Disjoint tree-compatible plane perfect matchings*

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Abstract

Two plane drawings of geometric graphs on the same set of points are called disjoint compatible if their union is plane and they do not have an edge in common. For a given set S of $2n$ points two plane drawings of perfect matchings M_1 and M_2 (which do not need to be disjoint nor compatible) are *disjoint tree-compatible* if there exists a plane drawing of a spanning tree T on S which is disjoint compatible to both M_1 and M_2.

We show that the graph of all disjoint tree-compatible perfect geometric matchings on $2n$ points in convex position is connected if and only if $2n \geq 10$. Moreover, in that case the diameter of this graph is either 4 or 5, independent of n.

1 Introduction

Two plane drawings of geometric graphs on the same set S of points are called *compatible* if their union is plane. The drawings are *disjoint compatible* if they are compatible and do not have an edge in common. For a fixed class $\mathcal{G}$, e.g. matchings, trees, etc., of plane geometric graphs on S the (disjoint) *compatibility graph* of S has the elements of $\mathcal{G}$ as the set of vertices and an edge between two elements of $\mathcal{G}$ if the two graphs are (disjoint) compatible. For example, it is well known that the (not necessarily disjoint) compatibility graph of plane perfect matchings is connected [4, 5]. Moreover, in [2] it is shown that there always exists a sequence of at most $O(\log n)$ compatible (but not necessarily disjoint) matchings between any two plane perfect matchings of a set of $2n$ points in general position, that is, the graph of perfect matchings is connected with diameter $O(\log n)$. On the other hand, Razen [8] provides an example of a point set where this diameter is $\Omega(\log n / \log \log n)$.

Disjoint compatible (perfect) matchings have been investigated in [2] for sets of $2n$ points in general position. The authors show that for odd n there exist isolated matchings and pose the following conjecture: For every perfect matching with an even number of edges there exists a disjoint compatible perfect matching. This conjecture was answered in the positive by Ishaque et al. [7] and it was mentioned that for even n it remains an open problem

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whether the disjoint compatibility graph is always connected. In [1] it is shown that for sets of $2n \geq 6$ points in convex position this disjoint compatibility graph is (always) disconnected.

Both concepts, compatibility and disjointness, are also used in combination with different geometric graphs. For example, in [5] it is shown that the flip-graph of all triangulations that admit a (compatible) perfect matching, is connected. It has also been shown that for every graph with an outerplanar embedding there exists a compatible plane perfect matching [3]. Considering plane trees and simple polygons, the same work provides bounds on the minimum number of edges a compatible plane perfect matching must have in common with the given graph. See also the survey [6] on the related concept of compatible graph augmentation.

In a similar spirit we can define a bipartite disjoint compatible graph, where the two sides of the bipartition represent two different graph classes. For example, let one side be all plane perfect matchings of S, $|S| = 2n$, while the other side consists of all plane spanning trees of S. Edges represent the pairs of matchings and trees whose union results in a plane, edge-disjoint drawing. Considering connectivity of this bipartite graph there trivially exist isolated vertices on the tree side - consider a spanning star, which can not have any disjoint compatible matching. Thus, the question remains whether there exists a bipartite connected subgraph which contains all vertices representing plane perfect matchings.

This point of view leads us to a new notion of adjacency for matchings. For a given set S of $2n$ points two plane drawings of perfect matchings M_1 and M_2 (which do not need to be disjoint nor compatible) are *disjoint tree-compatible* if there exists a plane drawing of a spanning tree T on S which is disjoint compatible to both, M_1 and M_2. The idea is that in the disjoint tree-compatible graph G_{2n} we have an edge between M_1 and M_2 as they are only two steps apart if we consider a disjoint and compatible transformation via T. Rephrasing the above question we ask whether G_{2n} is connected? Recall that the disjoint compatible graph for matchings alone is not connected (see [1, 2]) and that without disjointness the result of [5] already implies that the tree-compatible graph of matchings is connected.

