Efficiently stabbing convex polygons and variants of the Hadwiger-Debrunner (p, q)-theorem

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Abstract

Hadwiger and Debrunner showed that for families of convex sets in $\mathbb{R}^d$ with the property that among any p of them some q have a common point, the whole family can be stabbed with $p-q+1$ points if $p \geq q \geq d+1$ and $(d-1)p < d(q-1)$. This generalizes a classical result by Helly. We show how such a stabbing set can be computed for n convex polygons of constant size in the plane in $\mathcal{O}((p-q+1)n^{4/3}\log^{2+\epsilon}(n)+p^3)$ expected time. For convex polyhedra in $\mathbb{R}^3$, the method yields an algorithm running in $\mathcal{O}((p-q+1)n^{13/5+\epsilon}+p^4)$ expected time. We also show that analogous results of the Hadwiger and Debrunner (p, q)-theorem hold in other settings, such as convex sets in $\mathbb{R}^d \times \mathbb{Z}^k$ or abstract convex geometries.

1 Introduction

A classical result in convex geometry by Helly [10] states that if a collection of convex sets in $\mathbb{R}^d$ is such that any $d+1$ sets have a common intersection, then all sets do. In 1957, Hadwiger and Debrunner [9] considered a generalization of this setting. Let $\mathcal{F}$ be a collection of sets in $\mathbb{R}^d$ and let $p \geq q \geq d+1$ be integers. We say that $\mathcal{F}$ has the (p, q)-property if $|\mathcal{F}| \geq p$ and for every choice of p sets in $\mathcal{F}$ there exist q among them which have a common intersection. We further say that a set of points S stabs $\mathcal{F}$ if every set in $\mathcal{F}$ contains at least one point from S. Then the following holds:

▶ **Theorem 1.1** (Hadwiger and Debrunner). *Let $d \geq 1$ be an integer. Let p and q be integers such that $p \geq q \geq d+1$ and $(d-1)p < d(q-1)$, and let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$. Suppose that $\mathcal{F}$ has the (p, q)-property. Then $\mathcal{F}$ can be stabbed with $p-q+1$ points in $\mathbb{R}^d$.*

In 1992 Alon and Kleitman [3] proved that for any $p \geq q \geq d+1$, there exists a finite bound on the maximum number of points needed to stab a collection of convex sets with the (p, q)-property. However, the known bounds are probably far from being tight. There is a lot of work in this more general setting, both improving the bounds (e.g. [13]) as well as adapting to generalizations of convex sets (e.g. [12, 16]), and it is an interesting open problem to study algorithmic questions connected to these results. However, in this work we will focus on the setting of Theorem 1.1, where the bound is tight, and show how such a stabbing set can be efficiently computed for convex polytopes in dimensions 2 and 3. To make the presentation simpler, we focus on polytopes of constant size, although the techniques used can be partially adapted to work for polytopes of arbitrary size.

Helly's theorem has also been generalized to many other settings. In general, we say that a set system has *Helly number* h if the following holds: if any h sets in the set system have a common intersection, then the whole set system does. Helly numbers have been shown to exist for many set systems, such as convex sets in $\mathbb{R}^d \times \mathbb{Z}^k$ [4, 11] or abstract convex

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geometries (see [8] or Chapter III of [14]), which include subtrees of trees and ideals of posets. We will show that under some weak conditions, the existence of a Helly number implies a tight Hadwiger-Debrunner type result.

All of the skipped proofs and details can be found in the full version of this paper [7]

2 Stabbing convex polytopes

2.1 A proof of the Hadwiger-Debrunner (p,q)-theorem

We will first consider a proof of Theorem 1.1, taken from [15], which will naturally lead to an algorithm for finding stabbing points. The proof makes use of a lemma which can also be found in [15]. For a non-empty compact set S, let $lexmin(C)$ denote its lexicographical minimum point. Then we have the following:

▶ **Lemma 2.1** ([15]). *Let $\mathcal{F}$ be family of at least $d+1$ convex compact sets in $\mathbb{R}^d$, such that $I := \bigcap \mathcal{F}$ is non-empty. Let $x := lexmin(I)$. Then, there exist a subfamily $\mathcal{H} \subset \mathcal{F}$ of size d such that $x = lexmin(\bigcap \mathcal{H})$.*

We now sketch the idea of the proof of the Hadwiger-Debrunner theorem.

