On Minimal-Perimeter Lattice Animals*

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Abstract

A *lattice animal* is a connected set of cells on a lattice. The *perimeter* of a lattice animal A consists of all the cells that do not belong to A, but that have a least one neighboring cell of A. We consider *minimal-perimeter* lattice animals, that is, animals whose periemeter is minimal for all animals of the same area, and provide a set of conditions that are sufficient for a lattice to have the property that inflating all minimal-perimeter animals of a certain size yields (without repetitions) all minimal-perimeter animals of a new, larger size. We demonstrate this result for polyhexes (animals on the two-dimensional hexagonal lattice). In addition, we provide two efficient algorithms for counting minimal-perimeter polyhexes.

1 Introduction

An *animal* on a d-dimensional lattice is a connected set of lattice cells, where connectivity is through $(d-1)$-dimensional faces of the cells. Specifically, in two dimensions, connectivity is through lattice edges. Two animals are considered identical if one can be obtained from the other by *translation* only, without rotations or flipping.

Lattice animals attracted interest as combinatorial objects [4] and as key players in a model in statistical physics and chemistry [9]. In this paper, we consider lattices in two dimensions, specifically, the hexagonal, triangular, and square lattices, where animals are called polyhexes, polyiamonds, and polyominoes, respectively.

Let $A^{\mathcal{L}}(n)$ denote the number of animals of size n, that is, animals composed of n cells, on the lattice $\mathcal{L}$. A major research problem in the study of lattices is understanding the nature of $A^{\mathcal{L}}(n)$, either by finding a formula for it as a function of n, or by evaluating it for specific values of n. This problem is to this date still open for any nontrivial lattice $\mathcal{L}$. Redelmeier [7] introduced the first algorithm that generates (and counts) all polyominoes of a given size, with no polyomino being generated more than once. The first algorithm for counting lattice animals without generating all of them was introduced by Jensen [6]. Using his method, the number of animals on the 2-dimensional square, hexagonal, and triangular lattices were computed up to size 56, 46, and 75, respectively.

An important property of lattice animals is the size of their *perimeter* (sometimes called "site perimeter"). The perimeter of a lattice animal is defined as the set of empty cells adjacent to the cells of the animal.

In this paper, we consider *minimal-perimeter* animals, that is, animals with the minimal perimeter within all animals of the same size. Altshular et al. [1] and Sieben [8] characterized all polyominoes with maximum size for their perimeter, and provided a formula for the minimum perimeter of a polyomino of size n. Similar results for polyiamonds and polyhexes were given by Fülep and Sieben [5] and by Vainsencher and Bruckstein [10], respectively.

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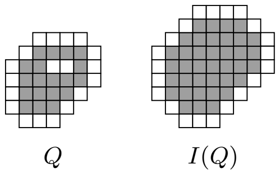


Figure 1 An example of a polyomino Q and its inflated polyomino $I(Q)$. Polyomino cells are colored in gray, perimeter cells are colored in white.

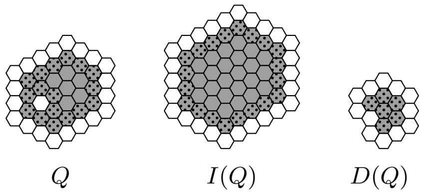


Figure 2 A polyhex Q, its inflated polyhex $I(Q)$, and its deflated polyhex $D(Q)$. The gray cells belong to Q, the white cells are its perimeter, and its border cells are marked with a pattern of dots.

Recently, we studied properties of minimal-perimeter polyominoes [2, 3]. A key notion in our findings was the *inflation* operation. Simply put, inflating a polyomino is adding to it all its perimeter cells (see Figure 1). We showed that inflating all minimal-perimeter polyominoes of some size yields all minimal-perimeter polyominoes of some larger size. In other words, the inflation operation induces a bijection between sets of minimal-perimeter polyominoes. Other results include efficient algorithms for counting minimal-perimeter polyominoes of a given size. In this paper, we provide a nontrivial generalization of this result to any lattice and find a sufficient set of conditions for such a bijection to exist. We also provide efficient counting algorithms for minimal-perimeter polyhexes.

2 Preliminaries

Let $\mathcal{L}$ be a lattice, and Q be an animal on $\mathcal{L}$. The *perimeter* of Q, denoted by $\mathcal{P}(Q)$, is the set of all empty lattice cells that are neighbors of at least one cell of Q. Similarly, the *border* of Q, denoted by $\mathcal{B}(Q)$, is the set of all cells of Q that are neighbors of at least one empty cell.