In this paper we show that the disjoint tree-compatible graph of perfect geometric matchings on $2n$ points in convex position is connected if and only if $2n \geq 10$. Moreover, in that case the diameter of this graph is either 4 or 5, independent of n.

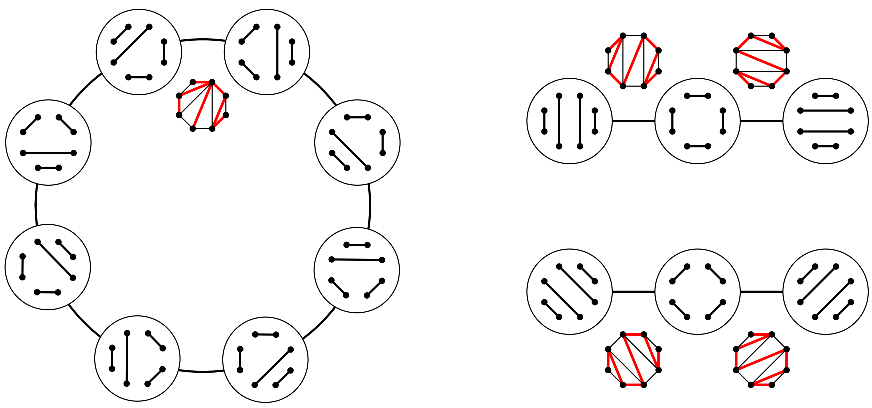

Figure 1 The disjoint tree-compatible graph G_8. Each vertex represents a plane matching on a convex point set of eight points. Two matchings are connected by an edge if they are tree-compatible (the trees are drawn in red).

1.1 Basic Definitions

Throughout this paper the point set S consists of $2n$ points in convex position. For simplicity we use the terms *matching* and *tree* for plane perfect matchings and plane spanning trees.

▶ **Definition 1.1.** Edges spanned by two neighbouring points on the boundary of the convex hull of S are called *perimeter edges*; all other edges spanned by S are called *diagonals*.

▶ **Definition 1.2.** A *perimeter matching* is a matching containing no diagonal. We label the sides of the convex hull of S alternately 'odd' and 'even'. Then the perimeter matching consisting of only odd perimeter edges is called *odd perimeter matching*, the one consisting of only even perimeter edges is called *even perimeter matching*.

2 Upper bound

We show that any matching on $2n \geq 10$ points in convex position has small distance to one of the two perimeter matchings which are themselves close to each other in the disjoint tree-compatible graph G_{2n}. When done carefully, this gives an upper bound of 5 on the diameter of G_{2n}.

First we introduce key notions of a semicycle, a cycle of edges and a rotation (see Figure 2).

▶ **Definition 2.1.** Let M be a matching on S. A set X of $k \geq 2$ matching edges is called a *k-semicycle* if the interior of the convex hull of X does not intersect any edges of M. Given a k-semicycle X, the perimeter of its convex hull (including the non-matching edges) is called its *k-cycle* and denoted by $\bar{X}$. A k-cycle $\bar{X}$ is called an *inside k-cycle* (or just an *inside cycle*) if $\bar{X}$ contains at least two diagonals, otherwise it is called a *k-ear* (or just an *ear*). Finally, given a semicycle X, we can obtain a matching $M' = \text{rot}(\bar{X})$ by *rotating* the cycle $\bar{X}$, that is, by omitting from M edges in X and including edges in $\bar{X} \setminus X$.

