Maximum Rectilinear Crossing Number of Uniform Hypergraphs

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Abstract

We improve the lower bound on the d-dimensional rectilinear crossing number of the complete d-uniform hypergraph having $2d$ vertices to $\Omega(2^d d)$ from $\Omega(2^d \sqrt{d})$. In [4], Anshu et al. conjectured that among all d-dimensional rectilinear drawings of a complete d-uniform hypergraph having n vertices, the number of crossing pairs of hyperedges is maximized if all its vertices are placed on the d-dimensional moment curve and proved this conjecture for $d = 3$. We prove that their conjecture is valid for $d = 4$ by using Gale transform. We establish that the maximum d-dimensional rectilinear crossing number of a complete d-partite d-uniform balanced hypergraph is $(2^{d-1}-1)\binom{n}{2}^d$, where n denotes the number of vertices in each part. We then prove that finding the maximum d-dimensional rectilinear crossing number of an arbitrary d-uniform hypergraph is NP-hard and give a randomized scheme to create a d-dimensional rectilinear drawing of a d-uniform hypergraph H producing the number of crossing pairs of hyperedges up to a constant factor of the maximum d-dimensional rectilinear crossing number of H.

1 Introduction

Let K_n^d denote the complete d-uniform hypergraph having n vertices and $\binom{n}{d}$ hyperedges. A d-uniform hypergraph is d-partite, if we can partition its vertex set into d disjoint parts such that each of the d vertices in each hyperedge belongs to a different part. This hypergraph is balanced if each part contains equal number of vertices. A balanced d-uniform d-partite hypergraph having n vertices in each part is complete if it has all n^d hyperedges and it is denoted by $K_{d\times n}^d$. A d-dimensional rectilinear drawing of a d-uniform hypergraph H in $\mathbb{R}^d$ places the vertices of H in general position (i.e., no $d+1$ points lie on a $(d-1)$-dimensional hypeplane) and the hyperedges are drawn as the convex hull of d corresponding vertices, i.e., $(d-1)$-simplices, where two hyperedges are said to cross each other if they are vertex disjoint and contain a common point in their relative interiors [6]. The d-dimensional rectilinear crossing number of H, denoted by $\overline{cr}_d(H)$, is the minimum number of crossing pairs of hyperedges among all d-dimensional rectilinear drawings of H. Dey and Pach [6] proved that H can have $O(n^{d-1})$ hyperedges if $\overline{cr}_d(H) = 0$ and Anshu et al. [4] proved that $\overline{cr}_d(K_{2d}^d) = \Omega(2^d)$ with the bound being later improved to $\Omega(2^d\sqrt{d})$ [8]. An extensive survey on rectilinear drawings of graphs (2-uniform hypergraphs) in $\mathbb{R}^2$ can be found in [12].

Inspired by [11], we define the maximum d-dimensional rectilinear crossing number of a d-uniform hypergraph H, denoted by $max\text{-}\overline{cr}_d(H)$, as the maximum number of crossing pairs of hyperedges among all d-dimensional rectilinear drawings of H. The d-dimensional moment curve γ is defined as $\gamma = \{(t, t^2, \ldots, t^d) : t \in \mathbb{R}\}$. Let $p_i = (t_i, t_i^2, \ldots, t_i^d)$ and $p_j = (t_j, t_j^2, \ldots, t_j^d)$ be two points on γ. We say that the point p_i precedes the point p_j ($p_i \prec p_j$) if $t_i < t_j$. We denote the convex hull of a point set P by $Conv(P)$. In [4], it was proved that placing all the vertices of K_{2d}^d on the d-dimensional moment curve gives a par-

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ticular d-dimensional rectilinear drawing of K^d_{2d} having c^m_d crossing pairs of hyperedges, where

$$c^m_d = \begin{cases} \binom{2d-1}{d-1} - \sum_{i=1}^{\frac{d}{2}} \binom{d}{i}\binom{d-1}{i-1} & if\ d\ is\ even \\ \binom{2d-1}{d-1} - 1 - \sum_{i=1}^{\lfloor \frac{d}{2} \rfloor} \binom{d-1}{i}\binom{d}{i} & if\ d\ is\ odd. \end{cases}$$