Proof idea of the Hadwiger-Debrunner theorem. Call a pair of integers (p, q) *admissible* if $p \geq q \geq d+1$ and $(d-1)p < d(q-1)$. Let (p, q) be an admissible pair, and let $\mathcal{F}$ be a family of compact convex sets in $\mathbb{R}^d$ with the (p, q)-property. Construct a point $x^*(\mathcal{F})$ defined as the lexicographically maximum point among all lexicographically minimum points in the intersection of d sets in $\mathcal{F}$. We choose it as one of our stabbing points. Now, remove all the sets stabbed by this point. It can be shown that the remaining sets either satisfy the $(p-d, q-d+1)$-property, where $(p-d, q-d+1)$ is admissible, or it consists of $p-q+k$ sets where some $k+1$ of them have a common intersection. In the first case, we can continue inductively, in the second case we can trivially stab the remaining sets using $p-q$ points. ◀

This proof naturally leads to an algorithm. In the following, we will assume that the convex sets are n polytopes of constant size. Similar ideas still work for general polytopes. Computing $x^*(\mathcal{F})$ can be done in $\mathcal{O}(n^d)$ time by computing all d-wise intersections, and it needs to be computed at most $p-q+1$ times. For the case where there are only $p-q+k$ sets remaining, it can be deduced from the proof the $p-q$ remaining stabbing points can be computed in $\mathcal{O}(p^{d+1})$ time. Thus, in the plane, we get a total runtime of $\mathcal{O}((p-q+1)n^2+p^3)$.

If p (and thus q), is small compared to n, the first term is the bottleneck in the computation time. In the following, we will show how to improve the runtime of this first part. The second term can be improved to $\mathcal{O}(p^2 \log p)$ by adapting the Bentley-Ottmann sweep line algorithm [5]. It is an interesting open problem whether further improvements are possible.

2.2 A more efficient algorithm for the planar case

We will now present an algorithm running in subquadratic time with respect to n. In this whole section, the collection $\mathcal{F}$ consists of n constant-size compact convex polygons in the plane and has the (p, q)-property, for some admissible pair (p, q).

In [6], Chan discovered a simple but remarkably powerful technique to reduce many optimization problems to a corresponding decision problem, with no blow-up in expected runtime. He proves the following lemma (stated in a slightly more general form here):

▶ **Lemma 2.2.** *Let $\alpha < 1$ and r be fixed constants. Suppose $f : \mathcal{P} \to \mathcal{Q}$ is a function that maps inputs to values in a totally ordered set (where elements can be compared in constant time), with the following properties:*

(1) For any input $P \in \mathcal{P}$ of constant size, $f(P)$ can be computed in constant time.

(2) For any input $P \in \mathcal{P}$ of size n and any $t \in \mathcal{Q}$, we can decide $f(P) \leq t$ in time $T(n)$.

(3) For any input $P \in \mathcal{P}$ of size n, we can construct inputs $P_1, \ldots, P_r \in \mathcal{P}$ each of size at most $\lceil \alpha n \rceil$, in time no more than $T(n)$, such that $f(P) = max\{f(P_1), \ldots, f(P_r)\}$.

Then for any input $P \in \mathcal{P}$ of size n, we can compute $f(P)$ in $\mathcal{O}(T(n))$ expected time, assuming that $T(n)/n^\epsilon$ is monotone increasing for some $\epsilon > 0$.

We can apply this technique to the computation of x^*. Here, each $P \in \mathcal{P}$ is a set of polygons, $\mathcal{Q}$ is the plane with lexicographical order, and f is x^* (which we define as $(-\infty, -\infty)$ if the intersection of the considered polygons is empty). For the sake of simplicity, we will assume that all points defined as the lexicographical minimum in the intersection of a pair of sets have different x-coordinates. We make the following observations:

1. For $\mathcal{F}$ a constant number of polygons, $x^*(\mathcal{F})$ can be computed in constant time by computing the lexicographical minimum of all pairs in $\mathcal{F}$. This verifies property (1).
2. For any $\mathcal{F}$ of size n, we can partition it into 3 disjoint subcollections S_1, S_2, S_3 of size between $\lfloor n/3 \rfloor$ and $\lceil n/3 \rceil$ each. Then, let $\mathcal{F}_1 := S_2 \cup S_3$, $\mathcal{F}_2 := S_1 \cup S_3$ and $\mathcal{F}_3 := S_1 \cup S_2$. Every set $\mathcal{F}_i$ is of size $|\mathcal{F}_i| \leq \lceil n \cdot 2/3 \rceil$. Moreover, every pair of sets of $\mathcal{F}$ appears in one $\mathcal{F}_i$. Thus, $x^*(\mathcal{F})$ is the lexicographical maximum of $\{x^*(\mathcal{F}_1), x^*(\mathcal{F}_2), x^*(\mathcal{F}_3)\}$. These collections can be constructed in $\mathcal{O}(n)$ time. This verifies property (3), assuming that $T(n) \in \Omega(n)$ (which it will be).

Thus, in order to apply Chan's framework, it remains to decide $x^*(\mathcal{F}) \leq_{lex} t$ for any t in subquadratic time. We can rephrase this as deciding whether there exist two intersecting sets in $\mathcal{F}$ whose intersection lies entirely to the right of a vertical line ℓ with x-coordinate t. We can make some simple observations to discard some of the sets in $\mathcal{F}$ in $\mathcal{O}(np)$ time:

- All sets which lie entirely to the left of ℓ can be safely ignored.
- If there are at least p sets lying entirely to the right of l, then by the (p, q)-property some $q > 2$ of them intersect and we can already answer the question in the affirmative.
- If there are fewer than p sets lying entirely to the right of l, then one can test in $\mathcal{O}(pn)$ whether the intersection of any of those with any other set in $\mathcal{F}$ lies entirely to the right of ℓ.

We are then left with answering the following question, which we define as the *Right Intersection Problem.*

▶ **Problem 2.3** (Right Intersection Problem). *Given n compact convex polygons of constant size and a vertical line ℓ intersecting all polygons, decide if there exist two polygons whose intersection is non-empty and lies strictly to the right of ℓ.*

Solving this problem trivially in quadratic time and applying Lemma 2.2 would result in no improvement in the runtime. However, we can solve it in subquadratic time.

▶ **Proposition 2.4.** *The Right Intersection Problem can be decided in $O(n^{4/3} \log^{2+\epsilon}(n))$ time, for any constant $\epsilon > 0$.*

Proof. It was shown in [1] that counting the number of pairwise intersections between n convex polygons of constant size can be done in $\mathcal{O}(n^{4/3}\log^{2+\epsilon}(n))$ time.

For any instance of the Right Intersection Problem, we can cut all polygons along ℓ and discard the parts lying on the left of ℓ in linear time.

The answer to the original instance is positive if and only if there are two polygons which have a non-empty intersection but do not intersect on ℓ. This can be decided by counting the number of pairwise intersecting polygons in $\mathcal{O}(n^{4/3}\log^{2+\epsilon}(n))$ time, counting the number of pairwise intersecting polygons on ℓ (this can be done in $\mathcal{O}(n\log(n))$ time), and then comparing these numbers. They differ if and only if some pair of polygons intersect exclusively to the right of ℓ. ◀

We can thus use Lemma 2.2 to get the following:

▶ **Proposition 2.5.** *Let (p,q) be an admissible pair for $d=2$ and let $\mathcal{F}$ be a family of n constant size compact convex polygons in the plane with the (p,q)-property. Then we can compute a set of at most $p-q+1$ points stabbing $\mathcal{F}$ in expected time*

$$\mathcal{O}((p-q+1)n^{4/3}\log^{2+\epsilon}(n)+p^3).$$

A similar method (using the method found in [1] for counting intersections of 3D convex polyhedra) can be applied for n convex polyhedra in $\mathbb{R}^3$, with a runtime of $\mathcal{O}((p-q+1)n^{13/5+\epsilon}+p^4)$.

