Augmenting Polygons with Matchings*

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Abstract

We study disjoint compatible noncrossing geometric matchings of simple polygons. That is, given a simple polygon P we want to draw a set of pairwise disjoint straight line edges with endpoints on the vertices of P such that these new edges neither cross nor contain any edge of the polygon. We prove NP-completeness of deciding whether there is such a perfect matching. For any n-vertex polygon we show that such a matching with $\leq (n-4)/8$ edges is not maximal, that is, it can be extended by another compatible matching edge. Complementing this we construct polygons with maximal matchings with $n/6$ edges. Finally we consider a related problem. We prove that it is NP-complete to decide whether a noncrossing geometric graph G admits a set of compatible noncrossing edges such that G together with these edges has minimum degree five.

1 Introduction

A geometric graph is a graph drawn in the plane with straight-line edges. Throughout this paper we additionally assume that all geometric graphs are noncrossing. Let G be a given (noncrossing) geometric graph G. We want to augment G with a geometric matching on the vertices of G such that no edges cross in the augmentation. We call such a (geometric) matching *compatible* with G. Note that our definition of a compatible matching implies that the matching is noncrossing and avoids the edges of G. Questions regarding compatible matchings were first studied by Rappaport et al. [13, 14]. Rappaport [13] proved that it is NP-hard to decide whether for a given geometric graph G there is a compatible matching M such that $G + M$ is a (spanning) cycle. Recently Akitaya et al. [3] confirmed a conjecture of Rappaport and proved that this holds even if G is a perfect matching. Note that in this case also M is necessarily a perfect matching. However, for some compatible perfect matchings M the union $G + M$ might be a collection of several disjoint cycles. There are graphs G that do not admit any compatible perfect matching, even when G is a matching. Such matchings were studied by Aichholzer et al. [1] who proved that each m-edge perfect matching G admits a compatible matching of size at least $\frac{4}{5}m$. Ishaque et al. [9] confirmed a conjecture of Aichholzer et al. [1] that any perfect matching G with an even number of edges admits a compatible perfect matching. For a geometric graph G let $d(G)$ denote the size of a largest compatible matching of G and for a family $\mathcal{F}$ of geometric graphs let $d(\mathcal{F}) = \min\{d(G) \mid G \in \mathcal{F}\}$. Aichholzer et al. [2] proved that for the families T_n and P_n of all n-vertex geometric trees, respectively n-vertex polygons, $\frac{1}{10}n \leq d(T_n) \leq \frac{1}{4}n$ and $\frac{n-3}{4} \leq d(P_n) \leq \frac{1}{3}n$ holds.

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This is an extended abstract of a presentation given at EuroCG'20. It has been made public for the benefit of the community and should be considered a preprint rather than a formally reviewed paper. Thus, this work is expected to appear eventually in more final form at a conference with formal proceedings and/or in a journal.

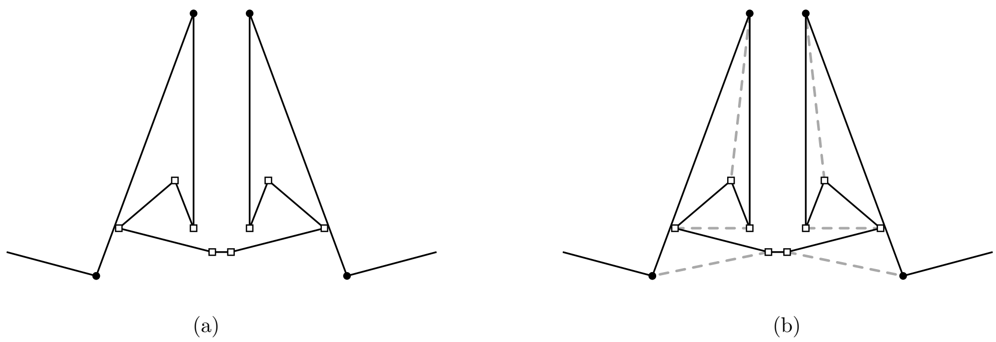

Figure 1 (a) This gadget allows for simulating a "bend" in the polygon without a vertex that needs to be matched. The construction is scaled such that the eight points marked with squares do not see any other point outside of the gadget (in particular, narrowing it horizontally). (b) A possible matching is shown.

