Empty Rainbow Triangles in k-colored Point Sets*

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Abstract

Let S be a set of n points in general position in the plane. Suppose that each point of S has been assigned one of $k \geq 3$ possible colors and that there is the same number, m, of points of each color class, so $n = km$. A triangle with vertices on S is empty if it does not contain points of S in its interior and it is rainbow if all its vertices have different colors. Let $f(k,m)$ be the minimum number of empty rainbow triangles determined by S. In this paper we show that $f(k,m) = \Theta(k^3)$. Furthermore we give a construction which does not contain an empty rainbow quadrilateral.

1 Introduction

A set of points in the plane is in *general* position if no three of its vertices are collinear. In this paper all sets of points are in general position. The well known Erdős-Szekeres theorem [15] states that for every positive integer $r > 3$ there exists a positive integer $n(r)$ such that every set of $n(r)$ or more points in the plane contains the vertices of a convex polygon of r vertices.

Let S be a set n points in the plane. A polygon with vertices on S is said to be *empty* if it does not contain a point of S in its interior. An r*-hole* of S is an empty convex r-gon spanned by points of S. In 1978, Erdős [14] asked if for every r, every sufficiently large set of points in the plane contains an r-hole. Klein [15] had already noted that every set of 5 points contains a 4-hole. Harboth [19] showed that every set of 10 points contains a 5-hole. Horton [20] constructed an arbitrarily large set of points without a 7-hole. The case for 6-holes remained open until Nicolás [25] and Gerken [18], independently showed that every sufficiently large point set contains a 6-hole.

Once the existence of a given r-hole is established, it is natural to ask what is the minimum number of r-holes in every set of n points in the plane. Katchalski and Meir [22] first considered this question for triangles. They showed that every set of n points determines $\Omega(n^2)$ empty triangles and provided an example of a point set determining $O(n^2)$ empty triangles. The lower and upper bounds on this number have been improved throughout the years [6, 13, 27, 17, 11, 3, 1]. The problem of determining the minimum number of r-holes for $r = 4, 5$ or 6 in every set of n points in the plane has also been considered in these papers.

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Colored variants of these problems where first studied by Devillers, Hurtado, Károlyi and Seara [12]. In these variants each point of S is given a color from a prescribed set. A point set is *k-colored* if every one of its points is assigned one of k available colors. We say that an r-hole on S is *monochromatic* if all its vertices are of the same color and *rainbow* if all its vertices are of different colors. Many chromatic variants on problems regarding r-holes in points sets have been studied since; see [7, 26, 2, 4, 8, 24, 5, 21, 9, 16, 28, 23]. In particular, Aichholzer, Fabila-Monroy, Flores-Peñaloza, Hackl, Huemer, and Urrutia showed that every 2-colored set of n points determines $\Omega(n^{5/4})$ empty monochromatic triangles [2]. This was later improved to $\Omega(n^{4/3})$ by Pach and Toth [26]. The best upper bound on this number is $O(n^2)$ and this is conjectured to be the right asymptotic value. If we take three colors, then there exist 3-colored point sets without a monochromatic triangle [12].

In this paper we consider the problem of counting the number of empty rainbow triangles in k-colored point sets in which there are the same number, m, of points of each color class. Let $f(k, m)$ be the minimum number empty rainbow triangles in such a point set. We give the following asymptotic tight bound for $f(k, m)$.

▶ **Theorem 1.1.** *Let $m \geq k$ be positive integers then*

$$f(k, m) = \Theta(k^3).$$

The lower bound is shown in Section 2 and the upper bound in Section 3. We point out that in contrast to the number of empty monochromatic triangles, the number of rainbow empty triangles does not necessarily grow with the number of points. Further we show that for every $k \geq 4$ there are k-colored point sets without a rainbow 4-hole.

2 Lower Bound

▶ **Theorem 2.1.** *Let $m \geq k$ be positive integers then*

$$f(k, m) \geq \frac{1}{6}k^3 - \frac{1}{2}k^2 + \frac{1}{3}k.$$

Proof. Let S be a k-colored set of points such that there are the $m \geq k$ points of each color class. Without loss of generality assume that no two points of S have the same x-coordinate. Assume that the set of colors is $\{1, \ldots, k\}$. For each $1 \leq i \leq k$, let p_i be the leftmost point of color i. Without loss of generality assume that when sorted by x-coordinate these points are $p_k, p_{k-1} \ldots, p_1$.