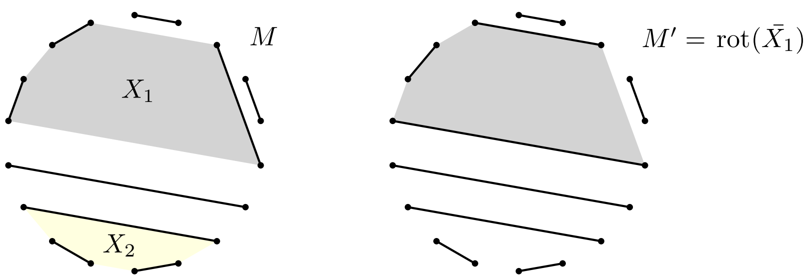

Figure 2 A matching M with convex hulls of two of its 3-semicycles X_1, X_2 shaded. The cycle $\bar{X}_1$ corresponding to X_1 is an inside cycle, since the boundary of the grey region contains at least two (in fact three) diagonals. The cycle $\bar{X}_2$ is an ear. Rotating $\bar{X}_1$, we obtain a matching $M' = \text{rot}(\bar{X}_1)$.

With this notation in place we show that any number of inside cycles can be simultaneously rotated in one step and that sufficiently long ears can be rotated in at most 3 steps.

▶ **Lemma 2.2.** *Let M, M' be two matchings whose symmetric difference is a union of disjoint inside cycles. Then M and M' are tree-compatible to each other.*

Proof idea. The idea is that the union of M and M' can be extended to a triangulation such that every inside cycle has at most two vertices of S which are neighbored in this triangulation to only two other points of the inside cycle. We argue that the added edges form a connected graph spanning all the nodes of S. Removing edges one by one we create a spanning tree edge-disjoint to both matchings (see Figure 3). See full version for a full proof.

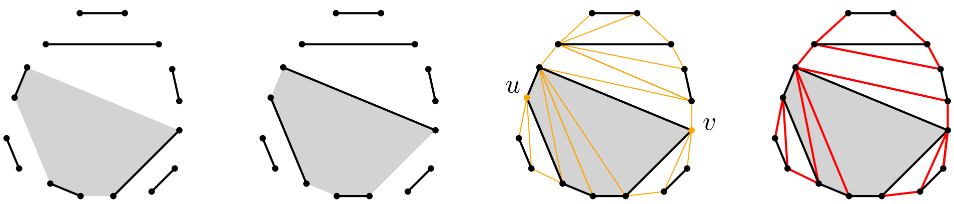

Figure 3 The union of two matchings differing at a single inside cycle (shaded) is extended to a suitable triangulation using yellow edges such that the yellow edges contain a spanning tree (red).

▶ **Lemma 2.3.** *Let M, M' be two matchings whose symmetric difference is a k-ear with $k \geq 6$. Then M and M' have distance at most 3 (in G_{2n}).*

Proof idea. The idea is to do three rotations as in Figure 4. See full version for a full proof.

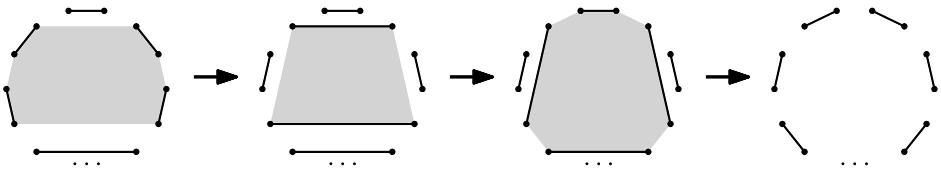

Figure 4 A 6-ear can be rotated in 3 steps (in each step we rotate the grey inside cycle).

▶ **Theorem 2.4.** *For $2n \geq 10$, the graph G_{2n} is connected and* $\mathrm{diam}(G_{2n}) \leq 5$.

