On the Number of Delaunay Triangles occurring in all Contiguous Subsequences

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Abstract

Given an ordered sequence of points $P = \{p_1, p_2, \ldots p_n\}$, we are interested in the number of different Delaunay triangles occurring when considering the Delaunay triangulations of all contiguous subsequences within P. While clearly point sets and orderings with $\Theta(n^2)$ Delaunay triangles exist, we prove that for an arbitrary point set in random order, the expected number of Delaunay triangles is $\Theta(n \log n)$.

1 Introduction

Given an ordered sequence of points $P = \{p_1, p_2, \ldots, p_n\}$, we consider for $1 \leq i < j \leq n$ the contiguous subsequences $P_{i,j} := \{p_i, p_{i+1}, \ldots, p_j\}$ and the set of Delaunay triangles $T_{i,j}$ that appear in the Delaunay triangulation $DT(P_{i,j})$ of the respective subset. We are interested in the size of the set $T := \bigcup_{i<j} T_{i,j}$ of distinct Delaunay triangles occurring over all contiguous subsequences.

There are sequences of points where $|T| = \Theta(n^2)$, e.g., see Figure 1. Here, points are ordered as shown in the Figure (note that the collinearity can easily be perturbed away). For $j > \frac{n}{2}$, any point p_j will be connected to all points $\{p_1, ..., p_{\frac{n}{2}}\}$ in $T_{1,j}$, so any such $T_{1,j}$ contains $\Theta(n)$ Delaunay triangles which are not contained in any $T_{1,j'}$ with $j' < j$, hence $|T| = \Theta(n^2)$. Note, though, that for this argument we only used linearly many contiguous subsequences. It is conceivable that the quadratically many contiguous subsequences create even a superquadratic number of distinct Delaunay triangles. We will show, though, that for an arbitrary point set in *random order*, $E[|T|] = \Theta(n \log n)$, and $|T| = O(n^2)$ for any order.

Applications and Motivation

Subcomplexes of the Delaunay triangulation have proven to be very useful for representing the shape of objects from a discrete sample in many contexts, see for example α-shapes [3],

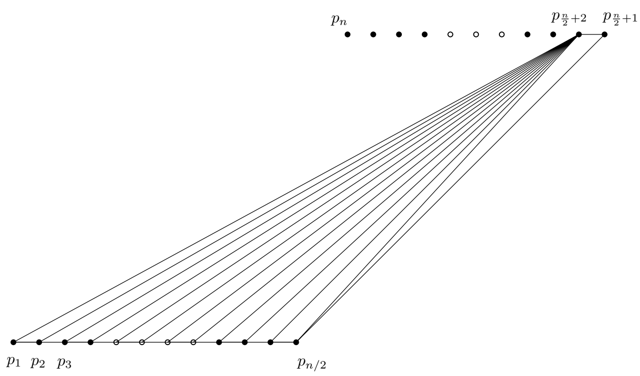

Figure 1 Example for a sequence of points with $|T| = \Theta(n^2)$, as in [4]

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the β-skeleton [5], or the crust [1]. If the samples are acquired over time, a subcomplex of the Delaunay triangulation of the samples within a contiguous time interval might allow for interesting insights into the data, see for example [2], where the authors use α-shapes to visualize the regions of storm event data within the United States between 1991 and 2000.

While in real world application scenarios, samples do not occur in truly random order, it has also been observed in [2] that the potentially huge size of T seems more like a pathological setting. Our result provides some sort of explanation for this observation. It also suggests the possibility of precomputing all Delaunay triangles occurring in all contiguous subsequences and indexing them with respect to time and possibly some other parameter (e.g., the α value in case of α-shapes, or the β values in case of the β-skeleton) for faster retrieval.

2 Counting Delaunay Edges and Triangles

Our proof will proceed in two steps. We first show that the expected number of Delaunay edges created when considering all contiguous subsequences is $\Theta(n \log n)$ for a uniformly random ordering of an arbitrary point set. Then we show that, for an arbitrary order, there is a linear dependence between the number of created Delaunay triangles and Delaunay edges.

As usual, we assume non-degeneracy of P, i.e., absence of four co-circular or three co-linear points. Also let us define the set of edges used in triangles of T as:

$$E_T := \{e \mid \exists\, t \in T : e \text{ edge of } t\}$$

We consider an arbitrary point set ordered uniformly at random and bound the expected size of E_T, that is, the number of edges that appear in at least one of the Delaunay triangulations $DT(P_{i,j})$. The following Lemma is a simple observation that helps to focus on a smaller subsequence when considering a potential Delaunay edge $\{p_i, p_j\}$.

▶ **Lemma 2.1.** *Any edge $e = \{p_i, p_j\} \in E_T$ (w.l.o.g. $i < j$) appears in $DT(P_{i,j})$.*

Proof. There exists some triangle $t \in T$ which uses e, so for suitable $a \leq i, b \geq j$, e appears in $DT(P_{a,b})$, i.e., there exists a disk with p_i, p_j on its boundary and its interior free of points from $P_{a,b}$. As $P_{i,j} \subseteq P_{a,b}$ this disk is also free of points from $P_{i,j}$, hence $e \in DT(P_{i,j})$. ◀

Essentially, Lemma 2.1 states that we only need to consider the minimal contiguous subsequence containing p_i and p_j to argue about the probability of an edge $\{p_i, p_j\}$ being present in E_T. Let us now bound the probability that an edge $e = \{p_i, p_j\}$, with $j > i + 1$, appears as Delaunay edge in some $DT(P_{i,j})$.

