see Algorithm 1.

Algorithm 1: Path-Greedy

```
Input: A graph G = (V, E), 1 < t ∈ ℝ⁺
Result: A t-spanner G' = (V, E') for G
E ← Sort all edges of G in non-decreasing order
E' = ∅
for (u, v) ∈ E (in sorted order) do
    if δ_G'(u, v) > t · δ_G(u, v) then
        Add (u, v) to E'
return G'
```

Surprisingly, this simple algorithm gives both theoretical and experimental results that are better than many more complicated state-of-the art algorithms. Farshi and Gudmundsson [6] experimented with implementations of several well known algorithms and showed that the Path-Greedy algorithm out-preformed other algorithms even when the theoretical bounds were similar. As it can be seen from Tables II-V in [6], the Path-Greedy algorithm achieved significantly better results than other algorithms including *θ-Graph*, *WSPD* based spanners, *sink-spanner*, by building smaller and lighter spanners with a lower maxumium degree regardless of the distribution or number of input points.

From a theoretical point of view, Filtser and Solomon [7] have narrowed the gap between the experimental results showing the superiority of the Path-Greedy algorithm and the known upper bounds, by showing that the Path-Greedy is nearly optimal in many cases.

The simplicity and efficiency of the Path-Greedy algorithm and the naïve implementation with runtime $O(n^3 \log n)$ encouraged researchers to find faster algorithms that mimic or approximate it. An $O(n \log^2 n)$ time algorithm approxmating Path-Greedy was given by Das and Narasimhan [5], and was later improved to an $O(n \log n)$ time algorithm by Gudmundsson et al. [8], while Bar-On and Carmi [2] and Bose et al. [3] showed constructions of the Path-Greedy itself in $O(n^2 \log n)$.

Contribution

In this paper, and specifically in section 2, we provide a simple alternative to the fairly involved algorithms that were previously described, by proving in subsection 2.1 that the Path-Greedy algorithm is also applicable in the case of transmission graphs and provides a t-spanner with $O(\frac{n}{(t-1)^{d-1}})$ edges and in-degree $O(\frac{1}{(t-1)^{d-1}})$ for every real $t > 1$. We then prove in subsection 2.2 that the weight of the resulted spanner can be bounded by a function of the *radius-ratio*. For simplicity, we conduct our analysis in $\mathbb{R}^2$, however, all of the results extend naturally to $\mathbb{R}^d$ for $d > 2$ with the appropriate bounds of the d-dimensional Path-Greedy algorithm.

2 Path-Greedy Analysis

2.1 In-Degree Bound

We begin by showing that the in-degree of the vertices in the t-spanner created by the Path-Greedy algorithm on transmission graphs is bounded by a constant which is a function of the *stretch-factor* t, thus essentially proving the bound on the size of the t-spanner. In order to do so, we prove that if p is the sink of a directed edge $e = (q, p)$, then the angle between e and every other edge e' directed towards p is bigger than a constant depending on t.

▶ **Lemma 1.** Let $TG = (D, E)$ be a transmission graph, with $D = \{d_1, ..., d_n\}$ where $d_i = (c_i, r_i)$, $c_i \in \mathbb{R}^2$, is the center and $r_i \in \mathbb{R}$ is the radius of the disk, and $(d_i, d_j) \in E$ is a directed edge in the graph if $||c_i, c_j|| \leq r_i$. And let $G = (D, E')$ be the result of using the *Path-Greedy* algorithm on the input graph TG with $1 < t \in \mathbb{R}$. Then G is a t-spanner of TG, and $|E'| = O(\frac{n}{t-1})$.

Proof. It is rather simple to discern that G is a t-spanner of TG due to the exhaustive nature of the algorithm, and so we are left with proving the bound on the size of $|E'|$.

We provide a bound on the in-degree of any vertex in G by showing that any two edges with a point p as their destination, form an angle bigger than a certain constant depending on t. The t-spanning property of the resulted graph follows directly from the algorithm.