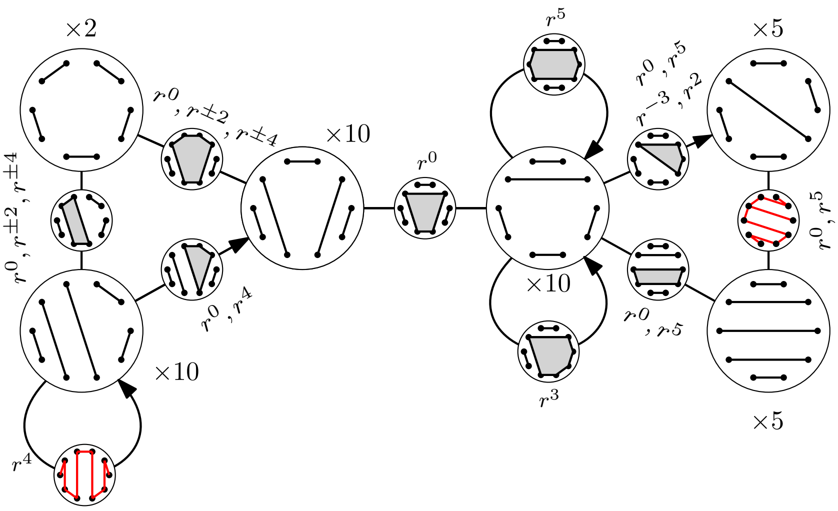

Figure 5 All 42 nodes of G_{10}, the letter r stands for a possible rotation by $2\pi/10$. Going against the arrows rotates in opposite direction.

Proof idea. When $2n = 10$, the claim can be checked in Figure 5. When $2n \geq 12$, the idea is to show that all matchings can be quickly transformed either to the odd perimeter matching O or to the even perimeter matching E (or to both – by Lemma 2.3 we have $\mathrm{dist}(O, E) \leq 3$). In particular, for a fixed matching M we denote by $\mathrm{d_{min}}(M)$ (resp. $\mathrm{d_{max}}(M)$) the distance from M to the closer (resp. further) perimeter matching. Then we prove that the non-perimeter matchings can be split into three classes S_1, S_2, S_3 with the following properties:

1. $\forall M \in S_1$ we have $\mathrm{d_{min}}(M) \leq 1$ (and hence $\mathrm{d_{max}}(M) \leq 1 + 3 = 4$);
2. $\forall M \in S_2$ we have $\mathrm{d_{min}}(M) \leq 2$ and $\mathrm{d_{max}}(M) \leq 3$;
3. $\forall M \in S_3$ we have $\mathrm{d_{max}}(M) \leq 3$ and $\forall M, M' \in S_3$ we have $\mathrm{dist}(M, M') \leq 4$.

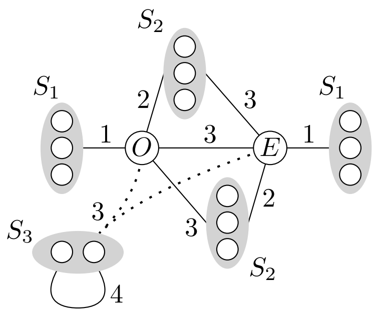

Figure 6 A partitioning of the non-perimeter matchings into sets S_1, S_2, S_3.

This guarantees that $\mathrm{diam}(G_{2n}) \leq 5$. See full version for a full proof.

3 Lower bound

In this chapter we prove the following theorem by constructing two matchings with distance at least 4.

▶ **Theorem 3.1.** *The diameter of the disjoint tree-compatible graph G_{2n} for $2n \geq 10$ can be lower bounded by 4.*

In the same way as we defined inside cycles and ears for cycles, we now introduce notions of inside semicycles and semiears for semicycles.

▶ **Definition 3.2.** Let M be a matching on S. A k-semicycle X is called an *inside k-semicycle* (or just an *inside semicycle*) if $\bar{X}$ contains at least two diagonals, otherwise it is called a *k-semiear* (or just a *semiear*).

▶ **Definition 3.3.** Let M and M' be two matchings in S. A *boundary area with k points* is an area within the convex hull of S restricted by edges in M and M' such that the matching edges intersect at least once and the points on the boundary of the area are adjacent on the boundary of the convex hull of S; see Figure 7.