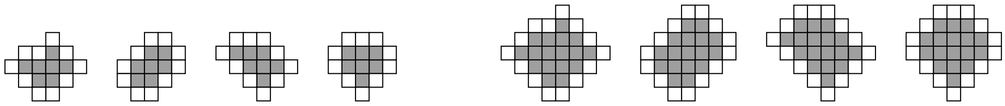
The *inflated* version of Q is defined as $I(Q) := Q \cup \mathcal{P}(Q)$. Similarly, the *deflated* version of Q is defined as $D(Q) := Q \backslash \mathcal{B}(Q)$. These operations are demonstrated in Figure 2.

Denote by $\epsilon^{\mathcal{L}}(n)$ the minimum size (number of cells) of the perimeter of n-cell animals on $\mathcal{L}$, and by $M_n^{\mathcal{L}}$ the set of all minimal-perimeter n-cell animals on $\mathcal{L}$.

Let $\mathcal{S}$ be the two-dimensional square lattice. As mentioned above, animals on $\mathcal{S}$ are usually called *polyominoes*. For this lattice, we know the following.

▶ **Theorem 2.1.** [2, Thm. 4] $\left|M_n^{\mathcal{S}}\right| = \left|M_{n+\epsilon^{\mathcal{S}}(n)}^{\mathcal{S}}\right|$ *(for $n \geq 3$).*

This theorem is a corollary of another theorem that states that the inflation operation induces bijections between sets of minimal-perimeter polyominoes. This is demonstrated in Figure 3. The proof of Theorem 2.1 is based directly on properties of the square lattice. In this paper, we present a nontrivial generalization of this theorem to animals on any lattice which fulfils a set of conditions. This result is stated in Theorem 3.1.



(a) All minimal-perimeter polyominoes of size 7 (up to rotations)

(b) All minimal-perimeter polyominoes of size 17 (up to rotations)

Figure 3 A demonstration of Theorem 2.1.

3 Inflation of Minimal-Perimeter Animals

Our main result consists of a set of conditions, which is sufficient for minimal-perimeter animals to satisfy a claim similar to the one stated in Theorem 2.1. Throughout this section, we consider animals on some specific lattice $\mathcal{L}$.

3.1 A Bijection

▶ **Theorem 3.1.** *Consider the following set of conditions for some lattice $\mathcal{L}$.*

(1) The function $\epsilon^{\mathcal{L}}(n)$ is weakly monotone increasing.
(2) There exists some constant $c \geq 0$, for which, for any minimal-perimeter animal Q on $\mathcal{L}$, we have that $|\mathcal{P}(Q)| = |\mathcal{B}(Q)| + c$ and $|\mathcal{P}(I(Q))| \leq |\mathcal{P}(Q)| + c$.
(3) If Q is a minimal-perimeter animal, then $D(Q)$ is a valid (connected) animal.

If all the above conditions hold for $\mathcal{L}$, then $\left|M_n^{\mathcal{L}}\right| = \left|M_{n+\epsilon^{\mathcal{L}}(n)}^{\mathcal{L}}\right|$. If these conditions are not satisfied for only a finite amount of sizes of animals on $\mathcal{L}$, then the claim holds for all sizes greater than some nominal size n_0. ◻

Remark. Obviously, no lattice fulfills condition (2) with $c < 0$, and only trivial lattices (e.g., the 1-dimensional lattice) fulfill it with $c = 0$.

The full proof is omitted due to lack of space. However, we detail here the main lemmata which are part of the proof, and which are interesting on their own right.

First, inflating a minimal-perimeter animal preserves this property of the animal.

▶ **Lemma 3.2.** *If Q is a minimal-perimeter animal, then $I(Q)$ is a minimal-perimeter animal as well.* ◻

Next, under the inflation operation, no two different minimal-perimeter animals are mapped into the same animal.

▶ **Lemma 3.3.** *Let Q_1, Q_2 be two different minimal-perimeter animals. Then, regardless of whether or not Q_1, Q_2 have the same size, the animals $I(Q_1)$ and $I(Q_2)$ are different as well.*

Finally, for any minimal-perimeter animal of size $n + \epsilon^{\mathcal{L}}(n)$ (that is, the size obtained by inflating a minimal-perimeter animal of size n), there exists only a single source animal of size n under the inflation map. Specifically, this source is the animal that is obtained by deflating the original animal.

▶ **Lemma 3.4.** *For any $Q \in M_{n+\epsilon^{\mathcal{L}}(n)}^{\mathcal{L}}$, we also have that $I(D(Q)) = Q$.*

Using the above lemmata, we can wrap up the proof of Theorem 3.1. In Lemma 3.2, we have shown that for any minimal-perimeter animal $Q \in M_n$, we have that $I(Q) \in M_{n+\epsilon^{\mathcal{L}}(n)}$. In addition, Lemma 3.3 states that the inflation of two different minimal-perimeter animals results in two other different minimal-perimeter animals. Combining the two lemmata, we

