Flips in higher order Delaunay triangulations*

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Abstract

We study the flip graph of higher order Delaunay triangulations. A triangulation of a set S of n points in the plane is order-k Delaunay if the circumcircle of every triangle encloses at most k points of S in its interior. The *flip graph* of S has one vertex for each possible triangulation of S, and an edge connects two vertices when the two corresponding triangulations can be transformed into each other by a *flip* (i.e., exchanging the diagonal of a convex quadrilateral by the other one). We show that, even though the order-k flip graph might be disconnected for $k \geq 3$, any order-k triangulation can be transformed into some different order-k triangulation by at most $k-1$ flips, such that the intermediate triangulations are of order at most $2k-2$, in the following settings: (1) for any $k \geq 0$ when S is in convex position, and (2) for any point set S when $k \leq 5$.

1 Introduction

Given a set S of points in the plane, a *triangulation* of S is a decomposition of the convex hull of S into triangles, such that each triangle has its three vertices in S. Despite the well-known fact that a point set S in the plane can have many different triangulations [7], most of the time the *Delaunay triangulation* is used, since its triangles are considered "well-shaped". A Delaunay triangulation of S, denoted $DT(S)$, is a triangulation where each triangle satisfies the *empty circle property*: the circumcircle of each triangle does not enclose other points of S (for a survey, see [3, 8]). When no four points of S are co-circular, $DT(S)$ is unique. However, when used to model terrains as a 3D surface, the Delaunay triangulation of points on the surface ignores the elevation information, potentially resulting in poor terrain models where important terrain features, such as valley or ridge lines, are ignored [6, 10]. This motivated Gudmundsson et al. [9] to propose *higher order Delaunay triangulations*. A triangulation T of S is an *order-k Delaunay triangulation*—or, simply, *order-k*—if the circumcircle [1] of each triangle of T contains at most k points of S in its interior. As soon as $k > 0$, one obtains a class of triangulations that, intuition suggests, still has well-shaped triangles for small values of k, but with potentially many triangulations to choose from.

A fundamental operation to locally modify triangulations is the *edge flip*. It consists of removing the edge shared by two triangles that form a convex quadrilateral, and inserting the other diagonal of the quadrilateral. A flip transforms a triangulation T into another

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[1] We refer to the *interior of a circumcircle.* as the interior of the disk defined by such circle

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triangulation T' that differs by exactly one edge and two triangles. The flip operation leads naturally to the definition of the *flip graph* of S. Each triangulation of S is represented by a vertex in this graph, and two vertices are adjacent if their corresponding triangulations differ by exactly one flip. The importance of flips in triangulations comes from the fact that the flip graph is connected [11]. In fact, it is known that $O(n^2)$ flips are enough to convert any triangulation of S into $DT(S)$ [12, 15]. In general, computing the distance in the flip graph between two given triangulations is a difficult problem [13, 14]. This has drawn considerable attention to the study of certain subgraphs of the flip graph, which define the flip graph of certain classes of triangulations. We refer to [5] for a survey.

Almost nothing is known about the flip graph of order-k triangulations, except that it is connected only for $k \leq 2$ [1]. Similarly, Abellanas et al. [2] showed that the flip graph of triangulations of point sets with edges of order k [2] is connected for $k \leq 1$, but can be disconnected for $k \geq 2$. On the other hand, they proved that for point sets in convex position the flip graph is connected provided one allows intermediary triangulations of order at most $3k$ [2]. However, their proof implies an exponential bound on the diameter of the flip graph.

In this paper we present several structural properties of the flip graph of order-k triangulations. For points in convex position, we show that for any $k > 2$ there exists point sets in convex position for which the flip graph is not connected. However, we prove that for any order-k triangulation there exists another order-k triangulation at distance at most $k-1$ in the flip graph of order-$(2k-2)$ triangulations. This shows that, while order-k triangulations are not connected via the flip operation, they become connected if a slightly larger neighborhood is considered. For points in generic (non-convex) position, we prove the same result for up to $k \leq 5$, although we conjecture that it holds for all k. Our results have implications on the flip distance between order-k triangulations, as well as on their efficient algorithmic enumeration.

Due to space limitations, most proofs are omitted.

