Representing Graphs by Polygons with Side Contacts in 3D*

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Abstract

A graph has a side-contact representation with polygons if its vertices can be represented by interior-disjoint polygons such that two polygons share a common side if and only if the corresponding vertices are adjacent. In this work we study representations in 3D. We show that every graph has a side-contact representation with polygons in 3D, while this is not the case if we additionally require that the polygons are convex: we show that every supergraph of K_5 and every nonplanar 3-tree does not admit a representation with convex polygons. On the other hand, $K_{4,4}$ admits such a representation, and so does every graph obtained from a complete graph by subdividing each edge once. Finally, we construct an unbounded family of graphs with average vertex degree $12 - o(1)$ that admit side-contact representations with convex polygons in 3D. Hence, such graphs can be considerably denser than planar graphs.

1 Introduction

A graph has a contact representation if its vertices can be represented by interior-disjoint geometric objects[1] such that two objects touch exactly if the corresponding vertices are adjacent. In concrete settings, one usually restricts the set of geometric objects (disks, lines, polygons, ...), the type of contact, and the embedding space. Numerous results about which graphs admit a contact representation of some type are known. Giving a comprehensive overview is out of scope for this extended abstract. We therefore mention only few results. By the Andreev–Koebe–Thurston circle packing theorem [3, 20] every planar graph has a contact representation by touching disks in 2D. Contact representations of graphs in 2D have since been considered for quite a variety of shapes, including triangles [4, 8, 13, 14, 19],

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[1] If the considered objects are not fully-dimensional in the considered space then *interior-disjoint* is meant with respect to the relative interior of the objects.

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axis-aligned rectangles [1, 5, 11], curves [15], or line segments [7, 6, 16] in 2D and balls [17], tetrahedra [2] or cubes [12, 18] in 3D. Evans et al. [10] showed that every graph has a contact representation in 3D in which each vertex is represented by a convex polygon and two polygons touch *in a corner* if and only if the corresponding two vertices are adjacent.

In this work we study contact representations with polygons in 3D where a contact between two polygons is realized by sharing a proper side that is not part of any other polygon of the representation. (To avoid confusion with the corresponding graph elements, we consistently refer to polygon vertices as *corners* and to polygon edges as *sides*.) The special case where we require that the polygons are convex is of particular interest. Note that we do not require that the polyhedral complex induced by the contact representation is a closed surface. In particular, not every polygon side has to be in contact with another polygon. By Steinitz's theorem [21], every 3-connected planar graph can be realized as a convex polyhedron, whose dual is also a planar graph. Thus all planar graphs have such a representation with convex polygons.

Results. We show that for the case of nonconvex polygons, every graph has a side-contact representation in 3D. For convex polygons, the situation is more intricate. We show that certain graphs do not have such a representation, in particular all nonplanar 3-trees and all supergraphs of K_5. On the other hand, many nonplanar graphs (for example, $K_{4,4}$) have such a representation. In particular, graphs that admit side-contact representations with convex polygons in 3D can be considerably denser than planar graphs. Due to lack of space, several proofs are only sketched or completely deferred to the full version of this work.

2 Representations with General Polygons

First we show that every graph can be represented by nonconvex polygons; see Figure 1.

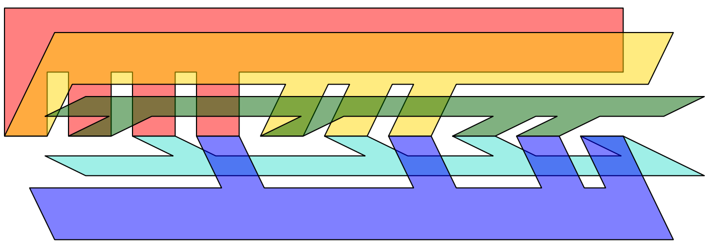


Figure 1 A realization of K_5 by nonconvex polygons with side contacts in 3D.

▶ **Proposition 2.1.** *Every graph can be realized by polygons with side contacts in 3D.*

Proof. To represent a graph G with n vertices, we start with n interior-disjoint rectangles such that there is a line segment s that acts as a common side of all these rectangles. We then cut away parts of each rectangle thereby turning it into a comb-shaped polygon as illustrated in Figure 1. This step ensures that for each pair (P, P') of polygons, there is a subsegment s' of s such that s' is a side of both P and P' that is disjoint from the remaining polygons. The result is a representation of K_n. To obtain a realization of G, it remains to remove side contacts that correspond to unwanted adjacencies, which is easy. ◀

If we additionally insist that each polygon shares all of its sides with other polygons, the polygons describe a closed volume. In this model, K_7 can be realized as the Szilassi polyhedron; see Figure 2. The tetrahedron and the Szilassi polyhedron are the only two known polyhedra in which each face shares a side with each other face [22]. Which other graphs can be represented in this way remains an open problem.