We now show that for $i \geq 3$ there are at least $(i^2 - 3i + 2)/2$ empty rainbow triangles having a point of color i as its rightmost point. Let $q_1 := p_i, q_2, \ldots, q_{i-2}$ be the first $i - 2$ points of color i when sorted by x-coordinate. For each $1 \leq j \leq i - 2$ do the following. Sort the points of S to the left of q_j counterclockwise by angle around q_j. Note that any two consecutive points in this order define an empty triangle with q_j as its rightmost point. Since the points $p_1, \ldots, p_{i-1}$ are to the left of q_j, there at least $i-2$ of these empty triangles such that the first point is of a color l distinct from i, and the next point is of a color distinct from l and i. Furthermore, for at least $(i - 2) - (j - 1) = i - j - 1$ of these triangles the next point is not of color i; thus they are rainbow. We have least

$$\sum_{j=1}^{i-1} i - j - 1 = \frac{i^2 - 3i + 2}{2}$$

empty rainbow triangles with a point of color i as its rightmost point.

Thus, S determines at least

$$\sum_{i=1}^{k} \frac{i^2 - 3i + 2}{2} = \frac{1}{6}k^3 - \frac{1}{2}k^2 + \frac{1}{3}k$$

empty rainbow triangles. ◀

3 Upper Bound

In this section we construct a k-colored point set which gives us an upper bound for $f(k,m)$.

3.1 The Empty Triangles of the Horton Set

In this section we define the point set introduced by Horton [20] and characterize its empty triangles. Let H be a set of n points in the plane with no two points having the same x-coordinate; sort its points by their x-coordinate so that $H := \{p_0, p_1, \ldots, p_{n-1}\}$. Let H_0 be the subset of the even-indexed points, and H_1 be the subset of the odd-indexed points. That is, $H_0 = \{p_0, p_2, \ldots\}$ and $H_1 = \{p_1, p_3, \ldots\}$. Let X and Y be two sets of points in the plane. We say that X is *high above* Y if: every line determined by two points in X is above every point in Y, and every line determined by two points in Y is below every point in X.

▶ **Definition 3.1.** H is a **Horton set** if
1. $|H| = 1$; or
2. $|H| \geq 2$; H_0 and H_1 are Horton sets; and H_1 is high above H_0.

Assume that H is a Horton set. We say that an edge $e := (p_i, p_j)$ is a *visible edge* of H if one of the following two conditions are met.
- Both i and j are even and for every even $i < l < j$, the point p_l is below the line passing through e. In this case we say that e is *visible from above.*
- Both i and j are odd and for every odd $i < l < j$, the point p_l is above the line passing through e. In this case we say that e is *visible from below.*

▶ **Lemma 3.2.** *The number of visible edges of H is less than $2n$.*

▶ **Lemma 3.3.** *Let p_i, p_j and p_l be the vertices of a triangle τ of H such that either*
- *(p_i, p_j) is an edge visible from below and $p_l \in H_0$; or*
- *(p_i, p_j) is an edge visible from above and $p_l \in H_1$.*

Then τ is empty. Moreover, every empty triangle of H with at least one vertex in each of H_0 and H_1 is of one these forms.

▶ **Corollary 3.4.** *[10] The number of empty triangles of H is at most $2n^2$.*

3.2 Blockers

Our strategy is to start with a Horton set H of k points and replace each point p_i of H with a cluster C_i of $m \geq k$ points. All of the points of C_i are of the same color and are at a distance of at most ε_1 from p_i. We choose ε_1 to be sufficiently small. Let S be the resulting set. Note that every rainbow triangle of S must have all its vertices in different clusters. Moreover, since each C_i is arbitrarily close to p_i we have the following. If τ is an empty triangle of S with vertices in clusters C_i, C_j and C_l then p_i,p_j and p_l are the vertices of an empty triangle in H. In principle, this gives up to m^3 empty triangles in S per empty triangle of H. However, we place the points within each cluster in such a way so that only very few of these triangles are actually empty.