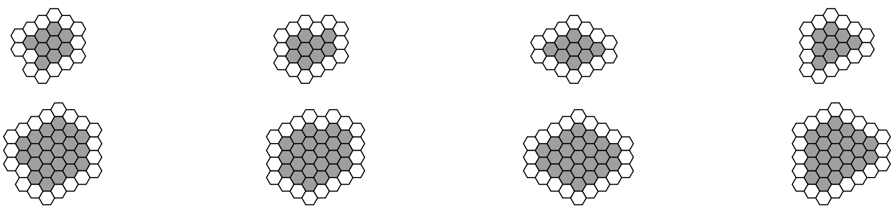


Figure 4 A demonstration of Theorem 3.1 for polyhexes. The top row contains all polyhexes in $M_9^{\mathcal{H}}$ (minimal-perimeter polyhexes of size 9) up to rotations, while the bottom row contains their inflated versions, all members (up to rotations as well) of $M_{23}^{\mathcal{H}}$.

obtain that $|M_n| \leq \left|M_{n+\epsilon^{\mathcal{L}}(n)}\right|$. On the other hand, in Lemma 3.4, we have shown that if $Q \in M_{n+\epsilon^{\mathcal{L}}(n)}$, then $I(D(Q)) = Q$, and, thus, for any animal in $M_{n+\epsilon^{\mathcal{L}}(n)}$, there is a unique source in M_n (specifically, $D(Q)$), whose inflation yields Q. Hence, $|M_n| \geq \left|M_{n+\epsilon^{\mathcal{L}}(n)}\right|$. Combining the two relations, we conclude that $|M_n| = \left|M_{n+\epsilon^{\mathcal{L}}(n)}\right|$.

We can also show that all the premises of Theorem 3.1 are fulfilled for animals on the hexagonal lattice, and, thus, inflating the set of all minimal-perimeter polyhexes of a certain size yields another set of minimal-perimeter polyhexes of another, larger, size. The proofs are omitted due to lack of space. This result is demonstrated in Figure 4.

For polyiamonds, the second condition in Theorem 3.1 does not hold, and indeed, inflating a minimal-perimeter polyiamond does not necessarily result in another minimal-perimeter polyiamond. However, if we slightly modify the triangular lattice to the one in which two cells which share only a vertex are also considered adjacent, then the theorem holds. This is not surprising since under the modified definition, the lattice is homomorphic to the hexagonal lattice. For cubical lattices in three or more dimensions, we know that the second condition does not hold. However, we do not know whether or not the bijection between sets of minimal-perimeter animals on these lattices exists.

3.2 Inflation Chains

Theorem 3.1 implies that there exist infinitely-many chains of sets of minimal-perimeter animals, each one obtained from the previous one by inflation, while the cardinalities of all sets in a single chain are identical. Obviously, there are sets of minimal-perimeter animals that are not created by inflating any other set. We call the size of animals in such sets an *inflation-chain root*. Using the definitions and proofs in the previous section, we are able to characterize which sizes are these roots. The result is stated in the following theorem, while the proof is omitted due to lack of space.

▶ **Theorem 3.5.** *Let $\mathcal{L}$ be a lattice for which the three premises of Theorem 3.1 are satisfied, and, in addition, the following condition holds.*

(4) The inflation operation preserves for an animal the property of having a maximum size for a given perimeter.

Then, if n is the minimum size for a minimal-perimeter size p, or equivalently, if there exists a perimeter size p, such that $n = \min\left\{n \in \mathbb{N} \mid \epsilon^{\mathcal{L}}(n) = p\right\}$, then n is an inflation-chain root. ◻

As mentioned above, we already provided a similar result for the restricted context of polyominoes [3]. We were not able to identify any lattice for which Theorem 3.1 holds while Theorem 3.5 does not hold, therefore, we suspect that the latter theorem can be inferred from the conditions of the former.

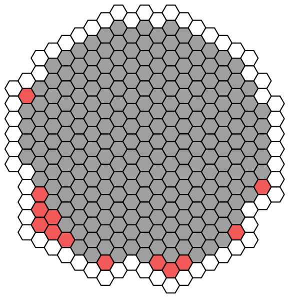


Figure 5 An example of the split of a minimal-perimeter polyhex into its body (grey) and caps (red). Note that the body is composed of 12 "lines" of cells, half of which are aligned with the main three directions (*e.g.*, the left and right edges), and the other half are toggling between two directions, thereby creating three other "lines" (*e.g.*, the top and bottom edges).

4 Counting Minimal-Perimeter Polyhexes

Following Theorem 3.5, one may ask how many minimal-perimeter animals exist for a given inflation-chain root. We describe here two efficient algorithms for counting minimal-perimeter polyhexes of a given size. In fact, it is sufficient to apply these algorithms only to the root of an inflation-chain in order to compute the number of minimal-perimeter polyhexes of the entire chain.