2 Preliminaries and general observations

Let S be a point set in the plane. The point set S is in *general position* if no three points of S lie on a line and no four points of S lie on a circle. For the rest of the paper we assume that the point set is in general position. Let T be a triangulation of S, and let $\triangle uyv$ be a triangle in T with vertices u,y,v. We will denote by $\bigcirc uyv$ the disk defined by the enclosed area of the *circumcircle* of $\triangle uyv$ (i.e., the unique circle going through u, y, and v), unless stated otherwise. Triangle $\triangle uyv$ is an *order-k triangle* if $\bigcirc uyv$ contains at most k points of S in its interior. A triangulation T where all triangles are order-k is an *order-k (Delaunay) triangulation*. Thus, T is not of order-k if $\bigcirc uxv$ contains more than k points in its interior for some $\triangle uxv$ in T. The set of all order-k triangulations of S will be denoted $\mathcal{T}_k(S)$.

Let $e = uv$ be an edge in T. Edge e is *flippable* if e is incident to two triangles $\triangle uxv$ and $\triangle uyv$ of T and $uxvy$ is a convex quadrilateral. The flippable edge e is *illegal* if $\bigcirc uxv$ contains y in its interior. Note that this happens if and only if $\bigcirc uyv$ contains x in its interior. Otherwise, it is called *legal*. The *angle-vector* $\alpha(T)$ of a triangulation T is the vector whose components are the angles of each triangle in T ordered in increasing order. Let $T' \neq T$ be another triangulation of S. We say that $\alpha(T) > \alpha(T')$ if $\alpha(T)$ is greater than $\alpha(T')$ in lexicographical order. It is well-known that if T' is the triangulation obtained by flipping an

[2] An edge uv is an *order-k edge* of S if there exists a disk that contains u and v on its boundary and at most k points of S in its interior. Observe that the edges of an order-k triangle are order-k edges.

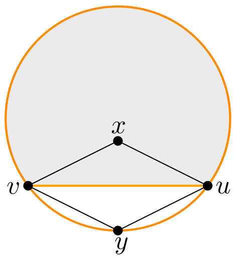


Figure 1 An illegal edge uv, with region $\LEFTcircle_y^{uv}$ in gray.

illegal edge of T, then $\alpha(T') > \alpha(T)$ [8]. Moreover, since $DT(S)$ maximizes the minimum angle, it follows that $DT(S)$ is the only triangulation where all the edges are legal [15]. This also implies that the flip graph is connected, since any triangulation can be transformed into the Delaunay triangulation. We will denote the flip graph of $\mathcal{T}_k(S)$ by $G(\mathcal{T}_k(S))$. Abe and Okamoto [1] observed that $G(\mathcal{T}_2(S))$ is connected as a consequence of the following lemma.

▶ **Lemma 2.1** (Abe and Okamoto [1]). *Let T be a triangulation of S, let uv be an illegal edge of T, and let $\triangle uvx$ and $\triangle uyv$ be the triangles incident to uv in T. If $\triangle uxv$ is of order k, and $\triangle uyv$ is of order l, then triangles $\triangle uxy$ and $\triangle xyv$ have orders k' and l', respectively, for k', l' with $k' + l' \leq k + l - 2$.*

When we refer to points in a certain region, we refer to points of S in that region.

For a triangle $\triangle uyv$, we will use $\LEFTcircle_y^{uv}$ to denote the open region bounded by edge uv and the arc of circle $\partial \bigcirc uyv$ that does not contain y. See Fig 1. Consider a triangulation T of order $k \geq 3$, and an illegal edge uv adjacent to triangles $\triangle uxv$ and $\triangle uyv$. Consider the triangulation T' resulting from flipping uv in T. Using that the interior of $\bigcirc uxv$ and $\bigcirc uyv$ contain at most k points each (including x and y), and the fact that the interior of $\bigcirc uxy$ contains at least $k+1$ points, a rather simple counting argument implies the following.

▶ **Observation 2.2.** *If $\bigcirc uxy$ contains more than $k \geq 3$ points in its interior then each region $\LEFTcircle_y^{ux} \setminus \bigcirc uxv$ and $\LEFTcircle_x^{uy} \setminus \bigcirc uyv$ contains at least 2 points.*

3 Points in convex position

First, we show that $G(\mathcal{T}_k(S))$ may not be connected and $k-1$ flips may be necessary to transform a triangulation in $\mathcal{T}_k(S)$ into some different order-k triangulation for $k > 2$.