Figure 2 The Szilassi polyhedron realizes K_7 by nonconvex polygons with side contacts in 3D [22].

3 Representations with Convex Polygons

We now consider the setting where each vertex of the given graph is represented by a convex polygon in 3D and two vertices of the given graph are adjacent if and only if their polygons intersect in a common side. (In most previous work, it was only required that the side of one polygon is contained in the side of the adjacent polygon. For example, Duncan et al. [9] showed that in this model every planar graph can be realized by hexagons in the plane and that hexagons are sometimes necessary.) Note that it is allowed to have sides that do not touch other polygons. Further, non-adjacent polygons may intersect at most in a common corner. We start with some simple observations.

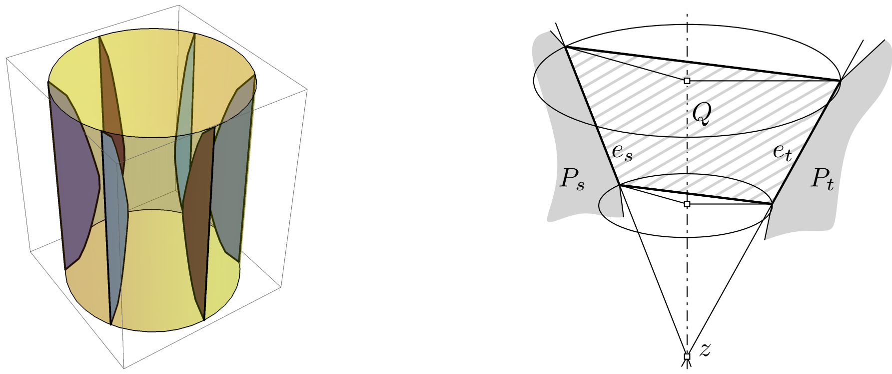
▶ **Proposition 3.1.** *Every planar graph can be realized by convex polygons with side contacts in 2D.*

Proof. Let G be a planar (embedded) graph with at least three vertices (for at most two vertices the statement is trivially true). Add to G a new vertex r and connect it to all vertices of some face. Let G' be a triangulation of the resulting graph. Then the dual $G^\star$ of G' is a cubic 3-connected planar graph. Using Tutte's barycentric method, draw $G^\star$ into a regular polygon with $\deg_{G'}(r)$ corners such that the face dual to r becomes the outer face. Note that the interior faces in this drawing are convex polygons. Hence the drawing is a side-contact representation of $G' - r$. To convert it to a representation of G, we may need to remove some side contacts, which can be easily achieved. ◀

Note that Proposition 3.1 also follows directly from the Andreev–Koebe–Thurston circle packing theorem. So for planar graphs, corner and side contacts behave similarly. For nonplanar graphs (for which we need the third dimension), the situation is different. Here, side contacts are more restrictive. We introduce the following notation. In a 3D representation of a graph G by polygons, we denote by P_v the polygon that represents vertex v of G.

▶ **Lemma 3.2.** *Let G be a graph. Consider a 3D side-contact representation of G with convex polygons. If G contains a triangle uvw, polygons P_v and P_w lie in the same halfspace with respect to the supporting plane of P_u.*

Proof. Due to their convexity, P_v and P_w must lie in the same halfspace with respect to the plane that supports P_u, otherwise P_v and P_w cannot share a side. In this case, the edge vw of G would not be represented; a contradiction. ◀



(a) The arrangement of the polygons $P_1, \ldots, P_n$. **(b)** Quadrilateral Q spanned by e_s and e_t.

Figure 3 Illustration for the proof of Proposition 3.4.

▶ **Proposition 3.3.** *For $n \geq 5$, K_n is not realizable by convex polygons with side contacts in 3D.*

Proof. Assume that K_n admits a 3D side-contact representation. Since every three vertices in K_n are pairwise connected, by Lemma 3.2, for every polygon of the representation, its supporting plane has the remaining polyhedral complex on one side. In other words, the complex we obtain is a subcomplex of a convex polyhedron. Consequently, the dual graph has to be planar, which rules out K_n for $n \geq 5$. ◀

▶ **Proposition 3.4.** *Let K'_n be the subdivision of the complete graph K_n in which every edge is subdivided with one vertex. For every n, K'_n has a side-contact representation with convex polygons in 3D.*