Let $e = (q, p), f = (r, p)$ be two edges in the set of spanner edges E', and let $\theta = \measuredangle qpr$. We assume w.l.o.g that $|qp| \geq |rp|$, and that $\overline{qp}$ is horizontal (see Figure 2), and also assume that $\theta \leq \frac{\pi}{4}$ since otherwise we are done. Let r' be the projection of r on $\overline{qp}$. Such a projection is possible and represents all possible cases due to our assumptions.

We now make the following observations:

- $|r'p| = |rp| \cdot \cos\theta$
- $|r'r| = |rp| \cdot \sin\theta$
- $|qr| < |r'r| + |qr'|$ (triangle inequality)

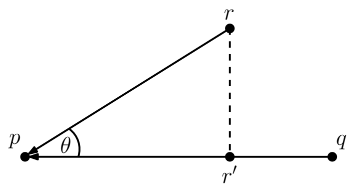

Figure 2 W.l.o.g $\overline{qp}$ is horizontal and at least as long as $\overline{rp}$

This gives us:

$$|qr| < |r'r| + |qr'| = |r'r| + (|qp| - |r'p|) = |rp| \cdot \sin\theta + |qp| - |rp| \cdot \cos\theta$$
$$= |qp| - |rp|(\cos\theta - \sin\theta),$$

which leads directly to:

$$\left(\frac{1}{\cos\theta - \sin\theta}\right)|qr| + |rp| < \left(\frac{1}{\cos\theta - \sin\theta}\right)|qp|.$$

Since we assume $\theta \leq \frac{\pi}{4}$, we get that $\overline{rq}$ is not the longest edge in Δqpr, and since we assume $\overline{rp} \leq \overline{qp}$ we get that $\overline{qr} \leq \overline{qp}$ as well, meaning that r is inside the disk centered at q. So, if $\left(\frac{1}{\cos\theta - \sin\theta}\right) \leq t$ we get that since the algorithm considered both $\overline{rp}$ and $\overline{qr}$ before $\overline{qp}$, the edge $\overline{qp}$ should not have been added to E' since at that point $\delta_G(q, r) \leq t \cdot |qr|$, which means that $\delta_G(q, r) + |rp| \leq t \cdot |qp|$, a contradiction to the choice of $\overline{qp}$ and $\overline{rp}$.

Thus, we get that the in-degree of any vertex $d \in D$ is at most $\frac{2\pi}{\theta}$. When t is big enough, it is clear that this degree is bounded by a constant, as the inequality $\frac{1}{\cos\theta - \sin\theta} \leq t$ is true for larger values of θ. But, as $t \to 1$, we get that $\theta \to 0$, meaning that the in-degree might be unbounded. We approximate $\frac{1}{\cos\theta - \sin\theta}$ using the Mclaurin series, and get that $\frac{1}{\cos\theta - \sin\theta} \approx 1 + \theta + O(\theta^2)$. It is now possible to see that as $t \to 1$ and after ommiting the negligible $O(\theta^2)$, we get that $\theta \leq t - 1$. So the in-degree of any vertex $d \in D$ is $O(\frac{1}{t-1})$, meaning that it is bounded by a constant depending on t, as reuiered.

◀

2.2 Weight Bound

In this section, we show a bound on the weight of *t-spanners* resulted by the Path-Greedy on transmission graphs. The bound is a function of the stretch factor t and a parameter called the *radius ratio*, which signifies the ratio between the largest and smallest radii amongst the given disks. More formally: let $G = (D, E')$ be the t-spanner computed by the Path-Greedy algorithm for the set D. Let $r_{max} = max\{r_i\}_{i=1}^n$, $r_{min} = min\{r_i\}_{i=1}^n$ and $\Psi = \frac{r_{max}}{r_{min}}$. Ψ is called the radius-ratio.

We show an upper bound on the total weight of the edges of G. That is, we show that $\sum_{e \in E'} |e|$ is

$$O((1 + \frac{1}{w}) \cdot \log n \cdot wt(MST(D))),$$

where $wt(MST(D))$ is the weight of the MST of the disk centers, and w is a constant that depends on the stretch factor t and the radius ratio.