4.1 Bulk Counting

An interesting property of minimal-perimeter polyhexes is the following. Any minimal-perimeter polyhex can be separated into two parts: (1) A dodecagon-like polyhex, with the same perimeter as the original minimal-perimeter polyhex; and (2) Some other cells on the edges of the dodecagon, which do not change the perimeter of the dodecagon. We call these two parts the "body" and the "caps" of the polyhex. This concept is demonstrated in Figure 5.

We can use this property for designing an efficient algorithm for counting minimal-perimeter polyhexes. The algorithm considers all possible bodies with a given perimeter, and calculates the number of different possible caps which yield minimal-perimeter polyhexes. The number of caps fitting each edge is computed by using a rather simple recursive formula, which is quite similar to the formula for Motzkin paths. The time complexity of this algorithm, $O(n^6)$, is pseudo-polynomial. (That is, it is polynomial in n, which is the only input to the algorithm.)

4.2 Column Counting

A polyhex is *convex* if any sequence of its cells along any of the three main directions of the hexagonal lattice is contiguous. A sample convex polyhex is shown in Figure 6.

A better algorithm is based on the following lemma.

▶ **Lemma 4.1.** *Minimal-perimeter polyhexes are convex.* ☐

Lemma 4.1 implies a simple algorithm for counting minimal-perimeter polyhexes. Simply put, we add cells to the polyhex, one column at a time, in a manner that preserves convexity along the vertical direction, while keeping track of the current area and perimeter. This process is demonstrated in Figure 7.

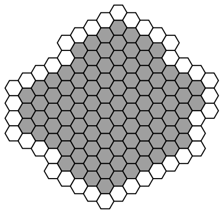


Figure 6 A sample convex polyhex.

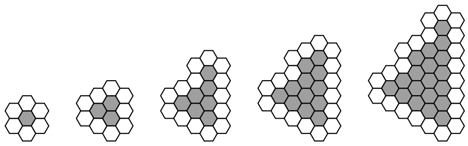


Figure 7 A demonstration of the column-counting algorithm. Each polyhex is generated by one iteration of the algorithm.

Note that in any stage of the algorithm, the possible completions of the animal depend only on the size and perimeter of the animal in the current state, and on the contents of the last two added columns. This property is exemplified in Figure 8. Using this property, we can memoize the results of the computations for each state and reuse them whenever we encounter again the same state. Using this method, we achieve, again, a pseudo-polynomial algorithm with a better running time of $O(n^4)$. The full complexity analysis of the algorithm is omitted due to lack of space.

4.3 Results

Running the column-counting algorithm overnight on a home laptop produced the results shown in Figure 9. One can clearly notice the patterns in the graph, which are very similar to those observed for polyominoes [3]. A natural question which arises from this graph is whether there exists a growth constant for the number of minimal-perimeter polyhexes (when we consider only the roots of the inflation chain). This question is relevant as well for polyominoes, and is still open.

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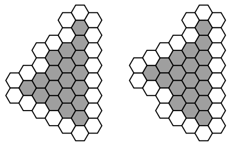

Figure 8 Two possible partial states of the algorithm (created from left to right). Since the size and perimeter, as well as the last two columns of both examples are identical, the possible completions of the polyhex into a minimal-perimeter polyhex are the same.

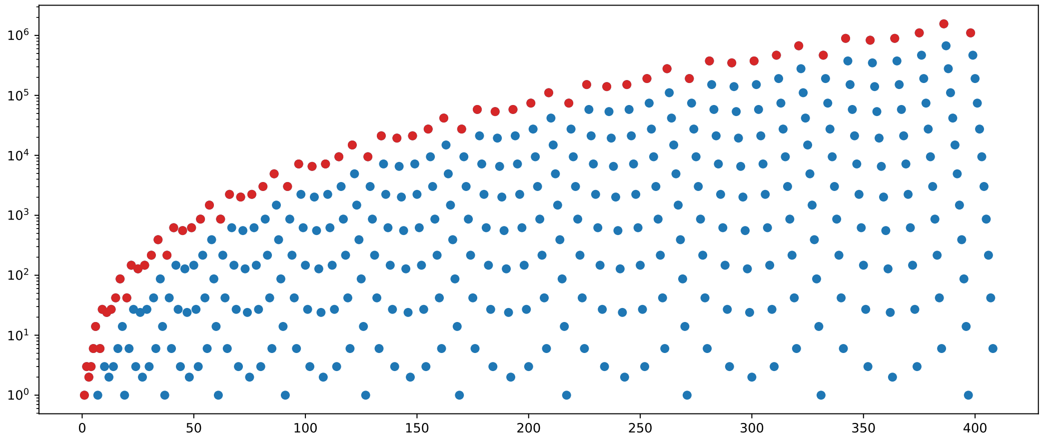

Figure 9 The number of minimal-perimeter polyhexes as a function of the size. Inflation chain roots are colored red.

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