Note that the best-known upper bound on $\overline{cr}_d(K^d_{2d})$ is c^m_d. In [4], it was conjectured that the maximum number of crossing pairs of hyperedges in any d-dimensional rectilinear drawing of K^d_{2d} is c^m_d for each $d \geq 2$. Note that we need at least $2d$ vertices to form a crossing pair of hyperedges since they need to be vertex disjoint and each set of $2d$ vertices creates distinct crossing pairs of hyperedges. This implies that any d-dimensional rectilinear drawing of K^d_n can have at most $c^m_d \binom{n}{2d}$ crossing pairs of hyperedges if the conjecture as mentioned earlier is correct. In [4], the authors also proved that $max\text{-}\overline{cr}_3(K^3_n) = 3\binom{n}{6}$.
We summarize the main results in the following.

▶ **Theorem 1.1.** *We have $\overline{cr}_d(K^d_{2d}) = \Omega\left(2^d d\right)$ for sufficiently large d.*

▶ **Theorem 1.2.** *We have $max\text{-}\overline{cr}_4(K^4_n) = 13\binom{n}{8}$ for $n \geq 8$.*

▶ **Theorem 1.3.** *We have $max\text{-}\overline{cr}_d(K^d_{d\times n}) = (2^{d-1}-1)\binom{n}{2}^d$ for $d \geq 2$ and $n \geq 2$.*

▶ **Theorem 1.4.** *For a fixed $d \geq 3$, finding the maximum d-dimensional rectilinear crossing number of an arbitrary d-uniform hypergraph is NP-hard.*

▶ **Theorem 1.5.** *Let $H = (V, E)$ be a d-uniform hypergraph. Let F be the total number of pairs of vertex disjoint hyperedges. There exists a d-dimensional rectilinear drawing D of H such that there are at least $\tilde{c_d} \cdot F$ crossing pairs of hyperedges in D, where $\tilde{c_d}$ is a positive constant.*

1.1 Techniques used

Let $A = \langle a_1, a_2, \ldots, a_n \rangle$ be a sequence of n points in $\mathbb{R}^d$ such that their affine hull is $\mathbb{R}^d$. The Gale transform [7] of A, denoted by $D(A)$, is a sequence of n vectors $\langle g_1, g_2, \ldots, g_n \rangle$ in $\mathbb{R}^{n-d-1}$.

Let the coordinate of a_i be $(x^i_1, x^i_2, \ldots, x^i_d)$. Let us consider the following matrix $M(A)$ having rank $d+1$.

$$M(A) = \begin{bmatrix} x^1_1 & x^2_1 & \cdots & x^n_1 \\ x^1_2 & x^2_2 & \cdots & x^n_2 \\ \vdots & \vdots & \vdots & \vdots \\ x^1_d & x^2_d & \cdots & x^n_d \\ 1 & 1 & \cdots & 1 \end{bmatrix}$$

Consider a basis of the null space of $M(A)$. Let $\{(b^1_1, b^1_2, \ldots, b^1_n), (b^2_1, b^2_2, \ldots, b^2_n), \ldots, (b^{n-d-1}_1, b^{n-d-1}_2, \ldots, b^{n-d-1}_n)\}$ be a basis of this null space. The vector g_i in the sequence $D(A)$ of n vectors is $g_i = (b^1_i, b^2_i, \ldots, b^{n-d-1}_i)$.

A linear separation of vectors in $D(A)$ is a partition of the vectors into $D^+(A)$ and $D^-(A)$ by a hyperplane passing through the origin. When $|D(A)|$ is even, a linear separation is called proper if $|D^+(A)| = |D^-(A)| = |D(A)|/2$.