▶ **Theorem 3.1.** *For any $k > 2$ there is a set S_k of $2k+2$ points in convex position such that $G(\mathcal{T}_k(S_k))$ is not connected. Moreover, there is a triangulation T_k in $\mathcal{T}_k(S_k)$ that is at least $k-1$ flips away from any other triangulation in $\mathcal{T}_k(S_k)$.*

Proof sketch. Set S_k is constructed as follows, see Fig. 2.a. Start with a horizontal segment uv and add points $S' = p_1, \ldots, p_k$ above it, and points $S'' = q_1, \ldots, q_k$ below it, such that q_i is the reflection of p_i with respect to the line through uv. Point p_1 is placed close enough to uv, and each next point p_{i+1} for $i = 1, \ldots, k-1$ is: (1) inside $\bigcirc uq_ip_i$, (2) below the line through $p_{i-1}p_i$, (3) above the line through uv, and (4) outside $\bigcirc up_{i-1}p_i$ (we set $p_0 = v$). The set S_k is $\{u, v\} \cup S' \cup S''$. Triangulation T_k of S_k is formed by all the triangles $\triangle up_ip_{i+1}$ and $\triangle uq_iq_{i+1}$ (where $p_0 = q_0 = v$). It turns out that any $\bigcirc up_ip_{i+1}$ (resp., $\bigcirc uq_iq_{i+1}$) contains exactly the k points of S'' (resp., of S') in its interior and no other point of S_k. Thus T_k is in $\mathcal{T}_k(S_k)$. We observe that any triangulation of S_k containing edge p_ip_t with $k \geq i > t+1$

is not of order k (the case for $q_i q_t$ is symmetric). Consider $T' \neq T_k$ in $\mathcal{T}_k(S_k)$. Thus, each edge in $T' \setminus T_k$ must have one endpoint in S' and one in S''. Thus, edge uv has to be flipped in order to transform T_k to T'. Triangle $\triangle(uq_1p_1)$ is of order $2k-2$. The second part of the statement follows from the observation that for any i, j with $k \geq i > 0$ and $k \geq j > 0$, the triangle up_iq_j is of order $2k-i-j$. Thus, $k-1$ flips are needed to get T', since some $\triangle up_iq_j$ has to be in T' with $i + j \geq k$. Otherwise uv is in T', a contradiction. ◀

Let S be a point set in convex position. Let T be an order-k triangulation of S. We say that T is *minimal* if flipping any illegal edge in T results in a triangulation that is not of order k. Let uv be a diagonal in T and let $\triangle uxv$ and $\triangle uyv$ be the triangles adjacent to it. Since S is in convex position, the diagonal uv in T partitions the triangulation T into two sub-triangulations that only share edge uv. Let T^x_{uv} (respectively, T^y_{uv}) denote the sub-triangulation that contains triangle $\triangle uxv$ (respectively, $\triangle uyv$). See Fig. 2.b.

We show that any order-k triangulation different from DT can be transformed into some other order-k triangulation by performing at most $k-1$ flips of illegal edges such that all the intermediate triangulations are of order $2k-2$.

▶ **Theorem 3.2.** *Let S be a point set in convex position and let $k \geq 2$. Let $T \neq DT(S)$ be in $\mathcal{T}_k(S)$. Then, there exists a triangulation T' in $\mathcal{T}_k(S)$ at flip distance at most $k-1$ in $G(\mathcal{T}_{2k-2}(S))$ from T such that $\alpha(T') > \alpha(T)$.*

Proof sketch. For $k = 2$ the theorem follows trivially, since $G(\mathcal{T}_2(S))$ is connected. Thus, we assume $k \geq 3$. Note that if T is not minimal, then there exists an illegal edge e such that the resulting triangulation T' after flipping e is a triangulation of order k with the property that $\alpha(T') > \alpha(T)$. Thus, we assume that T is a minimal triangulation. We observe that there must exist an illegal edge ac adjacent to triangle $\triangle abc$ such that T^b_{ac} consists of only legal edges: Since T is not an order-0 triangulation, there is an illegal edge in T. Note that if a triangle has two edges in the convex hull of S, its third edge must be legal in T, otherwise T would not be minimal. Since triangulations of polygons have at least two such triangles (often called *ears*), then there exists an edge ac in $\triangle abc$ such that T^b_{ac} consists of legal edges.

Let uv be an illegal edge in $\triangle uxv$ such that T^x_{uv} consists of legal edges. Let $\triangle uyv$ be the other triangle in T adjacent to uv. Consider the triangulation $T_1 = (T \setminus \{uv\}) \cup \{xy\}$. Note that $\alpha(T_1) > \alpha(T)$. Since T is minimal, T_1 is not an order-k triangulation. Thus, the only triangles that cannot be of order k in T_1 are the new triangles $\triangle uxy$ and $\triangle xyv$. Without loss of generality assume that $\triangle uxy$ is not of order k. By Lemma 2.1, it follows that $\triangle uxy$ is the only triangle that is not of order k. In addition, $\bigcirc uxy$ contains at most $2k-2$ points in its interior. By Obs. 2.2 it follows that ◐^{ux}_{y} has at least 2 points and at most $k-1$.