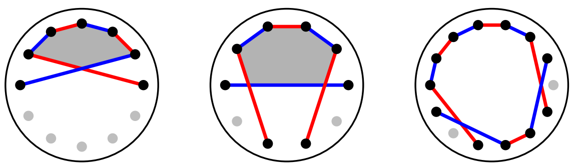

Figure 7 Boundary areas with five points (left) and four points (middle). The drawing on the right does not show a boundary area; not all points are neighbouring on the convex hull of S.

▶ **Definition 3.4.** A matching M on a set of $4k$ points is called a *2-semiear matching* if it consists of exactly k 2-semiears and an inside k-semicycle. A matching M on a set of $4k+2$ points is called a *near-2-semiear matching* if it consists of exactly k 2-semiears and an inside $(k+1)$-semicycle.

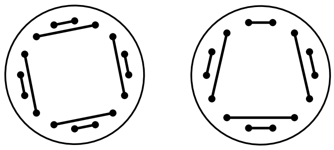

Figure 8 Left: A 2-semiear matching. Right: A near-2-semiear matching.

▶ Remark. Analogous to perimeter matchings we can distinguish between odd and even 2-semiear matchings, according to the values taken by the respective perimeter edges.

▶ **Lemma 3.5.** *Let M, M' be two matchings whose symmetric difference is an ear or a boundary area with at least three points. Then M and M' are not tree-compatible to each other.*

Proof idea. The idea is to apply a counting argument, a detailed proof is in the full version.

▶ **Lemma 3.6.** *Let M be a matching tree-compatible to an even 2-semiear-matching. Then M contains no odd perimeter edge.*

Proof idea. Adding an odd perimeter edge always yields either an ear or a boundary area with at least three points (cf. Figure 9), a detailed proof is in the full version.

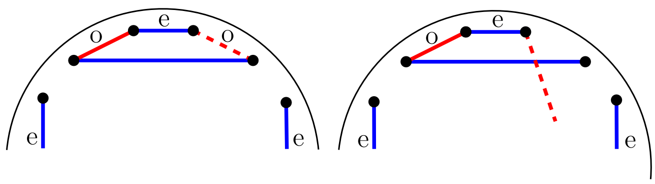

Figure 9 An (even) 2-semiear matching (in blue) and a (red) matching with at least one odd perimeter edge; the matchings create an ear (left) or a boundary area with three points (right).

▶ **Lemma 3.7.** *Let M be a matching tree-compatible to a near-2-semiear-matching M' consisting of k even and one odd perimeter edge. Then M contains at most one odd perimeter edge (the one in M').*

The proof works analogously to the proof of Lemma 3.6.

▶ **Lemma 3.8.** *Let M and M' be two tree-compatible matchings. Then M and M' have at least two perimeter edges in common.*

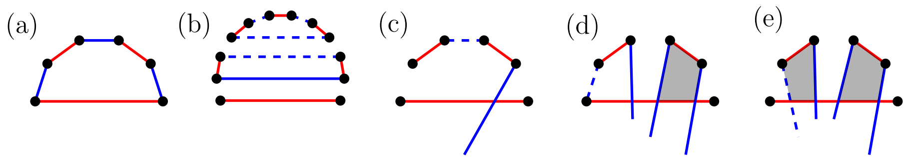

Figure 10 All possible cases for a semiear of size $k \geq 3$ in a matching M (depicted in red) and a second matching M' (depicted in blue) which does not use any of the perimeter edges in M.

Proof idea. We focus on proving that M and M' have one perimeter edge in common. Since we only use local arguments we can extend these to show that M and M' have at least two perimeter edges in common.

If M contains an ear of size at least three, then one of the perimeter edges of this ear is also in M'. Otherwise the union of the two matchings would give something like in Figure 10 which forbids a disjoint spanning tree.