We now define real numbers $\varepsilon_1 > \varepsilon_2 > \cdots > \varepsilon_k > 0$. Suppose that ε_r has been defined. We define ε_{r+1} small enough so that the following is satisfied for every triple of distinct indices i, j, l. Let q be a point at distance ε_r from p_l such that q is in the interior of every triangle with vertices p'_i, p'_j and p_l; where p'_i and p'_j are at a distance of at most ε_1 of p_i and p_j, respectively. Then q is in the interior of every triangle with vertices p'_i, p'_j and p'_l, where p'_l is any point at a distance of at most ε_{r+1} from p'_l. In this case we say that q blocks the triangle with vertices p'_i, p'_j and p'_l. For each point p_i of H, we place the points $q_1, \dots, q_{k-1}$ of C_i at a distance of $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{k-1}$ from p_i, respectively. We say that q_r is at *layer* r. The remaining points of C_i are placed at a distance of at most ε_k from p_i.

We now describe how these "blocker" points are placed for each point $p_i \in H$. Let $s_1, \dots, s_r$ be strings of 0's and 1's such that: $s_1 = \emptyset$; $s_{j+1} = s_j0$ or $s_{j+1} = s_j1$. Further let all H_{s_j} be Horton sets and $H_{s_1} = H$. Define recursively the sets H_{s_j0} and H_{s_j1} as Horton sets, such that $H_{s_j} = H_{s_j0} \cup H_{s_j1}$ and H_{s_j1} is high above H_{s_j0}. Let p_i be contained in every of the sets H_{s_j}. Note that $r \leq \lceil \log_2(k) \rceil$. For every $j = 1, \dots, r$ we place points q_{2j-1} and q_{2j} in layers $2j-1$ and $2j$, respectively as follows. Sort the points of $H \setminus \{p_i\}$ counterclockwise by angle around p_i

- Suppose that p_i is in H_{s_j0}. Place q_{2j-1}, just after the leftmost point of H_{s_j1} in counterclockwise order around p_i; place q_{2j}, just before the rightmost point of H_{s_j1} in counterclockwise order around p_i;
- suppose that p_i is in H_{s_j1}. Place q_{2j-1}, just before the leftmost point of H_{s_j0} in counterclockwise order around p_i; place q_{2j}, just after the rightmost point of H_{s_j0} in counterclockwise order around p_i;

For any two consecutive points of H in counterclockwise order around p_i, such that between them there is not yet a blocker, place a blocker in a new layer. Place the remaining points of C_i in any way but at a distance of at most ε_k from p_i.

Let S be the set that results from replacing each $p_i \in H$ with the cluster C_i.

▶ **Theorem 3.5.** *S determines $O(k^3)$ empty rainbow triangles.*

Proof. We classify the empty triangles of H as follows. Let τ be an empty triangle of H. Let s be the string of 0's and 1's such that the vertices of τ are contained in H_s but not in H_{s0} and H_{s1}. We say that τ is in layer $|s|$. Let p_i, p_j, p_l be the vertices of τ, such that p_j and p_l are both contained in H_{s0} or are both contained in H_{s1}. Then τ contains two blocker points at layers at most $2|s|$ in clusters C_j and C_l, respectively. Note τ also contains a blocker point in cluster C_i of layer at most $k-1$. Therefore, τ produces at most

$$4|s|^2k$$

empty rainbow triangles in S.

By Lemma 3.3, (p_j, p_l) is a visible edge of H_s. Since $|H_s| \leq \lceil k/2^{|s|} \rceil$, by Lemma 3.2 there are at most $2\lceil k/2^{|s|} \rceil \lceil k/2^{|s|+1} \rceil \leq 8\left(k^2/2^{2|s|}\right)$ empty triangles in H_s. Thus, for every $0 \leq r \leq \log_2(k)$ there at most $2^r 8\left(k^2/2^{2r}\right) = 8k^2/2^r$ empty triangles in H of layer r. Therefore, S contains at most