Proof sketch. For $n \leq 4$ the statement is true by Proposition 3.1. We sketch the construction of a representation for $n \geq 5$; see Figure 3. Let P be a convex polygon with $k = 2\binom{n}{2}$ corners, called $v_1, v_2, \ldots, v_k$, such that $v_1 v_k$ is a long side and the remaining corners form a flat convex chain connecting v_1 and v_k. We represent each high-degree vertex of K'_n by a copy of P. We arrange those copies in pairwise different vertical planes containing the z-axis such that all copies of $v_1 v_k$ are arranged vertically at the same height and at the same distance from the z-axis; and such that the convex chain of each copy of P faces the z-axis but does not intersect it. Consider two different copies P_s and P_t of P in this arrangement. They contain copies e_s and e_t of the same side e of P. It can be shown that e_s and e_t are coplanar. Moreover, they form a convex quadrilateral Q that does not intersect the arrangement except in e_s and e_t. We arbitrarily assign each side $v_{2i-1}v_{2i}$, $1 \leq i \leq k/2 = \binom{n}{2}$, to some edge st of K_n and use the quadrilateral Q spanned by e_s and e_t to represent the subdivision vertex of st in K'_n. As any two such quadrilaterals are vertically separated and hence disjoint, those $\binom{n}{2}$ quadrilaterals together with the n copies of P constitute a valid representation of K'_n. ◀

▶ **Proposition 3.5.** *$K_{4,4}$ is realizable by convex polygons with side contacts in 3D.*

Proof sketch. We sketch how to obtain a realization. Start with a box in 3D and intersect it with two rectangular slabs as indicated in Figure 4 on the left. We can now draw polygons on the faces of this complex such that each of the four vertical faces contains a polygon that has a side contact with a polygon on each of the four horizontal or slanted faces. The polygons on the slanted faces lie in the interior of the box and intersect each other. To remove this intersection, we pull out one corner of the original box; see Figure 4. ◀

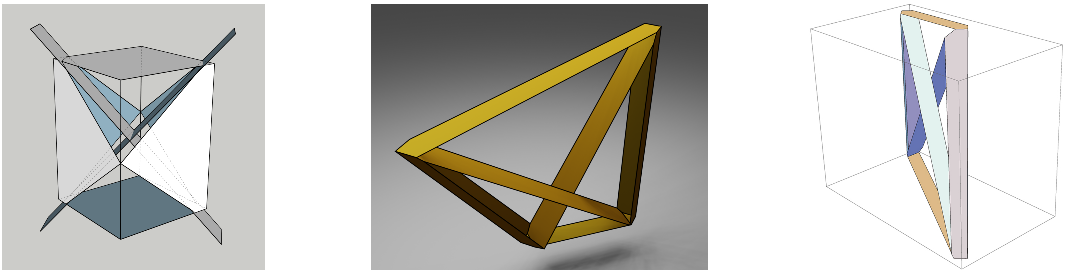


Figure 4 A realization of $K_{4,4}$ by convex polygons with side contacts in 3D.

In contrast to Proposition 3.5, we believe that the analogous statement does not hold for all bipartite graphs, i.e., we conjecture the following.

▶ **Conjecture 3.6.** *There exist values n and m such that the complete bipartite graph $K_{m,n}$ is not realizable by convex polygons with side contacts in 3D.*

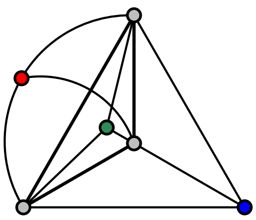
By Proposition 3.1, all planar 3-trees can be realized by convex polygons with side contacts (even in 2D). On the other hand, we can show that no nonplanar 3-tree has a realization in 3D. To this end, we prove the following two propositions, the first of which easily follows from the definition of 3-trees.

▶ **Lemma 3.7.** *A 3-tree is nonplanar if and only if it contains the graph depicted in Figure 5a as a subgraph.*

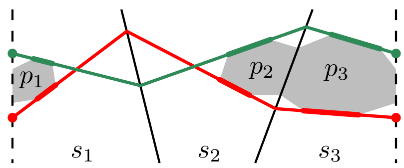
▶ **Lemma 3.8.** *The 3-tree depicted in Figure 5a cannot be realized by convex polygons with side contacts in 3D.*

▶ **Theorem 3.9.** *A 3-tree admits a side-contact representation with convex polygons in 3D if and only if it is planar.*

It is an intriguing question how dense graphs that admit a side-contact representation with convex polygons in 3D can be. In contrast to the results for corner contacts [10] and nonconvex polygons (Proposition 2.1) in 3D, we could not find a construction with a superlinear number of edges. The following construction yields the densest graphs we know.