A vector configuration is said to be totally cyclic if the zero vector can be expressed as a strictly positive linear combination of the vectors present in the configuration. Any totally cyclic vector configuration of n vectors in $\mathbb{R}^{n-d-1}$ that spans $\mathbb{R}^{n-d-1}$ can serve as a Gale transform of some point set having n points in $\mathbb{R}^d$ after proper scaling [9, 14].

▶ **Lemma 1.6.** *[9] Every set of $n-d-1$ vectors of $D(A)$ span $\mathbb{R}^{n-d-1}$ if and only if the points in A are in general position in $\mathbb{R}^d$.*

▶ **Lemma 1.7.** *[9] Consider a tuple $(i_1, i_2, \ldots, i_k)$, where $1 \leq i_1 < i_2 < \ldots < i_k \leq n$. The convex hull of $\{a_{i_1}, a_{i_2}, \ldots, a_{i_k}\}$ crosses the convex hull of $A \setminus \{a_{i_1}, a_{i_2}, \ldots, a_{i_k}\}$ if and only if there exists a linear separation of the vectors in $D(A)$ into $\{g_{i_1}, g_{i_2}, \ldots, g_{i_k}\}$ and $D(A) \setminus \{g_{i_1}, g_{i_2}, \ldots, g_{i_k}\}$.*

Let us consider a hyperplane h that is not parallel to any vector in $D(A)$ and not passing through the origin. For each $1 \leq i \leq n$, we extend the vector $g_i \in D(A)$ either in the direction of g_i or in its opposite direction until it cuts h at the point $\overline{g_i}$. We color $\overline{g_i}$ as *red* if the projection is in the direction of g_i, and *blue* otherwise. The sequence of points $\overline{D(A)} = \langle \overline{g_1}, \overline{g_2}, \ldots, \overline{g_n} \rangle$ in $\mathbb{R}^{n-d-2}$ along with the color is called an affine Gale diagram of A.

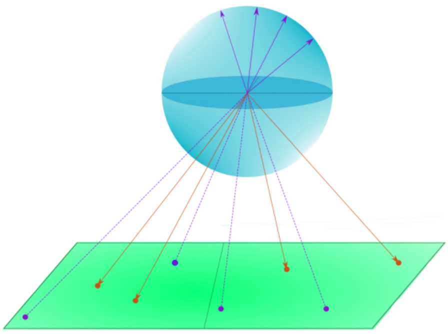

Figure 1 An affine Gale diagram of 8 points in $\mathbb{R}^4$.

Balanced $2m$-set: Let T be a set of n red and n blue points in $\mathbb{R}^2$ such that all the $2n$ points are in general position. A balanced $2m$-set ($1 \leq m \leq \lfloor n/2 \rfloor$) of T is a subset $X \subseteq T$ of size $2m$ that can be separated from the rest of the $(2n-2m)$ points by a line and X is balanced, i.e., it has an equal number of red and blue points.

We define a balanced 0-set to be a partition of T into an empty set and T. Note that there is only one balanced 0-set of a set.

▶ Observation 1. For a set A' of 8 points in general position in $\mathbb{R}^4$, there will always exist an affine Gale diagram $\overline{D(A')}$ having 4 red points and 4 blue points in $\mathbb{R}^2$ such that all the 8 points are in general position (shown in Figure 1). The total number of proper linear separations in $D(A')$ is equal to the sum of the total number of balanced 2-sets of $\overline{D(A')}$, the total number of balanced 4-sets of $\overline{D(A')}$ and 1.

We can think of order types as equivalence classes. Point configurations that have the same order type share many combinatorial and geometric properties [14]. Aichholzer et al. [1, 2] created a database which contains all order types of 8 points in general position in $\mathbb{R}^2$. We use those point sets in the proof of Theorem 1.2.

▶ **Lemma 1.8.** *[3] Let us consider d pairwise disjoint sets in $\mathbb{R}^d$, each consisting of two points, such that all $2d$ points are in general position. Then there exist 2 pairwise disjoint (d-1) simplices such that each simplex has one vertex from each set.*

We state the following lemmas which are used in the proof of Theorem 1.1.