We continue this line of research and consider the following problems. Given a geometric cycle, i.e., a polygon, we first show that it is NP-complete to decide whether the polygon admits a compatible perfect matching. Then we ask for the "worst" compatible matchings for a given polygon. That is, we search for small maximal compatible matchings.

The first studied problem can also be phrased as follows: Given a geometric cycle, can we add edges to obtain a cubic geometric graph? In the last section, we consider a related augmentation problem. Given a geometric graph, we show that it is NP-complete to decide whether the graph can be augmented to a graph of minimum degree five. The corresponding problem for the maximal vertex degree asks to add a *maximal* set of edges to the graph such that the maximal vertex degree is bounded by constant. This problem is also known to be NP-complete for maximum degree at most seven [10].

A survey of Hurtado and Tóth [8] discusses several other augmentation problems for geometric graphs. Besides the problems mentioned in that survey decreasing the diameter [5] and the continuous setting (where every point along the edges of an embedded graph is considered as a vertex) received considerable attention [4, 7].

2 Compatible perfect matchings in polygons

▶ **Theorem 2.1.** *Given a simple polygon, it is* NP*-complete to decide whether it admits a compatible perfect matching.*

Proof. The problem is obviously in NP, as a certificate one can merely provide the added edges. NP-hardness is shown by a reduction from POSITIVE PLANAR 1-IN-3-SAT. In this problem, shown to be NP-hard by Mulzer and Rote [11], we are given an instance of 3-SAT with a planar variable-clause incidence graph and no negative literals; the instance is considered satisfiable if and only if there is exactly one true variable per clause.

For a given 1-in-3-SAT formula, we take an embedding of its incidence graph and replace its elements by gadgets. We first show that finding compatible matchings for a set of disjoint simple polygons is hard and then show how to connect the individual polygons to obtain a single polygon.

Our construction relies on a gadget that restricts the possible matching edges of vertices. In particular, we introduce a polygonal chain, whose vertices need to be matched to each other. This is achieved by the *twin-peaks gadget* as shown in Fig. 1. The gadget is scaled such that the eight vertices in its interior (which are marked with squares in Fig. 1) do not