▶ **Lemma 2.2.** *For a potential edge $e = \{p_i, p_j\}$, $j > i+1$, we have $Pr[e \in DT(P_{i,j})] < \frac{6}{j-i}$.*

Proof. Observe that when considering the point set $P_{i,j}$, clearly $DT(P_{i,j})$ will be the same regardless of how the points in $P_{i,j}$ are ordered. All points in $P_{i,j}$ are equally likely to be p_i, or p_j. $DT(P_{i,j})$ is a planar graph with $j - i + 1 > 2$ nodes, and hence per Euler's formula contains at most $3(j - i + 1) - 6$ edges. $Pr[e \in DT(P_{i,j})]$ is thus bounded by the probability of two randomly chosen nodes in a graph with $j - i + 1$ nodes and at most $3(j - i + 1) - 6$ edges to be connected with an edge. By randomly choosing two nodes, we randomly choose one edge amongst all possible $\binom{j-i+1}{2}$ edges. The probability of that edge to be one of the $\leq 3(j - i + 1) - 6$ edges of $DT(P_{i,j})$ is $< \frac{6}{j-i}$. ◀

As due to Lemma 2.1 we have that $Pr[e \in E_T] = Pr[e \in DT(P_{i,j})])$, we can continue to bound the expected size of E_T.

▶ **Lemma 2.3.** *The expected size of E_T is $\Theta(n \log n)$.*

Proof. For the lower bound of $\Omega(n \log n)$ consider first for point p_1 the nearest neighbor in $P_{2,i}$ as i grows from 2 to n. It is well known that for random order of P, the nearest neighbor changes $\Theta(\log n)$ times in expectation. It is also well-known that the nearest neighbor graph of a point set is a subgraph of its Delaunay triangulation. Hence p_1 in expectation is involved in the creation of $\Omega(\log n)$ distinct Delaunay edges. The same argument can be applied to all other points, hence we get a $\Omega(n \log n)$ lower bound.

By linearity of expectation we can simply sum over all potential $\binom{n}{2}$ edges to obtain the upper bound on the expected size of E_T. We split the set of potential edges between those with neighboring nodes in the ordering (like p_i and p_{i+1}) and the remaining ones. The former always exist, but there are only linearly many of them, for the latter we use Lemma 2.2 to bound the probability of existence.

$$
\begin{aligned}
E[|E_T|] &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} Pr[\{p_i, p_j\} \in E_T] \\
&= \sum_{i=1}^{n-1} \left[Pr[\{p_i, p_{i+1}\} \in E_T] + \sum_{j=i+2}^{n} Pr[\{p_i, p_j\} \in DT(P_{i,j})] \right] \\
&\leq \sum_{i=1}^{n-1} \left[1 + \sum_{j=i+2}^{n} \frac{6}{j-i} \right] = (n-1) + 6 \sum_{i=1}^{n-1} \sum_{j=2}^{n-i} \frac{1}{j} \\
&\leq (n-1) + 6 \sum_{i=1}^{n-1} H_n = O(n \log n)
\end{aligned}
$$

◀

Note that in general, many of these edges are used by several Delaunay triangles, some may even be used by $\Theta(n)$ different Delaunay triangles, so it is not immediately obvious that the overall number of Delaunay edges linearly bounds the overall number of Delaunay triangles. Yet, the following Lemma shows why this is the case.

▶ **Lemma 2.4.** *$|T| \in \Theta(|E_T|)$.*

Proof. Consider some Delaunay triangle $t = p_a p_b p_c \in T$ (w.l.o.g. $a < b < c$). Due to Lemma 2.1, we have $t \in T_{a,c}$. Apart from t, there can exist at most one other triangle $t' \in T_{a,c}$ which uses the edge $\{p_a, p_c\}$. This way, we can charge every Delaunay triangle of T to some Delaunay edge of E_T, charging at most 2 Delaunay triangles to any Delaunay edge. So the overall number of Delaunay triangles is at most twice the overall number of Delaunay edges, hence $|T| \in O(|E_T|)$. The lower bound is obvious. ◀

As a corollary, Lemma 2.4 implies that $O(n^2)$ is also an upper bound for $|T|$ for arbitrary orderings of n points as there are only $O(n^2)$ possible edges.

▶ **Corollary 2.5.** *$|T| \in O(n^2)$ for arbitrary point sets and orderings.*

Finally we can state our main theorem.

▶ **Theorem 2.6.** *The expected number of different Delaunay triangles occurring in all contiguous subsequences of a (uniformly) randomly ordered point set of size n is $\Theta(n \log n)$.*

Proof. Follows from Lemmas 2.3 and 2.4. ◀

Note that our analysis relies on two main properties, namely, uniqueness of the triangulation of a point set, and that edges cannot disappear when removing points (except their endpoints, of course). It might be interesting to investigate other triangulations fulfilling this property.

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