▶ **Lemma 1.9.** *[8] Let C' be a set containing $d+4$ points in general position in $\mathbb{R}^d$. There exist $\lfloor (d+4)/2 \rfloor$ pairs of disjoint subsets of C', each of size $\lfloor (d+2)/2 \rfloor$ such that their union is C' and their convex hulls intersects.*

▶ **Lemma 1.10.** *[8] Consider a set C that contains $2d$ points in general position in $\mathbb{R}^d$. Let $C' \subset C$ be its subset such that $|C'| = d+4$. Let C'_1 and C'_2 be two disjoint subsets of C' such that $|C'_1| = c'_1, |C'_2| = c'_2, C'_1 \cup C'_2 = C'$ and $c'_1, c'_2 \geq \lfloor (d+2)/2 \rfloor$. If the $(c'_1 - 1)$-simplex formed by C'_1 and the $(c'_2 - 1)$-simplex formed by C'_2 form a crossing pair, then the $(d-1)$-simplex formed by a point set $B'_1 \supset C'_1$ and the $(d-1)$-simplex formed by a point set $B'_2 \supset C'_2$ satisfying $B'_1 \cap B'_2 = \emptyset$, $|B'_1|, |B'_2| = d$ and $B'_1 \cup B'_2 = C$ also form a crossing pair.*

2 Improved Lower Bound on $\overline{cr}_d(K^d_{2d})$

Proof of Theorem 1.1: Let $V = \{v_1, v_2, \ldots, v_{2d}\}$ denote the set of $2d$ points corresponding to the vertices of K^d_{2d} in its d-dimensional rectilinear drawing. Let E denote the set of $(d-1)$-simplices created by the corresponding hyperedges of K^d_{2d} in its particular d-dimensional rectilinear drawing. Let V' be any subset of V having $d+4$ points. Lemma 1.9 implies that there exist $\lfloor (d+4)/2 \rfloor$ pairs of subsets $\{V'_{i1}, V'_{i2}\}$ $(1 \leq i \leq \lfloor (d+4)/2 \rfloor)$ which satisfy the properties mentioned in Lemma 1.9. It follows from Lemma 1.10 that each such crossing pair of $(u_i - 1)$-simplex and $(v_i - 1)$-simplex can be extended to obtain at least $\binom{d-4}{d - \lfloor (d+2)/2 \rfloor} = \Omega\left(2^d/\sqrt{d}\right)$ crossing pairs of $(d-1)$-simplices formed by the hyperedges in E. Therefore, the total number of crossing pairs of hyperedges originated from a particular choice of V' is at least $\lfloor (d+4)/2 \rfloor \, \Omega\left(2^d/\sqrt{d}\right) = \Omega\left(2^d\sqrt{d}\right)$.

We can choose V' in $\binom{2d}{d+4} = \Theta\left(4^d/\sqrt{d}\right)$ ways. Note that if d is odd, a particular crossing pair of hyperedges can originate from at most $2\left(\binom{d}{\lfloor (d+2)/2 \rfloor}\binom{d}{\lceil (d+2)/2 \rceil + 2} + \binom{d}{\lceil (d+2)/2 \rceil}\binom{d}{\lfloor (d+2)/2 \rfloor + 2}\right) = \Theta\left(4^d/d\right)$ such subsets of $d+4$ points from V. If d is even, a particular crossing pair of hyperedges can originate from at most $2\left(\binom{d}{(d+2)/2}\binom{d}{(d+6)/2}\right) + \binom{d}{(d+4)/2}^2 = \Theta\left(4^d/d\right)$ such subsets of $d+4$ points from V. This implies that there exist at least $\left(\Omega\left(2^d\sqrt{d}\right)\Theta\left(4^d/\sqrt{d}\right)\right)/O\left(4^d/d\right) = \Omega\left(2^d d\right)$ distinct crossing pairs of hyperedges in any d-dimensional rectilinear drawing of K^d_{2d}. ◀

3 Maximum Rectilinear Crossing Number of Special Hypergraphs

Proof of Theorem 1.2: Let us consider all order types of the 8 points in general position in $\mathbb{R}^2$ [1, 2]. Let us denote the point sequence corresponding to the i^{th} ordertype with o_i. We also generate $\binom{8}{4} = 70$ possible colorings of a sequence of 8 points with 4 blue and 4 red. Each coloring is represented as an 8 bit binary string having an equal number of zeroes and ones. Let us denote the j^{th} coloring in lexicographical order by c_j. Formally, we consider the set $O_C = \{(o_i, c_j) : 1 \leq i \leq 3315 \ \& \ 1 \leq j \leq 70\}$.