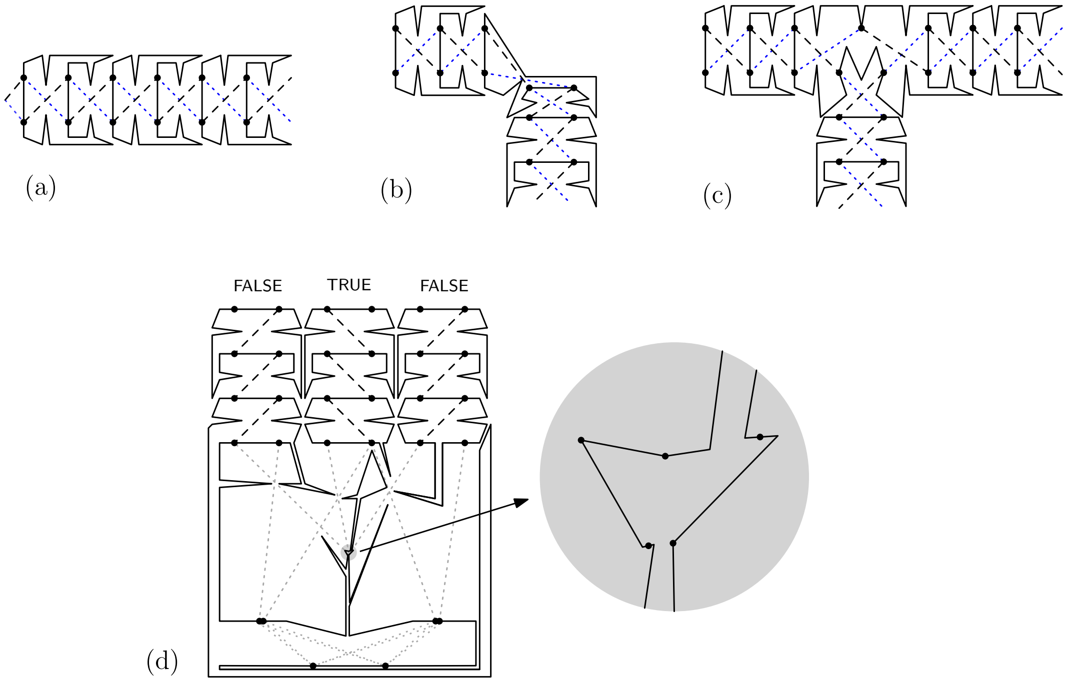

Figure 2 (a) A wire gadget and its two truth states (one in dashed, the other in dotted). (b) A bend in a wire gadget. (c) A split gadget that transports the truth setting of one wire to two other ones. This is used for representing the variables. (d) A clause gadget. The visibility among the vertices of degree two is indicated by the lighter lines. Exactly one vertex of degree two of the part in the circle must be connected to a wire above that carries the true state.

see any edges outside of the gadget. The two topmost vertices must have an edge to the vertices directly below as the vertices below do not see any other (non-adjacent) vertex. The remaining six "square" vertices do not have a geometric perfect matching on their own, so any perfect geometric matching containing them must connect them to the two bottommost vertices. Clearly, there is such a matching.

We now present the remaining gadgets (*wire*, *split*, and *clause*) for our reduction. The ideas are inspired by the reduction of Pilz [12]. In the following illustrations, vertices of degree two are drawn as a dot. Vertices in the figures without a dot represent a sufficiently small twin-peaks gadget.

The *wires* propagate the truth assignment of a variable. A wire consists of a sequence of polygons, each containing four vertices of degree two. There are only two possible global matchings for the vertices of degree two; see Fig. 2(a). A *bend* in a wire can be drawn as shown in Fig. 2(b). The truth assignment of a wire can be duplicated by a *split gadget*; see Fig. 2(c). A variable is represented by a cyclic wire with split gadgets. The *clause gadget* is illustrated in Fig. 2(d), where the wires enter from the top. The vertices there can be matched if and only if one of the vertices is connected to a wire that is in the true state. The vertices at the bottom of the gadget make sure that if there are exactly two wires in the false state, then we can add an edge to them. Hence, this set of polygons has a compatible perfect matching if and only if the initial formula was satisfiable.

It remains to "merge" the polygons of the construction to one simple polygon. The elements of the clause gadget are connected to the last polygons of the wires entering it. Observe that two neighboring polygons of a wiring gadget can be connected by adding four

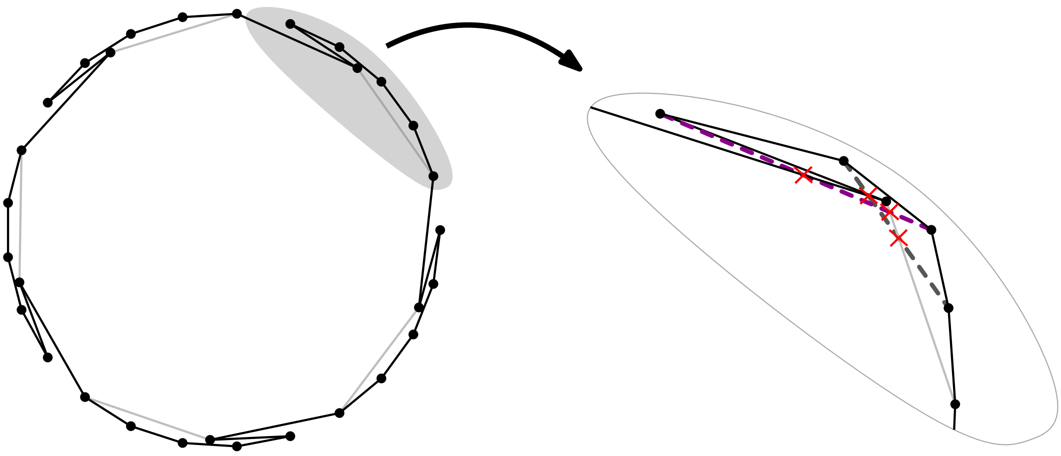

Figure 3 A polygon (black) with a maximal matching (gray) with only $\frac{n}{3}$ matched vertices. Notice that there is exactly one compatible edge between the six vertices in the gray area.