(a) A 3-tree that is not realizable by convex polygons with side contacts in 3D. The gray vertices form a 3-cycle.

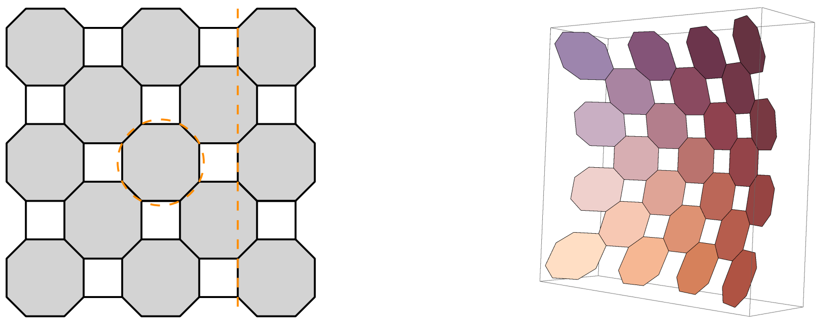


(b) Schematic drawing of a potential realization. Net of the three gray polygons and traces of the planes that contain the red and green polygons, which must touch each of the gray polygons. The line of intersection between two of the gray polygons is drawn twice (dashed).

Figure 5 Illustrations for the proof of Lemma 3.8.

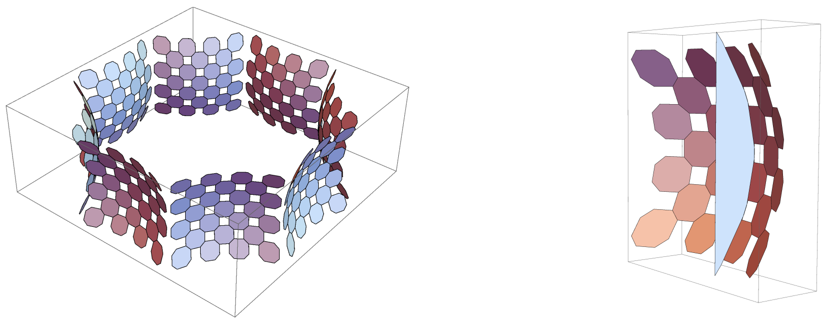
▶ **Theorem 3.10.** *There is an unbounded family of graphs with average vertex degree* $12-o(1)$ *that admit side-contact representations with convex polygons in 3D.*

Proof sketch. We first construct a contact representation of $m = \lceil\sqrt{n}\rceil$ regular octagons arranged as in a truncated square tiling; see Figure 6(a). Since the underlying geometric graph of the tiling is a Delaunay tessellation, we can lift the points to the paraboloid such that each octagon is lifted to coplanar points. We call the corresponding (scaled and rotated) polyhedral complex Γ; see Figure 6(b). Next we place $\lfloor\sqrt{n}\rfloor$ copies of Γ in a cyclic fashion as



(a) The plane octagon structure. **(b)** The complex Γ.

Figure 6 Proof of Theorem 3.10: construction of the complex Γ.



(a) The placement of the copies of Γ. **(b)** One vertical polygon.

Figure 7 Proof of Theorem 3.10: placement of the copies of Γ and vertical polygons.

shown in Figure 7(a) and we add vertical polygons in to generate a contact with the $\Theta(\sqrt{m})$ vertical sides of the octagons; see Figure 7(b). Finally, we introduce horizontal polygons in the "inner space" of our construction such that each of these polygons touches a specific side in each copy of Γ, as illustrated in Figure 8(a). A slight perturbation fixes the following two issues: First, many of the horizontal polygons lie on the same plane and intersect each other. Second, many sides of vertical polygons run along the faces of Γ. To fix these problems we modify the initial grid slightly; see Figure 8(b). ◀

4 Conclusion and Open Problems

Applying Turán's theorem [23] to Proposition 3.3 yields that the maximum number $e_{\text{cp}}(n)$ of edges in an n-vertex graph that admits a side-contact representation with convex polygons is at most $\frac{3}{8}n^2$. Theorem 3.10 gives a lower bound of $6n - o(n)$ for $e_{\text{cp}}(n)$. We tend to believe

(a) The location of a horizontal polygon as seen in a cross section.

(b) The modifications for Γ to separate nondisjoint faces.

Figure 8 Proof of Theorem 3.10: horizontal polygons and final modifications.

that the latter is closer to the truth than the former and conclude with the following open problem.

▶ **Question 4.1.** *What is the maximum number* $e_{\mathrm{cp}}(n)$ *of edges that an* n*-vertex graph admitting a side-contact representation with convex polygons can have?*

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