Consider a 4-dimensional rectilinear drawing of K_8^4. Let us denote these vertices by $V = \{v_1, v_2, \ldots, v_8\}$. Consider an affine Gale diagram $\overline{D(V)}$ of V having 4 red and 4 blue points such that all the 8 points are in general position in $\mathbb{R}^2$. Observation 1 ensures that the number of proper linear separations of $D(V)$ is equal to the sum of the total number of balanced 2-sets of $\overline{D(V)}$, the total number of balanced 4-sets of $\overline{D(V)}$ and 1. Note that $\overline{D(V)}$ is equivalent to one of the elements of O_C but all elements of O_C need not be a Gale diagram of some 8 points in $\mathbb{R}^4$. We find the maximum value of (total number of balanced 2-sets + the total number of balanced 4-sets) over all members of O_C by analyzing each of its members by computer[1]. Observation 1 implies that the maximum number of proper linear separations of $D(V)$ is $12 + 1 = 13$. Lemma 1.7 implies that the maximum number of crossing pairs of hyperedges in any 4-dimensional rectilinear drawing of K_8^4 is 13.

Consider a 4-dimensional rectilinear drawing of K_n^4 where all the vertices are placed on the 4-dimensional moment curve. This 4-dimensional rectilinear drawing of K_n^4 contains $13\binom{n}{8}$ crossing pairs of hyperedges. ◀

▶ **Lemma 3.1.** *Consider a 4-dimensional neighborly polytope P having n vertices such that all the vertices of P are in general position in $\mathbb{R}^4$. Consider a 4-dimensional rectilinear drawing of K_n^4 such that the vertices of K_n^4 are placed as the vertices of P. The number of crossing pairs of hyperedges in this 4-dimensional rectilinear drawing of K_n^4 is $13\binom{n}{8}$.*

Remarks: The proof of Lemma 3.1 is also computer assisted. We consider the set O_C. A member of O_C is a Gale diagram of a 4-dimensional neighborly polytope having 8 vertices if and only if any 2-set and 4-set is balanced and there is no monochromatic 3-set in it. We showed that the number of crossing pairs of hyperedges in a 4-dimensional rectilinear drawing of K_8^4 where its vertices are placed as the vertices of a 4-dimensional neighborly polytope is 13. Since any set of 8 vertices of P spans a neighborly sub-polytope, the result follows.

▶ **Lemma 3.2.** *The maximum d-dimensional rectilinear crossing number of $K_{d\times 2}^d$ is $2^{d-1} - 1$.*

Proof Sketch: Lemma 1.8 implies that in any d-dimensional rectilinear drawing of $K_{d\times 2}^d$, there exists a pair of disjoint simplices such that each simplex has one vertex from each part of $K_{d\times 2}^d$. We then give a placement of the vertices of $K_{d\times 2}^d$ that achieves this bound. For each i satisfying $1 \le i \le d$, let us denote the i^{th} part of the vertex set of $K_{d\times 2}^d$ by C_i. Let $\{p_{c_i}, p'_{c_i}\}$ denote the set of 2 vertices in C_i. In this particular drawing, the vertices of $K_{d\times 2}^d$ are placed on the d-dimensional moment curve such that they satisfy the following ordering on the d-dimensional moment curve. $p_{c_1} \prec p'_{c_1} \prec p_{c_2} \prec p'_{c_2} \ldots \prec p_{c_{d-1}} \prec p'_{c_{d-1}} \prec p_{c_d} \prec p'_{c_d}$.