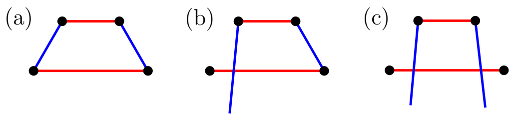

Figure 11 All possible cases for a 2-ear in a matching M (depicted in red) and a second matching M' (depicted in blue) which does not use the perimeter edges in M

So we can assume that M only has 2-semiears. If M' contains the perimeter edge of a 2-semiear of M, then we are done. So assume this is not the case. If we have a union of M and M', which looks locally like Figure 11(a) or Figure 11(b), then M and M' are not disjoint tree-compatible. So the only possibility that M and M' are disjoint tree-compatible and do not share a perimeter edge of a 2-semiear is depicted in Figure 11(c). Out of the 2-semiears of M we choose the one with no further semiear of M on one side of a diagonal d in M'. This is possible since the number of semiears is finite and the diagonals in M' cannot intersect each other, therefore there is an ordering of the 2-semiears in M. Since d is a diagonal, there exists a semiear E' on this side of the drawing in M'. Every edge of M on this side of d is a perimeter edge or intersects d, since there does not exist an semiear to this side of d in M. If E' is a 2-semiear and one diagonal in M intersects d, we get another blocking structure. This means that the perimeter edge of E' is also in M.

▶ **Corollary 3.9.** *Let S be of size $2n \geq 10$. For even n, the distance between an even 2-semiear matching and an odd 2-semiear matching is at least 4.*
For odd n, let M be a near-2-semiear matching with a single odd perimeter edge and M' be a near-2-semiear matching with a single even perimeter edge such that those two edges are incident in S. Then the distance between M and M' is at least 4.

Proof idea. We obtain the statement by applying Lemma 3.6 (for n even) or 3.7 (for n odd), respectively, and Lemma 3.8, cf. Figure 12. A detailed proof is in the full version.

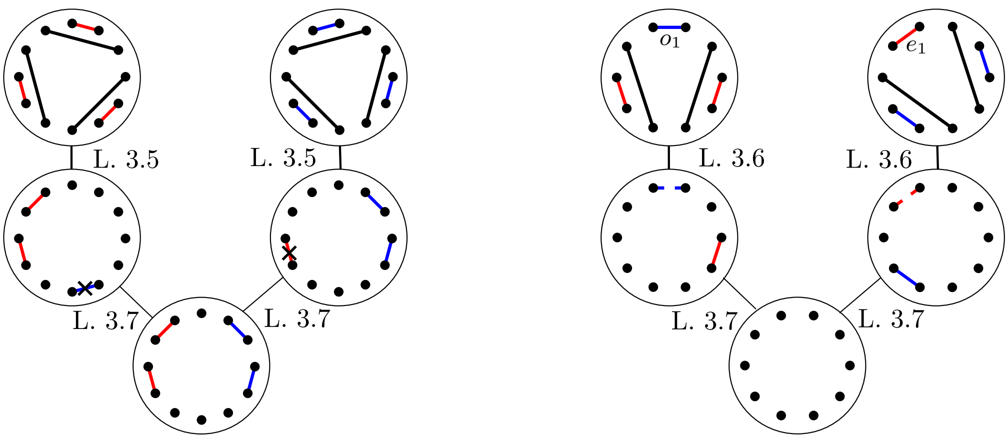

Figure 12 Illustrations, that the distance between two special 2-semiear matchings (left) and between two special near-2-semiear matchings (right) is at least 4. Even perimeter edges are drawn in red, odd ones are drawn in blue. The numbers next to the edges indicate which Lemma is applied.

4 Conclusion

We have shown that the diameter of the disjoint tree-compatible graph G_{2n} of disjoint tree-compatible matchings for points in convex position is 4 or 5 when $2n \geq 10$. Due to further computations, we conjecture that the diameter for all $2n \geq 18$ is 4. Further we obtained some results for the clique number of G_{2n}. Still, the question whether G_{2n} is connected for general point sets is open.

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