A set of directed edges E satisfy the w-gap property if for any two directed edges (p, q) and (r, s) in E, we have that $|pr| > min(|pq|, |rs|)$. I.e., the sources of any two edges are relatively far apart with respect to the length of the shorter of the two edges. In this section, we slightly change this definition and consider the distances between sinks instead of the distances between sources. Notice that the two definitions are equivalent by changing the direction of the edges.

In Lemma 2, we show that any two edges in E that form an angle of size at most θ, satisfy the w-gap property. That is, if $e, f \in E$ are 2 edges contained in the lines l_e and l_f respectively, and the angle between the two rays emenating from $l_e \cap l_f$ and that contain e and f is at most θ, then e and f satisfy the w-gap property, where w is a constant depending on θ, which in turn depends on t. Except for the additional constraint on w to be also smaller than $\frac{1}{\Psi}$, this lemma is similar to Lemma 15.1.1. in [11]. It is well known (Theorem 6.1.2 [11]) that for a set E of directed edges that satisfies the *w-gap property* we have that the total weight of E is less than $(1 + \frac{2}{w}) \cdot \log |P| \cdot wt(MST(P))$, where P is the set of the end-points of E.

▶ Lemma 2. Let $G = (D, E')$ be the t-spanner computed by the *Path-Greedy* algorithm for the set D. Let θ and w be two real numbers such that $0 < \theta < \frac{\pi}{4}$, $0 < w < \frac{\cos\theta - sin\theta}{2}$, $w < \frac{1}{\Psi}$, and $t \geq \frac{1}{\cos\theta - sin\theta - 2w}$. Let (q, p), (s, r) be two distinct directed edges in E, such that $\measuredangle(\overline{qp}, \overline{sr}) \leq \theta$. Then, (q, p) and (s, r) satisfy the *w-gap property.*

Proof. Assume w.l.o.g., that the *Path-Greedy* algorithm considers the edge (s, r) before it considers the edge (q, p), thus $|sr| \leq |qp|$. Assume towards contradiction that $|pr| \leq w|sr|$ (see figure 3 for an illustration). By *Lemma 6.4.1* from [11] we get:

1. $|qs| < |qp|$
2. $|rp| < |qp|$
3. $t|qs| + |sr| + t|rp| \leq t|qp|$.

By our choice of w, we have $w \leq \frac{1}{\Psi} = \frac{r_{min}}{r_{max}}$, and by our assumption we get, $|pr| \leq w|sr|$, thus $|pr| \leq \frac{r_{min}}{r_{max}}|sr| \leq r_{min}$. Therefore, we have that p is in the disk corresponding to r (since $|pr| \leq r_{min}$). Moreover, we have that s is in the disk corresponding to q, since $|qs| < |qp|$.

When the *Path-Greedy* algorithm considered the edge (q, p), the spanner already contained a t-spanning path from v to z, if z is in the disk corresponding to v and $|vz| < |pq|$. Therefore, when the *Path-Greedy* algorithm considered the edge (q, p) the spanner already contained a directed path from q to s of length at most $t|qs|$ and the edge (s, r) and a directed path from r to p of length at most $t|rp|$. This contradicts the fact that edge (q, p) has been added to the spanner, since by the lemma we have that $t|qs| + |sr| + t|rp| \leq t|qp|$. Thus, we conclude that $|pr| > w|sr|$. ◀

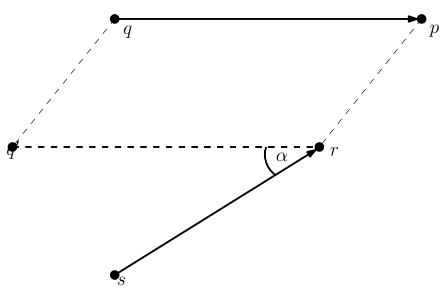

Figure 3 An illustration of the settings described in the proof above.