bends (using four twin-peaks gadgets). We can consider the incidence graph to be connected (otherwise the reduction splits into disjoint problems). Hence, we can always connect two disjoint polygons to one, until there is only a single polygon left. ◀

3 Compatible maximal matching in polygons

For a geometric graph G let $\mathrm{mm}(G)$ denote the size of a minimal maximal compatible matching of G and for a family $\mathcal{F}$ of geometric graphs let $\mathrm{mm}(\mathcal{F}) = \min\{\mathrm{mm}(G) \mid G \in \mathcal{F}\}$.

▶ **Theorem 3.1.** *Let P_n denote the family of all n-vertex polygons. Then $\mathrm{mm}(P_n) \geq \frac{n-4}{8}$ for all n and $\mathrm{mm}(P_n) \leq \frac{1}{6}n$ for infinitely many values of n.*

Proof. The construction in Fig. 3 shows that for infinitely many values of n there is an n-vertex polygon with a compatible maximal matching of size $\frac{n}{6}$. This shows $\mathrm{mm}(P_n) \leq \frac{n}{6}$ for infinitely many values of n.

It remains to prove the lower bound. Let P be an n-vertex polygon with a maximal compatible matching M. As the claim clearly holds for any triangle P we assume that $n \geq 4$. We shall subdivide the plane into cells with further edges as follows. First draw a rectangle enclosing P in the outer face. Then, for each reflex angle in $P + M$ (one after the other) draw a straightline edge starting at the incident vertex such that the edge cuts the reflex angle into two convex angles and stops when it hits some already drawn edge (but not a vertex). See Figure 4. The final drawing D contains at most $2 + |E(M)| + n$ bounded cells. Indeed, each edge on top of P subdivides some cell into two, where $|E(M)|$ such edges are in M and each vertex of P gives rise to at most one further such edge through a reflex angle. Moreover, all bounded cells in D are convex regions. Hence, any two unmatched vertices of P incident to a common cell F in D are connected by a side of P (within the boundary of F) as otherwise M is not maximal. This shows that each cell is incident to at most two unmatched vertices of P, since P is not a triangle. Each unmatched vertex of P is incident to exactly three bounded cells of D. Therefore, $3(n - |V(M)|) \leq 2\,(2 + |E(M)| + n)$ and hence $|E(M)| \geq (n-4)/8$. ◀

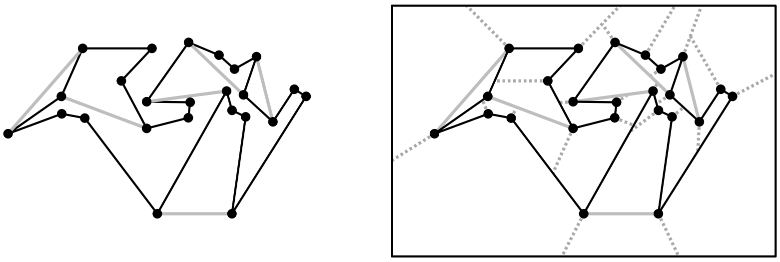

Figure 4 A polygon (black) with a maximal matching (gray) where each reflex angle is cut by a dotted edge.

4 Augmenting to Minimum Degree Five

In this section, we show that augmenting to a graph with minimum degree five is NP-complete. A related result states that it is NP-hard to decide whether a geometric graph can be augmented to a cubic graph [12].

▶ **Theorem 4.1.** *Given a geometric crossing-free graph G, it is* NP*-complete to decide whether there is a set of compatible edges E such that $G + E$ has minimum degree five.*

Proof. The problem is obviously in NP, a certificate provides the added edges. NP-hardness is shown by a reduction from MONTONE PLANAR RECTILINEAR 3-SAT. In this problem, shown to be NP-hard by de Berg and Khosravi [6], we are given an instance of monotone (meaning that each clause has only negative or only positive variables) 3-SAT with a planar incidence graph. In this graph, the variable and clause gadgets are represented by rectangles. All variable rectangles lie on a horizontal line. The clauses with positive variables lie above the variables and the ones with negative variables below. The edges connecting the clause gadgets to the variable gadgets are vertical line segments and no edges cross. See Fig. 5 (a).