3 Other Hadwiger-Debrunner type results

By taking a close look at our proof for the Hadwiger-Debrunner (p,q)-theorem, we can observe that we made use of relatively few properties of compact convex sets. These properties are (i) closure under intersection, (ii) existence of a lexicographically minimum point, (iii) Helly's theorem as well as (iv) the fact that the set of all points lexicographically smaller than some point y is convex (this last property doesn't appear explicitly but is needed in the proof of Lemma 2.1). We define *Ordered-Helly systems* as set systems with analogue properties:

▶ **Definition 3.1** (Ordered-Helly system)**.** An Ordered-Helly system $\mathfrak{S}$ is a tuple $(\mathcal{B},\mathcal{C},\mathcal{D},h,\preceq)$ consisting of a *base set* $\mathcal{B}$ with a total order $\preceq$, a set $\mathcal{C}\subset\mathcal{P}(\mathcal{B})$, whose members are called *convex sets*, a set $\mathcal{D}\subset\mathcal{C}$, whose members are called *compact sets*, and an integer $h\geq 2$, called the *Helly-number* of $\mathfrak{S}$, with the following properties:

1. $\mathcal{D}$ is closed under intersections (Intersection closure);
2. For all non-empty $S\in\mathcal{D}$, there exists $x\in S$ which is minimal with respect to $\preceq$ (Attainable minimum);
3. If $\mathcal{F}\subset\mathcal{C}$ is a finite family of sets in $\mathcal{C}$ where any h members of $\mathcal{F}$ have non-empty common intersection, then all of $\mathcal{F}$ has non-empty intersection (Helly property);
4. For all $t\in\mathcal{B}$, we have $\{x\in\mathcal{B}\mid x\preceq t \text{ and } x\neq t\}\in\mathcal{C}$ (Convex order).

We can thus carry out an analogue proof and derive (p,q)-theorems in these systems. A similar algorithm to the previous case can also be used to stab such a system, assuming we have access to a few oracles. Let h be the Helly number of an Ordered-Helly system. We say that a pair of integers (p,q) is h-admissible if $p\geq q\geq h$ and $(h-2)p<(h-1)(q-1)$. We then get an analogue of Lemma 2.1 as well as the following:

▶ **Theorem 3.2.** *Let $\mathfrak{S}$ be an Ordered-Helly system. Let (p,q) be an h-admissible pair and let $\mathcal{F}$ be a family of non-empty sets in $\mathcal{D}$ with the (p,q)-property. Then $\mathcal{F}$ can be stabbed with $p-q+1$ elements of $\mathcal{B}$.*

It should be mentioned that the existence of a Helly number alone is not enough to show such a result, see [2] for an example of a set system with Helly number 2 but no general (p, q)-theorem.

It can be shown that both convex sets in $\mathbb{R}^d \times \mathbb{Z}^k$ as well as abstract convex geometries are Ordered-Helly sytems, so we immediately get (p, q)-theorems for all of them. In the following, we give a non-exhaustive list of results that can be obtained this way:

▶ **Corollary 3.3** (Mixed-Integer Hadwiger-Debrunner)**.**
Let $d, k \geq 0$. Let (p, q) be a $((d+1)2^k)$-admissible pair. Let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d \times \mathbb{Z}^k$ with the (p, q)-property. Then $\mathcal{F}$ can be stabbed with $p - q + 1$ points in $\mathbb{R}^d \times \mathbb{Z}^k$.

▶ **Corollary 3.4.** *Let T be a tree and let $\mathcal{F}$ be a collection of subtrees of T (represented as sets of vertices). Let (p, q) be a 2-admissible pair. Let $F \subset \mathcal{N}$ be a family of non-empty subtrees of T with the (p, q)-property. Then $\mathcal{F}$ can be stabbed with $p - q + 1$ vertices.*

For a finite poset $(E, \leq)$, we say that a set $S \subset E$ is an ideal of E if for all $x \in S$ and all $y \in E$, $y \leq w \Rightarrow y \in S$. Let $h(E)$ be the maximum length of an antichain in E.

▶ **Corollary 3.5.** *Let $(E, \leq)$ be a finite poset. Let (p, q) be an $h(E)$-admissible pair and let $\mathcal{F}$ be a family of non-empty ideals of E with the (p, q)-property. Then $\mathcal{F}$ can be stabbed with $p - q + 1$ elements of E.*

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