References

1 Ingo Althöfer, Gautam Das, David P. Dobkin, Deborah Joseph, and José Soares. On sparse spanners of weighted graphs. *Discrete & Computational Geometry*, 9:81–100, 1993. URL: https://doi.org/10.1007/BF02189308, doi:10.1007/BF02189308.

2 Gali Bar-On and Paz Carmi. \delta -greedy t-spanner. In *Algorithms and Data Structures - 15th International Symposium, WADS 2017, St. John's, NL, Canada, July 31 - August 2, 2017, Proceedings*, pages 85–96, 2017. URL: https://doi.org/10.1007/978-3-319-62127-2_8, doi:10.1007/978-3-319-62127-2_8.

3 Prosenjit Bose, Paz Carmi, Mohammad Farshi, Anil Maheshwari, and Michiel H. M. Smid. Computing the greedy spanner in near-quadratic time. *Algorithmica*, 58(3):711–729, 2010. URL: https://doi.org/10.1007/s00453-009-9293-4, doi:10.1007/s00453-009-9293-4.

4 Gautam Das, Paul J. Heffernan, and Giri Narasimhan. Optimally sparse spanners in 3-dimensional euclidean space. In *Proceedings of the Ninth Annual Symposium on Computational GeometrySan Diego, CA, USA, May 19-21, 1993*, pages 53–62, 1993. URL: https://doi.org/10.1145/160985.160998, doi:10.1145/160985.160998.

5 Gautam Das and Giri Narasimhan. A fast algorithm for constructing sparse euclidean spanners. *Int. J. Comput. Geometry Appl.*, 7(4):297–315, 1997. URL: https://doi.org/10.1142/S0218195997000193, doi:10.1142/S0218195997000193.

6 Mohammad Farshi and Joachim Gudmundsson. Experimental study of geometric *t*-spanners. *ACM Journal of Experimental Algorithmics*, 14, 2009. URL: https://doi.org/10.1145/1498698.1564499, doi:10.1145/1498698.1564499.

7 Arnold Filtser and Shay Solomon. The greedy spanner is existentially optimal. In *Proceedings of the 2016 ACM Symposium on Principles of Distributed Computing, PODC 2016, Chicago, IL, USA, July 25-28, 2016*, pages 9–17, 2016. URL: https://doi.org/10.1145/2933057.2933114, doi:10.1145/2933057.2933114.

8 Joachim Gudmundsson, Christos Levcopoulos, and Giri Narasimhan. Fast greedy algorithms for constructing sparse geometric spanners. *SIAM J. Comput.*, 31(5):1479–1500, 2002. URL: https://doi.org/10.1137/S0097539700382947, doi:10.1137/S0097539700382947.

9 Haim Kaplan, Wolfgang Mulzer, Liam Roditty, and Paul Seiferth. Spanners and reachability oracles for directed transmission graphs. In *31st International Symposium on Computational Geometry, SoCG 2015, June 22-25, 2015, Eindhoven, The Netherlands*, pages 156–170, 2015. URL: https://doi.org/10.4230/LIPIcs.SOCG.2015.156, doi:10.4230/LIPIcs.SOCG.2015.156.

10 Haim Kaplan, Wolfgang Mulzer, Liam Roditty, and Paul Seiferth. Spanners for directed transmission graphs. *SIAM J. Comput.*, 47(4):1585–1609, 2018. URL: https://doi.org/10.1137/16M1059692, doi:10.1137/16M1059692.

11 Giri Narasimhan and Michiel H. M. Smid. *Geometric spanner networks*. Cambridge University Press, 2007.

12 David Peleg and Liam Roditty. Relaxed spanners for directed disk graphs. *CoRR*, abs/0912.2815, 2009. URL: http://arxiv.org/abs/0912.2815, arXiv:0912.2815.

13 David Peleg and Liam Roditty. Localized spanner construction for ad hoc networks with variable transmission range. *TOSN*, 7(3):25:1–25:14, 2010. URL: https://doi.org/10.1145/1807048.1807054, doi:10.1145/1807048.1807054.