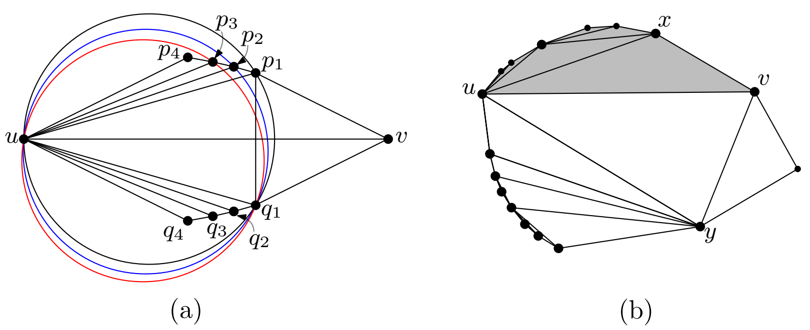
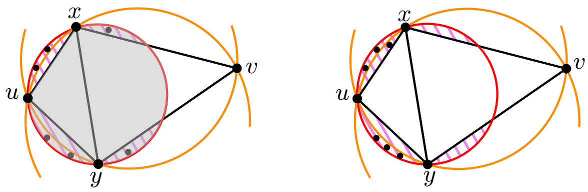


Figure 2 (a) An order-k triangulation at distance at least $k-1$ from other order-k triangulations (k=4). (b) The gray area corresponds to T^x_{uv}.



(a) (b)

Figure 3 In both cases, $k = 5$. (a) There are four points in the gray region $\bigcirc uxy \setminus \ominus^{ux}_{y}$. (b) There are exactly three points in $\ominus^{ux}_{y}$ and exactly three points in $\ominus^{uy}_{x}$.

By induction on the number of points in $\ominus^{ux}_{y}$ we show that T_1 can be transformed into an order-k triangulation T' by flipping at most $k-2$ illegal edges. Hence, $\alpha(T') > \alpha(T_1) > \alpha(T)$. Moreover, the triangulations from T_1 to T' are of order $2k-2$, implying our result. ◀

If T is an order-k triangulation, then the edges of T have order k. There are $O(kn)$ edges of order k [2, 9]. It follows from Theorem 3.2 that T can be transformed into $DT(S)$ by a sequence of at most $O((2k-2)n) = O(kn)$ flips, since all the flipped edges are illegal and of order $2k-2$, which implies that no order-$(2k-2)$ edge is flipped twice.

4 General point sets

Consider a general point set S. Using a much more involved approach than the one for convex point sets, we can obtain an analogous result for triangulations of order $k = 3, 4$ or 5.

In order to prove this result, we consider a triangulation $T \in \mathcal{T}_k(S)$. If there is an illegal edge in T whose flip results in a new order-k triangulation, we are done. If not, $k > 2$ and T is a minimal triangulation. Thus, for any illegal edge uv in T, flipping uv produces a new and unique triangle $\triangle uxy$ that is not of order k. Since $k = 3, 4, 5$, we notice that there are only two cases to consider for the number of points in each region of $\bigcirc uxy$. For $k = 3, 4$, since $\triangle uxy$ is not of order k, by Obs. 2.2 it follows that one of the regions $\bigcirc uxy \setminus \ominus^{ux}_{y}$ and $\bigcirc uxy \setminus \ominus^{uy}_{x}$ contains $k-1$ points in its interior. For $k = 5$, if none of the regions $\bigcirc uxy \setminus \ominus^{ux}_{y}$ and $\bigcirc uxy \setminus \ominus^{uy}_{x}$ contains $k-1$ points of S in its interior then, by Obs. 2.2 and the fact that T is of order k, it follows that each region of $\ominus^{ux}_{y}$ and $\ominus^{uy}_{x}$ contains 3 points of S. See Fig. 3. Finally, for each of these two cases we show that the statement holds using the fact that when flipping certain illegal edges, the circumcircles of the new triangles lie in the union $\bigcirc uxv \cup \bigcirc uyv$.

In addition, since there are $O(kn)$ order-k edges (see [2, 9]), it follows that for $k \leq 5$, any order-k triangulation can be transformed into $DT(S)$ by a sequence of at most $O(kn)$ triangulations of order $2k-2$. Moreover, using the reverse search framework of Avis and Fukuda [4] and the pre-processing method of order-k triangulations given by Silveira and van Kreveld [16], all order-k triangulations can be enumerated in polynomial expected time per triangulation.

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