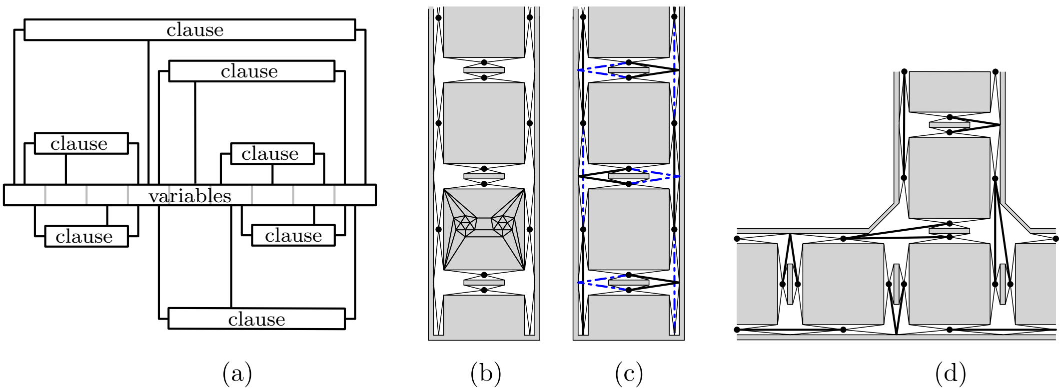

Figure 5 (a) A montone rectilinear representation of a planar 3SAT instance. (b) A wire gadget (vertices of degree four are drawn with dots) and its (c) two truth states (one in bold, the other dotted). Vertices incident to a gray region have degree at least five. This can be achieved by adding edges and vertices (of degree at least five) inside the gray regions – as shown in one gray square. (d) A split gadget with edges for the truth assignment (bold).

For a given 3-SAT formula, we take an embedding of its incident graph (as discussed) and replace its elements by gadgets. Again, we have a *wire gadget* that propagates the truth assignments; see Fig. 5(b–c). It consists of a sequence of similar subgraphs, each containing

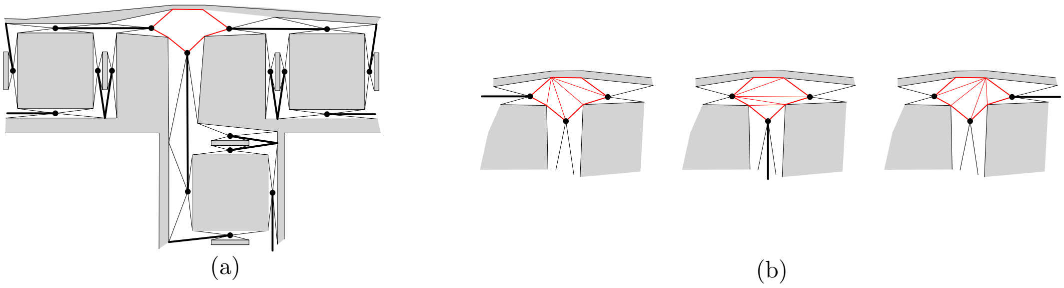

Figure 6 (a) A clause gadget, the three bold segments represent that the corresponding literals are set to true. The central 7-gon can be augmented to a subgraph of degree at least five if and only if at least one literal is true. (b) The three possibilities to augment the 7-gon if one literal is true.

four vertices of degree four (the other vertices have at least degree five). The main idea is that we need to add an edge to each of the vertices of degree four surrounding the big gray squares. But due to blocked visibilities this can only be achieved by a "windmill" pattern which has to align with the neighboring parts. Thus, we have exactly two ways to add edges in order to augment the wire to a graph with minimum degree five. The truth assignment of a wire can be duplicated by the *split gadget* shown in Fig. 5(d). The *clause gadget* is illustrated in Fig. 6(a). The wires enter from left, right and below (respectively above). The 7-gon in the middle of the clause gadget can be augmented to a subgraph with minimum degree five if and only if it is connected to at least one wire in the true state. See also Fig. 6(b). ◀

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