$$\sum_{r=0}^{\lceil \log_2(k) \rceil} \left(4r^2k\right)\left(8k^2/2^r\right) = 32k^3 \sum_{r=0}^{\lceil \log_2(k) \rceil} \frac{r^2}{2^r} = 192k^3$$

empty rainbow triangles. ◀

4 Empty rainbow 4-gons

A natural generalization is to consider empty rainbow r-gons for $r \geq 4$. Empty r-gons can also be non-convex, in contrast to r-holes, which are convex. If there does not exist an empty rainbow 4-gon, then also empty rainbow r-gons, $r \geq 5$, do not exist. We construct arbitrary large colored point sets which do not contain any empty rainbow 4-gon. Before constructing our point set we observe that the colored point set in Figure 1 does not contain any empty rainbow 4-gon. Note, that we can place arbitrary many further red points between the red points on the lines, such that the point set still does not contain an empty rainbow 4-gon. A more detailed explanation for this can be found in the full version.

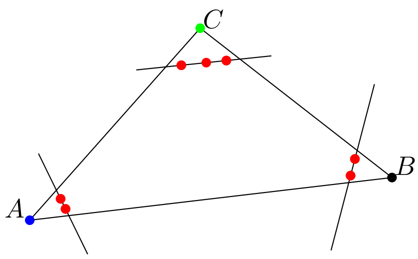

Figure 1 Colored point set without an empty rainbow 4-gon.

We use this observation to construct our point set. First we take a regular $(k-1)$-gon, P, with vertices $p_1, \ldots, p_{k-1}$; we replace every point p_i with a cluster C_i of m points with color i. Let P' be a copy of P with vertices $p'_1, \ldots, p'_{k-1}$, which is rotated by $\frac{360°}{2(k-1)} = \frac{180°}{k-1}$. So $p_1, p'_1, p_2, \ldots, p_{k-1}, p'_{k-1}$ form a regular $2(k-1)$-gon. Let ε be sufficiently small. For every $1 \leq i \leq k-1$, we place the points of C_i at a distance of at most ε from p_i. We place at least $2(k-3)$ points of color k on the line segment $p'_{i-1}p'_i$, for $1 \leq i \leq k-1$ with $p'_0 = p'_{k-1}$, so that the following holds. Let q_1, q_2 be any two consecutive points of P distinct from p_i. In the triangle with vertices p_i, q_1 and q_2 there are at least two points of $p'_{i-1}p'_i$ of color k. Further these points have at least distance ε to the lines $\overline{p_i q_1}$ and $\overline{p_i q_2}$. The construction for $k = 6$ is depicted in Figure 2. Note, that the clusters C_i are drawn enlarged.

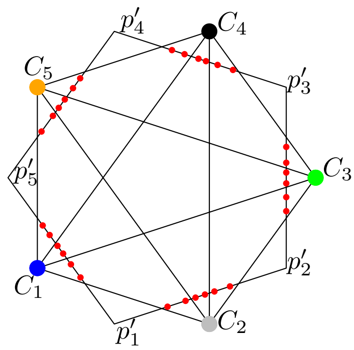

Figure 2 Construction for a 6-colored point set without empty rainbow 4-gons.

▶ **Theorem 4.1.** *All clusters C_i, $1 \leq i \leq k-1$, along with the points with color k describe a k-colored point set without an empty rainbow 4-gon.*

A detailed proof can be found in the full version of the paper. Note that in this construction we have that $m \geq 2k^2 - 8k + 6$. Further the points placed on the line segments $p'_{i-1}p'_i$

can be moved slightly such that the point set still does not contain an empty rainbow quadrilateral but that the points are in general position.

5 Open Problems

We finish the paper with two open problems.

For our results we relied heavily on the fact that $m \geq k$; it would be interesting to obtain sharp bounds of $f(m,k)$ when $m < k$.

▶ Problem 1. Compute $f(m,k)$ for $m < k$.

We constructed a k-colored point set with the same number of points in each color class and without (convex or non-convex) empty rainbow 4-gon. This point set contains many empty monochromatic 4-gons. This leads us to the following question.

▶ Problem 2. Does every sufficiently large k-colored ($k \geq 4$) point set with the same number of points in each color class contain an empty rainbow 4-gon or an empty monochromatic 4-gon?

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