The only pair of hyperedges $\{A, B\}$ that does not form a crossing is the following. For $d = 4$, it is shown in Figure 2.

$A = \{p_{c_1}, p'_{c_2}, p_{c_3}, p'_{c_4}, \ldots, p_{c_{d-1}}, p'_{c_d}\}, B = \{p'_{c_1}, p_{c_2}, p'_{c_3}, p_{c_4}, \ldots, p'_{c_{d-1}}, p_{c_d}\}$. ◀

Proof Sketch of Theorem 1.3: For each i satisfying $1 \le i \le d$, let C_i be the i^{th} color class of the vertex set of $K_{d\times n}^d$. Let $\{p_1^i, p_2^i, \ldots, p_n^i\}$ be the set of n vertices in C_i. Consider the following arrangement of the vertices of $K_{d\times n}^d$ on the d-dimensional moment curve.

- Any vertex of C_i precedes any vertex of C_j if $i < j$.
- For each i satisfying $1 \le i \le d$, $p_l^i \prec p_m^i$ if $l < m$.

[1] https://github.com/ayan-iiitd/maximum-rectilinear-crossing-number-of-uniform-hypergraphs.git

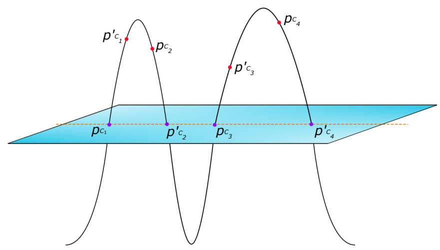

Figure 2 Non-crossing pair of hyperedges of $K^4_{4\times 2}$.

Lemma 3.2 implies that this particular d-dimensional rectilinear drawing of $K^d_{d\times n}$ contains $(2^{d-1}-1)\binom{n}{2}^d$ crossing pairs of hyperedges. ◀

4 Maximum Rectilinear Crossing Number of General Hypergraphs

MAX-E_d-Set-Splitting: Consider a collection F of subsets of a finite set X. Each set in F has cardinality d ($d \geq 2$). It is NP-hard to find a partition of X such that it maximizes the number of subsets of F which are not contained in either of the partitions [10].

Proof Sketch of Theorem 1.4: We are given a d-uniform hypergraph $H = (V, E)$ and a constant integer c'. We create a d-uniform hypergraph $\tilde{H} = (\tilde{V}, \tilde{E})$, where

$\tilde{V} = V \cup \{v'_0, v'_1, v'_2, \ldots, v'_{t(d-1)}\}$ where $t = \binom{|E|}{2} + 1$.

$\tilde{E} = \cup_i\{e_i\} \cup E$ where $e_i = \{v'_0, v'_{(i-1)(d-1)+1}, v'_{(i-1)(d-1)+2}, \ldots, v'_{(i-1)(d-1)+(d-1)}\}$ for each i satisfying $1 \leq i \leq t$.

We prove that $\tilde{H}$ has a d-dimensional rectilinear drawing D having at least tc' crossing pairs of hyperedges if and only if there exists a partition of V into two parts such that at least c' hyperedges of E contains at least one vertex from both the parts. ◀

Proof Sketch of Theorem 1.5: Let $V = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of H. Pick an uniformly random permutation of V. Place the vertices on the d-dimensional moment curve according to that order. ◀

5 Discussions and Open Problems

▶ Conjecture 1. Among all d-dimensional ($d > 4$) rectilinear drawings of K^d_n, the number of crossing pairs of hyperedges gets maximized if all the vertices of K^d_n are placed as the vertices of a neighborly d-polytope that are in general position in $\mathbb{R}^d$.

Theorem 1.5 shows that there is a randomized approximation algorithm which in expectation provides a $\tilde{c_d} = \dfrac{c^m_d}{\binom{2d-1}{d-1}}$ guarantee. It is an interesting open problem to derandomize such algorithm. For $d = 2$, Bald et al. [5] derandomized the algorithm by Verbitsky [13].

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